Reducibility of valence-3 Killing tensors in Weyl's class of stationary and axially symmetric spacetimes

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Stationary and axially symmetric spacetimes play an important role in astrophysics, particularly in the theory of neutron stars and black holes. The static vacuum subclass of these spacetimes is known as Weyl's class, and contains the Schwarzschild spacetime as its most prominent example. This paper is going to study the space of Killing tensor fields of valence 3 for spacetimes of Weyl's class. Killing tensor fields play a crucial role in physics since they are in correspondence to invariants of the geodesic motion (i.e. constants of the motion). It will be proven that in static and axially symmetric vacuum spacetimes the space of Killing tensor fields of valence 3 is generated by Killing vector fields and quadratic Killing tensor fields. Using this result, it will be proven that for the family of Zipoy-Voorhees metrics, valence-3 Killing tensor fields are always generated by Killing vector fields and the metric.

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I. INTRODUCTION

Consider a manifold M with Lorentzian metric g, and its cotangent space T^*M endowed with its natural symplectic form. A Killing tensor field K of valence d on M is a symmetric $(0,d)$-tensor such that

$$\nabla_{(a}K_{b_1\ldots b_d)} = 0. \tag{1}$$

The Lorentzian metric provides an isomorphism between T^*M and TM, and we are therefore going to identify co- and contravariant tensor fields as well as the corresponding homomorphisms with mixed co- and contravariant indices.

Killing tensor fields are in 1-to-1 correspondence to (first) integrals (or *Hamiltonian invariants*) polynomial in the momenta p (or the velocities $\dot{\gamma}$) of the Hamiltonian motion for the Hamiltonian function $H = g^{ij}p_ip_j = g(\dot{\gamma},\dot{\gamma})$. In the language of integrals, the Killing tensor equation takes the form

$$\{I_K, H\} = X_H(I_K) \equiv 0 \tag{2}$$

where $\{\cdot,\cdot\}$ denotes the usual Poisson bracket on T^*M, and where $X_H(I_K)$ is the derivative of the function $I_K = K(\dot{\gamma},\ldots,\dot{\gamma})$ in the direction of the Hamiltonian vector field X_H.Two integrals are said to be in involution if they commute with respect to the Poisson bracket.

Studying the existence of polynomial integrals is interesting from at least two perspectives. Firstly, the existence of integrals can help in answering natural questions about the behavior of trajectories, i.e. the behavior of free falling particles in physically motivated Hamiltonian systems (e.g. by the famous Liouville-Arnold theorem one can integrate the system by quadratures under certain additional assumptions).

Secondly, asking for the existence of integrals is a natural geometric requirement. Metrics meeting this requirement may lead to physically interesting examples, as for example in the case of the Kerr metric possessing the Carter constant [1,2], an integral quadratic in momenta in addition to energy and axial symmetry. Integrals polynomial in the momenta are of particular interest since they represent a generalization of constants of the motion that emerge from the action of one-parameter groups of diffeomorphisms. According to [3], the example of Kerr-de Sitter spacetimes is at present the only known example of integrable spacetimes with an additional integral of higher-than-linear degree, among the class of stationary and axially symmetric spacetimes.

Static and axially symmetric vacuum (StAV) spacetimes form Weyl's class; they are special cases of stationary and axially symmetric vacuum (SAV) spacetimes [4,5]. Recently, some attention has been drawn to the Zipoy-Voorhees family, which belongs to this class [6,7]. Numerical studies [4,8–10] suggested integrability for some StAV metrics, while later studies provided contradicting evidence [11–13]. Note that while these studies considered fixed values of the parameter δ of the Zipoy-Voorhees metric, in this paper we are going to consider arbitrary δ in the case of the Zipoy-Voorhees metric. Therefore the result for the Zipoy-Voorhees family is in line with the evidence contradicting integrability of the family.

The methods used in this paper are not restricted to the concrete setting of Weyl's class. It is therefore likely that the methods are suitable for other examples of parametrized metrics in two dimensions. It seems probable that such examples include 2-dimensional ones with potential, since this is closest to the case studied here. The study of integrals in 2-dimensional manifolds is a classical problem in

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differential geometry and goes at least back to Darboux and Koenigs [14]. For instance, some nonexistence results have been obtained on the 2-torus for cubic and quartic integrals in [15–17], while for higher degrees almost nothing is known. On the 2-sphere, however, some integrable examples are known. For instance, a new integrable system has been presented, with the additional integral being of cubic degree, by Dullin and Matveev [18].

