New approach to curved projective superspace

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We present a new formulation of curved projective superspace. The $4D\ \mathcal{N}=2$ supermanifold $\mathcal{M}^{4|8}$ (four bosonic and eight Grassmann coordinates) is extended by an auxiliary SU(2) manifold, which involves introducing a vielbein and related connections on the full $\mathcal{M}^{7|8}=\mathcal{M}^{4|8}\times \mathrm{SU}(2)$. Constraints are chosen so that it is always possible to return to the central basis where the auxiliary SU(2) manifold largely decouples from the curved manifold $\mathcal{M}^{4|8}$ describing $4D\ \mathcal{N}=2$ conformal supergravity. We introduce the relevant projective superspace action principle in the analytic subspace of $\mathcal{M}^{7|8}$ and construct its component reduction in terms of a five-form $\mathcal{J}$ living on $\mathcal{M}^4\times\mathcal{C}$, with $\mathcal{C}$ a contour in SU(2). This approach is inspired by and generalizes the original approach, which can be identified with a complexified version of the central gauge of the formulation presented here.

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I. INTRODUCTION

It is well-known that theories of eight supercharges in various dimensions possess natural on shell representations (such as the hypermultiplet) that do not admit off shell representations with a finite number of auxiliary fields—at least, not without a central charge. In fact, a no-go theorem guarantees that the most general charged hypermultiplet cannot be lifted to a finite off shell representation (see e.g. [1] for a clear discussion with references). Both harmonic and projective superspace solve this problem in the same way: the hypermultiplet is lifted to an off shell multiplet by introducing an *infinite* number of auxiliary fields in a controlled way. For harmonic superspace [1,2], these auxiliary fields correspond to Fourier modes on an auxiliary S^2 manifold, and the hypermultiplet is associated with a globally defined function on S^2. For projective superspace [3], the auxiliary fields appear as components of a Taylor (or Laurent) expansion in a coordinate ζ parametrizing the space $\mathbb{C}P^1$. (For recent reviews, see [4] and [5].) As a result, both superspaces actually allow the direct construction of the most general off shell actions involving hypermultiplets. Of equal importance is the way in which both superspaces allow superfield gauge prepotentials for Yang-Mills theories,[1] which are necessary for performing quantum calculations in a manifestly supersymmetric way.

These two approaches are not actually too dissimilar and make use of the superspace introduced by Rosly [13] [Hartwell and Howe have also discussed the so-called $(\mathcal{N},p,q)$ superspaces [14,15], which provide generalizations to higher $\mathcal{N}$.] Proposed relations between harmonic and projective superspaces have also been discussed in [16] and [17,18]. Our concern here will mainly be with 4D $\mathcal{N}=2$ projective superspace.

The incorporation of a curved supermanifold into projective superspace, a necessary step for the description of general supergravity-matter systems, was explicitly accomplished first in five dimensions in a series of papers by Kuzenko and Tartaglino-Mazzucchelli [19]. It was subsequently extended to dimensions 2 through 6 by various collaborations involving Kuzenko, Linch, Lindström, Roček, and Tartaglino-Mazzucchelli [20–23].[2] (Because we are interested here in 4D $\mathcal{N}=2$ supersymmetry, we will make frequent reference to the four-dimensional references [20], but many important features were already present in [19].) The formulation of curved projective superspace presented in these works we will refer to as *conventional projective superspace*.

A key ingredient of the conventional approach is to understand the role of superconformal projective multiplets of weight n, which are the natural objects of interest in projective superspace [28,29] (see [5] for a pedagogical discussion). In curved space, such superfields $\mathcal{Q}^{(n)}(z,v^i)$ are holomorphic in v^i on some open domain of $\mathbb{C}^{2*}\equiv \mathbb{C}^2\backslash\{0\}$, homogeneous in v^i of degree n, $\mathcal{Q}^{(n)}(z,cv^i)=c^n\mathcal{Q}^{(n)}(z,v)$ and transform under the superconformal gauge transformations as

$$\delta\mathcal{Q}^{(n)}=\xi^A\mathcal{D}_A\mathcal{Q}^{(n)}+n\Lambda_{\mathrm{D}}\mathcal{Q}^{(n)}-\lambda^i{}_j v^j\frac{\partial}{\partial v^i}\mathcal{Q}^{(n)}, \quad (1.1)$$

where the covariant derivatives $\mathcal{D}_A$ are built from the supervielbein and other connections of some curved

[1]The early work in harmonic superspace [6] (see also the monograph [1] for references) stimulated many manifestly supersymmetric calculations in $\mathcal{N}=2$ super Yang-Mills theories. Projective supergraphs and their applications have been discussed in [7–12].

[2]Corresponding constructions of harmonic superspace in other dimensions, which preceded the projective constructions, can be found in [24–27].

supermanifold $\mathcal{M}^{4|8}$, with $\Lambda_{\rm D}$ and $\lambda^i{}_j$, respectively, the dilatation and $\mathrm{SU}(2)_R$ gauge parameters. The $\mathrm{SU}(2)_R$ transformation can be rewritten

$$\delta_\lambda \mathcal{Q}^{(n)} = -\lambda^{++} D^{--} \mathcal{Q}^{(n)} + n\lambda^0 \mathcal{Q}^{(n)}, \quad \lambda^0 = \lambda^{ij} \frac{v_i u_j}{(v,u)},$$
$$D^{--} = \frac{u^i}{(v,u)} \frac{\partial}{\partial v^i}, \quad \lambda^{++} = \lambda^{ij} v_i v_j, \quad (v,u) = v^j u_j. \tag{1.2}$$

The parameter u_i appearing in (1.2) is an arbitrary coordinate, required only to obey $(v,u) \neq 0$ in the region of interest. Given this prescription, it is consistent to impose the covariant analyticity constraint[3]

$$v_i \mathcal{D}_\alpha{}^i \mathcal{Q}^{(n)} = v_i \bar{\mathcal{D}}_{\dot\alpha}{}^i \mathcal{Q}^{(n)} = 0. \tag{1.3}$$

This implies that $\mathcal{Q}^{(n)}$ depends on only half the Grassmann coordinates of superspace, in much the same way as chiral multiplets in $\mathcal{N} = 1$ superspace depend (essentially) on θ and not $\bar\theta$.

Once the means to minimally couple supergravity is understood, the curved extension of many flat space results becomes possible. This is done by generalizing the natural action principle of flat projective superspace [3,29,32]

$$S = -\frac{1}{2\pi} \oint_{\mathcal{C}} v_i \mathrm{d} v^i \int \mathrm{d}^4 x \mathrm{d}^4 \theta^+ \mathcal{L}^{++},$$
$$\theta^{\alpha +} = \theta^{\alpha i} v_i, \qquad \bar\theta^{\dot\alpha +} = \bar\theta^{\dot\alpha i} v_i, \tag{1.4}$$

where $\mathcal{L}^{++}$ is a weight-two projective multiplet Lagrangian, and $\mathcal{C}$ is some contour in $\mathbb{C}P^1$. A full description of the action requires both elements as different contours can lead to different actions. The component form can be written

$$S = -\frac{1}{2\pi} \oint_{\mathcal{C}} v_i \mathrm{d} v^i \int \mathrm{d}^4 x \mathcal{L}^{--},$$
$$\mathcal{L}^{--} = \frac{1}{16} \frac{u_i u_j u_k u_l}{(v,u)^4} D^{ij} \bar D^{kl} \mathcal{L}^{++}, \tag{1.5}$$

in terms of an additional coordinate u_i; however, the result is actually independent of u_i, except for the requirement that $(v,u) \neq 0$ along the contour $\mathcal{C}$. The extension to the curved case was given in [20] as

$$S = -\frac{1}{2\pi} \oint_{\mathcal{C}} v_i \mathrm{d} v^i \int \mathrm{d}^4 x e \mathcal{L}^{--},$$
$$\mathcal{L}^{--} = \frac{1}{16} \frac{u_i u_j u_k u_l}{(v,u)^4} \mathcal{D}^{ij} \bar{\mathcal{D}}^{kl} \mathcal{L}^{++} + \cdots \tag{1.6}$$

An additional requirement of constant u_i turned out to be useful to impose. The elided terms in the above expression for $\mathcal{L}^{--}$ were determined by requiring independence under small shifts of the constant u_i. Large classes of actions can then be constructed directly from (1.6) by choosing $\mathcal{L}^{++}$ to be built out of fundamental arctic, antarctic, vector and tensor multiplets: the resulting actions include general supergravity-matter systems [20]. The coupling to conformal supergravity naturally occurs automatically because of the super-Weyl invariance of the action [20].

There are some curious features about this formulation. First, as noted in [20], the coordinates v^i are effectively invariant under SU(2) transformations. Second, the manifold is $\mathcal{M}^{4|8} \times \mathbb{C}P^1$ but the action and constraints are clearly formulated in a *central gauge* (or *central basis* in the language of [1]) where $\mathcal{M}^{4|8}$ and $\mathbb{C}P^1$ are largely decoupled. One is not permitted to make $\mathbb{C}P^1$-dependent Lorentz transformations (for example) or arbitrary diffeomorphisms on $\mathbb{C}P^1$. Finally, an auxiliary coordinate u_i must be introduced to evaluate the action, subject only to the condition that $(v,u) \neq 0$ along $\mathcal{C}$. (Such a constant u_i exists for any contour.) In the original flat superspace approach of [3,32], the coordinate u_i could actually be chosen to vary along the contour; in the curved superspace approach, it was chosen constant for convenience.

In this paper, we will shed some light on these features by presenting a modified version of curved projective superspace where we emphasize similarities with the harmonic superspace approach [1]. The main idea will be to introduce a supermanifold $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$, that admits gauge transformations and diffeomorphisms involving both the coordinates z^M of $\mathcal{M}^{4|8}$ and the coordinates $v^{i\pm}$ of SU (2), placing them on an equal footing.[4] Because our fields will always be chosen to depend only on $\mathbb{C}P^1 \cong \mathrm{SU}(2)/\mathrm{U}(1)$, the supermanifold will effectively be $\mathcal{M}^{4|8} \times \mathbb{C}P^1$. We will assume that, as in harmonic superspace, there exists a central basis (or central gauge) where $\mathcal{M}^{4|8}$ and $\mathbb{C}P^1$ largely decouple.

We will find that the coordinates $v^{i\pm}$ indeed transform under SU(2) diffeomorphisms; however, upon restriction to a central gauge they can be interpreted as inert. This in turn explains the two curious features mentioned above. In the new framework, the role of the coordinate u_i will be played by the complex conjugate v_i^- of v^{i+}, so that $v^{i+} v_i^- = 1$. The conventional formulation of projective superspace will arise after a complexification of $v^{i+} \to v^i$ and

[3]Such superfields $\mathcal{Q}^{(n)}$ with these properties can be understood as generalizations of complex $\mathcal{O}(n)$ superfields $\mathcal{G}^{(n)} = v_{i_1} \cdots v_{i_n} \mathcal{G}^{i_1 \cdots i_n}(z)$ whose components $\mathcal{G}^{i_1 \cdots i_n}$ transform as symmetric tensors of SU(2), with the constraint $\mathcal{D}_\alpha^{(j} \mathcal{G}^{i_1 \cdots i_n)} = \bar{\mathcal{D}}_{\dot\alpha}^{(j} \mathcal{G}^{i_1 \cdots i_n)} = 0$ [30,31].

[4]A similar approach was sketched by Hartwell and Howe [15].

$v_i^- \to u_i/(v,u)$, which is always possible provided (v,u) is nonzero along the contour $\mathcal{C}$ of interest.

Although a full discussion would be beyond the scope of this paper (but see the conclusion for a few additional comments), there is a deep relationship between the harmonic superspace approach to supergravity and the projective superspace approach that we are describing here. Both (as usually formulated) involve the same supermanifold $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$ [with the SU(2) factor effectively $\mathbb{C}P^1$], and we will emphasize the similarity in later sections by employing the same language (e.g. harmonic coordinates $v^{i\pm}$ and harmonic derivatives $D^{\pm\pm}$ and D^0) for the auxiliary SU(2) manifold. What will differ will be the fields employed and the action principle. The advantage of emphasizing the common aspects of the two approaches will be that the projective approach we present obviously admits diffeomorphisms on the auxiliary manifold just as in the analytic basis of harmonic superspace. This leads to some important conceptual advantages, which we will discuss in the conclusion.

This paper is organized as follows. In Sec. II, we review the properties of the SU(2) manifold that will augment the usual supermanifold $\mathcal{M}^{4|8}$. Many of the important features of the full superspace will already be apparent when considering just the SU(2) manifold itself. Section III presents the structure of the supermanifold $\mathcal{M}^{4|8}\times \mathrm{SU}(2)$, upon which projective superspace can be placed. In Sec. IV, we present three action principles on $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$ involving, respectively, integration over all, half, or 3/4 of the Grassmann coordinates. The most important of these is the analytic superspace action involving half the Grassmann coordinates (the others can always be reduced to it) so we give its component reduction in Sec. V. This yields an interesting surprise: in a general gauge, the component action can always be written as the integral of a five-form $\mathcal{J}$ living on $\mathcal{M}^4 \times \mathcal{C}$, where $\mathcal{M}^4$ is the spacetime manifold and $\mathcal{C}$ is the contour in SU(2).[5] When restricted to the central gauge, the five-form leads to a component action similar to (1.6) with one intriguing difference. In the Conclusion, we briefly speculate on possible advantages of this new extended formulation.

Three appendices are included. Appendix A covers details of the superspace curvatures that are not included in Sec. III. Appendix B briefly reviews how to formulate invariant integrals over submanifolds, which is necessary for constructing invariant actions over 1/2 or 3/4 of the Grassmann coordinates. Appendix C presents the details of the component reduction of the analytic superspace action.

The notation and conventions for the SU(2) manifold are largely those of [1] and are straightforwardly related to those employed in [20]. The conventions for $\mathcal{N} = 2$ superspace, spinors, σ-matrices, and so on follow [34].

[5]A five-form description of the flat projective superspace action was also discussed by Biswas and Siegel [33].

II. GEOMETRIC PROPERTIES OF SU(2)

In this section, we provide a compact review of the geometric properties of the auxiliary SU(2) manifold, following mainly the approach commonly used in harmonic superspace [1]. As harmonic and projective superspace utilize the same auxiliary manifold, there is no obstruction to exploiting the same technology in both; in fact, a common notation can help accentuate the meaningful differences between them.

As in [20], we are not actually interested in SU(2) but rather the projective space $\mathbb{C}P^1$. This will come about because, as in harmonic superspace, we will always be dealing with quantities of fixed charge under the diagonal U(1) subgroup of SU(2). In other words, the effective space will actually be the coset $\mathrm{SU}(2)/\mathrm{U}(1) \cong \mathbb{C}P^1 \cong S^2$. Afterwards, we will highlight how complexifying SU(2) to $\mathrm{SL}(2,\mathbb{C})$ naturally recovers the formulation of [20].

A. The relations $\mathbf{SU(2)/U(1)} \cong S^2 \cong \mathbb{C}P^1$

Let us begin with the usual formulation of $\mathbb{C}P^1 \cong \mathbb{C}^{2*}/\mathbb{C}^*$. Introduce two complex coordinates v^i for $i = \underline{1}, \underline{2}$, with complex conjugates $\bar{v}_i = (v^i)^*$. These are homogeneous coordinates on $\mathbb{C}P^1$ under the identification

$$v^i \sim c v^i, \qquad c \in \mathbb{C}^*. \tag{2.1}$$

The north chart of $\mathbb{C}P^1$ is where $v^{\underline{1}}$ is nonzero, while the south chart possesses nonzero $v^{\underline{2}}$. We denote the point $v^i \sim (1,0)$ as the *north pole* and $v^i \sim (0,1)$ as the *south pole*.[6]

The space $\mathbb{C}P^1$ can alternatively be described within the space $\mathrm{SU}(2) \cong \mathbb{C}^{2*}/\mathbb{R}_+$. The normalized harmonic variables

$$v^{i+} := \frac{v^i}{|v|}, \qquad v_i^- := \frac{\bar{v}_i}{|v|},$$
$$|v^2| = (v,\bar{v}) \equiv v^k \bar{v}_k, \qquad v^{i+} v_i^- = 1, \tag{2.2}$$

can be used to construct a generic SU(2) group element

$$\mathbf{g} = \begin{pmatrix} v^{\underline{1}+} & -v_{\underline{2}}^- \\ v^{\underline{2}+} & v_{\underline{1}}^- \end{pmatrix} = \begin{pmatrix} v^{\underline{1}+} & -v^{\underline{1}-} \\ v^{\underline{2}+} & -v^{\underline{2}-} \end{pmatrix},$$
$$\mathbf{g}^{-1} = \mathbf{g}^\dagger, \qquad \det \mathbf{g} = 1. \tag{2.3}$$

$\mathbb{C}P^1$ is then identified as $\mathrm{SU}(2)/\mathrm{U}(1)$ by imposing the equivalence relation

[6]Note that some references (e.g. [5]) define the north pole to lie at $v^i \sim (0,1)$ and the south pole at $v^i \sim (1,0)$. In that convention, the north chart is generated by stereographic projection from the north pole, and so the north pole lies outside the north chart.

$$v^{i+} \sim e^{i\alpha} v^{i+}, \qquad e^{i\alpha} \in \mathrm{U}(1). \tag{2.4}$$

We can use the inhomogeneous coordinate $\zeta = v^{\underline{2}}/v^{\underline{1}}$ of $\mathbb{C}P^1$ to parametrize the harmonics. The harmonics $v_i^{\pm}$ are given in terms of ζ and the phase $e^{i\psi} := v^{\underline{1}}/|v^{\underline{1}}|$ by

$$\begin{aligned} v^{i+} &= (v^{\underline{1}+}, v^{\underline{2}+}) = \frac{e^{i\psi}}{\sqrt{1+\zeta\bar\zeta}}(1,\zeta), \\ v_i^{-} &= (v_{\underline{1}}^{-}, v_{\underline{2}}^{-}) = \frac{e^{-i\psi}}{\sqrt{1+\zeta\bar\zeta}}(1,\bar\zeta). \end{aligned} \tag{2.5}$$

The coordinates $\zeta, \bar\zeta$ describe the north chart of the Riemann sphere, so the coordinates $y^{\underline{m}} = (\zeta, \bar\zeta, \psi)$ may be called the north chart of SU(2). In what follows, we will frequently present quantities in terms of this chart.

Following [1], complex conjugation can be extended by an additional antipodal map on S^2. The new complex conjugation is denoted with a $\tilde{}$ and acts as $\widetilde{v^{i\pm}} = -v_i^{\pm}$, equivalently $\widetilde{v_i^{\pm}} = v^{i\pm}$. This is exactly the smile conjugation of the conventional formulation of projective superspace [20].

B. Vielbeins and covariant derivatives of SU(2)

The three derivative operations D^{++}, D^{--} and D^0 correspond to the right action of SU(2) on $\mathbf{g}$. Following [1], they are conventionally defined on the harmonic coordinates as

$$\begin{aligned} D^{++} &:= v_i^{+}\frac{\partial}{\partial v_i^{-}}, \qquad D^{--} := v^{i-}\frac{\partial}{\partial v^{i+}}, \\ D^0 &:= v^{i+}\frac{\partial}{\partial v^{i+}} - v_i^{-}\frac{\partial}{\partial v_i^{-}}, \end{aligned} \tag{2.6}$$

but can also be written in terms of the homogeneous coordinates v^i and $\bar v_i$, $D^{++} = v_i \frac{\partial}{\partial \bar v_i}$, $D^{--} = \bar v^i \frac{\partial}{\partial v^i}$, and $D^0 = v^i \frac{\partial}{\partial v^i} - \bar v_i \frac{\partial}{\partial \bar v_i}$ or in terms of the inhomogeneous coordinate ζ and the phase ψ,

$$\begin{aligned} D^{++} &= e^{2i\psi}\left((1+\zeta\bar\zeta)\partial_{\bar\zeta} - \frac{i}{2}\zeta\partial_\psi\right), \\ D^{--} &= -e^{-2i\psi}\left((1+\zeta\bar\zeta)\partial_{\zeta} + \frac{i}{2}\bar\zeta\partial_\psi\right), \\ D^0 &= -i\partial_\psi. \end{aligned} \tag{2.7}$$

They possess the commutation relations

$$\begin{aligned} [D^{++}, D^{--}] &= D^0, \qquad [D^0, D^{++}] = 2D^{++}, \\ [D^0, D^{--}] &= -2D^{--}, \end{aligned} \tag{2.8}$$

and one can interpret D^0 as a charge generator, with D^{++} and D^{--} respectively carrying charge $+2$ and -2. It will be convenient to denote the charges on these derivatives by an index $\underline{a}$ and to introduce a convention for lowering this index. A convenient definition is

$$\begin{aligned} D_{\underline{a}} &= (D_{++}, D_{--}, D_0), \qquad D_{++} := -D^{--}, \\ D_{--} &:= D^{++}, \qquad D_0 := D^0. \end{aligned} \tag{2.9}$$

Then the algebra of these covariant derivatives can be written as $[D_{\underline{a}}, D_{\underline{b}}] = -T_{\underline{a}\underline{b}}{}^{\underline{c}} D_{\underline{c}}$ for a constant torsion tensor. Associated with these are three vielbeins $\mathcal{V}^{\underline{a}} = \mathrm{d}y^{\underline{m}} \mathcal{V}_{\underline{m}}{}^{\underline{a}}$, given by (using different conventions than [1])

$$\begin{aligned} \mathcal{V}^{++} &= v_i^{+}\mathrm{d}v^{i+}, \qquad \mathcal{V}^{--} = v_i^{-}\mathrm{d}v^{i-}, \\ \mathcal{V}^0 &= v_i^{-}\mathrm{d}v^{i+} = v_i^{+}\mathrm{d}v^{i-}. \end{aligned} \tag{2.10}$$

In the homogeneous coordinate system, these are

$$\begin{aligned} \mathcal{V}^{++} &= \frac{1}{(v,\bar v)} v_i \mathrm{d}v^i, \qquad \mathcal{V}^{--} = \frac{1}{(v,\bar v)} \bar v_i \mathrm{d}\bar v^i, \\ \mathcal{V}^0 &= \frac{1}{(v,\bar v)}\frac{1}{2}(\bar v_i \mathrm{d}v^i - v^i \mathrm{d}\bar v_i), \end{aligned} \tag{2.11}$$

and in the inhomogeneous coordinate system by

$$\begin{aligned} \mathcal{V}^{++} &= \frac{e^{2i\psi}}{1+\zeta\bar\zeta}\mathrm{d}\zeta, \qquad \mathcal{V}^{--} = \frac{e^{-2i\psi}}{1+\zeta\bar\zeta}\mathrm{d}\bar\zeta, \\ \mathcal{V}^0 &= i\mathrm{d}\psi + \frac{1}{2}\frac{1}{1+\zeta\bar\zeta}(\bar\zeta\mathrm{d}\zeta - \zeta\mathrm{d}\bar\zeta). \end{aligned} \tag{2.12}$$

The Cartan structure equations are[7]

$$\begin{aligned} \mathrm{d}\mathcal{V}^{++} &= 2\mathcal{V}^{++}\wedge\mathcal{V}^0, \qquad \mathrm{d}\mathcal{V}^{--} = -2\mathcal{V}^{--}\wedge\mathcal{V}^0, \\ \mathrm{d}\mathcal{V}^0 &= \mathcal{V}^{++}\wedge\mathcal{V}^{--}. \end{aligned} \tag{2.13}$$

The covariant derivative can be written in the usual way, $D_{\underline{a}} = \mathcal{V}_{\underline{a}}{}^{\underline{m}}\partial_{\underline{m}}$, in terms of the inverse vielbein. One can verify these relations by checking that $\mathrm{d} = \mathcal{V}^{\underline{a}} D_{\underline{a}} = \mathrm{d}y^{\underline{m}}\partial_{\underline{m}}$ acts as an exterior derivative on any function of the SU(2) coordinates $y^{\underline{m}}$. Note that under the $\tilde{}$ conjugation, the derivatives and vielbeins are real, $\tilde{D}_{\underline{a}} = D_{\underline{a}}$ and $\tilde{\mathcal{V}}^{\underline{a}} = \mathcal{V}^{\underline{a}}$.[8]

The isometries of SU(2) correspond to the left action on the group element $\mathbf{g}$. These can be denoted by generators $\hat I^i{}_j$ which act as

[7]We use the superspace conventions for forms so that exterior derivatives act from the right (see e.g. [35]).