The procedure taken in this paper is a new approach to the question of existence of integrals. It combines two major previous lines of action:

(i) It is inspired by ideas from prolongation-projection methods well-known in the theory of overdetermined partial differential equation (PDE) systems. However, it does not follow the algorithmic procedure used in [11].
(ii) It follows ideas outlined in [19], but takes a somehow converse track that could be described as "bottom-up" in contrast to the "top-down" approach taken in [19]. The advantage of this direction of reasoning is that it avoids solving the leading-degree Poisson equation. Instead, our approach rather begins with solving simple geometric orthogonality relations.

A. Static and axially symmetric vacuum spacetimes

Stationary and axially symmetric vacuum spacetimes possess two commuting Killing vector fields, one being spacelike and the other being timelike. Such spacetimes can be brought into the following standard form [5] by the aid of suitable coordinate transformations. The coordinates are called Lewis-Papapetrou coordinates [8,10].

$$g = e^{2U}(e^{-2\gamma}(dx^2+dy^2)+R^2 d\phi^2) - e^{-2U}(dt-\omega d\phi)^2 \tag{3}$$

We will restrict our attention to vacuum spacetimes and therefore require the Ricci tensor of g to be identically zero. This is a fair assumption for the movement of test particles around astrophysical objects as long as electromagnetic fields are ignored. For SAV spacetimes, Ricci flatness is encoded in a set of equations which are called the Ernst equations. In the static case, we require $\omega = 0$. Then, the Ernst equations read as follows:

$$\begin{aligned} R_yU_y + R_xU_x + RU_{yy} + RU_{xx} &= 0\\ \Delta R = R_{xx} + R_{yy} &= 0\\ 2RU_x^2 - 2R_y\gamma_y + 2R_x\gamma_x + R_{xx} - R_{yy} - 2RU_y^2 &= 0\\ 2RU_xU_y + R_y\gamma_x + R_x\gamma_y + R_{xy} &= 0. \end{aligned}$$

The equations break up into two sets of two equations each, which we shall refer to as primary and secondary equations. The primary equations give restrictions on U and R. Provided R is nonconstant, $\Delta R = 0$ allows setting $R = x > 0$ by a change of coordinates [5]. If R is constant, this change of coordinates is impossible, but one can show that $\Delta\gamma = 0$ holds and the metric is flat. In case of nonconstant R, the secondary equations enable us to express derivatives of γ in terms of derivatives of U, allowing us to eliminate them, and finally γ, from the equations. We obtain the relations:

$$\begin{aligned} U_{yy} &= -U_{xx} - \frac{1}{x}U_x \qquad & \gamma_x &= -xU_x^2 + xU_y^2\\ R &= x & \gamma_y &= -2xU_xU_y. \end{aligned} \tag{4}$$

By definition, stationarity and axial symmetry can be described by the global symmetry group $h = \mathbb{R}\times\mathbb{S}^1$. The action of h is *Hamiltonian* and we denote the moment map by μ. h acts freely, and we may pass to the symplectic quotient $Q_{\mathrm{red}} = \mu^{-1}(0)/h$, which inherits a symplectic form from the initial spacetime (with an additional compactness assumption, this is the *Marsden-Weinstein quotient*, see e.g. [20]). In Lewis-Papapetrou coordinates, h acts along coordinate directions and we will be able to identify the reduced coordinates easily. The 4-dimensional problem then is reformulated as a 2-dimensional problem with metric g_{red}, and the Hamiltonian $H = T + V$ splits into a kinetic term $T = H_{\mathrm{red}} = g^{ij}_{\mathrm{red}}p_ip_j$ along with a potential V, which is polynomial in p_ϕ and p_t. As a consequence, we distinguish Hamiltonian integrals that commute with H, from *metric integrals* that commute with T only. Note that the highest-degree component with respect to (p_x, p_y) of a Hamiltonian integral is metric (i.e. commutes with T). The metric and the potential on the reduced space read

$$g_{\mathrm{red}} = e^{2U}(e^{-2\gamma}(dx^2+dy^2)) \tag{5a}$$

$$V = R^{-2}e^{-2U}p_\phi^2 - e^{2U}p_t^2. \tag{5b}$$

B. Main results

We consider integrals of the general form

$$I(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{i}\sum_{k=0}^{3-i} a_{i,j,k}(x,y)p_x^jp_y^{i-j}p_\phi^kp_t^{3-i-k}. \tag{6}$$

Such integrals are in involution with the p_ϕ and p_t. Since the Hamiltonian defined by Eq. (5) does not mix momenta p_x, p_y with p_ϕ, p_t, the components of (6) of odd and even parity in (p_x, p_y) can be considered separately. We prove reducibility of degree-3 integrals in spacetimes of Weyl's class.