[8]The metric on SU(2) can be chosen as $\mathrm{d}s^2_{\mathrm{SU(2)}} = \mathrm{Tr}(\mathrm{d}\mathbf{g}^{-1}\otimes\mathrm{d}\mathbf{g}) = 2\mathcal{V}^{++}\otimes\mathcal{V}^{--} - 2\mathcal{V}^0\otimes\mathcal{V}^0 = 2\mathrm{d}v^{i+}\otimes\mathrm{d}v_i^{-}$, although we will not use it explicitly in what follows.

$$\hat{I}^i{}_j = -v^{i+}\frac{\partial}{\partial v^{j+}} + v_j^-\frac{\partial}{\partial v_i^-} + \frac{1}{2}\delta^i_j\left(v^{k+}\frac{\partial}{\partial v^{k+}} - v_k^-\frac{\partial}{\partial v_k^-}\right). \tag{2.14}$$

These leave the covariant derivatives invariant, $[\hat{I}^i{}_j, D_{\underline{a}}] = 0$. One can further verify that an isometry with constant parameters $\lambda^j{}_i$ can be rewritten as

$$\begin{aligned}\delta_I &= \lambda^j{}_i\hat{I}^i{}_j = \lambda^{\underline{a}}D_{\underline{a}}\\ &= -\lambda^{++}D^{--} + \lambda^0 D^0 + \lambda^{--}D^{++},\end{aligned} \tag{2.15}$$

where $\lambda^{\pm\pm}$ and λ^0 are coordinate-dependent transformations given by

$$\lambda^{\pm\pm} \coloneqq v_i^\pm v_j^\pm \lambda^{ij}, \qquad \lambda^0 \coloneqq v_i^+ v_j^- \lambda^{ij}. \tag{2.16}$$

(It is sometimes convenient to denote $\lambda^0 = \lambda^{+-}$ in analogy with $\lambda^{\pm\pm}$.) The appearance of the minus sign in (2.15) was the reason for introducing the sign in (2.9).

If we now restrict to the space $S^2 \cong \mathrm{SU}(2)/\mathrm{U}(1)$, then the covariant derivatives $D_{\underline{a}}$ possess a different interpretation. D^0 can be identified with the rotation generator on the tangent space of S^2, while D^{++} and D^{--} can be identified with the covariant holomorphic and antiholomorphic derivatives. Then a scalar function $f^{(q)}$ of fixed D^0 charge on SU(2) is reinterpreted as a function of fixed spin weight on S^2 (see e.g. the discussion in [36]). In what follows, although we will always remain with an explicit SU(2) manifold, we will only be dealing with such functions $f^{(q)}$, and so it will always be possible to reinterpret calculations as being performed on the space $\mathrm{SU}(2)/\mathrm{U}(1) \cong S^2 \cong \mathbb{C}P^1$.

C. Harmonic and holomorphic tensors on $\mathbb{C}P^1$

There are two interesting classes of tensors on $\mathbb{C}P^1 \cong S^2$. The first are the so-called harmonic functions, which are globally defined functions on SU(2) with fixed D^0 charge. These are given by

$$f^{(q)} = \sum_{n=0}^{\infty} f^{(i_1\ldots i_{n+q}j_1\ldots j_n)} v_{i_1}^+ \cdots v_{i_{n+q}}^+ v_{j_1}^- \cdots v_{j_n}^-, \tag{2.17}$$

with $D^0 f^{(q)} = q f^{(q)}$ (assuming $q \geq 0$, but similarly for $q < 0$) and are extensively discussed in [1].

The second interesting class are the functions $\mathcal{Q}^{(q)}$ with fixed D^0 charge but annihilated by D^{++},

$$D^0\mathcal{Q}^{(q)} = q\mathcal{Q}^{(q)}, \qquad D^{++}\mathcal{Q}^{(q)} = 0. \tag{2.18}$$

The most general class of such functions is not globally defined on SU(2). If they are nonsingular near the north pole, they are called *arctic* and possess an expansion[9]

$$\mathcal{Q}^{(q)} = (v^{\underline{1}+})^q\mathcal{Q}(\zeta) = (v^{\underline{1}+})^q\sum_{n=0}^{\infty}\mathcal{Q}_n\zeta^n. \tag{2.19}$$

Their conjugates $\tilde{\mathcal{Q}}^{(q)}$ are nonsingular near the south pole and are called *antarctic*. They possess an expansion

$$\tilde{\mathcal{Q}}^{(q)} = (v^{\underline{2}+})^q\tilde{\mathcal{Q}}(\zeta) = (v^{\underline{2}+})^q\sum_{n=0}^{\infty}(-1)^n\bar{\mathcal{Q}}_n\zeta^{-n}. \tag{2.20}$$

It will be convenient to refer to functions $\mathcal{Q}^{(q)}$ satisfying (2.18) as *holomorphic* although strictly speaking they are generically holomorphic only on an open domain of $\mathrm{SU}(2)/\mathrm{U}(1)$.

Of course, it is possible for such functions to be both holomorphic and globally defined. These generally have an expansion of the form $\mathcal{G}^{(q)} = \mathcal{G}^{(i_1\ldots i_q)}v_{i_1}^+\cdots v_{i_q}^+$ and can be real only if q is even.

D. Integration measures and global SU(2) invariance

The most straightforward integration over the auxiliary manifold SU(2) is accomplished using the usual Haar measure. Given some globally defined function $f^{(0)}(v^+, v^-)$, one can define the action integral

$$\begin{aligned}S &= \int \mathrm{d}v f^{(0)} = \frac{i}{4\pi^2}\int_0^{2\pi}\mathrm{d}\psi\int\frac{\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}}{(1+\zeta\bar{\zeta})^2}f^{(0)}\\ &= \frac{i}{2\pi}\int\frac{\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}}{(1+\zeta\bar{\zeta})^2}f^{(0)},\end{aligned} \tag{2.21}$$

normalized so that $\int \mathrm{d}v = 1$. Integrals of the above type are encountered when using harmonic superspace, which is concerned with globally defined functions. Since we will be dealing instead with holomorphic functions, the natural integration principle will involve a one-dimensional contour integral on SU(2), with the contour avoiding regions where the functions become singular. The natural integrand is a one-form $\omega = \mathrm{d}y^{\underline{m}}\omega_{\underline{m}} = \mathcal{V}^{\underline{a}}\omega_{\underline{a}}$ and the corresponding integral is

$$S = \frac{1}{2\pi}\oint_C \mathcal{V}^{\underline{a}}\omega_{\underline{a}}. \tag{2.22}$$

Because we are actually interested in contours in $\mathbb{C}P^1 \cong S^2$, we will always assume $\omega_0 = 0$ so that the resulting action is given by

[9]Superfields in projective superspace with such expansions were introduced in [3]. The arctic or antarctic nomenclature appeared later in [7].

$$S = -\frac{1}{2\pi}\oint_{\mathcal{C}} \mathcal{V}^{++}\omega^{--} + \frac{1}{2\pi}\oint_{\mathcal{C}} \mathcal{V}^{--}\omega^{++}$$
$$= -\frac{1}{2\pi}\oint_{\mathcal{C}} v_i^+ \mathrm{d}v^{i+}\omega^{--} + \frac{1}{2\pi}\oint_{\mathcal{C}} v_i^- \mathrm{d}v^{i-}\omega^{++}. \quad (2.23)$$

For later convenience we have "raised" the indices on $\omega_{\underline{a}}$, using the same convention as in (2.9), so that the D^0 charges of the integrands are clear.

A natural question to ask is whether the contour action (2.23) is invariant under SU(2) isometries. It turns out that the answer is yes, provided the integrands ω^{--} and ω^{++} obey a certain condition. First let us establish Stokes' theorem. Suppose $\omega = \mathrm{d}\Lambda^{(0)}$ for some function $\Lambda^{(0)}$. Then we must have

$$0 = -\frac{1}{2\pi}\oint_{\mathcal{C}} \mathcal{V}^{++}D^{--}\Lambda^{(0)} + \frac{1}{2\pi}\oint_{\mathcal{C}} \mathcal{V}^{--}D^{++}\Lambda^{(0)}. \quad (2.24)$$

If $\Lambda^{(0)}$ is holomorphic, this reduces to

$$0 = -\frac{1}{2\pi}\oint_{\mathcal{C}} \mathcal{V}^{++}D^{--}\Lambda^{(0)}, \quad \text{if } D^{++}\Lambda^{(0)} = 0.$$

These two results are quite important, so let us discuss their form in an explicit coordinate basis. If τ is the coordinate parametrizing the contour, one can show that

$$\frac{1}{2\pi}\oint_{\mathcal{C}} \mathcal{V}^{++}D^{--}\Lambda^{(0)} = -\frac{1}{2\pi}\oint_{\mathcal{C}} \mathrm{d}\tau\frac{\mathrm{d}\zeta}{\mathrm{d}\tau}\frac{\partial\Lambda^{(0)}}{\partial\zeta}. \quad (2.25)$$

If $\Lambda^{(0)}$ is holomorphic, then the right-hand side vanishes as a total derivative. If not, we find that

$$-\frac{1}{2\pi}\oint_{\mathcal{C}} \mathrm{d}\tau\frac{\mathrm{d}\zeta}{\mathrm{d}\tau}\frac{\partial\Lambda^{(0)}}{\partial\zeta} = \frac{1}{2\pi}\oint_{\mathcal{C}} \mathrm{d}\tau\frac{\mathrm{d}\bar\zeta}{\mathrm{d}\tau}\frac{\partial\Lambda^{(0)}}{\partial\bar\zeta}$$
$$= \frac{1}{2\pi}\oint_{\mathcal{C}} \mathcal{V}^{--}D^{++}\Lambda^{(0)}. \quad (2.26)$$

This establishes (2.24).

Now let us calculate $\delta_I S$. The vielbein one-forms are necessarily invariant under the isometry while $\omega_{\underline{a}}$ transforms as $\delta_I\omega_{\underline{a}} = \lambda^{\underline{b}}D_{\underline{b}}\omega_{\underline{a}}$. This implies, using the explicit form (2.16) of the parameters $\lambda^{\underline{a}}$,

$$\delta_I\omega^{--} = -D^{--}(\lambda^{++}\omega^{--}) + \lambda^{--}D^{++}\omega^{--},$$
$$\delta_I\omega^{++} = D^{++}(\lambda^{--}\omega^{++}) - \lambda^{++}D^{--}\omega^{++}. \quad (2.27)$$

This leads, using (2.24), to

$$\delta_I\int \mathcal{V}^{++}\omega^{--} = -\int \mathcal{V}^{--}D^{++}(\lambda^{++}\omega^{--}) + \int \mathcal{V}^{++}\lambda^{--}D^{++}\omega^{--},$$
$$\delta_I\int \mathcal{V}^{--}\omega^{++} = \int \mathcal{V}^{++}D^{--}(\lambda^{--}\omega^{++}) - \int \mathcal{V}^{--}\lambda^{++}D^{--}\omega^{++}, \quad (2.28)$$

and one can see that the difference between these two terms vanishes (and so $\delta_I S = 0$) precisely when[10]

$$D^{++}\omega^{--} = D^{--}\omega^{++}. \quad (2.29)$$

This is merely the tangent space version of the condition that ω is closed.

E. Extension to local SU(2) transformations

Up until now, we restricted our attention to SU(2) isometries. These preserved the form of the SU(2) vielbein $\mathcal{V}$ and were generated by constant parameters $\lambda^i{}_j$. In principle, there is no reason why we cannot perform *local* SU(2) transformations of the form (2.15) but with parameters ξ^{++}, ξ^{--} and ξ^0 subject only to the condition that $\xi^{\pm\pm}$ and ξ^0 have D^0 charges ± 2 and 0, respectively. That is, we can take

$$\delta = \xi^{\underline{a}}D_{\underline{a}} = -\xi^{++}D^{--} + \xi^0 D^0 + \xi^{--}D^{++} \quad (2.30)$$

but with e.g. ξ^{++} not necessarily of the form $\xi^{ij}v_i^+ v_j^+$. Such SU(2) diffeomorphisms can be interpreted as diffeomorphisms on S^2 (generated by $\xi^{\pm\pm}$) along with local U(1) frame rotations (generated by ξ^0).

Under such a local transformation, the vielbeins transform in the usual way, $\delta\mathcal{V}^{\underline{a}} = \mathrm{d}\xi^{\underline{a}} + \mathcal{V}^{\underline{b}}\xi^{\underline{c}}T_{\underline{c}\,\underline{b}}{}^{\underline{a}}$, leading to

$$\delta\mathcal{V}^{++} = \mathrm{d}\xi^{++} - 2\mathcal{V}^0\xi^{++} + 2\mathcal{V}^{++}\xi^0, \quad (2.31a)$$

$$\delta\mathcal{V}^{--} = \mathrm{d}\xi^{--} + 2\mathcal{V}^0\xi^{--} - 2\mathcal{V}^{--}\xi^0, \quad (2.31b)$$

$$\delta\mathcal{V}^0 = \mathrm{d}\xi^0 + \mathcal{V}^{++}\xi^{--} - \mathcal{V}^{--}\xi^{++}. \quad (2.31c)$$

One can check that the above transformations are consistent with the definitions (2.10).

Now let us briefly discuss the consequences of requiring that the contour action (2.23) remain invariant under such diffeomorphisms. A general diffeomorphism on ω can always be written as $\delta_\xi\omega = \mathrm{d}(\imath_\xi\omega) + \imath_\xi\mathrm{d}\omega$. The first term

[10]It is possible to have purely holomorphic one-forms ω that obey $\omega^{++} = 0$ and $D^{++}\omega^{--} = 0$. The one-forms we consider in projective superspace will generally not be purely holomorphic, but will instead carry some small nonvanishing ω^{++} piece.

vanishes along the contour integral so we conclude that ω must be closed. In the tangent frame, with the condition $\omega_0 = 0$, this leads to (2.29), and so the condition for invariance under SU(2) isometries is the same condition as for full diffeomorphism invariance. This closure condition has an obvious geometric interpretation. An arbitrary diffeomorphism of an integral $\oint_{\mathcal{C}} \omega$ can be interpreted as a small deformation of the contour $\mathcal{C}$, and this can generically vanish only if the flux of $\mathrm{d}\omega$ through any $\delta\mathcal{C}$ vanishes in the vicinity of $\mathcal{C}$. This leads to the well-known condition in projective superspace that the integrals $\oint_{\mathcal{C}} \omega$ depend only on the topology of the contour and how it winds around singularities of ω.

F. The complexified SU(2) and the emergence of a projective structure

Our final topic in this opening section is to address how the SU(2) framework we have been discussing can be related to the $\mathbb{C}P^1$ framework that one encounters in the conventional formulation of projective superspace coupled to supergravity. The key idea is to complexify SU(2) and to treat v^i and $\bar{v}_i$ as independent coordinates. Beginning with the representation (2.2) for the harmonic coordinates, complexify $\bar{v}_i \to u_i$. In doing so, it is convenient to modify the definitions of the harmonics so that

$$v^{i+} = v^i, \qquad v_i^- = \frac{u_i}{(v,u)},$$
$$v^{i+} v_i^- = 1, \qquad v_i^- \neq (v^{i+})^*. \tag{2.32}$$

We have shifted the entirety of the (v,u) factor into the second harmonic because $\sqrt{(v,u)}$ is not well-defined. This can be interpreted as a local complex D^0 gauge transformation, converting all quantities of fixed D^0 charge q into quantities of degree q in v^i and degree 0 in u_i. In other words, the $+$ and $-$ labels on the harmonics (as well as any other quantities) now denote their homogeneity under the projective transformation

$$v^i \to c v^i, \qquad c \in \mathbb{C}. \tag{2.33}$$

The resulting group element $\mathbf{g}$ given in (2.3) still obeys $\det \mathbf{g} = 1$ but is no longer unitary. In other words, we have complexified SU(2) to $\mathrm{SL}(2,\mathbb{C})$.

It is straightforward to extend the entirety of the previous discussion to $\mathrm{SL}(2,\mathbb{C})$. Instead of dealing with operators and functions of fixed D^0 charge, we have fixed homogeneity under (2.33) and *invariance* under $u_i \to c u_i$. One can introduce derivatives

$$D^{++} = (v,u) v_i \frac{\partial}{\partial u_i}, \qquad D^{--} = \frac{u_i}{(v,u)} \frac{\partial}{\partial v^i},$$
$$D^0 = v^i \frac{\partial}{\partial v^i} - u_i \frac{\partial}{\partial u_i}, \tag{2.34}$$

and their corresponding vielbeins

$$\mathcal{V}^{++} = v_i \mathrm{d}v^i, \qquad \mathcal{V}^{--} = \frac{u_i \mathrm{d}u^i}{(v,u)^2}, \qquad \mathcal{V}^0 = \frac{u_i \mathrm{d}v^i}{(v,u)}. \tag{2.35}$$

It is natural to convert all holomorphic functions $\mathcal{Q}^{(q)}$ to new quantities of definite homogeneity in v^i and independent of u_i, $\mathcal{Q}'^{(q)}(cv) = c^q \mathcal{Q}'^{(q)}(v)$. These are related to the original $\mathcal{Q}^{(q)}$ by the same complex D^0 transformation, and we will drop the primes when it is clear from context which quantities we are discussing.

Finally, the complex version of the contour integral (2.23) takes the form

$$S = -\frac{1}{2\pi} \oint_{\mathcal{C}} \mathcal{V}^{++} \omega^{--} + \frac{1}{2\pi} \oint_{\mathcal{C}} \mathcal{V}^{--} \omega^{++}, \tag{2.36}$$

where ω^{--} and ω^{++} are respectively degrees -2 and $+2$ in v^i, degree zero in u_i, and related by the complex version of (2.29). Under a local $\mathrm{SL}(2,\mathbb{C})$ diffeomorphism, the coordinates v^i and u_i transform as

$$\delta v^i = \xi^0 v^i - \frac{\xi^{++}}{(v,u)} u^i,$$
$$\delta u_i = -\xi^0 u_i + (v,u) \xi^{--} v_i, \tag{2.37}$$

while ω^{++} and ω^{--} transform as

$$\delta\omega^{--} = -\xi^{++} D^{--} \omega^{--} - 2\xi^0 \omega^{--} + \xi^{--} D^{++} \omega^{--},$$
$$\delta\omega^{++} = -\xi^{++} D^{--} \omega^{++} + 2\xi^0 \omega^{++} + \xi^{--} D^{++} \omega^{++}. \tag{2.38}$$

The parameters $\xi^{\pm\pm}$ and ξ^0 are each assumed to be of degree zero in u_i while possessing homogeneity of the indicated degree in v^i.

The major advantage of the complexified SU(2) is that we may choose v^i and u_i to have entirely uncorrelated behavior along the contour. In particular, one can take u_i to be *fixed*, subject only to the condition that $(v,u) \neq 0$ along the contour. This can be interpreted as deforming the contour $\mathcal{C}$ within $\mathrm{SL}(2,\mathbb{C})$. After such a choice, the gauge freedom (2.37) is no longer arbitrary, but is restricted by the requirement that δu_i is similarly constant. This implies certain constraints on the functions ξ^{--} and ξ^0. (This residual freedom was discussed in [20].) The advantage of this choice is that the second contour integral in (2.36) *automatically* vanishes even if ω^{++} is nonzero. This is a consequence of the property that a total contour derivative is simplified from (2.24) to

$$0 = -\frac{1}{2\pi}\oint_{\mathcal{C}} \mathcal{V}^{++} D^{--} \Lambda^0(v,u), \tag{2.39}$$

where we emphasize that $\Lambda^0(v,u)$ may depend on fixed u_i (with degree zero).

Although taking u_i to be constant can simplify the contour integrals, we have found it useful to remain with a real SU(2) manifold in defining our formulation of projective superspace. This guarantees, for example, that the harmonics are always well-defined; there is no requirement that the contour avoid the location where $(v,u) = 0$. It also permits full SU(2) diffeomorphisms, rather than the restricted $\mathrm{SL}(2,\mathbb{C})$ diffeomorphisms that leave u_i constant. Nonetheless, starting from a real SU(2) manifold it is always possible to complexify to $\mathrm{SL}(2,\mathbb{C})$ and then to adopt the choice of constant u_i where needed.

III. PROJECTIVE SUPERSPACE AND $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$

In this section, we will describe how to construct a covariant projective superspace generalizing the work of [20]. We will do this first by constructing a direct product of the supermanifold $\mathcal{M}^{4|8}$ and SU(2), and then splicing together the tangent space action of $I^i{}_j$ on $\mathcal{M}^{4|8}$ with the isometry transformation on SU(2). The resulting construction will correspond to that given in the usual version of projective superspace. We will then show how to lift to a general gauge. Finally, we will comment briefly on the admissible types of primary analytic superfields.

A. Conformal superspace on $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$: A bottom-up construction

Let us begin with a conventional supermanifold $\mathcal{M}^{4|8}$ with local coordinates $z^M = (x^m, \theta^\mu{}_\imath, \bar\theta_{\dot\mu}{}^\imath)$ with $m = 0,1,2,3$, $\mu = 1,2$, $\dot\mu = 1,2$ and $\imath = \underline{1},\underline{2}$. The associated superspace vielbein is given by $E_M{}^A = (E_M{}^a, E_M{}^\alpha{}_i, E_{M\dot\alpha}{}^i)$. We will assume we are working with conformal superspace [34], so that the supermanifold possesses the full superconformal structure group, but the framework we present here would work equally well with SU(2) or U(2) superspace where the superconformal transformations take the form of super-Weyl transformations [20,37].

In conformal superspace, the covariant derivative $\nabla_A = (\nabla_a, \nabla_\alpha{}^i, \bar\nabla^{\dot\alpha}{}_i)$ is defined implicitly by[11]

$$\partial_M = E_M{}^A \nabla_A + \mathcal{V}_M{}^j{}_i I^i{}_j + \frac{1}{2}\Omega_M{}^{ab} M_{ba} + A_M \mathbb{A} + B_M \mathbb{D} + F_M{}^{\alpha i} S_{\alpha i} + F_{M\dot\alpha i} \bar S^{\dot\alpha i} + F_M{}^a K_a, \tag{3.1}$$

which can equivalently be written

$$\nabla_A = E_A{}^M \Big(\partial_M - \mathcal{V}_M{}^k{}_j I^j{}_k - \frac{1}{2}\Omega_M{}^{bc} M_{cb} - A_M \mathbb{A} - B_M \mathbb{D} - F_M{}^{\beta j} S_{\beta j} - F_{M\dot\beta j} \bar S^{\dot\beta j} - F_M{}^b K_b\Big). \tag{3.2}$$

M_{ab} is the Lorentz generator, $\mathbb{A}$ and $I^i{}_j$ are the U(1) and SU(2) R-symmetry generators, $\mathbb{D}$ is the dilatation generator, $S_{\alpha i}$ and $\bar S^{\dot\alpha i}$ are S-supersymmetry generators, and K_a is the special conformal generator. Their algebra is summarized in [34].

Now we wish to combine this structure with the SU(2) manifold with covariant derivatives D^{++}, D^{--}, and D^0. The only nontrivial step is to decide how the action of $I^i{}_j$ should be manifested on functions $\mathcal{F}(z, v^+, v^-)$ depending also on the SU(2) coordinates, which we choose as in (2.14). The operator $I^i{}_j$ acts as the isometry generator on the SU(2) manifold. At this stage, we immediately recover the construction of [20], since a general supergravity $\mathrm{SU}(2)_R$ transformation is given by

$$\lambda^j{}_i I^i{}_j \mathcal{F} = -\lambda^{++} D^{--}\mathcal{F} + \lambda^{--} D^{++}\mathcal{F} + \lambda^0 D^0 \mathcal{F}, \tag{3.3}$$

for arbitrary local $\lambda^i{}_j(z)$ independent of the harmonics. Specializing this equation to holomorphic functions $\mathcal{Q}^{(n)}(z, v^+)$ of fixed D^0 charge n recovers the transformation law (1.2), up to the complexification of SU(2) to $\mathrm{SL}(2,\mathbb{C})$ discussed in Sec. II F.