Definition 1: Let I be a polynomial integral of degree d. We say that I is *reducible* (by one degree) if there are polynomial integrals $I_1, \ldots, I_m$ of degree at most $d-1$ such that I is a linear combination of products of the integrals I_i.

We say that I is *totally reducible* if there is a representation of this form such that the I_i are integrals linear in momenta or the Hamiltonian.

For spacetimes in Weyl's class, we prove reducibility of degree-3 integrals by one degree. In addition, total reducibility of degree-3 integrals is shown for the family of Zipoy-Voorhees spacetimes (a subfamily of Weyl's class).

Theorem 1: Let M be a spacetime in Weyl's class. Then any integral (6) of third degree on M is reducible.

Reducibility of a degree-3 polynomial integral means that these integrals can be written via products of lower-degree integrals. In the language of Killing tensors, this means that any valence-3 Killing tensor field can be written via symmetrized products of Killing vector fields and quadratic Killing fields.

For a concrete example, consider the Zipoy-Voorhees class of spacetimes. Their metrics are static and axially symmetric, and parametrized by a parameter $\delta \geq 0$ [6]. Therefore, this family forms a subset of Weyl's class.

$$g = \left(\frac{x+1}{x-1}\right)^{\delta}\left(\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2-1} dx^2 + \left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}\frac{x^2-y^2}{1-y^2}dy^2 + (x^2-1)(1-y^2)dz^2\right) - \left(\frac{x-1}{x+1}\right)^{\delta} dt^2. \tag{7}$$

The resulting metric for $\delta = 0$ is flat. The value $\delta = 1$ gives the Schwarzschild metric (that admits one additional quadratic integral). We allow arbitrary $\delta \geq 0$.

Proposition 1: Let M_{ZV} be a Zipoy-Voorhees metric with parameter $\delta > 0, \delta \neq 1$. Let I be an integral (6) of third degree on M_{ZV}. Then I is totally reducible, i.e. generated by linear integrals (i.e. Killing vector fields) and the Hamiltonian (i.e. the metric).

II. METHOD

The basic procedure is as follows:

(i) Reduce the 4-dimensional problem without potential to finding integrals on a 2-dimensional Hamiltonian manifold with potential (symplectic reduction).

(ii) The existence of integrals is encoded in equations that emerge from the Poisson equation $\{H, I_K\} = 0$ as coefficients with respect to momenta. Splitting according to the degree in momenta (p_x, p_y) yields three polynomials in p_ϕ and p_t. If we decompose these polynomials further with respect to momenta, we obtain three blocks of equations.

(iii) Use the equations obtained from zeroth degree in momenta (p_x, p_y) and solve them as far as possible, obtaining one function α to parametrize the integral (this is the case without an additional integral present. If there is an additional linear integral, Lemma 3 applies).

(iv) From the block obtained from the degree-2 polynomial, extract two integrability conditions for α.

(v) The remaining system of equations is an overdetermined system of PDE involving the metric which is described by one function U in two variables. We consider derivatives of U as being new, independent unknowns. By taking derivatives (prolongation), and then eliminating higher derivatives of U (projection), we end up with an ordinary differential equation.

(vi) For the remaining ordinary differential equation (ODE) we show that its only solution corresponds to flat space. This allows us to conclude that degree-3 integrals are always reducible.

The computer algebra computations for this paper have been performed using Maple 18 (™ Waterloo Maple Inc.).

Equation (2) is the condition for a function I to be an integral. Since we take I to be polynomial in momenta of degree d, (2) is a polynomial in momenta of degree $d+1$. We are going to consider the system of PDE obtained from the coefficients of (2) with respect to momenta. Symplectic reduction with respect to the symmetry group (stationarity, axial symmetry) suggests to regard p_ϕ and p_t as parameters. We distinguish the equations of the PDE system by the momenta monomials to which they appeared as a coefficient. For Weyl's class, the equations can then be arranged in a treelike [4] structure. We write down the Hamiltonian in the following form:

$$H = T + \underbrace{V^{\phi\phi} p_\phi^2 + V^{tt} p_t^2}_{=V} \tag{8}$$

where $T \equiv H_{\mathrm{red}}$ is the reduced Hamiltonian, i.e. a homogeneous polynomial in p_x, p_y, and where V^{ab} are the smooth coefficient functions of $p_a p_b$ $(a, b \in \{\phi, t\})$. The integral I can be decomposed accordingly. We denote

$$I = I^{(\mathrm{d})} + \underbrace{I_\phi^{(\mathrm{d\text{-}1})} p_\phi + I_t^{(\mathrm{d\text{-}1})} p_t}_{=I^{(\mathrm{d\text{-}1})}} + \underbrace{I_{\phi\phi}^{(\mathrm{d\text{-}2})} p_\phi^2 + I_{t\phi}^{(\mathrm{d\text{-}2})} p_t p_\phi + I_{tt}^{(\mathrm{d\text{-}2})} p_t^2}_{=I^{(\mathrm{d\text{-}2})}} + \cdots + I_{tt\ldots t}^{(0)} p_t^d, \tag{9}$$

where each $I^{(\mathrm{k})}$ is of degree k in the momenta p_x, p_y. We require the metric to be nonflat such that we can choose coordinates with $R = x$. In this case we have three blocks of equations coming from the respective polynomials [this is step (ii) of the above list]. We can extract the equations from the polynomials which are obtained by splitting (2) according to the degree with respect to (p_x, p_y):

$$\{T, I^{(3)}\} = 0 \quad \text{degree 4} \tag{10a}$$

$$\{T, I^{(1)}\} + \{V, I^{(3)}\} = 0 \quad \text{degree 2} \tag{10b}$$

$$\{V, I^{(1)}\} = 0 \quad \text{degree 0}. \tag{10c}$$

The equations of even parity in (p_x, p_y) split off from this system and form a separate, decoupled system. Equation (10a) is the condition that must hold for an integral $I^{(3)}$ on the reduced manifold with Hamiltonian $T = H_{\mathrm{red}}$. However, only some of these integrals ascent to integrals upstairs on the initial manifold. This is due to the restrictions (10b) and (10c). For a better understanding of these equations (or the equations obtained from them as coefficients with respect to momenta), we will characterize them as defining equations for $I^{(3)}$ and $I^{(1)}$. But first let us split the system further by considering coefficients with respect to (p_t, p_ϕ). The polynomial (10a) does not split since it is already of degree 4 in momenta (p_x, p_y). The polynomial (10b) splits into three parts:

$$\begin{aligned} \{T, I^{(1)}_{\phi\phi}\} + \{V^{\phi\phi}, I^{(3)}\} &= 0 \\ 0 &= \{T, I^{(1)}_{t\phi}\} \\ \{T, I^{(1)}_{tt}\} + \{V^{tt}, I^{(3)}\} &= 0. \end{aligned}$$

We write the equations in this form to hint at the fact that the equations can be divided into two groups that can be treated separately. This will become clear when we include (10c). The second of the equations says that $I^{(1)}_{t\phi}$ is a metric integral on the reduced space. In fact, we will see that it even has to be an integral on the initial spacetime, and therefore is not of interest for our considerations. The polynomial (10c) splits into five parts:

$$\begin{aligned} \{V^{\phi\phi}, I^{(1)}_{\phi\phi}\} &= 0 \\ 0 &= \{V^{\phi\phi}, I^{(1)}_{t\phi}\} \\ \{V^{tt}, I^{(1)}_{\phi\phi}\} + \{V^{\phi\phi}, I^{(1)}_{tt}\} &= 0 \\ 0 &= \{V^{tt}, I^{(1)}_{t\phi}\} \\ \{V^{tt}, I^{(1)}_{tt}\} &= 0. \end{aligned}$$

The second and fourth of these relations tell us that $I^{(1)}_{t\phi}$ is an integral not only on the reduced, but also on the initial space. We can isolate this subsystem from the remaining one and solve it separately (this procedure is possible in general for Weyl's class). This can easily be done and is equivalent to finding Killing vector fields of the spacetime under consideration.