At this stage, we have two different ways in which $I^i{}_j$ can act. It can act on a function $\mathcal{F}(z, v^+, v^-)$ as an SU(2) isometry, or it can act on an SU(2) tensor independent of $v^{i\pm}$, such as $E_M{}^\alpha{}_i(z)$, as a tangent space rotation. Now we wish to eliminate the latter in favor of the former so that the operator acts in only one way. Consider for definiteness some superfield q^i with a single SU(2) index, independent of $v^{i\pm}$ and transforming covariantly under $\mathrm{SU}(2)_R$. (For example, q^i could be $E_M{}^{\alpha i}$.) If we interpret q^i as a component of $q^+ = q^i v_i^+$, then the action of $\mathrm{SU}(2)_R$ on q^i, treating v_i^+ as invariant, is

$$\delta_\lambda q^+ = \lambda^i{}_j q^j v_i^+ = -\lambda^{++} D^{--} q^+ + \lambda^0 q^+. \tag{3.4}$$

This is exactly the same transformation rule as (3.3), corresponding to an isometry transformation on the SU(2) manifold. If we exchange all quantities with $\mathrm{SU}(2)_R$ indices for scalar functions on the SU(2) manifold, e.g.

$$E_M{}^{\alpha i} \Rightarrow E_M{}^{\alpha\pm}, \qquad E_{M\dot\alpha}{}^i \Rightarrow E_{M\dot\alpha}{}^\pm, \tag{3.5}$$

then $I^i{}_j$ can be interpreted as *always* acting as (3.3). In particular, the SU(2) connection can be rewritten as

[11] We have relabeled the SU(2) connection $\Phi_M{}^i{}_j$ of [34] to $\mathcal{V}_M{}^i{}_j$.

$$\mathcal{V}_M{}^i{}_j I^j{}_i = -\mathcal{V}_M{}^{++} D^{--} + \mathcal{V}_M{}^0 D^0 + \mathcal{V}_M{}^{--} D^{++},$$
$$\mathcal{V}_M{}^{\pm\pm} \coloneqq v_i^{\pm} v_j^{\pm} \mathcal{V}_M{}^{ij}, \qquad \mathcal{V}_M{}^0 \coloneqq v_i^{+} v_j^{-} \mathcal{V}_M{}^{ij}. \tag{3.6}$$

Note that $\mathcal{V}_M{}^{\pm\pm}$ and $\mathcal{V}_M{}^0$ do not transform as scalar functions under the SU(2) isometry, but rather as connections,

$$\delta_\lambda \mathcal{V}_M{}^{++} = \partial_M \lambda^{++} - 2\mathcal{V}_M{}^0 \lambda^{++} + 2\mathcal{V}_M{}^{++}\lambda^0, \tag{3.7a}$$

$$\delta_\lambda \mathcal{V}_M{}^{--} = \partial_M \lambda^{--} + 2\mathcal{V}_M{}^0 \lambda^{--} - 2\mathcal{V}_M{}^{--}\lambda^0, \tag{3.7b}$$

$$\delta_\lambda \mathcal{V}_M{}^{0} = \partial_M \lambda^{0} + \mathcal{V}_M{}^{++} \lambda^{--} - \mathcal{V}_M{}^{--}\lambda^{++}. \tag{3.7c}$$

This is exactly how the SU(2) vielbeins $\mathcal{V}_{\underline{m}}{}^{\pm\pm}$ and $\mathcal{V}_{\underline{m}}{}^0$ transform under SU(2) diffeomorphisms (see (2.31)) but with the arbitrary $\xi^{\pm\pm}$, ξ^0 parameters replaced with $\lambda^{\pm\pm}$, λ^0. Before interpreting this further, let us make a few additional comments.

The implicit expression (3.1) for the covariant derivative can be rewritten

$$\begin{aligned}\partial_M = {} & E_M{}^{\underline{\alpha}-}\nabla_{\underline{\alpha}}^{+} - E_M{}^{\underline{\alpha}+}\nabla_{\underline{\alpha}}^{-} + E_M{}^a\nabla_a \\ & - \mathcal{V}_M{}^{++}D^{--} + \mathcal{V}_M{}^{--}D^{++} + \mathcal{V}_M{}^0 D^0 \\ & + \frac{1}{2}\Omega_M{}^{ab}M_{ba} + A_M\mathbb{A} + B_M\mathbb{D} \\ & + F_M{}^{\underline{\alpha}+}S_{\underline{\alpha}}^{-} - F_M{}^{\underline{\alpha}-}S_{\underline{\alpha}}^{+} + F_M{}^a K_a,\end{aligned} \tag{3.8}$$

where we use

$$E_M{}^{\underline{\alpha}\pm} = E_M{}^{\underline{\alpha}i} v_i{}^{\pm}, \qquad \nabla_{\underline{\alpha}}^{\pm} = v_i^{\pm}\nabla_{\underline{\alpha}}{}^i,$$
$$F_M{}^{\underline{\alpha}\pm} = F_M{}^{\underline{\alpha}i} v_i^{\pm}, \qquad S_{\underline{\alpha}}^{\pm} = v_i^{\pm} S_{\underline{\alpha}}{}^i, \tag{3.9}$$

for the spinor vielbeins, S-supersymmetry connections, and their corresponding operators. We have introduced a new compact notation

$$\psi^{\underline{\alpha}} = (\psi^\alpha, \bar\psi^{\dot\alpha}), \qquad \psi_{\underline{\alpha}} = (\psi_\alpha, \bar\psi_{\dot\alpha}), \tag{3.10}$$

to deal collectively with the left- and right-handed vielbeins, spinor derivatives, etc. It is helpful to introduce some further notation to simplify the first line of (3.8). As in the previous section, we wish to treat the $\pm\pm$ and 0 indices of the SU(2) derivatives as tangent space indices and to lower them using the same conventions (2.9), with $D_{\underline{a}} \coloneqq (D_{++}, D_{--}, D_0)$. It will also be useful to introduce a convention for lowering the $\pm$ on $\nabla_{\underline{\alpha}}^{\pm}$, and similarly for the S-supersymmetry generator:

$$\nabla_{\underline{\alpha}\mp} \coloneqq \pm\nabla_{\underline{\alpha}}^{\pm}, \qquad S_{\underline{\alpha}\mp} \coloneqq \mp S_{\underline{\alpha}}^{\pm}. \tag{3.11}$$

Now introducing $\nabla_A = (\nabla_a, \nabla_{\underline{\alpha}\pm})$ and $K_A = (K_a, S_{\underline{\alpha}\pm})$, we can rewrite (3.8) as

$$\begin{aligned}\partial_M = {} & E_M{}^A\nabla_A + \mathcal{V}_M{}^{\underline{a}}D_{\underline{a}} + \frac{1}{2}\Omega_M{}^{ab}M_{ba} \\ & + A_M\mathbb{A} + B_M\mathbb{D} + F_M{}^A K_A.\end{aligned} \tag{3.12}$$

Recalling that the partial derivatives $\partial_{\underline{m}}$ can be written in a similar way,

$$\begin{aligned}\partial_{\underline{m}} &= \mathcal{V}_{\underline{m}}{}^{\underline{a}} D_{\underline{a}} \\ &= -\mathcal{V}_{\underline{m}}{}^{++}D^{--} + \mathcal{V}_{\underline{m}}{}^0 D^0 + \mathcal{V}_{\underline{m}}{}^{--}D^{++},\end{aligned} \tag{3.13}$$

a new unified notation becomes apparent. Let $z^{\underline{M}}$ denote the full set of coordinates $z^{\underline{M}} = (z^M, y^{\underline{m}})$ and introduce a unified covariant derivative $\nabla_{\underline{A}} = (\nabla_A, D_{\underline{a}})$. Then (3.12) and (3.13) can be written

$$\begin{aligned}\partial_{\underline{M}} = {} & E_{\underline{M}}{}^{\underline{A}}\nabla_{\underline{A}} + \frac{1}{2}\Omega_{\underline{M}}{}^{ab}M_{ba} + A_{\underline{M}}\mathbb{A} \\ & + B_{\underline{M}}\mathbb{D} + F_{\underline{M}}{}^A K_A,\end{aligned} \tag{3.14}$$

where the full supervielbein is given by

$$E_{\underline{M}}{}^{\underline{A}} = \begin{pmatrix} E_M{}^A & \mathcal{V}_M{}^{++} & \mathcal{V}_M{}^{--} & \mathcal{V}_M{}^0 \\ 0 & \mathcal{V}_\zeta{}^{++} & \mathcal{V}_\zeta{}^{--} & \mathcal{V}_\zeta{}^0 \\ 0 & \mathcal{V}_{\bar\zeta}{}^{++} & \mathcal{V}_{\bar\zeta}{}^{--} & \mathcal{V}_{\bar\zeta}{}^0 \\ 0 & \mathcal{V}_\psi{}^{++} & \mathcal{V}_\psi{}^{--} & \mathcal{V}_\psi{}^0 \end{pmatrix} \tag{3.15}$$

and the other connections live purely on $\mathcal{M}^{4|8}$,

$$\Omega_{\underline{M}}{}^{ab} = (\Omega_M{}^{ab}, 0, 0, 0), \qquad A_{\underline{M}} = (A_M, 0, 0, 0),$$
$$B_{\underline{M}} = (B_M, 0, 0, 0), \qquad F_{\underline{M}}{}^A = (F_M{}^A, 0, 0, 0). \tag{3.16}$$

This rearrangement is equivalent to that proposed in [15].

These identifications (3.15) are completely consistent so long as two conditions are obeyed. First, the only SU(2) diffeomorphisms that we may perform are those that are isometries on the SU(2) manifold. Then the full SU(2) vielbeins $\mathcal{V}^{\underline{a}} = \mathrm{d}z^{\underline{M}}\mathcal{V}_{\underline{M}}{}^{\underline{a}}$ transform as (2.31) with the special choice of $\xi^{\underline{a}} = \lambda^{\underline{a}}$ with $\lambda^i{}_j$ depending on z^M alone. Second, the only z^M diffeomorphisms and other gauge transformations (i.e. Lorentz, $\mathrm{U}(1)_R$, S-supersymmetry and special conformal) that are allowed are those that do not depend on $v^{i\pm}$. This ensures the zeros in the identifications (3.15) and (3.16) as well as the decompositions (3.9).

As a final check, we can invert (3.14) to find the covariant derivative $\nabla_{\underline{A}}$:

$$\begin{aligned}\nabla_A = E_A{}^M\Big(& \partial_M - \mathcal{V}_M{}^{\underline{a}}D_{\underline{a}} - \frac{1}{2}\Omega_M{}^{bc}M_{cb} \\ & - A_M\mathbb{A} - B_M\mathbb{D} - F_M{}^B K_B\Big),\end{aligned} \tag{3.17a}$$

$$\nabla_{\underline{a}} \equiv D_{\underline{a}} = \mathcal{V}_{\underline{a}}{}^{\underline{m}} \partial_{\underline{m}}. \tag{3.17b}$$

The first equation exactly matches (3.2).

The algebra of the redefined operators retains its original form but with minor modifications involving the exchange of e.g. $I^i{}_j$ for $D^{\pm\pm}$ and D^0 and $S_\beta{}^i$ for $S_\beta{}^\pm$, and will be given in a general gauge in the next subsection.

It is evident that starting from this formulation of conformal superspace on $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$, there is no intrinsic barrier to performing $v^{i\pm}$-dependent gauge transformations and diffeomorphisms. These will move us away from the original gauge where (3.9), (3.15), and (3.16) hold and where the SU(2) vielbein $\mathcal{V}_{\underline{m}}{}^{\underline{a}}$ takes the simple form (2.10). Of course, we can always return to this gauge. We refer to it as the *central gauge* (or central basis) in analogy with the terminology employed within the harmonic superspace literature.[12] In the next section, we will extend this construction to a completely general gauge.

B. Conformal superspace on $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$: The top-down construction

In contrast to the preceding treatment where we spliced together SU(2) with the supermanifold $\mathcal{M}^{4|8}$ of conformal superspace, we can simply postulate the structure of the new superspace $\mathcal{M}^{7|8} = \mathcal{M}^{4|8} \times \mathrm{SU}(2)$ and impose all the relevant constraints. This will have the benefit of not requiring that we begin in central gauge, although central gauge always remains a possibility.

The supermanifold $\mathcal{M}^{7|8} = \mathcal{M}^{4|8} \times \mathrm{SU}(2)$ possesses local coordinates $z^{\underline{M}} = (z^M, y^{\underline{m}}) = (x^m, \theta^{\underline{\mu}\pm}, \zeta, \bar\zeta, \psi)$. For convenience, we have labeled the Grassmann coordinates $\theta^{\underline{\mu}\imath}$ by $\imath = \pm$ to facilitate a later discussion of analytic gauge. (We emphasize that $\imath$ is a world index and so does not correspond to any notion of charge; we could just as well have used $\imath = 1, 2$.)

The covariant derivatives $\nabla_{\underline{A}} = (\nabla_A, \nabla_{\underline{a}}) = (\nabla_a, \nabla_{\underline{\alpha}\pm}, \nabla_{\pm\pm}, \nabla_0)$ are defined implicitly as in (3.14). The supervielbein is required to be invertible, and its components can be labeled as

$$E_{\underline{M}}{}^{\underline{A}} = \begin{pmatrix} E_M{}^A & E_M{}^{++} & E_M{}^{--} & E_M{}^0 \\ E_\zeta{}^A & E_\zeta{}^{++} & E_\zeta{}^{--} & E_\zeta{}^0 \\ E_{\bar\zeta}{}^A & E_{\bar\zeta}{}^{++} & E_{\bar\zeta}{}^{--} & E_{\bar\zeta}{}^0 \\ E_\psi{}^A & E_\psi{}^{++} & E_\psi{}^{--} & E_\psi{}^0 \end{pmatrix}. \tag{3.18}$$

We make no assumptions about whether the vielbeins and connections are globally defined on $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$. In fact, we generically need (at least) two charts for SU(2).

[12] Note that the central gauge is not unique; any harmonic-independent gauge transformation, z^M-diffeomorphism or SU(2) isometry will take us from one central gauge to another.

Again we have a prescription for raising the $\pm$ tangent space indices, $\nabla_{\underline{\alpha}\mp} = \pm\nabla_{\underline{\alpha}}^{\pm}$, $\nabla_{\mp\mp} = \pm\nabla^{\pm\pm}$, $S_{\underline{\alpha}\mp} = \mp S_{\underline{\alpha}}^{\pm}$, and $\nabla_0 = \nabla^0$, so that they correspond to the ∇^0 charge of the operator. Now let us summarize the algebra of the various operators. The Lorentz generator is normalized to obey

$$\begin{aligned}
[M_{ab}, M_{cd}] &= -2\eta_{c[a} M_{b]d} + 2\eta_{d[a} M_{b]c}, \\
[M_{ab}, \nabla_c] &= \eta_{bc} \nabla_a - \eta_{ac} \nabla_b, \\
[M_{ab}, \nabla_\gamma^\pm] &= (\sigma_{ab})_\gamma{}^\beta \nabla_\beta^\pm, \\
[M_{ab}, \bar\nabla^{\dot\gamma\pm}] &= (\bar\sigma_{ab})^{\dot\gamma}{}_{\dot\beta} \bar\nabla^{\dot\beta\pm}.
\end{aligned} \tag{3.19}$$

The action of the dilatation and $\mathrm{U}(1)_R$ generators is

$$\begin{aligned}
&[\mathbb{D}, \nabla_\alpha^\pm] = \frac{1}{2}\nabla_\alpha^\pm, \qquad [\mathbb{D}, \bar\nabla^{\dot\alpha\pm}] = \frac{1}{2}\bar\nabla^{\dot\alpha\pm}, \\
&[\mathbb{D}, S_\alpha^\pm] = -\frac{1}{2} S_\alpha^\pm, \qquad [\mathbb{D}, \bar S^{\dot\alpha\pm}] = -\frac{1}{2}\bar S^{\dot\alpha\pm}, \\
&[\mathbb{D}, \nabla_a] = \nabla_a, \qquad [\mathbb{D}, K_a] = -K_a, \\
&[\mathbb{A}, \nabla_\alpha^\pm] = -i\nabla_\alpha^\pm, \qquad [\mathbb{A}, \bar\nabla^{\dot\alpha\pm}] = +i\bar\nabla^{\dot\alpha\pm}, \\
&[\mathbb{A}, S_\alpha^\pm] = +iS_\alpha^\pm, \qquad [\mathbb{A}, \bar S^{\dot\alpha\pm}] = -i\bar S^{\dot\alpha\pm}.
\end{aligned} \tag{3.20}$$

The special conformal and S-supersymmetry generators obey

$$\begin{aligned}
[K_a, \nabla_b] &= 2\eta_{ab}\mathbb{D} - 2M_{ab}, \\
\{S_\beta^\pm, \nabla_\alpha^\pm\} &= \pm 4\epsilon_{\beta\alpha}\nabla^{\pm\pm}, \\
\{\bar S^{\dot\beta\pm}, \bar\nabla^{\dot\alpha\pm}\} &= \mp 4\epsilon^{\dot\beta\dot\alpha}\nabla^{\pm\pm}, \\
\{S_\beta^\mp, \nabla_\alpha^\pm\} &= \pm(2\epsilon_{\beta\alpha}\mathbb{D} - 2M_{\beta\alpha} - i\epsilon_{\beta\alpha}\mathbb{A}) - 2\epsilon_{\beta\alpha}\nabla^0, \\
\{\bar S^{\dot\beta\mp}, \bar\nabla^{\dot\alpha\pm}\} &= \mp(2\epsilon^{\dot\beta\dot\alpha}\mathbb{D} - 2M^{\dot\beta\dot\alpha} + i\epsilon^{\dot\beta\dot\alpha}\mathbb{A}) + 2\epsilon^{\dot\beta\dot\alpha}\nabla^0, \\
[K_a, \nabla_\alpha^\pm] &= i(\sigma_a)_{\alpha\dot\beta}\bar S^{\dot\beta\pm}, \\
[K_a, \bar\nabla^{\dot\alpha\pm}] &= i(\bar\sigma_a)^{\dot\alpha\beta} S_\beta^\pm, \\
[S_\alpha^\pm, \nabla_a] &= i(\sigma_a)_{\alpha\dot\beta}\bar\nabla^{\dot\beta\pm}, \\
[\bar S^{\dot\alpha\pm}, \nabla_a] &= i(\bar\sigma_a)^{\dot\alpha\beta}\nabla_\beta{}^\pm, \\
[\nabla^{\pm\pm}, S_{\underline{\alpha}}^\pm] &= 0, \qquad [\nabla^{\mp\mp}, S_{\underline{\alpha}}^\pm] = S_{\underline{\alpha}}^\mp, \\
[\nabla^0, S_{\underline{\alpha}}^\pm] &= \pm S_{\underline{\alpha}}^\pm.
\end{aligned} \tag{3.21}$$

Up to this point, we have only been discussing the algebra of the gauge generators with themselves and with the covariant derivatives $\nabla_{\underline{A}}$. These dictate how the connections transform under the corresponding symmetries. (An explicit discussion of this can be found, for example, in [34].) What remains is to specify the algebra of the covariant derivatives themselves, corresponding to the torsion and curvatures on the supermanifold. The various constraints imposed will dictate the supergeometry.

Begin by specifying the algebra of the SU(2) covariant derivatives with the spinor derivatives:

$$[\nabla^{++},\nabla^{--}]=\nabla^0,\qquad [\nabla^0,\nabla^{\pm\pm}]=\pm 2\nabla^{\pm\pm},$$
$$[\nabla^{\pm\pm},\nabla^{\pm}_{\underline{\alpha}}]=0,\qquad [\nabla^{\mp\mp},\nabla^{\pm}_{\underline{\alpha}}]=\nabla^{\mp}_{\underline{\alpha}},$$
$$[\nabla^0,\nabla^{\pm}_{\underline{\alpha}}]=\pm\nabla^{\pm}_{\underline{\alpha}}. \tag{3.22}$$

These conditions imply that the SU(2) part of the manifold is flat, possessing only constant torsion and no curvature, and are necessary for the existence of a central gauge where the SU(2) manifold (almost) decouples. In other words, if we did not impose these constraints, then we would be introducing new degrees of freedom. For the algebra of the spinor covariant derivatives, we impose

$$\{\nabla^{\pm}_{\underline{\alpha}},\nabla^{\pm}_{\underline{\beta}}\}=0. \tag{3.23}$$

This is an integrability condition for the existence of analytic superfields, which we will discuss shortly. The remainder of the dimension-1 curvatures can be written

$$\{\nabla^{\pm}_{\alpha},\bar\nabla^{\mp}_{\dot\beta}\}=\mp 2i\nabla_{\alpha\dot\beta},$$
$$\{\nabla^{\pm}_{\alpha},\nabla^{\mp}_{\beta}\}=\pm 2\epsilon_{\alpha\beta}\bar{\mathcal{W}},$$
$$\{\bar\nabla^{\dot\alpha\pm},\bar\nabla^{\dot\beta\mp}\}=\pm 2\epsilon^{\dot\alpha\dot\beta}\mathcal{W}. \tag{3.24}$$

The first equation of (3.24) is a conventional constraint and serves to define $\nabla_{\alpha\dot\beta}=(\sigma^a)_{\alpha\dot\beta}\nabla_a$. As a consequence, the vector covariant derivative has vanishing algebra with the SU(2) derivatives, $[\nabla^{\pm\pm},\nabla_a]=[\nabla^0,\nabla_a]=0$, and obeys the other algebraic properties given in Eqs. (3.19)–(3.21). The second and third equations involve a chiral primary operator $\mathcal{W}$ and its conjugate antichiral primary operator $\bar{\mathcal{W}}$, which are constrained by

$$[\nabla^{\pm\pm},\mathcal{W}]=[\nabla^0,\mathcal{W}]=[\bar\nabla^{\pm}_{\dot\alpha},\mathcal{W}]=0,$$
$$[\nabla^{\pm\pm},\bar{\mathcal{W}}]=[\nabla^0,\bar{\mathcal{W}}]=[\nabla^{\pm}_{\alpha},\bar{\mathcal{W}}]=0,$$
$$\{\nabla^{\gamma+},[\nabla^{+}_{\gamma},\mathcal{W}]\}=\{\bar\nabla^{+}_{\dot\gamma},[\bar\nabla^{\dot\gamma+},\bar{\mathcal{W}}]\}. \tag{3.25}$$

The solution corresponding to conformal superspace involves specifying $\mathcal{W}$ in terms of a superfield $W_{\alpha\beta}$,

$$\mathcal{W}=\frac{1}{2}W^{\alpha\beta}M_{\beta\alpha}+\frac{1}{4}\nabla^{\beta+}W_{\beta}{}^{\alpha}S^{-}_{\alpha}-\frac{1}{4}\nabla^{\beta-}W_{\beta}{}^{\alpha}S^{+}_{\alpha}+\frac{1}{4}\nabla^{\dot\alpha\beta}W_{\beta}{}^{\alpha}K_{\alpha\dot\alpha}, \tag{3.26a}$$

$$\bar{\mathcal{W}}=\frac{1}{2}\bar W_{\dot\alpha\dot\beta}M^{\dot\beta\dot\alpha}+\frac{1}{4}\bar\nabla^{-}_{\dot\beta}\bar W^{\dot\beta}{}_{\dot\alpha}\bar S^{\dot\alpha+}-\frac{1}{4}\bar\nabla^{+}_{\dot\beta}\bar W^{\dot\beta}{}_{\dot\alpha}\bar S^{\dot\alpha-}+\frac{1}{4}\nabla_{\alpha\dot\beta}\bar W^{\dot\beta}{}_{\dot\alpha}K^{\dot\alpha\alpha}. \tag{3.26b}$$

These operators obey (3.25) provided $W_{\alpha\beta}$ is primary and obeys the constraints

$$\nabla^{\pm\pm}W_{\alpha\beta}=\nabla^0W_{\alpha\beta}=\bar\nabla^{\pm}_{\dot\gamma}W_{\alpha\beta}=0,$$
$$\nabla^{\alpha\beta}W_{\alpha\beta}=\bar\nabla^{\dot\alpha\dot\beta}\bar W_{\dot\alpha\dot\beta}, \tag{3.27}$$

where we have introduced the abbreviations

$$\nabla^{\alpha\beta}:=2\nabla^{(\alpha+}\nabla^{\beta)-}=-2\nabla^{(\alpha-}\nabla^{\beta)+},$$
$$\bar\nabla^{\dot\alpha\dot\beta}:=2\bar\nabla^{(\dot\alpha+}\bar\nabla^{\dot\beta)-}=-2\bar\nabla^{(\dot\alpha-}\bar\nabla^{\dot\beta)+}. \tag{3.28}$$

In other words, $W_{\alpha\beta}$ is a chiral primary superfield inert under covariant SU(2) derivatives.

The dimension-3/2 curvatures can be written

$$[\nabla^{\pm}_{\beta},\nabla_{\alpha\dot\alpha}]=-2\epsilon_{\beta\alpha}\bar{\mathcal{W}}^{\pm}_{\dot\alpha},$$
$$[\bar\nabla^{\pm}_{\dot\beta},\nabla_{\alpha\dot\alpha}]=-2\epsilon_{\dot\beta\dot\alpha}\mathcal{W}^{\pm}_{\alpha}. \tag{3.29}$$

The operators $\mathcal{W}^{\pm}_{\underline{\alpha}}$ are given by $\mathcal{W}^{\pm}_{\alpha}=-\frac{i}{2}[\nabla^{\pm}_{\alpha},\mathcal{W}]$ and $\bar{\mathcal{W}}^{\pm}_{\dot\alpha}=-\frac{i}{2}[\bar\nabla^{\pm}_{\dot\alpha},\mathcal{W}]$ with explicit forms

$$\mathcal{W}^{+}_{\alpha}=-\frac{i}{8}\nabla^{\dot\gamma\beta}\nabla^{+}_{\alpha}W_{\beta}{}^{\gamma}K_{\gamma\dot\gamma}+\frac{i}{16}(\nabla^{+})^2W_{\alpha}{}^{\gamma}S^{-}_{\gamma}+\frac{i}{8}\nabla^{+}_{\alpha}\nabla^{\beta-}W_{\beta}{}^{\gamma}S^{+}_{\gamma}+\frac{1}{4}\nabla_{\dot\gamma}{}^{\beta}W_{\beta\alpha}\bar S^{\dot\gamma+}$$
$$-\frac{i}{4}\nabla^{\beta+}W^{\gamma}{}_{\alpha}M_{\gamma\beta}-\frac{i}{4}\nabla^{\beta+}W_{\beta\alpha}\left(\mathbb{D}-\frac{i}{2}\mathbb{A}-\nabla^0\right)+\frac{i}{2}\nabla^{\beta-}W_{\beta\alpha}\nabla^{++}+\frac{i}{2}W_{\alpha}{}^{\beta}\nabla^{+}_{\beta},$$
$$\bar{\mathcal{W}}^{+}_{\dot\alpha}=+\frac{i}{8}\nabla^{\dot\beta\gamma}\bar\nabla^{+}_{\dot\alpha}\bar W_{\dot\beta}{}^{\dot\gamma}K_{\gamma\dot\gamma}+\frac{i}{16}(\bar\nabla^{+})^2\bar W_{\dot\alpha}{}^{\dot\gamma}\bar S^{-}_{\dot\gamma}-\frac{i}{8}\bar\nabla^{+}_{\dot\alpha}\bar\nabla^{\dot\beta-}\bar W_{\dot\beta}{}^{\dot\gamma}\bar S^{+}_{\dot\gamma}+\frac{1}{4}\nabla_{\gamma}{}^{\dot\beta}\bar W_{\dot\beta\dot\alpha}S^{\gamma+}$$
$$-\frac{i}{4}\bar\nabla^{+}_{\dot\beta}\bar W_{\dot\gamma\dot\alpha}\bar M^{\dot\gamma\dot\beta}+\frac{i}{4}\bar\nabla^{\dot\beta+}\bar W_{\dot\beta\dot\alpha}\left(\mathbb{D}+\frac{i}{2}\mathbb{A}-\nabla^0\right)-\frac{i}{2}\bar\nabla^{\dot\beta-}\bar W_{\dot\beta\dot\alpha}\nabla^{++}-\frac{i}{2}\bar W_{\dot\alpha}{}^{\dot\beta}\bar\nabla^{+}_{\dot\beta},$$
$$\mathcal{W}^{-}_{\underline{\alpha}}=[\nabla^{--},\mathcal{W}^{+}_{\underline{\alpha}}]. \tag{3.30}$$

Above we have introduced $(\nabla^\pm)^2 := \nabla^{\gamma\pm}\nabla_\gamma^\pm$ and $(\bar\nabla^\pm)^2 := \bar\nabla^\pm_{\dot\gamma}\bar\nabla^{\dot\gamma\pm}$. Note that these operators obey the rules $[\nabla^{\pm\pm}, \mathcal{W}^\pm_{\underline\alpha}] = 0$ and $[\nabla^{\pm\pm}, \mathcal{W}^\mp_{\underline\alpha}] = \mathcal{W}^\pm_{\underline\alpha}$ as a consequence of (3.25).

The dimension-2 curvatures $[\nabla_b, \nabla_a]$ are a bit more complicated. Writing

$$[\nabla_{\beta\dot\beta}, \nabla_{\alpha\dot\alpha}] = -\mathcal{F}_{\beta\dot\beta\alpha\dot\alpha} = -2\epsilon_{\dot\beta\dot\alpha}\mathcal{F}_{\underbrace{\beta\alpha}} + 2\epsilon_{\beta\alpha}\mathcal{F}_{\underbrace{\dot\beta\dot\alpha}}, \quad (3.31)$$

the antiselfdual and selfdual components of $\mathcal{F}_{ba}$ are

$$\mathcal{F}_{\underbrace{\beta\alpha}} = \frac{1}{4}\{\nabla^+_{(\beta}, [\nabla^-_{\alpha)}, \mathcal{W}]\},$$
$$\mathcal{F}_{\underbrace{\dot\beta\dot\alpha}} = \frac{1}{4}\{\bar\nabla^+_{(\dot\beta}, [\bar\nabla^-_{\dot\alpha)}, \bar{\mathcal{W}}]\}. \quad (3.32)$$

The curvatures $\mathcal{F}_{ba}$ must be invariant under the SU(2) derivatives, $[\nabla^{\pm\pm}, \mathcal{F}_{ba}] = [\nabla^0, \mathcal{F}_{ba}] = 0$. The explicit expressions for $\mathcal{F}_{ba}$ won't be of much use to us here, so we will not discuss them explicitly. Instead, we collect them, along with the other curvatures, in Appendix A.

We note that under the generalized ~ conjugation, the derivatives transform as in [1]:

$$\widetilde{\nabla^\pm_\alpha} = -\bar\nabla^\pm_{\dot\alpha}, \qquad \widetilde{\bar\nabla^\pm_{\dot\alpha}} = \nabla^\pm_\alpha. \quad (3.33)$$

Finally, observe that this superspace admits a full set of gauge transformations, $\delta = \xi^{\underline{A}}\nabla_{\underline{A}} + \frac{1}{2}\lambda^{ab}M_{ba} + \lambda\mathbb{A} + \Lambda\mathbb{D} + \epsilon^A K_A$, where each of the parameters may depend arbitrarily on the coordinates $z^{\underline{M}}$.

Now let us argue that we can always recover the central gauge of the previous section. Because it is obvious that we can always start from the central gauge in constructing $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$, we will only give a sketch of a proof. As a consequence of the algebra (3.22), one can always adopt a gauge where $\nabla^{\pm\pm}$ and ∇^0 are given by their forms in the central gauge in terms of $v^{i\pm}$. This implies that the superspace vielbein takes the form (3.15) and the other connections the form (3.16). It is easy to prove that $\Omega_M{}^{ab}$, A_M, B_M and $F_M{}^a$ are independent of the SU(2) coordinates: one merely needs that the corresponding curvature components $R_{\underline{n}M}$ all vanish in this gauge. For the S-supersymmetry connection $F_M{}^{\underline{\alpha}\pm}$, the vanishing of $R(S)_{\underline{n}M}{}^{\underline{\alpha}\pm}$ implies that $F_M{}^{\underline{\alpha}\pm} = v_i^\pm F_M{}^{\underline{\alpha}i}$ as expected. For the vielbein $E_M{}^A$, similar arguments imply that $E_M{}^a$ is harmonic independent while $E_M{}^{\underline{\alpha}\pm} = E_M{}^{\underline{\alpha}i} v_i^\pm$. Finally, a similar argument with $\mathcal{V}_M{}^{\underline{a}}$ establishes that they are given by $\mathcal{V}_M{}^{\pm\pm} = \mathcal{V}_M{}^{ij} v_i^\pm v_j^\pm$ and $\mathcal{V}_M{}^0 = \mathcal{V}_M{}^{ij} v_i^+ v_j^-$.

C. Consequences of analyticity

In this paper, we will not present specific actions (e.g. explicit models involving hypermultiplets), so we will not have much need for an extended discussion of the types of superfields possible in this superspace. However, it is clear that if we wish to use the superspace $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$ for projective multiplets like those discussed in the introduction, then we must discuss (at least briefly) the consequences of imposing analyticity on superfields.

Due to the integrability conditions (3.23), it is admissible to have primary analytic superfields Ψ,

$$S^\pm_{\underline\alpha}\Psi = K_a\Psi = 0, \qquad \nabla^+_{\underline\alpha}\Psi = 0. \quad (3.34)$$

Consistency with the algebra implies that Ψ is a Lorentz scalar, invariant under $\mathrm{U}(1)_R$, and obeys

$$\nabla^0\Psi = \mathbb{D}\Psi, \qquad \nabla^{++}\Psi = 0. \quad (3.35)$$

The first condition implies Ψ must have ∇^0 charge equal to its conformal dimension; for definiteness, denote both quantities by n. The second condition ensures that in the central gauge Ψ is a holomorphic tensor on (an open domain of) $\mathbb{C}P^1$. These are exactly the same conditions (up to the complexification discussed in Sec. II F) as those for admissible projective multiplets $\mathcal{Q}^{(n)}$ in the usual formulation of projective superspace [20]. These conditions also match those found for superconformal projective multiplets in flat projective superspace [28].

An interesting feature of the superspace $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$ is that it *forbids* analytic superfields of the general harmonic type. Primary analytic superfields must be holomorphic on an open domain of SU(2)/U(1). We will briefly comment on this further in the conclusion.

IV. SUPERSPACE ACTION PRINCIPLES ON $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$

The original supermanifold $\mathcal{M}^{4|8}$ came equipped with two natural action principles, involving respectively integrals over the full superspace and the chiral superspace,

$$\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar\theta E\mathcal{L}, \qquad \int \mathrm{d}^4x\mathrm{d}^4\theta\mathcal{E}\mathcal{L}_c. \quad (4.1)$$

Here E and $\mathcal{E}$ were defined respectively as

$$E = \mathrm{sdet}E_M{}^A, \qquad \mathcal{E} = \mathrm{sdet}\begin{pmatrix} E_m{}^a & E_m{}^{\iota\alpha}{}_i \\ E_\mu{}^{\iota a} & E_\mu{}^{\iota\alpha}{}_i \end{pmatrix}; \quad (4.2)$$

the superspace Lagrangian $\mathcal{L}$ was required to be a conformal primary scalar superfield of vanishing dilation and $\mathrm{U}(1)_R$ weight, inert under $\mathrm{SU}(2)_R$,

$$\mathbb{D}\mathcal{L} = \mathbb{A}\mathcal{L} = I^i{}_j\mathcal{L} = K_A\mathcal{L} = 0; \quad (4.3)$$

and the chiral Lagrangian $\mathcal{L}_c$ was required to be a conformal primary chiral scalar superfield, inert under $\mathrm{SU}(2)_R$, with certain weights,

$$\mathbb{D}\mathcal{L}_c = 2\mathcal{L}_c, \qquad \mathbb{A}\mathcal{L}_c = 4i\mathcal{L}_c,$$
$$\bar{\nabla}^{\dot{\alpha}}{}_i\mathcal{L}_c = I^i{}_j\mathcal{L}_c = K_A\mathcal{L}_c = 0. \tag{4.4}$$

These properties of the respective Lagrangians can be proven e.g. by applying the results of Appendix B.

After extending the superspace to $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$, other possibilities emerge. The ones we will discuss below fall into three classes: full superspace integrals involving integrals over both S^2 and over a contour $\mathcal{C}$, analytic superspace integrals over a contour $\mathcal{C}$, and chiral-analytic superspace integrals over a contour $\mathcal{C}$.

A. Full superspace integrals

One can extend the full superspace action to include an integral over $\mathrm{SU}(2)/\mathrm{U}(1)$. In the central basis,

$$\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}E \int \mathrm{d}v\mathcal{L}^0, \tag{4.5}$$

where $\mathrm{d}v$ is the standard measure on the S^2,

$$\mathrm{d}v := \frac{i}{2\pi}\frac{\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}}{(1+\zeta\bar{\zeta})^2}, \tag{4.6}$$

and $\mathcal{L}^0$ is assumed to have vanishing D^0 charge, vanishing Weyl and $\mathrm{U}(1)_R$ weights, and to be globally defined on SU(2), but otherwise to be unconstrained. In a generic gauge, this action is written

$$\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}\mathrm{d}^2\zeta E^0\mathcal{L}^0, \tag{4.7}$$

using the abbreviation $\mathrm{d}^2\zeta := \frac{i}{2\pi}\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}$ for the complex coordinates on the S^2. The rest of the usual S^2 measure is contained in the full superspace measure

$$E^0 = \mathrm{sdet}\begin{pmatrix} E_M{}^A & E_M{}^{++} & E_M{}^{--} \\ E_\zeta{}^A & E_\zeta{}^{++} & E_\zeta{}^{--} \\ E_{\bar{\zeta}}{}^A & E_{\bar{\zeta}}{}^{++} & E_{\bar{\zeta}}{}^{--} \end{pmatrix}. \tag{4.8}$$

The full superspace action can also be extended to involve an integral over a contour $\mathcal{C}$. The natural choice is a purely holomorphic contour, given in the central gauge by[13]

$$-\frac{1}{2\pi}\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}E \oint_{\mathcal{C}} \mathcal{V}^{++}\mathcal{L}^{--}, \tag{4.9}$$

where $\mathcal{L}^{--}$ has vanishing Weyl and $\mathrm{U}(1)_R$ weights, but is required to be holomorphic with D^0 charge -2,

$$D^{++}\mathcal{L}^{--} = 0, \qquad D^0\mathcal{L}^{--} = -2\mathcal{L}^{--}. \tag{4.10}$$

Extending to a generic gauge is straightforward. Letting τ be the coordinate parametrizing the contour, we introduce the action

$$-\frac{1}{2\pi}\oint_{\mathcal{C}}\mathrm{d}\tau\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}E^{++}\mathcal{L}^{--}. \tag{4.11}$$

where

$$E^{++} = \mathrm{sdet}\begin{pmatrix} E_M{}^A & E_M{}^{++} \\ E_\tau{}^A & E_\tau{}^{++} \end{pmatrix}, \tag{4.12}$$

with $E_\tau{}^{\underline{A}}$ corresponding to the pullback of the one-form $E^{\underline{A}}$ to the contour.[14] Applying the results of Appendix B, $\mathcal{L}^{--}$ must be a covariantly holomorphic primary superfield with vanishing Weyl and $\mathrm{U}(1)_R$ weights and ∇^0 charge -2:

$$0 = \nabla^{++}\mathcal{L}^{--} = K_A\mathcal{L}^{--} = \mathbb{D}\mathcal{L}^{--} = \mathbb{A}\mathcal{L}^{--},$$
$$\nabla^0\mathcal{L}^{--} = -2\mathcal{L}^{--}. \tag{4.13}$$

Within projective superspace, the natural quantities are holomorphic on $\mathrm{SU}(2)/\mathrm{U}(1)$, so the action principle (4.9) [or (4.11) in its generic form] is more commonly encountered than (4.5) [or (4.7) in its generic form]. In fact, as we will shortly review, the action principle (4.9) can also efficiently encapsulate the other relevant action principles involving integrals over smaller superspaces. Let us describe these other possibilities next.

B. Analytic superspace integrals

As discussed in the Introduction, the natural action principle in projective superspace involves a contour integral and a Grassmann integration over $\theta^{\mu+} = v_i^+\theta^{\mu i}$ and $\bar{\theta}^{\dot{\mu}+} = v_i^+\bar{\theta}^{\dot{\mu}i}$. In flat projective superspace, such actions take the form [29,32]

$$-\frac{1}{2\pi}\oint_{\mathcal{C}} v_i^+\mathrm{d}v^{i+}\int \mathrm{d}^4x\mathrm{d}^4\theta^+\mathcal{L}^{++}$$
$$= -\frac{1}{2\pi}\oint_{\mathcal{C}} v_i^+\mathrm{d}v^{i+}\int \mathrm{d}^4x(D^-)^4\mathcal{L}^{++}, \tag{4.14}$$

where $\mathcal{L}^{++}$ is a holomorphic analytic Lagrangian, $D^{++}\mathcal{L}^{++} = D^+_{\underline{\alpha}}\mathcal{L}^{++} = 0$.

The curved generalization of the analytic superspace integral (4.14) is naturally written

$$-\frac{1}{2\pi}\oint_{\mathcal{C}}\mathrm{d}\tau\int \mathrm{d}^4x\mathrm{d}^4\theta^+\mathcal{E}^{--}\mathcal{L}^{++}, \tag{4.15}$$

where the measure is

[13] This action principle is used as the universal action principle in the conventional formulation of projective superspace [20]. We will discuss shortly why this form is actually universal.

[14] For example, $E_\tau{}^{++} \equiv \dot{\zeta}E_\zeta{}^{++} + \dot{\bar{\zeta}}E_{\bar{\zeta}}{}^{++}$, where $\dot{} := \mathrm{d}/\mathrm{d}\tau$.

$$\mathcal{E}^{--} = \mathrm{sdet}\begin{pmatrix} E_m{}^a & E_m{}^{\underline{\alpha}+} & E_m{}^{++} \\ E_{\underline{\mu}+}{}^a & E_{\underline{\mu}+}{}^{\underline{\alpha}+} & E_{\underline{\mu}+}{}^{++} \\ E_\tau{}^a & E_\tau{}^{\underline{\alpha}+} & E_\tau{}^{++} \end{pmatrix}. \tag{4.16}$$

The action is invariant under all gauge transformations provided $\mathcal{L}^{++}$ is a covariantly holomorphic, analytic, conformal primary superfield, with vanishing $\mathrm{U}(1)_R$ weight and equal Weyl and ∇^0 weights,

$$\nabla^{++}\mathcal{L}^{++} = \nabla_{\underline{\alpha}}^{+}\mathcal{L}^{++} = K_A\mathcal{L}^{++} = \mathbb{A}\mathcal{L}^{++} = 0,$$
$$\mathbb{D}\mathcal{L}^{++} = \nabla^0\mathcal{L}^{++} = 2\mathcal{L}^{++}. \tag{4.17}$$

The integral (4.15) is the natural action principle in projective superspace on $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$. We will discuss its component reduction in Sec. V. For now, we wish to establish the relationship between analytic superspace actions (4.15) and full superspace actions (4.11). Begin by recalling two relationships between $\mathcal{N} = 1$ full superspace and chiral superspace integrals, which are respectively written

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta E\mathcal{L}, \qquad \int \mathrm{d}^4x\mathrm{d}^2\theta\mathcal{E}\mathcal{L}_c. \tag{4.18}$$

The first relationship is that any full superspace integral can be written as a chiral superspace integral as

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta E\mathcal{L} = -\frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathcal{E}\bar\nabla^2\mathcal{L}$$
$$= -\frac{1}{4}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathcal{E}(\bar{\mathcal{D}}^2 - 8R)\mathcal{L}. \tag{4.19}$$

We have written the chiral integrand in two ways; the first expression is appropriate for $\mathcal{N} = 1$ conformal superspace [38] while the second involves the conventional formulation of $\mathcal{N} = 1$ Poincaré (old minimal) superspace.[15] The second relationship can be written

$$\int \mathrm{d}^4x\mathrm{d}^4\theta\mathcal{E}\mathcal{L}_c = -4\int \mathrm{d}^4x\mathrm{d}^4\theta E\frac{X}{\bar\nabla^2 X}\mathcal{L}_c$$
$$= -4\int \mathrm{d}^4x\mathrm{d}^4\theta E\frac{X}{(\bar{\mathcal{D}}^2 - 8R)X}\mathcal{L}_c, \tag{4.20}$$

where X is a real primary superfield of dimension 2. [The proof follows by applying (4.19) to the right-hand side.] In this expression, $\bar\nabla^2 X$ is chiral and primary and so the second integrand is primary. The third integrand involves the same expression in Poincaré (old minimal) superspace. This last expression is especially useful because we can adopt the Weyl gauge where $X = 1$, in which case the above equality simplifies to

$$\int \mathrm{d}^4x\mathrm{d}^4\theta\mathcal{E}\mathcal{L}_c = \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4\theta\frac{E}{R}\mathcal{L}_c. \tag{4.21}$$

It turns out that two analogous relationships can be constructed between full superspace and analytic superspace, both over a contour $\mathcal{C}$. The first relationship we will establish is the analogue of (4.19),

$$-\frac{1}{2\pi}\oint_{\mathcal{C}}\mathrm{d}\tau\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar\theta E^{++}\mathcal{L}^{--}$$
$$= -\frac{1}{2\pi}\oint_{\mathcal{C}}\mathrm{d}\tau\int \mathrm{d}^4x\mathrm{d}^4\theta^+\mathcal{E}^{--}(\nabla^+)^4\mathcal{L}^{--}. \tag{4.22}$$

To prove this, we go to the analytic gauge where the covariant derivative $\nabla_{\underline{\alpha}-} \equiv \nabla_{\underline{\alpha}}^{+}$ is simply given by $\partial/\partial\theta^{\underline{\alpha}-}$. This is always possible to do because of the constraints (3.23). This fixes the gauge up to $\theta^{\underline{\mu}-}$-independent gauge transformations. In this gauge, E^{++} is equal to $\mathcal{E}^{--}$; the difference in apparent ∇^0 charges of the two quantities arises because in the analytic gauge, any ∇^0 gauge transformation must be accompanied by a special diffeomorphism to maintain that gauge. The integral becomes

$$-\frac{1}{32\pi}\oint_{\mathcal{C}}\mathrm{d}\tau\int \mathrm{d}^4x\mathrm{d}^4\theta^+\partial_-^\alpha\partial_{\alpha-}\bar\partial_{\dot\alpha-}\bar\partial_-^{\dot\alpha}(\mathcal{E}^{--}\mathcal{L}^{--}). \tag{4.23}$$

$\mathcal{E}^{--}$ is itself analytic in this gauge,

$$\partial_{\underline{\alpha}-}\mathcal{E}^{--} = \mathcal{E}^{--}(\partial_{\underline{\alpha}-}\mathcal{E}_{\underline{N}}{}^{\underline{B}})\mathcal{E}_{\underline{B}}{}^{\underline{N}}(-)^N$$
$$= \mathcal{E}^{--}T_{\underline{\alpha}-\underline{N}}{}^{\underline{B}}\mathcal{E}_{\underline{B}}{}^{\underline{N}}(-)^N = 0. \tag{4.24}$$

As a result, we find

$$-\frac{1}{32\pi}\oint_{\mathcal{C}}\mathrm{d}\tau\int \mathrm{d}^4x\mathrm{d}^4\theta^+\mathcal{E}^{--}\partial_-^\alpha\partial_{\alpha-}\bar\partial_{\dot\alpha-}\bar\partial_-^{\dot\alpha}\mathcal{L}^{--}, \tag{4.25}$$

with the integrand equal to $(\nabla^+)^2(\bar\nabla^+)^2\mathcal{L}^{--}$ in this gauge. Rewriting in a gauge-invariant way recovers (4.22).

In projective superspace, the expression analogous to (4.20) is

$$-\frac{1}{2\pi}\oint_{\mathcal{C}}\mathrm{d}\tau\int \mathrm{d}^4x\mathrm{d}^4\theta^+\mathcal{E}^{--}\mathcal{L}^{++}$$
$$= -\frac{1}{2\pi}\oint_{\mathcal{C}}\mathrm{d}\tau\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar\theta E^{++}\frac{X}{(\nabla^+)^4X}\mathcal{L}^{++}, \tag{4.26}$$

where X is a real superfield of conformal dimension 2 and invariant under the SU(2) derivatives. $(\nabla^+)^4X$ is a real conformal primary of dimension 4 and so the integrand on the right-hand side is a real primary superfield of vanishing weight. The advantage of the right-hand side is that it can be formulated directly in the central gauge. Indeed, an equivalent formulation appeared in [20] (mirroring an identical construction in 5D [19]) where it was used to

[15]We use the conventions of [35]. See also [39,40], where different conventions are employed.

define analytic integration in the central gauge. There the particular choice $X = W\bar{W}$ was made, where W was an abelian vector multiplet. Moving to the central gauge where $E^{++} = E\mathcal{V}_\tau^{++}$, one finds

$$-\frac{1}{2\pi}\oint_{\mathcal{C}} \mathrm{d}\tau \int \mathrm{d}^4x\mathrm{d}^4\theta^+ \mathcal{E}^{--}\mathcal{L}^{++} = -\frac{16}{2\pi}\oint_{\mathcal{C}} \mathcal{V}^{++}\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}\frac{EW\bar{W}}{(\nabla^+)^2W(\bar{\nabla}^+)^2\bar{W}}\mathcal{L}^{++}. \tag{4.27}$$

If one degauges conformal superspace to SU(2) superspace, $(\nabla^+)^2W$ becomes $((\mathcal{D}^+)^2 + 4S^{++})W$. The super-Weyl gauge $W = 1$ leads to the final relation

$$-\frac{1}{2\pi}\oint_{\mathcal{C}} \mathrm{d}\tau \int \mathrm{d}^4x\mathrm{d}^4\theta^+ \mathcal{E}^{--}\mathcal{L}^{++} = -\frac{1}{2\pi}\oint_{\mathcal{C}} \mathcal{V}^{++}\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}\frac{E}{(S^{++})^2}\mathcal{L}^{++}. \tag{4.28}$$

The expression on the right is a particularly elegant form of the analytic action principle [20], permitting easy manipulation in the central gauge. The similarity with the $\mathcal{N} = 1$ analogue (4.21) is especially striking.

Before moving on to another possible action principle, we should comment why we did not consider analytic integrals over the full S^2, which would presumably lead to a curved harmonic superspace action principle. From the discussion in Sec. III C, we know that any analytic primary Lagrangian obeying $\nabla^+_{\underline{\alpha}}\mathcal{L}^{(q)} = 0$ must also be covariantly holomorphic $\nabla^{++}\mathcal{L}^{(q)} = 0$. This condition is difficult to reconcile with harmonic integration on the S^2, where we expect any integrand to be globally defined. Even if this barrier could be overcome, one still finds an essential difficulty in the equality between the ∇^0 charge and the Weyl weight. If one is to construct the curved superspace generalization of a harmonic superspace integral, the leading term should be $(\nabla^-)^4\mathcal{L}^{(q)}$, which suggests the choice $q = 4$. However, the Weyl weight of $\mathcal{L}^{(q)}$ requires $q = 2$, and so the charge is inconsistent with harmonic integration. We will comment further about the resolution to the problem of covariant harmonic superspace in the conclusion.

C. Chiral-analytic superspace

The final action principle we will discuss is a curious one because it involves an integration over 3/4 of the Grassmann variables, with a complex conformal primary Lagrangian $\mathcal{L}^0$ that is *chiral-analytic*, $\bar{\nabla}^+_{\dot{\alpha}}\mathcal{L}^0 = 0$. Such a Lagrangian would, in the analytic gauge, be independent of $\bar{\theta}^{\dot{\mu}-}$. Provided that the Lagrangian is holomorphic with certain weights, $\nabla^{++}\mathcal{L}^0 = 0$, $\mathbb{D}\mathcal{L}^0 = \mathcal{L}^0$ $\mathbb{A}\mathcal{L}^0 = 2\mathcal{L}^0$, and $\nabla^0\mathcal{L}^0 = 0$, then the following action is invariant:

$$-\frac{1}{2\pi}\oint_{\mathcal{C}} \mathrm{d}\tau \int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^2\bar{\theta}^+\mathcal{E}^0\mathcal{L}^0 + \text{H.c.}, \tag{4.29}$$

where the measure is

$$\mathcal{E}^0 = \mathrm{sdet}\begin{pmatrix} E_m{}^a & E_m{}^{\alpha\pm} & E_m{}^{\dot{\alpha}+} & E_m{}^{++} \\ E_{\mu\pm}{}^a & E_{\mu\pm}{}^{\alpha\pm} & E_{\mu\pm}{}^{\dot{\alpha}+} & E_{\mu\pm}{}^{++} \\ E_{\dot{\mu}+}{}^a & E_{\dot{\mu}+}{}^{\alpha\pm} & E_{\dot{\mu}+}{}^{\dot{\alpha}+} & E_{\dot{\mu}+}{}^{++} \\ E_\tau{}^a & E_\tau{}^{\alpha\pm} & E_\tau{}^{\dot{\alpha}+} & E_\tau{}^{++} \end{pmatrix}. \tag{4.30}$$

Such chiral-analytic actions are naturally a higher derivative and have been discussed recently in [41] in the context of curved projective superspace, as well as [42] in the context of flat harmonic superspace.

To evaluate such actions, one can convert them to analytic integrals by integrating over the two $\theta^{\mu-}$ coordinates:

$$-\frac{1}{2\pi}\oint_{\mathcal{C}} \mathrm{d}\tau \int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^2\bar{\theta}^+\mathcal{E}^0\mathcal{L}^0 = \frac{1}{8\pi}\oint_{\mathcal{C}} \mathrm{d}\tau \int \mathrm{d}^4x\mathrm{d}^4\theta^+\mathcal{E}^{--}(\nabla^+)^2\mathcal{L}^0. \tag{4.31}$$

The integrand $(\nabla^+)^2\mathcal{L}^0$ satisfies all the required properties of an analytic superspace Lagrangian. Alternatively, one can lift a chiral-analytic superspace integral to full superspace in the same way as Eqs. (4.26)–(4.28). For example, using the antichiral field strength $\bar{W}$ of a vector multiplet, one has in the central gauge

$$-\frac{1}{2\pi}\oint_{\mathcal{C}} \mathrm{d}\tau \int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^2\bar{\theta}^+\mathcal{E}^0\mathcal{L}^0 = \frac{2}{\pi}\oint_{\mathcal{C}} \mathcal{V}^{++}\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}E\frac{\bar{W}}{(\bar{\nabla}^+)^2\bar{W}}\mathcal{L}^0 \tag{4.32}$$

or imposing the Weyl-U(1) gauge $\bar{W} = 1$,

$$-\frac{1}{2\pi}\oint_{\mathcal{C}} \mathrm{d}\tau \int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^2\bar{\theta}^+\mathcal{E}^0\mathcal{L}^0 = \frac{1}{2\pi}\oint_{\mathcal{C}} \mathcal{V}^{++}\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}E\frac{1}{S^{++}}\mathcal{L}^0. \tag{4.33}$$

This formulation of the chiral-analytic projective superspace action appeared in [41]. Finally, we mention that one can convert a chiral-analytic integral to a chiral superspace integral by integrating over $\theta^{\underline{\mu}+}$. This is easiest in the central gauge:

$$-\frac{1}{2\pi}\oint_{\mathcal{C}} \mathrm{d}\tau \int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^2\bar{\theta}^+\mathcal{E}^0\mathcal{L}^0 = \frac{1}{8\pi}\int \mathrm{d}^4x\mathrm{d}^4\theta\mathcal{E}\oint_{\mathcal{C}} \mathcal{V}^{++}(\bar{\nabla}^-)^2\mathcal{L}^0. \quad (4.34)$$

The simplest proof of this is to convert the full superspace integral on the right-hand side of (4.32) to a chiral superspace integral while remaining in the central gauge.

V. COMPONENT REDUCTION OF ANALYTIC SUPERSPACE ACTION

Our goal in this section is to perform the component reduction of the general analytic superspace action

$$S = -\frac{1}{2\pi}\oint_{\mathcal{C}} \mathrm{d}\tau \int \mathrm{d}^4x\mathrm{d}^4\theta^+\mathcal{E}^{--}\mathcal{L}^{++} \quad (5.1)$$

in the central gauge. We begin by noting that the action can be evaluated at $\theta^{\underline{\mu}-}=0$. Along this submanifold, it is possible to adopt a gauge where $\nabla_{\underline{\alpha}+} = \partial/\partial\theta^{\underline{\alpha}+}$, corresponding to

$$\mathcal{E}^{--} = \mathrm{sdet}\begin{pmatrix} E_m{}^a & E_m{}^{++} & E_m{}^{\underline{\alpha}+} \\ E_\tau{}^a & E_\tau{}^{++} & E_\tau{}^{\underline{\alpha}+} \\ 0 & 0 & \delta_{\underline{\mu}}{}^{\underline{\alpha}} \end{pmatrix} = \mathrm{sdet}\begin{pmatrix} E_m{}^a & E_m{}^{++} \\ E_\tau{}^a & E_\tau{}^{++} \end{pmatrix} \equiv e^{++}, \quad (5.2)$$

so our goal is to evaluate

$$S = -\frac{1}{2\pi}\frac{1}{16}\int \mathrm{d}^4x \oint_{\mathcal{C}} \mathrm{d}\tau(\partial_+)^2(\bar{\partial}_+)^2(e^{++}\mathcal{L}^{++}). \quad (5.3)$$

At this stage, we emphasize that $\theta^{\underline{\mu}\pm}$-independent gauge transformations are still permitted in the gauge $\nabla_{\underline{\alpha}+} = \partial/\partial\theta^{\underline{\alpha}+}$. In other words, the gauge of the *component fields* at $\theta^{\underline{\mu}\pm}=0$ remains completely unfixed. Naturally, one expects the resulting action should take its simplest form if we adopt the central gauge at $\theta^{\underline{\mu}\pm}=0$, and we will do this at the very end. However, it is not easy to impose central gauge at the component level prior to taking the $\theta^{\underline{\mu}+}$ derivatives, so we will remain in a more general gauge for the time being.

To organize the calculation, it is convenient to write the integrand as a five-form:

$$S = -\frac{1}{2\pi}\frac{1}{16}\int_{\mathcal{M}^4\times\mathcal{C}}(\partial_+)^2(\bar{\partial}_+)^2(\hat{e}^{++}\mathcal{L}^{--}), \quad (5.4)$$

where $\hat{e}^{++}$ is the volume five-form

$$\hat{e}^{++} = \mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}\tau e^{++} = \frac{1}{4!}\epsilon_{abcd}E^a\wedge E^b\wedge E^c\wedge E^d\wedge E^{++}. \quad (5.5)$$

Taking the $\theta^{\underline{\mu}+}$ derivatives of this five-form proves to be simpler than the corresponding calculation with the determinant. Expanding out the action, one finds

$$S = -\frac{1}{2\pi}\int_{\mathcal{M}^4\times\mathcal{C}}\mathcal{J}, \quad (5.6)$$

where the five-form $\mathcal{J}$ is

$$\begin{aligned}\mathcal{J} &= \hat{e}^{++}(\nabla^-)^4\mathcal{L}^{++} - \frac{1}{8}\partial^\alpha_+\hat{e}^{++}\nabla^-_\alpha(\bar{\nabla}^-)^2\mathcal{L}^{++} - \frac{1}{8}\bar{\partial}_{\dot{\alpha}+}\hat{e}^{++}\bar{\nabla}^{\dot{\alpha}-}(\nabla^-)^2\mathcal{L}^{++} \\ &\quad + \frac{1}{16}(\partial_+)^2\hat{e}^{++}(\bar{\nabla}^-)^2\mathcal{L}^{++} + \frac{1}{16}(\bar{\partial}_+)^2\hat{e}^{++}(\nabla^-)^2\mathcal{L}^{++} + \frac{1}{8}\partial_{\alpha+}\bar{\partial}_{\dot{\alpha}+}\hat{e}^{++}[\nabla^{\alpha-},\bar{\nabla}^{\dot{\alpha}-}]\mathcal{L}^{++} \\ &\quad - \frac{1}{8}\partial^\alpha_+(\bar{\partial}_+)^2\hat{e}^{++}\nabla^-_\alpha\mathcal{L}^{++} - \frac{1}{8}\bar{\partial}_{\dot{\alpha}+}(\partial_+)^2\hat{e}^{++}\bar{\nabla}^{\dot{\alpha}-}\mathcal{L}^{++} + \frac{1}{16}(\partial_+)^2(\bar{\partial}_+)^2\hat{e}^{++}\mathcal{L}^{++}.\end{aligned} \quad (5.7)$$

In the above expression, we have replaced $\partial_{\underline{\alpha}+}\to\nabla_{\underline{\alpha}+}$ for all the derivatives acting upon the analytic Lagrangian $\mathcal{L}^{++}$. This is allowed because after projecting to $\theta^{\underline{\mu}+}=\theta^{\underline{\mu}-}=0$ (implicitly assumed above) the result holds in a general component gauge. To recover the explicit expression for $\mathcal{J}$, one must evaluate each of the $\theta^{\underline{\mu}+}$ derivatives of $\hat{e}^{++}$. This can be done systematically, although the resulting formulae grow quite complicated as the number of spinor derivatives increases. The results are given in Eqs. (C9)–(C12) of Appendix C, where some details of the calculation are also included. We emphasize that upon using Eqs. (C9)–(C12), the result for $\mathcal{J}$ is given in a general component gauge.

Some comments should now be made about the nature of this five-form:

(i) It is invariant under all gauge transformations, up to an exact form. This is a direct consequence of its origin from a gauge-invariant superspace action, but it can be checked explicitly. A straightforward calculation shows, for example, that $\mathcal{J}$ transforms under S-supersymmetry, $\delta = \eta^{\underline{\alpha}+}S^-_{\underline{\alpha}} - \eta^{\underline{\alpha}-}S^+_{\underline{\alpha}}$, into an exact form involving $\eta^{\alpha+}$. Therefore, strictly speaking, $\mathcal{J}$ is *not* a conformal primary five-form, although its integral is invariant.

(ii) Viewed as a five-form in superspace, $\mathcal{J}$ is closed. In principle, this can also be established by an explicit computation but is a direct consequence of its construction. Under an arbitrary diffeomorphism on $\mathcal{M}^{4|8}\times \mathrm{SU}(2)$, $\mathcal{J}$ transforms as

$$\delta_\xi \mathcal{J} = \mathrm{d}(\imath_\xi \mathcal{J}) + \imath_\xi \mathrm{d}\mathcal{J}. \tag{5.8}$$

The first term vanishes upon integration over the bosonic manifold $\mathcal{M}^4\times\mathcal{C}$, while the second must vanish for arbitrary ξ because the original action was invariant under diffeomorphisms of all types. This implies that $\mathcal{J}$ is closed.

These two features are indicative of the superform approach to supersymmetric invariants [43], known within the superspace literature as the ectoplasm method [44,45] (see also [46]). There one usually encounters a superform expanded entirely in terms of the supervielbein—in our case, this would mean $\mathcal{J}=\frac{1}{5!}E^{\underline{A}_1}\wedge\cdots\wedge E^{\underline{A}_5}\mathcal{J}_{\underline{A}_5\cdots\underline{A}_1}$—but this is by no means a necessary requirement (see e.g. the discussion at the end of [44]). In fact, the five-form we have found above, given by (5.7) upon substituting (C9)–(C12), involves the explicit appearance of S-supersymmetry and special conformal connections.

A dramatic simplification of $\mathcal{J}$ occurs if we now adopt the central gauge for the $\theta^{\underline{\mu}\pm}=0$ components of the connections. We leave the details again to Appendix C and merely summarize that the action can then be written $S=\int \mathrm{d}^4x e\mathcal{L}$ where the Lagrangian $\mathcal{L}$ involves a contour integral with two distinct integrands,

$$\mathcal{L} = -\frac{1}{2\pi}\oint_{\mathcal{C}}\mathcal{V}^{++}\mathcal{L}^{--} + \frac{1}{2\pi}\oint_{\mathcal{C}}\mathcal{V}^{--}\mathcal{L}^{++}, \tag{5.9}$$

where

$$\begin{aligned}\mathcal{L}^{--} =& \frac{1}{16}(\nabla^-)^2(\bar\nabla^-)^2\mathcal{L}^{++} - \frac{i}{8}(\bar\psi_m^-\bar\sigma^m)^\alpha\nabla_\alpha^-(\bar\nabla^-)^2\mathcal{L}^{++} - \frac{i}{8}(\psi_m^-\sigma^m)_{\dot\alpha}\bar\nabla^{\dot\alpha-}(\nabla^-)^2\mathcal{L}^{++}\\ &+\frac{1}{4}((\psi_n^-\sigma^{nm})^\alpha\bar\psi_m{}^{\dot\alpha-} + \psi_n{}^{\alpha-}(\bar\sigma^{nm}\bar\psi_m^-)^{\dot\alpha} - i\mathcal{V}_m^{--}(\sigma^m)_{\alpha\dot\alpha})[\nabla_\alpha^-,\bar\nabla_{\dot\alpha}^-]\mathcal{L}^{++}\\ &+\frac{1}{4}(\psi_m^-\sigma^{mn}\psi_n^-)(\nabla^-)^2\mathcal{L}^{++} + \frac{1}{4}(\bar\psi_m^-\bar\sigma^{mn}\bar\psi_n^-)(\bar\nabla^-)^2\mathcal{L}^{++}\\ &-\left(\frac{1}{2}\epsilon^{mnpq}(\psi_m^-\sigma_n\bar\psi_p^-)\psi_q^{\alpha-} - 2(\psi_m^-\sigma^{mn})^\alpha\mathcal{V}_n^{--}\right)\nabla_\alpha^-\mathcal{L}^{++}\\ &+\left(\frac{1}{2}\epsilon^{mnpq}(\bar\psi_m^-\bar\sigma_n\psi_p^-)\bar\psi_{q\dot\alpha}^- - 2(\bar\psi_m^-\bar\sigma^{mn})_{\dot\alpha}\mathcal{V}_n^{--}\right)\bar\nabla^{\dot\alpha-}\mathcal{L}^{++} + 3\epsilon^{mnpq}(\psi_m^-\sigma_n\bar\psi_p^-)\mathcal{V}_q^{--}\mathcal{L}^{++},\end{aligned} \tag{5.10}$$

$$\begin{aligned}\mathcal{L}^{++} =& -[3D + 4f_a{}^a - 4(\bar\psi_m^-\bar\sigma^{mn}\hat{\bar\phi}_n^+) + 4(\psi_m^-\sigma^{mn}\hat\phi_n^+) - 3\epsilon^{mnpq}(\psi_m^-\sigma_n\bar\psi_p^-)\mathcal{V}_q^{++}]\mathcal{L}^{++}\\ &+\left[\frac{3}{2}\chi^{\alpha+} - i(\bar\phi_m^+\bar\sigma^m)^\alpha + 2(\psi_m^-\sigma^{mn})^\alpha\mathcal{V}_n^{++}\right]\nabla_\alpha^-\mathcal{L}^{++} - \left[\frac{3}{2}\chi_{\dot\alpha}^+ - i(\phi_m^+\sigma^m)_{\dot\alpha} + 2(\bar\psi_m^-\bar\sigma^{mn})_{\dot\alpha}\mathcal{V}_n^{++}\right]\bar\nabla^{\dot\alpha-}\mathcal{L}^{++}\\ &-\frac{i}{4}\mathcal{V}_m^{++}(\bar\sigma^m)^{\dot\alpha\alpha}[\nabla_\alpha^-,\bar\nabla_{\dot\alpha}^-]\mathcal{L}^{++}.\end{aligned} \tag{5.11}$$

The component fields appearing above are defined in [34] and correspond to the matter content of $\mathcal{N}=2$ conformal supergravity. These consist of (i) five fundamental connections—the vierbein $e_m{}^a$, the gravitini $\psi_m{}^\alpha{}_i$, the $\mathrm{SU}(2)_R$ and $\mathrm{U}(1)_R$ connections $\mathcal{V}_m{}^i{}_j$ and A_m, and the dilatation connection b_m; (ii) covariant auxiliary fields W_{ab}, $\chi_{\alpha i}$, and D; and (iii) composite connections $\omega_m{}^{ab}$, $\phi_m{}^{\alpha i}$ and $f_m{}^a$, given in terms of the other fields, which are associated respectively with Lorentz, S-supersymmetry and special conformal gauge symmetries. In the expression for $\mathcal{L}^{++}$, we have used the symbol $\hat\phi_m{}^{\alpha+}$ to denote the gravitino-dependent part of the S-supersymmetry connection. It is given by

$$\begin{aligned}\hat\phi_{m\alpha}{}^j &:= \phi_{m\alpha}{}^j + \frac{i}{4}(\sigma_m\bar\chi^j)_\alpha = \frac{i}{2}\left(\sigma^{pn}\sigma_m - \frac{1}{3}\sigma_m\bar\sigma^{pn}\right)_{\alpha\dot\beta}\left(\mathcal{D}_p\bar\psi_n{}^{\dot\beta j} + \frac{i}{4}\bar W_{ab}(\bar\sigma^{ab}\bar\sigma_p\psi_n{}^j)^{\dot\beta}\right),\\ \hat{\bar\phi}_m{}^{\dot\alpha}{}_j &:= \bar\phi_m{}^{\dot\alpha}{}_j + \frac{i}{4}(\bar\sigma_m\chi_j)^{\dot\alpha} = \frac{i}{2}\left(\bar\sigma^{pn}\bar\sigma_m - \frac{1}{3}\bar\sigma_m\sigma^{pn}\right)^{\dot\alpha\beta}\left(\mathcal{D}_p\psi_{n\beta j} - \frac{i}{4}W_{ab}(\sigma^{ab}\sigma_p\bar\psi_{nj})_\beta\right).\end{aligned} \tag{5.12}$$

Note that $\mathcal{L}^{++}$ vanishes in the rigid limit, where we recover the five-form of [33].

Introducing

$$\omega^{--} \equiv \int \mathrm{d}^4x e \mathcal{L}^{--}, \qquad \omega^{++} \equiv \int \mathrm{d}^4x e \mathcal{L}^{++}, \tag{5.13}$$

it is a straightforward exercise to demonstrate that ω is closed as a one-form on SU(2),

$$\begin{aligned} &D^{++}\omega^{--} = D^{--}\omega^{++} \Leftrightarrow \\ &e D^{++}\mathcal{L}^{--} = e D^{--}\mathcal{L}^{++} + \text{total } x \text{ derivative}. \end{aligned} \tag{5.14}$$

This is a direct consequence of our construction, but it can also be checked explicitly.

The importance of two distinct integrands can be attributed to the fact that $\mathcal{L}^{--}$ is not holomorphic, even up to a total derivative. The presence of the $\mathcal{L}^{++}$ term is necessary in order for the full action to be invariant under all of the component gauge transformations. These include not only S-supersymmetry and Q-supersymmetry but also SU(2) diffeomorphisms that leave us in the central basis. Recall that these act as

$$\delta_\lambda = -\lambda^{++}D^{--} + \lambda^0 D^0 + \lambda^{--}D^{++}, \tag{5.15}$$

where $\lambda^{\pm\pm}$ and λ^0 are given by (2.16), now with $\lambda^i{}_j$ potentially depending on x. Invariance under δ_λ can actually be used to uniquely determine $\mathcal{L}^{--}$ and $\mathcal{L}^{++}$ starting from the leading term in $\mathcal{L}^{--}$.

At this stage, we should mention that the action (5.9) is actually invariant under another group of transformations—arbitrary diffeomorphisms on the SU(2) manifold,

$$\begin{aligned} \delta v^{i+} &= -\xi^{++}v^{i-} + \xi^0 v^{i+}, \\ \delta v_i^- &= \xi^{--}v_i^+ - \xi^0 v_i^-, \end{aligned} \tag{5.16}$$

where $\xi^{\pm\pm}$ and ξ^0 are x independent but otherwise arbitrary. This implies an invariance of the action under small deformations of the contour $\mathcal{C}$.

The component action (5.9) can be compared with the original expression (4.13) in [20] [where SU(2) superspace was used] as well as the later result (4.13) in [47] (using conformal superspace). Both expressions involve only the first contour integral with $\mathcal{L}^{--}$. This earlier formulation can be interpreted in our language as involving a complex SU(2) manifold [i.e. an SL$(2,\mathbb{C})$ manifold] as discussed in Sec. II F. This involves making a certain complexification of the harmonic variables $v^{i\pm}$,

$$\begin{pmatrix} v^{i+} \\ v_i^- \end{pmatrix} \to \begin{pmatrix} v^i \\ u_i/(v,u) \end{pmatrix}, \tag{5.17}$$

where $u_i \neq (v^i)^*$. Then it is possible to choose a contour in SL$(2,\mathbb{C})$ where v^i varies with u_i fixed, with the requirement that (v,u) be nonzero. In such a case, $\mathcal{V}^{--} = 0$ on the SL$(2,\mathbb{C})$ manifold and so the second contour integral vanishes automatically even through $\mathcal{L}^{++}$ is nonzero. Moreover, if we take the rigid limit with nonconstant u_i, it is easy to see that $\mathcal{L}^{++}$ vanishes even though $\mathcal{V}^{--}$ is nonzero. Thus we recover both the original flat space formulation of [3,32] with arbitrary u_i as well as the curved formulation of [20] with fixed u_i.

We emphasize that the original derivation of $\mathcal{L}^{--}$ in [20] was based on a very similar observation to (5.14). The method there was to construct $\mathcal{L}^{--}$ iteratively by first specifying the leading term, analogous to $(\nabla^-)^4\mathcal{L}^{++}$, and then to add the terms needed to ensure that $\mathcal{L}^{--}$ was independent of the fixed coordinate u_i, up to a total contour derivative (analogous to $D^{--}\mathcal{L}^{++}$) and a total spacetime derivative. More explicitly, let us consider the complexified version of the expression (5.10) for $\mathcal{L}^{--}$ in the central gauge,

$$\begin{aligned} \mathcal{L}^{--} &= (\nabla^-)^4\mathcal{L}^{++}(v) + \cdots \\ &= \frac{1}{16}\frac{u_i u_j u_k u_l}{(v,u)^4}\nabla^{ij}\bar{\nabla}^{kl}\mathcal{L}^{++}(v) + \cdots \end{aligned} \tag{5.18}$$

Following the same argument as [20], the action must be invariant under constant shifts δu_i, which can be parametrized as

$$\delta u_i = \alpha u_i + \beta v_i, \tag{5.19}$$

in terms of x-independent parameters α and β. (This is possible since v_i and u_i are linearly independent along the contour.) The parameters α and β must depend on the contour coordinate τ in order for δu_i to be τ independent, but the precise relationship will not concern us here. The important feature is that $\delta v_i^- = \beta v_i^+/(v,u)$ and so the transformation (5.19) can be interpreted as the SL$(2,\mathbb{C})$ diffeomorphism $\delta = \xi^{--}D^{++}$ with $\xi^{--} = \beta/(v,u)$. This acts only on v_i^-. It follows that

$$\delta\mathcal{L}^{--} = \xi^{--}D^{++}\mathcal{L}^{--}. \tag{5.20}$$

Now in order for this to vanish under the contour integral, it must be that (5.14) holds for some choice of function $\mathcal{L}^{++}$. This allows one to iteratively determine all contributions to $\mathcal{L}^{--}$ starting from the leading term (5.18). This uniquely specifies $\mathcal{L}^{--}$ and $\mathcal{L}^{++}$ in (5.10) and (5.11). Now assuming that $\mathcal{L}^{--}$ has been so constructed, one has

$$\delta\mathcal{L}^{--} = \xi^{--}D^{--}\mathcal{L}^{++} + \text{total } x \text{ derivative}. \tag{5.21}$$

Using $D^{--}\delta u_i = 0$, one can prove $D^{--}\beta \propto D^{--}\xi^{--} = 0$, and so one recovers

$$\delta\mathcal{L}^{--} = D^{--}(\xi^{--}\mathcal{L}^{++}) + \text{total } x \text{ derivative}. \tag{5.22}$$

The remaining contour can then be discarded and invariance under (5.19) confirmed.

A natural question to ask is what happens if we keep an $SL(2,\mathbb{C})$ manifold but allow u_i to vary along the contour, as in [3,32]. We may still demand the invariance of the action under (5.19), but now *there is no need for any constraint to be imposed on α or β*. We find as before (5.20). This leads [using $\delta\mathcal{V}^{++} = \delta(v_i \mathrm{d}v^i) = 0$] up to a total x derivative to

$$\delta \oint_C \mathcal{V}^{++}\mathcal{L}^{--} = \oint_C \mathcal{V}^{++}\xi^{--}D^{--}\mathcal{L}^{++}, \qquad (5.23)$$

which does not vanish automatically. But now the second contour integral is not zero, so we must analyze its variation. This involves calculating $\delta\mathcal{V}^{--}$ using the expression for the complexified vielbeins (2.35). The result is $\delta\mathcal{V}^{--} = \mathrm{d}\xi^{--} + 2\xi^{--}\mathcal{V}^0$, the same expression as (2.31) found on the real SU(2) manifold. This leads to

$$\begin{aligned}\delta \oint_C \mathcal{V}^{--}\mathcal{L}^{++} &= \oint_C (\delta\mathcal{V}^{--}\mathcal{L}^{++} + \mathcal{V}^{--}\xi^{--}D^{++}\mathcal{L}^{++}) \\ &= \oint_C (\mathrm{d}\xi^{--}\mathcal{L}^{++} + 2\xi^{--}\mathcal{V}^0\mathcal{L}^{++} \\ &\quad + \xi^{--}\mathcal{V}^{--}D^{++}\mathcal{L}^{++}) \end{aligned} \qquad (5.24)$$

and the difference between (5.23) and (5.24) is, after rewriting $\mathcal{V}^{\underline{a}}D_{\underline{a}}\mathcal{L}^{++} = \mathrm{d}\mathcal{L}^{++}$ and discarding a total derivative,

$$\begin{aligned}-\delta \oint_C \mathcal{V}^{++}\mathcal{L}^{--} + \delta \oint_C \mathcal{V}^{--}\mathcal{L}^{++} &= \oint_C (\mathrm{d}\xi^{--}\mathcal{L}^{++} + \xi^{--}\mathrm{d}\mathcal{L}^{++}) \\ &= 0. \end{aligned} \qquad (5.25)$$

This is a happy state of affairs. The expression (5.9), which we derived using a real SU(2) manifold in the central gauge, proves to generalize to an $SL(2,\mathbb{C})$ manifold in the central gauge, *no matter the behavior of u_i along the contour*, so long as $(v,u) \neq 0$. In practice, one expects the calculation either with constant u_i or with $u_i = \bar{v}_i$ to be convenient: both correspond to special cases of a more general formulation involving the auxiliary manifold $SL(2,\mathbb{C})$. That we can make arbitrary shifts (5.19) ensures that one can analytically continue from $u_i = \bar{v}_i$ to $u_i =$ constant (and back again) without any difficulty. This ensures the formulation presented here and the conventional formulation [20] are equivalent.

VI. CONCLUSION

In this paper we have constructed curved projective superspace using the supermanifold $\mathcal{M}^{4|8} \times SU(2)$. This approach generalizes previous work [20] in four dimensions, which we have interpreted as the central gauge of the superspace $\mathcal{M}^{4|8} \times SL(2,\mathbb{C})$, the complexified version of the superspace taken here. This approach to curved projective superspace can straightforwardly be extended to dimensions 2 through 6 using the existing body of work [19,21–23].

In particular, a recent paper [48] has explored superforms in 6D curved superspace [23], motivated partly by an attempt to construct the component form associated with the 6D projective superspace action principle. It seems to us that an interpretation of 6D projective superspace along the lines we have taken here should be possible. We reiterate here that the five-form $\mathcal{J}$ corresponding to the component Lagrangian of the 4D analytic projective superspace action, which we gave implicitly in (5.7) upon substituting Eqs. (C9)–(C12), rather curiously does not possess the standard form $\mathcal{J} = \frac{1}{5!}E^{\underline{A}_1} \wedge \cdots \wedge E^{\underline{A}_5}\mathcal{J}_{\underline{A}_5\cdots\underline{A}_1}$ of an expansion purely in terms of the supervielbeins. It is plausible that this is a source of the difficulties observed in [48]. Another intriguing feature of [48] was its use of pure spinor Lorentz harmonics to drastically simplify the study of the complex of differential forms; perhaps a curved superspace which implements such Lorentz harmonics directly within the superfields could have powerful applications.

To keep our construction as simple as possible, we have avoided introducing a Yang-Mills connection on $\mathcal{M}^{4|8} \times SU(2)$, but there is no barrier to doing so. This was already discussed in the conventional formulation [20], and the extension to the formulation here is completely straightforward. Similarly, we have not discussed the various possible actions one can construct involving covariantly arctic, antarctic, tensor and vector multiplets. These have been discussed elsewhere in the conventional approach; see [20] where the vector multiplet action and off shell supergravity-matter actions with a tensor multiplet compensator were constructed in curved superspace. Their construction in the general gauge is similarly straightforward.

The main benefit of this new extended formulation is that it transparently admits the existence of an analytic gauge where (at least locally) $\nabla^+_{\underline{\alpha}} = \partial/\partial\theta^{\underline{\alpha}-}$ and $D^{++} = v_i^+\partial/\partial v_i^-$; in such a gauge, covariantly analytic superfields are characterized simply by their independence of v_i^- and $\theta^{\underline{\alpha}-}$. It is well-known in harmonic superspace that the analytic gauge (known as the *analytic basis* in the harmonic context) plays a critical role when one constructs the supergravity prepotentials [49,50]. It seems likely that the analytic gauge should help resolve the problem of finding supergravity prepotentials in projective superspace, a partial solution of which was presented in [19]. Presumably, it would follow closely the approach utilized in [50], where the supergravity prepotentials in harmonic superspace were explicitly derived from the constraints on the algebra of covariant derivatives. Perhaps the harmonic and projective approaches could even be related to each other, as was the case with the gauge prepotentials [16,17]. We intend to revisit this subject in the near future.

Another interesting feature is that it provides a window into a covariant formulation of harmonic superspace. We have mentioned in passing that harmonic superspace seems to be less feasible in the curved superspace $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$, at least when the SU(2) manifold is identified with the R-symmetry group. The main barrier is that analyticity in the Grassmann coordinates imposes holomorphy in the SU(2) coordinates, but this negates the possibility of using globally defined superfields. This issue has been noted in harmonic superspace—one is forced to distinguish between the SU(2) of the harmonics and the SU(2) of the superconformal group (see Chapter 9 of [1])—and a solution has also been suggested: one should complexify the S^2 of harmonic superspace to two copies of $\mathbb{C}P^1$, with the superconformal SU(2) group acting only on one of them. A similar observation was made in [17] and elaborated upon in [18] where it was suggested to complexify harmonic superspace in a similar way to recover projective superspace. Based on this observation, it seems feasible to construct curved harmonic superspace using the curved superspace $\mathcal{M}^{4|8} \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ (effectively $\mathcal{M}^{4|8} \times \mathbb{C}P^1 \times \mathbb{C}P^1$). As discussed in [18], harmonic superfields can be interpreted as biholomorphic superfields on $\mathbb{C}P^1 \times \mathbb{C}P^1$, restricted to possess a harmonic expansion on the subspace where the $\mathbb{C}P^1$ manifolds are identified. This will be explored in a subsequent publication [51].

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APPENDIX A: CURVATURES OF CONFORMAL SUPERSPACE ON $\mathcal{M}^{4|8} \times \mathrm{SU}(2)$

1. Torsion

The torsion two-forms are defined by

$$T^a := \mathrm{d}E^a + E^b \wedge \Omega_b{}^a + E^a \wedge B, \tag{A1a}$$

$$T^{\alpha\pm} := \mathrm{d}E^{\alpha\pm} + \frac{1}{2}E^{\alpha\pm} \wedge B - iE^{\alpha\pm} \wedge A + E^{\beta\pm} \wedge \Omega_\beta{}^\alpha + iE^b \wedge F^{\pm}_{\dot\gamma}(\bar\sigma_b)^{\dot\gamma\alpha}, \tag{A1b}$$

$$T^{\dot\alpha\pm} := \mathrm{d}E^{\dot\alpha\pm} + \frac{1}{2}E^{\dot\alpha\pm} \wedge B + iE^{\dot\alpha\pm} \wedge A - E^{\dot\beta\pm} \wedge \Omega_{\dot\beta}{}^{\dot\alpha} - iE^b \wedge F_\gamma{}^{\pm}(\bar\sigma_b)^{\dot\alpha\gamma}, \tag{A1c}$$

$$T^{\pm\pm} := \mathrm{d}E^{\pm\pm} + 4E^{\underline{\beta}\pm} \wedge F^{\pm}_{\underline{\beta}}, \tag{A1d}$$

$$T^0 := \mathrm{d}E^0 + 2E^{\underline{\beta}+} \wedge F^{-}_{\underline{\beta}} + 2E^{\underline{\beta}-} \wedge F^{+}_{\underline{\beta}}. \tag{A1e}$$

The nonvanishing components of the torsion tensor can be grouped by dimension:

(i) Dimension 0

$$T_{\gamma\pm\dot\beta\mp}{}^a = \pm 2i(\sigma^a)_{\gamma\dot\beta},$$
$$T_{\pm\pm\underline{\beta}\mp}{}^{\underline{\alpha}\pm} = \mp\,\delta_{\underline{\beta}}{}^{\underline{\alpha}}, \qquad T_{0\underline{\beta}\pm}{}^{\underline{\alpha}\pm} = \pm\delta_{\underline{\beta}}{}^{\underline{\alpha}},$$
$$T_{0\pm\pm}{}^{\pm\pm} = \pm 2, \qquad T_{--++}{}^0 = 1. \tag{A2a}$$

(ii) Dimension 1

$$T_{\dot\gamma\pm\beta\dot\beta}{}^{\alpha\pm} = i\epsilon_{\dot\gamma\dot\beta}W_\beta{}^\alpha,$$
$$T_{\gamma\pm\beta\dot\beta}{}^{\dot\alpha\pm} = -i\epsilon_{\gamma\beta}\bar W_{\dot\beta}{}^{\dot\alpha}. \tag{A2b}$$

(iii) Dimension 3/2

$$T_{\gamma\dot\gamma\beta\dot\beta}{}^{\alpha\pm} = \frac{1}{2}\epsilon_{\dot\gamma\dot\beta}\nabla^{\alpha\pm}W_{\gamma\beta},$$
$$T_{\gamma\dot\gamma\beta\dot\beta}{}^{\dot\alpha\pm} = \frac{1}{2}\epsilon_{\gamma\beta}\bar\nabla^{\dot\alpha\pm}\bar W_{\dot\gamma\dot\beta},$$
$$T_{\beta\pm\alpha\dot\alpha}{}^{\pm\pm} = -i\epsilon_{\beta\alpha}\bar\nabla^{\dot\phi\pm}\bar W_{\dot\phi\dot\alpha},$$
$$T_{\dot\beta\pm\alpha\dot\alpha}{}^{\pm\pm} = i\epsilon_{\dot\beta\dot\alpha}\nabla^{\phi\pm}W_{\phi\alpha},$$
$$T_{\beta\pm\alpha\dot\alpha}{}^0 = -\frac{i}{2}\epsilon_{\beta\alpha}\bar\nabla^{\dot\phi\mp}\bar W_{\dot\phi\dot\alpha},$$
$$T_{\dot\beta\pm\alpha\dot\alpha}{}^0 = \frac{i}{2}\epsilon_{\dot\beta\dot\alpha}\nabla^{\phi\mp}W_{\phi\alpha}. \tag{A2c}$$

(iii) Dimension 2

$$T_{\beta\dot\beta\alpha\dot\alpha}{}^{\pm\pm} = -\frac{1}{4}\epsilon_{\dot\beta\dot\alpha}\nabla^{\gamma\pm}\nabla^{\pm}_{\gamma}W_{\beta\alpha} + \frac{1}{4}\epsilon_{\beta\alpha}\bar\nabla^{\pm}_{\dot\gamma}\bar\nabla^{\dot\gamma\pm}\bar W_{\dot\beta\dot\alpha},$$
$$T_{\beta\dot\beta\alpha\dot\alpha}{}^0 = -\frac{1}{4}\epsilon_{\dot\beta\dot\alpha}\nabla^{\gamma+}\nabla^{-}_{\gamma}W_{\beta\alpha} + \frac{1}{4}\epsilon_{\beta\alpha}\bar\nabla^{+}_{\dot\gamma}\bar\nabla^{\dot\gamma-}\bar W_{\dot\beta\dot\alpha}. \tag{A2d}$$

Some subtleties arise when one compares these equations to those in [34]. For example, there one finds (relabeling $\Phi^j{}_i \to \mathcal{V}^j{}_i$)

$$T^\alpha{}_i = \mathrm{d}E^\alpha{}_i + E^\alpha{}_j \wedge \mathcal{V}^j{}_i + \frac{1}{2}E^\alpha{}_i \wedge B - iE^\alpha{}_i \wedge A + E^\beta{}_i \wedge \Omega_\beta{}^\alpha + iE^b \wedge F_{\dot\gamma i}\bar\sigma_b^{\dot\gamma\alpha}. \tag{A3}$$

There is an apparent discrepancy in the second term, which is absent in the corresponding equation for $T^{\alpha\pm}$. This is because here the tensor $\mathcal{V}^j{}_i$ is no longer interpreted as part of the vielbein and so the formal definition of the torsion two-form differs. However, what does not differ is the actual equation one finds for $\mathrm{d}E^\alpha{}_i$. From [34], one finds the *constraint*

$$T^\alpha{}_i = -iE^b \wedge E_{\dot\gamma i}(\bar\sigma_b)^{\dot\gamma\beta} W_\beta{}^\alpha - \frac{1}{8} E^b \wedge E^c (\sigma_{cb})^{\gamma\beta} \nabla^\alpha{}_i W_{\gamma\beta}, \tag{A4}$$

which should be equated to (A3) to give a constraint on $\mathrm{d}E^\alpha{}_i$. In our framework here, we have instead

$$T^{\alpha\pm} = -iE^b \wedge E^\pm_{\dot\beta}(\bar\sigma_b)^{\dot\beta\beta} W_\beta{}^\alpha - \frac{1}{8} E^b \wedge E^c (\sigma_{cb})^{\gamma\beta} \nabla^{\alpha\pm} W_{\gamma\beta} \mp E^{\alpha\mp} \wedge E^{\pm\pm} \pm E^{\alpha\pm} \wedge E^0. \tag{A5}$$

This should be equated with (A1b) to find a constraint for $\mathrm{d}E^{\alpha\pm}$. In the central basis, the two equations for $\mathrm{d}E^\alpha{}_i$ are identical. The "additional" terms in the second line of (A5) are the same as the terms "missing" in (A1b); this swapping amounts merely to a redefinition of the torsion two-form. Moreover, this redefinition does not change the values of the tangent space components $T_{CB}{}^A$, so the same algebra of covariant derivatives holds in both approaches.

A similar alteration happens in the definitions of $T^{\pm\pm}$ and T^0 when compared with the SU(2) curvature $R(\mathcal{V})^i{}_j$ given in [34]. Nevertheless, the values of $T_{CB}{}^{\pm\pm}$ and $T_{CB}{}^0$ are identical in the central basis to $R(\mathcal{V})_{CB}{}^{ij} v_i^\pm v_j^\pm$ and $R(\mathcal{V})_{CB}{}^{ij} v_i^+ v_j^-$.

This swapping of terms between the constraints on and the definition of the torsion tensor occurs also when one compares the curvature $R(P)_{nm}{}^a$ from the tensor calculus formulation of conformal supergravity with the torsion tensor $T_{nm}{}^a$. These differ by a term proportional to $\psi_{mj}\sigma^a\bar\psi_n{}^j$. In the component formulation, this bilinear appears in the definition of $R(P)_{nm}{}^a$ (which is set to zero). In the supergravity formulation, it appears in the constraint equation from the nonzero component $T_\gamma{}^k{}_{\dot\beta j}{}^a = 2i\delta^k_j(\sigma^a)_{\gamma\dot\beta}$. However, the curvature $[\nabla_b, \nabla_a]$ is the same in both approaches, as is the equation for $\mathrm{d}e^a$, which is used to determine the spin connection.

2. Lorentz curvature

The conformal Lorentz curvature two-form is

$$R^{ba} = \mathrm{d}\Omega^{ba} + \Omega^{bc} \wedge \Omega_c{}^a - 2E^{[b} \wedge F^{a]} + 4E^{\beta-} \wedge F^{\alpha+}(\sigma^{ba})_{\alpha\beta} - 4E^{\beta+} \wedge F^{\alpha-}(\sigma^{ba})_{\alpha\beta} + 4E^{\dot\beta+} \wedge F^{\dot\alpha-}(\bar\sigma^{ba})_{\dot\alpha\dot\beta} - 4E^{\dot\beta-} \wedge F^{\dot\alpha+}(\bar\sigma^{ba})_{\dot\alpha\dot\beta} \tag{A6}$$

and may be canonically decomposed as $R_{DC\beta\dot\beta\alpha\dot\alpha} = 2\epsilon_{\dot\beta\dot\alpha} R_{DC\beta\alpha} - 2\epsilon_{\beta\alpha} R_{DC\dot\beta\dot\alpha}$. It is simplest to express the curvature results in terms of these components. We group the nonvanishing components by dimension.

(i) Dimension 1

$$R_{\delta+\gamma-\dot\beta\dot\alpha} = -2\epsilon_{\delta\gamma}\bar W_{\dot\beta\dot\alpha}, \qquad R_{\dot\delta+\dot\gamma-\beta\alpha} = 2\epsilon_{\dot\delta\dot\gamma} W_{\beta\alpha}. \tag{A7a}$$

(ii) Dimension 3/2

$$R_{\delta\mp\gamma\dot\gamma\dot\beta\dot\alpha} = \mp\frac{i}{2}\epsilon_{\delta\gamma}\bar\nabla^\pm_{\dot\beta}\bar W_{\dot\alpha\dot\gamma} \mp \frac{i}{2}\epsilon_{\delta\gamma}\bar\nabla^\pm_{\dot\alpha}\bar W_{\dot\beta\dot\gamma}, \tag{A7b}$$

$$R_{\dot\delta\mp\gamma\dot\gamma\beta\alpha} = \mp\frac{i}{2}\epsilon_{\dot\delta\dot\gamma}\nabla^\pm_\beta W_{\alpha\gamma} \mp \frac{i}{2}\epsilon_{\dot\delta\dot\gamma}\nabla^\pm_\alpha W_{\beta\gamma}. \tag{A7c}$$

(iii) Dimension 2

$$R_{\delta\dot\delta\gamma\dot\gamma\beta\alpha} = -\frac{1}{8}\epsilon_{\dot\delta\dot\gamma}(\epsilon_{\delta\beta}\epsilon_{\gamma\alpha} + \epsilon_{\delta\alpha}\epsilon_{\gamma\beta})\nabla_{\phi\rho}W^{\rho\phi} + \frac{1}{4}\epsilon_{\dot\delta\dot\gamma}\nabla_{\beta\alpha}W_{\delta\gamma} + \epsilon_{\delta\gamma}\bar W_{\dot\delta\dot\gamma}W_{\beta\alpha}, \tag{A7d}$$

$$R_{\delta\dot\delta\gamma\dot\gamma\dot\beta\dot\alpha} = +\frac{1}{8}\epsilon_{\delta\gamma}(\epsilon_{\dot\delta\dot\beta}\epsilon_{\dot\gamma\dot\alpha} + \epsilon_{\dot\delta\dot\alpha}\epsilon_{\dot\gamma\dot\beta})\bar\nabla_{\dot\phi\dot\rho}\bar W^{\dot\rho\dot\phi} - \frac{1}{4}\epsilon_{\delta\gamma}\bar\nabla_{\dot\beta\dot\alpha}\bar W_{\dot\delta\dot\gamma} - \epsilon_{\dot\delta\dot\gamma}W_{\delta\gamma}\bar W_{\dot\beta\dot\alpha}. \tag{A7e}$$

3. Dilatation and $\mathrm{U}(1)_R$ curvatures

The conformal field strengths for dilatations and chiral rotations are

$$R(\mathbb{D}) = \mathrm{d}B + 2E^a \wedge F_a - 2E^{\underline\alpha-} \wedge F^+_{\underline\alpha} + 2E^{\underline\alpha+} \wedge F^-_{\underline\alpha}, \tag{A8}$$

$$R(\mathbb{A}) = \mathrm{d}A + iE^{\alpha-} \wedge F^+_\alpha - iE^{\alpha+} \wedge F^-_\alpha - iE^{\dot\alpha-} \wedge F^+_{\dot\alpha} + iE^{\dot\alpha+} \wedge F^-_{\dot\alpha}. \tag{A9}$$

We group the nonvanishing components by dimension.

(i) Dimension 3/2

$$R(\mathbb{D})_{\beta\mp\alpha\dot\alpha} = \pm\frac{i}{2}\epsilon_{\beta\alpha}\bar\nabla^{\dot\phi\pm}\bar W_{\dot\phi\dot\alpha}, \tag{A10a}$$

$$R(\mathbb{D})_{\dot\beta\mp\alpha\dot\alpha} = \mp\frac{i}{2}\epsilon_{\dot\beta\dot\alpha}\nabla^{\phi\pm}W_{\phi\alpha}, \tag{A10b}$$

$$R(\mathbb{A})_{\beta\mp\alpha\dot\alpha} = \mp\frac{1}{4}\epsilon_{\beta\alpha}\bar\nabla^{\dot\phi\pm}\bar W_{\dot\phi\dot\alpha}, \tag{A10c}$$

$$R(\mathbb{A})_{\dot\beta\mp\alpha\dot\alpha} = \mp\frac{1}{4}\epsilon_{\dot\beta\dot\alpha}\nabla^{\phi\pm}W_{\phi\alpha}. \tag{A10d}$$

(ii) Dimension 2

$$R(\mathbb{D})_{\beta\dot\beta\alpha\dot\alpha} = \frac{1}{8}\epsilon_{\dot\beta\dot\alpha}(\nabla_\beta{}^\phi W_{\phi\alpha} + \nabla_\alpha{}^\phi W_{\phi\beta}) - \frac{1}{8}\epsilon_{\beta\alpha}(\bar\nabla_{\dot\beta\dot\phi}\bar W^{\dot\phi}{}_{\dot\alpha} + \bar\nabla_{\dot\alpha\dot\phi}\bar W^{\dot\phi}{}_{\dot\beta}), \quad \text{(A10e)}$$

$$R(\mathbb{A})_{\beta\dot\beta\alpha\dot\alpha} = -\frac{i}{16}\epsilon_{\dot\beta\dot\alpha}(\nabla_\beta{}^\phi W_{\phi\alpha} + \nabla_\alpha{}^\phi W_{\phi\beta}) - \frac{i}{16}\epsilon_{\beta\alpha}(\bar\nabla_{\dot\beta\dot\phi}\bar W^{\dot\phi}{}_{\dot\alpha} + \bar\nabla_{\dot\alpha\dot\phi}\bar W^{\dot\phi}{}_{\dot\beta}). \quad \text{(A10f)}$$

4. Special superconformal curvatures

The special superconformal curvatures $R(K)^A$, consisting of S-supersymmetry $R(S)^{\underline{\alpha}\pm}$ and special conformal curvatures $R(K)^a$, are defined by

$$R(K)^a = \mathrm{d}F^a - F^b \wedge \Omega_b{}^a - F^a \wedge B + 2i(\sigma^a)_{\alpha\dot\alpha}(F^{\alpha-} \wedge F^{\dot\alpha+} - F^{\alpha+} \wedge F^{\dot\alpha-}), \quad \text{(A11)}$$

$$R(S)^{\alpha\pm} = \mathrm{d}F^{\alpha\pm} - \frac{1}{2}F^{\alpha\pm} \wedge B + iF^{\alpha\pm} \wedge A + F^{\beta\pm} \wedge \Omega_\beta{}^\alpha \mp F^{\alpha\pm} \wedge E^0 \pm F^{\alpha\mp} \wedge E^{\pm\pm} - iF^b \wedge E_{\dot\alpha}{}^\pm(\bar\sigma_b)^{\dot\alpha\alpha}, \quad \text{(A12)}$$

$$R(S)^{\dot\alpha\pm} = \mathrm{d}F^{\dot\alpha\pm} - \frac{1}{2}F^{\dot\alpha\pm} \wedge B - iF^{\dot\alpha\pm} \wedge A - F^{\dot\beta\pm} \wedge \Omega_{\dot\beta}{}^{\dot\alpha} \mp F^{\dot\alpha\pm} \wedge E^0 \pm F^{\dot\alpha\mp} \wedge E^{\pm\pm} + iF^b \wedge E_\alpha{}^\pm(\bar\sigma_b)^{\dot\alpha\alpha}. \quad \text{(A13)}$$

We give the nonvanishing components of $R(S)_{\underline{C}\underline{B}}{}^{\underline{\alpha}\pm}$ grouped by dimension.

(i) Dimension 3/2

$$R(S)_{\dot\gamma+\dot\beta-}{}^{\alpha\pm} = \frac{1}{2}\epsilon_{\dot\gamma\dot\beta}\nabla^{\phi\pm}W_\phi{}^\alpha, \quad \text{(A14a)}$$

$$R(S)_{\gamma+\beta-}{}^{\dot\alpha\pm} = \frac{1}{2}\epsilon_{\gamma\beta}\bar\nabla^{\dot\phi\pm}\bar W_{\dot\phi}{}^{\dot\alpha}. \quad \text{(A14b)}$$

(ii) Dimension 2

$$R(S)_{\gamma\pm\beta\dot\beta}{}^{\alpha\pm} = \frac{1}{2}\epsilon_{\gamma\beta}\nabla^{\dot\phi\alpha}\bar W_{\dot\phi\dot\beta}, \quad \text{(A14c)}$$

$$R(S)_{\dot\gamma\pm\beta\dot\beta}{}^{\dot\alpha\pm} = \frac{1}{2}\epsilon_{\dot\gamma\dot\beta}\nabla^{\dot\alpha\phi}W_{\phi\beta}, \quad \text{(A14d)}$$

$$R(S)_{\dot\gamma\pm\beta\dot\beta}{}^{\alpha\pm} = \pm\frac{i}{4}\epsilon_{\dot\gamma\dot\beta}\nabla_\beta^\mp\nabla^{\phi\pm}W_\phi{}^\alpha, \quad \text{(A14e)}$$

$$R(S)_{\gamma\pm\beta\dot\beta}{}^{\dot\alpha\pm} = \pm\frac{i}{4}\epsilon_{\gamma\beta}\bar\nabla_{\dot\beta}^\mp\bar\nabla_{\dot\phi}^\pm\bar W^{\dot\phi\dot\alpha}, \quad \text{(A14f)}$$

$$R(S)_{\dot\gamma\mp\beta\dot\beta}{}^{\alpha\pm} = \pm\frac{i}{8}\epsilon_{\dot\gamma\dot\beta}(\nabla^\pm)^2W_\beta{}^\alpha, \quad \text{(A14g)}$$

$$R(S)_{\gamma\mp\beta\dot\beta}{}^{\dot\alpha\pm} = \pm\frac{i}{8}\epsilon_{\gamma\beta}(\bar\nabla^\pm)^2\bar W_{\dot\beta}{}^{\dot\alpha}. \quad \text{(A14h)}$$

(iii) Dimension 5/2

$$R(S)_{\gamma\dot\gamma\beta\dot\beta}{}^{\alpha\pm} = \frac{1}{4}\epsilon_{\gamma\beta}(i\nabla^{\dot\phi\alpha}\bar\nabla^\pm_{(\dot\gamma}\bar W_{\dot\beta)\dot\phi} + \bar W_{\dot\gamma\dot\beta}\nabla^{\phi\pm}W_\phi{}^\alpha) \pm \frac{1}{16}\epsilon_{\dot\gamma\dot\beta}(\nabla^\pm)^2\nabla^\mp_{(\gamma}W_{\beta)}{}^\alpha, \quad \text{(A28)}$$

$$R(S)_{\gamma\dot\gamma\beta\dot\beta}{}^{\dot\alpha\pm} = -\frac{1}{4}\epsilon_{\dot\gamma\dot\beta}(i\nabla^{\dot\alpha\phi}\nabla^\pm_{(\gamma}W_{\beta)\phi} + W_{\gamma\beta}\bar\nabla^\pm_{\dot\phi}\bar W^{\dot\phi\dot\alpha}) \mp \frac{1}{16}\epsilon_{\gamma\beta}(\bar\nabla^\pm)^2\bar\nabla^\mp_{(\dot\gamma}\bar W_{\dot\beta)}{}^{\dot\alpha}. \quad \text{(A29)}$$

The nonvanishing components of $R(K)_{\underline{C}\underline{B}\alpha\dot\alpha} = R(K)_{\underline{C}\underline{B}}{}^a(\sigma_a)_{\alpha\dot\alpha}$ are given by

$$R(K)_{\gamma+\beta-\alpha\dot\alpha} = \epsilon_{\gamma\beta}\nabla_{\alpha\dot\phi}\bar W^{\dot\phi}{}_{\dot\alpha}, \quad \text{(A15a)}$$

$$R(K)_{\dot\gamma+\dot\beta-\alpha\dot\alpha} = -\epsilon_{\dot\gamma\dot\beta}\nabla_{\dot\alpha}{}^\phi W_{\phi\alpha}, \quad \text{(A15b)}$$

$$R(K)_{\gamma\mp\beta\dot\beta\alpha\dot\alpha} = \pm\frac{i}{2}\epsilon_{\gamma\beta}\bar\nabla^\pm_{\dot\beta}\nabla_{\alpha\dot\phi}\bar W^{\dot\phi}{}_{\dot\alpha}, \quad \text{(A15c)}$$

$$R(K)_{\dot\gamma\mp\beta\dot\beta\alpha\dot\alpha} = \pm\frac{i}{2}\epsilon_{\dot\gamma\dot\beta}\nabla^\pm_\beta\nabla_{\dot\alpha}{}^\phi W_{\phi\alpha}, \quad \text{(A15d)}$$

and

$$R(K)_{\gamma\dot\gamma\beta\dot\beta\alpha\dot\alpha} = -\frac{1}{8}\epsilon_{\dot\gamma\dot\beta}\nabla_{\dot\alpha}{}^\phi\nabla_{\gamma\beta}W_{\phi\alpha} - \frac{1}{8}\epsilon_{\gamma\beta}\nabla_\alpha{}^{\dot\phi}\bar\nabla_{\dot\gamma\dot\beta}\bar W_{\dot\phi\dot\alpha} + \frac{1}{4}\epsilon_{\dot\gamma\dot\beta}\nabla_\gamma{}^{\dot\phi}(\bar W_{\dot\phi\dot\alpha}W_{\beta\alpha}) + \frac{1}{4}\epsilon_{\dot\gamma\dot\beta}\nabla_\beta{}^{\dot\phi}(\bar W_{\dot\phi\dot\alpha}W_{\gamma\alpha}) + \frac{1}{4}\epsilon_{\gamma\beta}\nabla_{\dot\gamma}{}^\phi(W_{\phi\alpha}\bar W_{\dot\beta\dot\alpha}) + \frac{1}{4}\epsilon_{\gamma\beta}\nabla_{\dot\beta}{}^\phi(W_{\phi\alpha}\bar W_{\dot\gamma\dot\alpha}) + \frac{i}{4}\epsilon_{\dot\gamma\dot\beta}(\bar\nabla^{\dot\phi-}\bar W_{\dot\phi\dot\alpha})(\nabla^+_{(\gamma}W_{\beta)\alpha}) - \frac{i}{4}\epsilon_{\dot\gamma\dot\beta}(\bar\nabla^{\dot\phi+}\bar W_{\dot\phi\dot\alpha})(\nabla^-_{(\gamma}W_{\beta)\alpha}) + \frac{i}{4}\epsilon_{\gamma\beta}(\nabla^{\phi+}W_{\phi\alpha})(\bar\nabla^-_{(\dot\gamma}\bar W_{\dot\beta)\dot\alpha}) - \frac{i}{4}\epsilon_{\gamma\beta}(\nabla^{\phi-}W_{\phi\alpha})(\bar\nabla^+_{(\dot\gamma}\bar W_{\dot\beta)\dot\alpha}). \quad \text{(A15e)}$$

APPENDIX B: INTEGRATION OVER SUBMANIFOLDS

In this appendix, we briefly review some elements of integration theory over submanifolds. A complementary discussion can be found in [20].

Let $\mathcal{M}$ be a supermanifold of dimension D with local coordinates z^M, $M = 1, \ldots, \mathrm{D}$. We denote the grading of a coordinate z^M by $(-)^M$. The manifold possesses a vielbein $E_M{}^A$, and we can introduce an integral over a Lagrangian $\mathcal{L}$ in the usual way as

$$S = \int \mathrm{d}^D z E \mathcal{L}. \tag{B1}$$

Provided that $\mathcal{L}$ transform as a scalar field under diffeomorphisms, $\delta_\xi \mathcal{L} = \xi^M \partial_M \mathcal{L}$, the action S is invariant. If the manifold possesses an additional local symmetry group $\mathcal{H}$ with generators $X_{\underline{a}}$, under which the vielbein transforms as

$$\delta_{\mathcal{H}} E_M{}^A = E_M{}^B g^{\underline{c}} f_{\underline{c}B}{}^A, \tag{B2}$$

with structure constants $f_{\underline{c}B}{}^A$ (see the discussion in e.g. [34]) then the action S is invariant provided $\mathcal{L}$ transforms as

$$\delta_{\mathcal{H}} \mathcal{L} = -(-)^A g^{\underline{b}} f_{\underline{b}A}{}^A \mathcal{L}. \tag{B3}$$

Now suppose we are given a submanifold $\mathfrak{M}$ of dimension d with local coordinates $\mathfrak{z}^m$, $m = 1, \ldots, d$. We have in mind a situation where the original coordinates z^M can be decomposed (at least in the vicinity of $\mathfrak{M}$) as $z^M = (\mathfrak{z}^m, y^\mu)$ with the submanifold $\mathfrak{M}$ corresponding to the surface with $y^\mu = 0$. We make no assumptions about whether $\mathfrak{z}^m$ and y^μ are bosonic or fermionic; in fact, we are interested in cases where both consist of bosonic and fermionic coordinates. We decompose the vielbein and its inverse as

$$E_M{}^A = \begin{pmatrix} \mathcal{E}_m{}^a & E_m{}^\alpha \\ E_\mu{}^a & E_\mu{}^\alpha \end{pmatrix}, \qquad E_A{}^M = \begin{pmatrix} E_a{}^m & E_a{}^\mu \\ E_\alpha{}^m & \phi_\alpha{}^\mu \end{pmatrix}, \tag{B4}$$

with the assumption that both $\mathcal{E}_m{}^a$ and $\phi_\alpha{}^\mu$ are invertible, with inverses $\mathcal{E}_a{}^m$ and $\phi_\mu{}^\alpha$, respectively. This allows one to compactly specify all the remaining components of the vielbein and its inverse in terms of these quantities, and $E_m{}^\alpha$ and $E_\alpha{}^m$:

$$E_M{}^A = \left(\begin{array}{c|c} \mathcal{E}_m{}^a & E_m{}^\alpha \\ \hline -\phi_\mu{}^\beta E_\beta{}^n \mathcal{E}_n{}^a & \phi_\mu{}^\alpha - \phi_\mu{}^\beta E_\beta{}^n E_n{}^\alpha \end{array} \right),$$
$$E_A{}^M = \left(\begin{array}{c|c} \mathcal{E}_a{}^m - \mathcal{E}_a{}^n E_n{}^\beta E_\beta{}^m & -\mathcal{E}_a{}^n E_n{}^\beta \phi_\beta{}^\mu \\ \hline E_\alpha{}^m & \phi_\alpha{}^\mu \end{array} \right). \tag{B5}$$

No assumptions need to be made about $E_m{}^\alpha$ or $E_\alpha{}^m$. One can check that

$$E \equiv \mathrm{sdet} E_M{}^A = \mathrm{sdet} \mathcal{E}_m{}^a \, \mathrm{sdet} \phi_\mu{}^\alpha = \frac{\mathrm{sdet} \mathcal{E}_m{}^a}{\mathrm{sdet} \phi_\alpha{}^\mu}, \tag{B6}$$

although we won't make use of this feature.

Now consider the action $\mathcal{S}$ over the submanifold $\mathfrak{M}$ with Lagrangian $\mathcal{L}$:

$$\mathcal{S} = \int \mathrm{d}^d \mathfrak{z} \mathcal{E} \mathcal{L}, \qquad \mathcal{E} = \mathrm{sdet} \mathcal{E}_m{}^a. \tag{B7}$$

This is invariant under $\mathfrak{z}^m$ diffeomorphisms provided $\mathcal{L}$ transforms as a scalar function. If we impose $f_{\underline{c}\beta}{}^a = 0$, then (B2) implies $\delta_H \mathcal{E} = (-)^a g^{\underline{b}} f_{\underline{b}a}{}^a$. So a set of sufficient conditions for $\mathcal{H}$-invariance is

$$\delta_{\mathcal{H}} \mathcal{L} = -(-)^a g^{\underline{b}} f_{\underline{b}a}{}^a \mathcal{L}, \qquad f_{\underline{c}\beta}{}^a = 0. \tag{B8}$$

It turns out that $\mathcal{S}$ can also be made invariant under diffeomorphisms generated by ξ^μ. The easiest way to see this is to note that because $E_\alpha{}^\mu \equiv \phi_\alpha{}^\mu$ is invertible, it is possible to construct a one-to-one relation between any diffeomorphism in ξ^μ and a covariant diffeomorphism generated by $\xi'^\alpha = \xi^\mu \phi_\mu{}^\alpha$ modulo a certain diffeomorphism in $\mathfrak{z}^m$ and an $\mathcal{H}$ gauge transformation. Recall that a covariant diffeomorphism is given by

$$\delta_\xi = \xi^A \nabla_A = \xi^A E_A{}^M \partial_M - \xi^A H_A{}^{\underline{b}} X_{\underline{b}}, \tag{B9}$$

where $H_M{}^{\underline{a}}$ is the connection associated with the group $\mathcal{H}$. Taking $\xi^A = (0, \xi'^\alpha) = (0, \xi^\mu \phi_\mu{}^\alpha)$, one finds

$$\begin{aligned} \xi'^\alpha \nabla_\alpha &= \xi'^\alpha \phi_\alpha{}^\mu \partial_\mu + \xi'^\alpha E_\alpha{}^m \partial_m - \xi'^\alpha H_\alpha{}^{\underline{b}} X_{\underline{b}} \\ &= \xi^\mu \partial_\mu + \xi^\mu \phi_\mu{}^\alpha E_\alpha{}^m \partial_m - \xi^\mu \phi_\mu{}^\alpha H_\alpha{}^{\underline{b}} X_{\underline{b}}. \end{aligned} \tag{B10}$$

Since we have already established invariance under $\mathfrak{z}^m$ diffeomorphisms and $\mathcal{H}$ gauge transformations, we need only check covariant diffeomorphisms generated by arbitrary ξ^α. This will establish invariance under the full set of diffeomorphisms. To prove invariance under covariant diffeomorphisms with parameter ξ^α, observe that

$$\delta \mathcal{E}_m{}^a = \mathcal{E}_m{}^b \xi^\gamma T_{\gamma b}{}^a + E_m{}^\beta \xi^\gamma T_{\gamma\beta}{}^a. \tag{B11}$$

We will restrict our attention to superspaces where $T_{\gamma\beta}{}^a = 0$ so only the first term in $\delta \mathcal{E}_m{}^a$ contributes. Noting that $\delta \mathcal{L} = \xi^\alpha \nabla_\alpha \mathcal{L}$, it follows that the remaining sufficient conditions for invariance of the action (B7) are

$$\nabla_\alpha \mathcal{L} = -(-)^b T_{\alpha b}{}^b \mathcal{L}, \qquad T_{\gamma\beta}{}^a = 0. \tag{B12}$$

APPENDIX C: COMPONENT ACTION DERIVATION

In this appendix, we describe how to derive the component action of

$$S = -\frac{1}{2\pi}\oint_{\mathcal{C}} d\tau \int d^4x d^4\theta^+ \mathcal{E}^{--}\mathcal{L}^{++}. \qquad (C1)$$

The integral can be understood as evaluated at $\theta^{\underline{\mu}-} = 0$, since these Grassmann variables do not appear in the measure. To evaluate the action, it helps to exploit the $\theta^{\underline{\mu}+}$-dependent parts of our gauge transformations (including covariant diffeomorphisms) to fix the gauge[16] $\nabla_{\underline{\alpha}+} = \partial/\partial\theta^{\underline{\alpha}+}$. Now the analytic superspace vielbein is given by

$$\mathcal{E}_{\underline{M}}{}^{\underline{A}}|_{\theta^-=0} = \begin{pmatrix} e_m{}^a & \mathcal{V}_m{}^{++} & \frac{1}{2}\psi_m{}^{\underline{\alpha}+} \\ e_\tau{}^a & \mathcal{V}_\tau{}^{++} & \frac{1}{2}\psi_\tau{}^{\underline{\alpha}+} \\ 0 & 0 & \delta_{\underline{\mu}}{}^{\underline{\alpha}} \end{pmatrix}. \qquad (C2)$$

In this equality, we have relabeled the components of the one-forms $\mathcal{E}^{\underline{A}}$ by e^a, $\mathcal{V}^{++}$, and $\frac{1}{2}\psi^{\underline{\alpha}+}$ to simplify the notation that will follow.[17] Its determinant $\mathcal{E}^{--}$ is equal in this gauge to e^{++} given by

$$e^{++} = \det\begin{pmatrix} e_m{}^a & \mathcal{V}_m{}^{++} \\ e_\tau{}^a & \mathcal{V}_\tau{}^{++} \end{pmatrix}. \qquad (C3)$$

This determinant is over the five-by-five component vielbein describing both the base manifold with coordinates x^m and the SU(2) contour with coordinate τ.

The easiest way to evaluate the component action is to rewrite S as

$$S = -\frac{1}{2\pi}\frac{1}{16}\int_{\mathcal{M}^4\times\mathcal{C}} (\partial_+)^2(\bar\partial_+)^2(\hat e^{++}\mathcal{L}^{++}), \qquad (C4)$$

where $\hat e^{++}$ is the volume five-form

$$\begin{aligned}\hat e^{++} &= dx^0\wedge dx^1\wedge dx^2\wedge dx^3\wedge d\tau e^{++} \\ &= \frac{1}{4!}\epsilon_{abcd}e^a\wedge e^b\wedge e^c\wedge e^d\wedge\mathcal{V}^{++}. \qquad (C5)\end{aligned}$$

In order to evaluate successive spinor derivatives of $\hat e^{++}$, one must work out the rules for spinor differentiation of the one-forms e^a and $\mathcal{V}^{++}$ in the gauge where $\nabla_{\underline{\alpha}+} = \partial_{\underline{\alpha}+}$. These can be derived by using the relations for the corresponding curvatures T^a and T^{++}. For example, from the definition of T^a, one can show that

$$\begin{aligned}T_{\nu+M}{}^a = -(-)^M T_{\nu+\dot\beta-}{}^a E_M{}^{\dot\beta-} = \partial_{\nu+}e_M{}^a \Rightarrow \\ \partial_{\nu+}e^a = -i(\sigma^a)_{\nu\dot\beta}\bar\psi^{\dot\beta-}. \qquad (C6)\end{aligned}$$

Similar relations can be used to define the spinor derivative of any one-form. The ones we will need are

$$\begin{aligned}&\partial_{\alpha+}e^a = -i(\sigma^a)_{\alpha\dot\beta}\bar\psi^{\dot\beta-}, \\ &\partial_{\alpha+}\mathcal{V}^{++} = 2\phi^+_\alpha + \frac{3i}{2}e^b(\sigma_b)_{\alpha\dot\beta}\bar\chi^{\dot\beta+}, \qquad \partial_{\alpha+}\mathcal{V}^{--} = 0, \\ &\partial_{\alpha+}\psi^{\beta-} = -2\delta_\alpha{}^\beta\mathcal{V}^{--}, \qquad \partial_{\alpha+}\bar\psi^{\dot\beta-} = 0, \\ &\partial_{\alpha+}\phi^{\beta+} = -2e^cR(S)_{c\alpha+}{}^{\beta+}, \\ &\partial_{\alpha+}\bar\phi^{\dot\beta+} = \frac{3}{2}\psi^-_\alpha\bar\chi^{\dot\beta+} - 2e^cR(S)_{c\alpha+}{}^{\dot\beta+} + 2if^b(\sigma_b)_\alpha{}^{\dot\beta}, \\ &\partial_{\alpha+}f^b = -e^cR(K)_{c\alpha+}{}^b - \frac{1}{2}\psi^{\gamma-}R(K)_{\gamma-\alpha+}{}^b, \qquad (C7)\end{aligned}$$

as well as their complex conjugates,

$$\begin{aligned}&\bar\partial_{\dot\alpha+}e^a = i(\sigma^a)_{\beta\dot\alpha}\psi^{\beta-}, \\ &\bar\partial_{\dot\alpha+}\mathcal{V}^{++} = 2\bar\phi^+_{\dot\alpha} - \frac{3i}{2}e^b(\sigma_b)_{\beta\dot\alpha}\chi^{\beta+}, \qquad \bar\partial_{\dot\alpha+}\mathcal{V}^{--} = 0, \\ &\bar\partial_{\dot\alpha+}\bar\psi^{\dot\beta-} = -2\delta_{\dot\alpha}{}^{\dot\beta}\mathcal{V}^{--}, \qquad \bar\partial_{\dot\alpha+}\psi^{\beta-} = 0, \\ &\bar\partial_{\dot\alpha+}\bar\phi^{\dot\beta+} = -2e^cR(S)_{c\dot\alpha+}{}^{\dot\beta+}, \\ &\bar\partial_{\dot\alpha+}\phi^{\beta+} = \frac{3}{2}\bar\psi^-_{\dot\alpha}\chi^{\beta+} - 2e^cR(S)_{c\dot\alpha+}{}^{\beta+} - 2if^b(\sigma_b)^\beta{}_{\dot\alpha}, \\ &\bar\partial_{\dot\alpha+}f^b = -e^cR(K)_{c\dot\alpha+}{}^b - \frac{1}{2}\bar\psi^{\dot\gamma-}R(K)_{\dot\gamma-\dot\alpha+}{}^b. \qquad (C8)\end{aligned}$$

As with the other connections, we label the superfield connections F^A by their component names, $F^A = (f^a, \frac{1}{2}\phi^{\underline{\alpha}\pm})$.

Applying these rules and using the explicit expressions for the curvatures $R(K)$ and $R(S)$ where needed, one can derive all the spinor derivatives of $\hat e^{++}$. Suppressing the explicit $\wedge$ symbol from now on, we find

$$\begin{aligned}\partial^\alpha_+\hat e^{++} &= \epsilon_{abcd}e^ae^be^c\left(\frac{i}{6}\bar\psi^-_{\dot\beta}\mathcal{V}^{++}(\bar\sigma^d)^{\dot\beta\alpha} + \frac{1}{12}e^d\phi^{\alpha+}\right), \\ \bar\partial_{\dot\alpha+}\hat e^{++} &= \epsilon_{abcd}e^ae^be^c\left(\frac{i}{6}\psi^{\beta-}\mathcal{V}^{++}(\sigma^d)_{\beta\dot\alpha} + \frac{1}{12}e^d\bar\phi^+_{\dot\alpha}\right). \qquad (C9)\end{aligned}$$

The second spinor derivatives are

[16] This is just the superspace analogue of Riemann normal coordinates for the Grassmann coordinates [52]. For an extensive discussion of using normal coordinates to derive component actions, see [20].

[17] A precise notation would reserve these labels for the component projections $E^{\underline{A}}|_{\theta=0}$, but it is convenient to use the same labels for the full superfields.

$$
\begin{aligned}
(\partial_+)^2\hat{e}^{++} &= -2ie^a e^b \bar{\psi}^{\dot{\alpha}-}\bar{\psi}^{\dot{\beta}-}\mathcal{V}^{++}(\bar{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} + \frac{2i}{3}e^a e^b e^c \bar{\psi}^{\dot{\beta}-}\phi^{\beta+}\epsilon_{abcd}(\sigma^d)_{\beta\dot{\beta}} - \frac{1}{4}e^a e^b e^c e^d \bar{\psi}^-_{\dot{\beta}}\bar{\chi}^{\dot{\beta}+}\epsilon_{abcd},\\
(\bar{\partial}_+)^2\hat{e}^{++} &= 2ie^a e^b \psi^{\alpha-}\psi^{\beta-}\mathcal{V}^{++}(\sigma_{ab})_{\alpha\beta} + \frac{2i}{3}e^a e^b e^c \psi^{\beta-}\bar{\phi}^{\dot{\beta}+}\epsilon_{abcd}(\sigma^d)_{\beta\dot{\beta}} - \frac{1}{4}e^a e^b e^c e^d \psi^{\beta-}\chi^+_\beta\epsilon_{abcd},\\
\partial_{\alpha+}\bar{\partial}_{\dot{\alpha}+}\hat{e}^{++} &= -\frac{i}{3}\epsilon_{abcd}e^a e^b e^c(\mathcal{V}^{--}\mathcal{V}^{++}(\sigma^d)_{\alpha\dot{\alpha}} + \psi^{\beta-}\phi^+_\alpha(\sigma^d)_{\beta\dot{\alpha}} + \bar{\psi}^{\dot{\beta}-}\bar{\phi}^+_{\dot{\alpha}}(\sigma^d)_{\alpha\dot{\beta}})\\
&\quad + \frac{i}{6}e^a e^b e^c e^d f^f(\sigma_f)_{\alpha\dot{\alpha}}\epsilon_{abcd} + \frac{1}{2}e^a e^b \psi^{\beta-}\bar{\psi}^{\dot{\beta}-}\mathcal{V}^{++}(\sigma^c)_{\alpha\dot{\beta}}(\sigma^d)_{\beta\dot{\alpha}}\epsilon_{abcd}.
\end{aligned}
\tag{C10}
$$

The terms with three spinor derivatives are

$$
\begin{aligned}
\partial^\alpha_+(\bar{\partial}_+)^2\hat{e}^{++} &= \epsilon_{abcd}e^a e^b e^c e^d\left[\frac{1}{2}\mathcal{V}^{--}\chi^{\alpha+} + \frac{1}{4}\psi^{\alpha-}D - \frac{1}{6}\psi^{\beta-}(T_{de}{}^0 - R(\mathbb{D})_{de})(\sigma^{de})_\beta{}^\alpha\right]\\
&\quad - \epsilon_{abcd}e^a e^b e^c\left[\frac{4i}{3}\bar{\phi}^+_{\dot{\beta}}\mathcal{V}^{--}(\bar{\sigma}^d)^{\dot{\beta}\alpha} - \frac{4}{3}\psi^{\beta-}f^e(\sigma^d\bar{\sigma}_e)_\beta{}^\alpha + i\bar{\psi}^-_{\dot{\beta}}\psi^{\beta-}\chi^+_\beta(\bar{\sigma}^d)^{\dot{\beta}\alpha}\right]\\
&\quad + e^a e^b\Big[8i\psi^{\beta-}\mathcal{V}^{--}\mathcal{V}^{++}(\sigma_{ab})_\beta{}^\alpha - 2\bar{\psi}^-_{\dot{\alpha}}\psi^{\beta+}\bar{\phi}^{\dot{\beta}+}\epsilon_{abcd}(\bar{\sigma}^c)^{\dot{\alpha}\alpha}(\sigma^d)_{\beta\dot{\beta}}\\
&\quad + 4i\psi^{\beta-}\psi^{\gamma-}\phi^{\alpha+}(\sigma_{ab})_{\beta\gamma}\Big] + 4e^a\bar{\psi}^{\dot{\beta}-}\psi^{\beta-}\psi^{\alpha-}\mathcal{V}^{++}(\sigma_a)_{\beta\dot{\beta}},\\
\bar{\partial}_{\dot{\alpha}+}(\partial_+)^2\hat{e}^{++} &= \epsilon_{abcd}e^a e^b e^c e^d\left[-\frac{1}{2}\mathcal{V}^{--}\chi^+_{\dot{\alpha}} - \frac{1}{4}\bar{\psi}^-_{\dot{\alpha}}D + \frac{1}{6}\bar{\psi}^-_{\dot{\beta}}(T_{de}{}^0 - R(\mathbb{D})_{de})(\bar{\sigma}^{de})^{\dot{\beta}}{}_{\dot{\alpha}}\right]\\
&\quad + \epsilon_{abcd}e^a e^b e^c\left[\frac{4i}{3}\phi^{\beta+}\mathcal{V}^{--}(\sigma^d)_{\beta\dot{\alpha}} - \frac{4}{3}\bar{\psi}^-_{\dot{\beta}}f^e(\bar{\sigma}^d\sigma_e)^{\dot{\beta}}{}_{\dot{\alpha}} - i\psi^{\beta-}\bar{\psi}^-_{\dot{\beta}}\bar{\chi}^{\dot{\beta}+}(\sigma^d)_{\beta\dot{\alpha}}\right]\\
&\quad + e^a e^b[8i\bar{\psi}^-_{\dot{\beta}}\mathcal{V}^{--}\mathcal{V}^{++}(\bar{\sigma}_{ab})^{\dot{\beta}}{}_{\dot{\alpha}} - 2\psi^{\beta-}\bar{\psi}^{\dot{\beta}+}\phi^{\gamma+}\epsilon_{abcd}(\bar{\sigma}^c)_{\beta\dot{\alpha}}(\sigma^d)_{\gamma\dot{\beta}}\\
&\quad - 4i\bar{\psi}^{\dot{\beta}-}\bar{\psi}^{\dot{\gamma}-}\bar{\phi}^+_{\dot{\alpha}}(\bar{\sigma}_{ab})_{\dot{\beta}\dot{\gamma}}\Big] - 4e^a\bar{\psi}^{\dot{\beta}-}\psi^{\beta-}\bar{\psi}^-_{\dot{\alpha}}\mathcal{V}^{++}(\sigma_a)_{\beta\dot{\beta}}.
\end{aligned}
\tag{C11}
$$

The highest term involves four spinor derivatives:

$$
\begin{aligned}
(\partial_+)^2(\bar{\partial}_+)^2\hat{e}^{++} &= 2D\epsilon_{abcd}e^a e^b e^c e^d\mathcal{V}^{--} + e^a e^b e^c\Big[8\psi^{\beta-}\bar{\psi}^{\dot{\beta}-}(\sigma_c)_{\beta\dot{\beta}}(T_{ab}{}^0 - R(\mathbb{D})_{ab}) + \frac{32}{3}f^d\mathcal{V}^{--}\epsilon_{abcd}\\
&\quad + 4i\bar{\psi}^-_{\dot{\alpha}}\mathcal{V}^{--}\chi^+_\alpha(\bar{\sigma}^d)^{\dot{\alpha}\alpha}\epsilon_{abcd} - 4i\psi^{\alpha-}\mathcal{V}^{--}\bar{\chi}^{\dot{\alpha}+}(\sigma^d)_{\alpha\dot{\alpha}}\epsilon_{abcd}\Big]\\
&\quad + e^a e^b\Big[32i\bar{\psi}^{\dot{\beta}-}\bar{\phi}^{\dot{\gamma}+}\mathcal{V}^{--}(\bar{\sigma}_{ab})_{\dot{\beta}\dot{\gamma}} + 32i\psi^{\beta-}\phi^{\gamma+}\mathcal{V}^{--}(\sigma_{ab})_{\beta\gamma}\\
&\quad + 12i\bar{\psi}^{\dot{\beta}-}\bar{\psi}^{\dot{\gamma}-}\psi^{\alpha-}\chi^+_\alpha(\bar{\sigma}_{ab})_{\dot{\beta}\dot{\gamma}} - 12i\psi^{\beta-}\psi^{\gamma-}\bar{\psi}^-_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}+}(\sigma_{ab})_{\beta\gamma} + 32\psi^{\beta-}\bar{\psi}^{\dot{\beta}-}f_a(\sigma_b)_{\beta\dot{\beta}}\Big]\\
&\quad + 16e^a\psi^{\alpha-}\bar{\psi}^{\dot{\alpha}-}[\psi^{\beta-}\phi^+_\beta - \bar{\psi}^-_{\dot{\beta}}\bar{\phi}^{\dot{\beta}+} + 3\mathcal{V}^{--}\mathcal{V}^{++}](\sigma_a)_{\alpha\dot{\alpha}}.
\end{aligned}
\tag{C12}
$$

In the above expressions, we note that the curvatures $T_{ab}{}^0$ and $R(\mathbb{D})_{ab}$ were actually found by spinor differentiation of covariant fields such as $\chi^{\alpha+}$ and $\bar{\chi}^{\dot{\alpha}+}$ that appeared at lower dimensions, using the explicit expressions for T^0 and $R(\mathbb{D})$ in terms of $W_{\alpha\beta}$ and $\bar{W}_{\dot{\alpha}\dot{\beta}}$.

The component action can then be written as

$$
S = -\frac{1}{2\pi}\oint_{\mathcal{C}} \mathrm{d}\tau \int \mathrm{d}^4x\mathrm{d}^4\theta^+\mathcal{E}^{--}\mathcal{L}^{++} = -\frac{1}{2\pi}\int_{\mathcal{M}^4\times\mathcal{C}}\mathcal{J},
\tag{C13}
$$

where $\mathcal{J}$ is a five-form given by (5.7). The full expression for the five-form $\mathcal{J}$ is quite complicated in the general component gauge. In practice, one should always analyze component actions in the central gauge. Recall in this gauge

$$e^a = \mathrm{d}x^m e_m{}^a, \qquad \psi^{\alpha+} = \mathrm{d}x^m \psi_m{}^{\alpha+}, \text{etc.} \tag{C14}$$

while only the connections $\mathcal{V}^{\pm\pm}$ and $\mathcal{V}^0$ possess a $\mathrm{d}\tau$ component,

$$\mathcal{V}^{\pm\pm} = \mathrm{d}x^m \mathcal{V}_m{}^{\pm\pm} + \mathrm{d}\tau \mathcal{V}_\tau{}^{\pm\pm},$$
$$\mathcal{V}^0 = \mathrm{d}x^m \mathcal{V}_m{}^0 + \mathrm{d}\tau \mathcal{V}_\tau{}^0. \tag{C15}$$

Because the integral selects out only the component $\mathcal{J}$ involving $\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}\tau$, only those components of $\mathcal{J}$ involving at least one of $\mathcal{V}^{\pm\pm}$ and $\mathcal{V}^0$ can contribute. Now one can make a dramatic simplification by going to the central gauge:

$$[\partial_+^\alpha \hat{e}^{++}]_{\mathrm{CG}} \sim \frac{i}{6}\epsilon_{abcd} e^a e^b e^c \bar\psi^-_{\dot\beta} \mathcal{V}^{++} (\bar\sigma^d)^{\dot\beta\alpha}, \qquad [\bar\partial_{\dot\alpha+} \hat{e}^{++}]_{\mathrm{CG}} \sim \frac{i}{6}\epsilon_{abcd} e^a e^b e^c \psi^{\beta-} \mathcal{V}^{++} (\sigma^d)_{\beta\dot\alpha},$$
$$[(\partial_+)^2 \hat{e}^{++}]_{\mathrm{CG}} \sim -2i e^a e^b \bar\psi^{\dot\alpha-} \bar\psi^{\dot\beta-} \mathcal{V}^{++} (\bar\sigma_{ab})_{\dot\alpha\dot\beta}, \qquad [(\bar\partial_+)^2 \hat{e}^{++}]_{\mathrm{CG}} \sim 2i e^a e^b \psi^{\alpha-} \psi^{\beta-} \mathcal{V}^{++} (\sigma_{ab})_{\alpha\beta},$$
$$[\partial_{\alpha+}\bar\partial_{\dot\alpha+} \hat{e}^{++}]_{\mathrm{CG}} \sim e^a e^b \left(\frac{1}{2}\psi^{\beta-}\bar\psi^{\dot\beta-}\mathcal{V}^{++}(\sigma^c)_{\alpha\dot\beta}(\sigma^d)_{\beta\dot\alpha} - \frac{i}{3} e^c \mathcal{V}^{--}\mathcal{V}^{++}(\sigma^d)_{\alpha\dot\alpha}\right)\epsilon_{abcd},$$
$$\begin{aligned}[\partial_+^\alpha (\bar\partial_+)^2 \hat{e}^{++}]_{\mathrm{CG}} \sim{}& \frac{1}{2}\epsilon_{abcd} e^a e^b e^c e^d \mathcal{V}^{--}\chi^{\alpha+} - \frac{4i}{3}\epsilon_{abcd} e^a e^b e^c \bar\phi^+_{\dot\beta} \mathcal{V}^{--} (\bar\sigma^d)^{\dot\beta\alpha} \\ &+ 8i e^a e^b \psi^{\beta-}\mathcal{V}^{--}\mathcal{V}^{++}(\sigma_{ab})_\beta{}^\alpha + 4 e^a \bar\psi^{\dot\beta-}\psi^{\beta-}\psi^{\alpha-}\mathcal{V}^{++}(\sigma_a)_{\beta\dot\beta},\end{aligned}$$
$$\begin{aligned}[\bar\partial_{\dot\alpha+} (\partial_+)^2 \hat{e}^{++}]_{\mathrm{CG}} \sim{}& -\frac{1}{2}\epsilon_{abcd} e^a e^b e^c e^d \mathcal{V}^{--}\chi^+_{\dot\alpha} + \frac{4i}{3}\epsilon_{abcd} e^a e^b e^c \phi^{\beta+} \mathcal{V}^{--} (\sigma^d)_{\beta\dot\alpha} \\ &+ 8i e^a e^b \bar\psi^-_{\dot\beta}\mathcal{V}^{--}\mathcal{V}^{++}(\bar\sigma_{ab})^{\dot\beta}{}_{\dot\alpha} - 4 e^a \bar\psi^{\dot\beta-}\psi^{\beta-}\bar\psi^-_{\dot\alpha}\mathcal{V}^{++}(\sigma_a)_{\beta\dot\beta},\end{aligned}$$
$$\begin{aligned}[(\partial_+)^2 (\bar\partial_+)^2 \hat{e}^{++}]_{\mathrm{CG}} \sim{}& e^a e^b e^c (2 e^d \mathcal{V}^{--} D + 4i\bar\psi^-_{\dot\alpha}\mathcal{V}^{--}\chi^+_\alpha (\bar\sigma^d)^{\dot\alpha\alpha} - 4i\psi^{\alpha-}\mathcal{V}^{--}\bar\chi^{\dot\alpha+}(\sigma^d)_{\alpha\dot\alpha})\epsilon_{abcd} \\ &+ \frac{32}{3} e^a e^b e^c f^d \mathcal{V}^{--}\epsilon_{abcd} + 48 e^a \psi^{\alpha-}\bar\psi^{\dot\alpha-}\mathcal{V}^{--}\mathcal{V}^{++}(\sigma_a)_{\alpha\dot\alpha} \\ &+ 32i e^a e^b \bar\psi^{\dot\beta-}\bar\phi^{\dot\gamma+}\mathcal{V}^{--}(\bar\sigma_{ab})_{\dot\beta\dot\gamma} + 32i e^a e^b \psi^{\beta-}\phi^{\gamma+}\mathcal{V}^{--}(\sigma_{ab})_{\beta\gamma}.\end{aligned}$$

Converting the five-form into its corresponding integral density gives

$$\int_{\mathcal{M}^4\times\mathcal{C}} \mathcal{J} = \oint_{\mathcal{C}} \mathrm{d}\tau \int \mathrm{d}^4x\, e(\mathcal{V}_\tau^{++}\mathcal{L}^{--} - \mathcal{V}_\tau^{--}\mathcal{L}^{++}), \tag{C16}$$

where

$$\begin{aligned}\mathcal{L}^{--} ={}& \frac{1}{16}(\nabla^-)^2(\bar\nabla^-)^2\mathcal{L}^{++} - \frac{i}{8}(\bar\psi^-_m\bar\sigma^m)^\alpha\nabla^-_\alpha(\bar\nabla^-)^2\mathcal{L}^{++} - \frac{i}{8}(\psi^-_m\sigma^m)_{\dot\alpha}\bar\nabla^{\dot\alpha-}(\nabla^-)^2\mathcal{L}^{++} \\ &+ \frac{1}{4}((\psi^-_n\sigma^{nm})^\alpha\bar\psi_m{}^{\dot\alpha-} + \psi_n{}^{\alpha-}(\bar\sigma^{nm}\bar\psi^-_m)^{\dot\alpha} - i\mathcal{V}_m^{--}\sigma^m_{\alpha\dot\alpha})[\nabla^-_\alpha, \bar\nabla^-_{\dot\alpha}]\mathcal{L}^{++} \\ &+ \frac{1}{4}(\psi^-_m\sigma^{mn}\psi^-_n)(\nabla^-)^2\mathcal{L}^{++} + \frac{1}{4}(\bar\psi^-_m\bar\sigma^{mn}\bar\psi^-_n)(\bar\nabla^-)^2\mathcal{L}^{++} \\ &- \left(\frac{1}{2}\epsilon^{mnpq}(\psi^-_m\sigma_n\bar\psi^-_p)\psi^{\alpha-}_q - 2(\psi^-_m\sigma^{mn})^\alpha\mathcal{V}_n^{--}\right)\nabla^-_\alpha\mathcal{L}^{++} \\ &+ \left(\frac{1}{2}\epsilon^{mnpq}(\bar\psi^-_m\bar\sigma_n\psi^-_p)\bar\psi^-_{q\dot\alpha} - 2(\bar\psi^-_m\bar\sigma^{mn}\right)_{\dot\alpha}\mathcal{V}_n^{--})\bar\nabla^{\dot\alpha-}\mathcal{L}^{++} \\ &+ 3\epsilon^{mnpq}(\psi^-_m\sigma_n\bar\psi^-_p)\mathcal{V}_q^{--}\mathcal{L}^{++}\end{aligned} \tag{C17}$$

and

$$\mathcal{L}^{++} = -\Big[3D + \frac{3i}{2}(\bar{\psi}^-_m\bar{\sigma}^m\chi^+) - \frac{3i}{2}(\psi^-_m\sigma^m\bar{\chi}^+) + 4f_a{}^a - 4(\bar{\psi}^-_m\bar{\sigma}^{mn}\bar{\phi}^+_n) + 4(\psi^-_m\sigma^{mn}\phi^+_n) - 3\epsilon^{mnpq}(\psi^-_m\sigma_n\bar{\psi}^-_p)\mathcal{V}^{++}_q]\mathcal{L}^{++}$$
$$+\Big[\frac{3}{2}\chi^{\alpha+} - i(\bar{\phi}^+_m\bar{\sigma}^m)^\alpha + 2(\psi^-_m\sigma^{mn})^\alpha\mathcal{V}^{++}_n\Big]\nabla^-_\alpha\mathcal{L}^{++} - \Big[\frac{3}{2}\chi^+_{\dot\alpha} - i(\phi^+_m\sigma^m)_{\dot\alpha} + 2(\bar{\psi}^-_m\bar{\sigma}^{mn})_{\dot\alpha}\mathcal{V}^{++}_n\Big]\bar{\nabla}^{\dot\alpha-}\mathcal{L}^{++}$$
$$-\frac{i}{4}\mathcal{V}^{++}_m(\bar{\sigma}^m)^{\dot\alpha\alpha}[\nabla^-_\alpha,\bar{\nabla}^-_{\dot\alpha}]\mathcal{L}^{++}. \tag{C18}$$

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