The remaining equations from (10c) can be interpreted in a nice way as scalar product relations for the components of $I^{(1)}$. For instance,

$$\begin{aligned} \{V^{\phi\phi}, I^{(1)}_{\phi\phi}\} &= V^{\phi\phi}_x b^{\phi\phi}_1 + V^{\phi\phi}_y b^{\phi\phi}_2 = e^{2U-2\gamma}\langle \nabla V^{\phi\phi}, b^{\phi\phi}\rangle \\ &= \langle dV^{\phi\phi}, b^{\phi\phi}\rangle \end{aligned}$$

where $I^{(1)}_{\phi\phi} = b^{\phi\phi}_1 p_x + b^{\phi\phi}_2 p_y$ and where $\nabla V^{\phi\phi}$ denotes the gradient vector corresponding to the differential $dV^{\phi\phi}$. The polynomial (10c) therefore gives rise to a set of scalar product relations:

$$\begin{aligned} \langle \nabla V^{\phi\phi}, b^{\phi\phi}\rangle &= 0 \\ \langle \nabla V^{tt}, b^{\phi\phi}\rangle + \langle \nabla V^{\phi\phi}, b^{tt}\rangle &= 0 \\ \langle \nabla V^{tt}, b^{tt}\rangle &= 0. \end{aligned}$$

This allows us to solve (10c) directly for $b^{\phi\phi}$ and b^{tt},

$$b^{\phi\phi} = \alpha_1 \nabla^\perp V^{\phi\phi}, \qquad b^{tt} = \alpha_2 \nabla^\perp V^{tt}$$

where we introduce the shorthand notation $\nabla^\perp f = e^{2U-2\gamma}(-f_y, f_x)$ for a function f, i.e. $\nabla^\perp f$ is the vector field rotated by $\pi/2$ compared to ∇f. Defining the angle Ψ between ∇V^{tt} and $\nabla V^{\phi\phi}$,

$$\cos\Psi = \frac{\langle \nabla V^{tt}, \nabla V^{\phi\phi}\rangle}{\|\nabla V^{\phi\phi}\|\|\nabla V^{tt}\|},$$

the second of the three scalar product relations can be brought into the form

$$(\alpha_2 - \alpha_1)\sin\Psi = 0. \tag{11}$$

This is step (iii) in the list given at the beginning of this section. We summarize.

Lemma 1: Either the metric potentials are such that $\nabla V^{\phi\phi}$ and ∇V^{tt} are parallel, or the parameter functions α_1 and α_2 are equal.

We now turn to an interpretation of (10b). Consider $\{V, I^{(3)}\}$ and denote $I^{(3)} = I^{ijk} p_i p_j p_k$. Then,

$$\begin{aligned} \{V, I^{(3)}\} &= 3(V_x I^{xij} p_i p_j + V_y I^{yij} p_i p_j) \\ &= 3(V_k I^{kij} p_i p_j), \end{aligned}$$

and analogously for $\{V^{\phi\phi}, I^{(3)}\}$ and $\{V^{tt}, I^{(3)}\}$. With this in mind, we interpret (10b) as defining equations for the tensor field $K^{(3)}(\nabla V, \cdot, \cdot)$. There are two more equations than components of $K^{(3)}$ and this allows us to find independent expressions for $K^{(3)}(\nabla V^{\phi\phi}, \cdot, \cdot)$ as well as $K^{(3)}(\nabla V^{tt}, \cdot, \cdot)$. Then, if dV^{tt} and $dV^{\phi\phi}$ are linearly independent, we can determine $K^{(3)}$ in terms of derivatives of the function $\alpha = \alpha_1 = \alpha_2$. We have the following obvious identities:

$$K^{(3)}(\nabla V^{\phi\phi}; \nabla V^{\phi\phi}, \nabla V^{tt}) = K^{(3)}(\nabla V^{tt}; \nabla V^{\phi\phi}, \nabla V^{\phi\phi}) \tag{12a}$$

$$K^{(3)}(\nabla V^{\phi\phi}; \nabla V^{tt}, \nabla V^{tt}) = K^{(3)}(\nabla V^{tt}; \nabla V^{\phi\phi}, \nabla V^{tt}). \tag{12b}$$

We proceed as follows:

(1) Determine $K^{(3)}$ in terms of α and its derivatives, if $\sin\Psi \neq 0$.

(2) Determine derivatives of α using the symmetry in the arguments of $K^{(3)}$. Then derive an integrability condition for α.

(3) Combine the integrability condition with (10a) and the Ernst equations. Show that the system does not have any solutions, using algebraic manipulations as well as prolongation-projection arguments for the system.

First, however, consider the case $\sin\Psi = 0$. In order to do this, we begin with a closer look at Killing vectors.

A. Killing vector fields

Assuming there is an additional linear integral on the 4-dimensional spacetime, we characterize the existence of linear integrals in terms of the rank of the 2×3-matrix whose columns are given by gradients of the potential components $V^{\phi\phi}$, $V^{t\phi}$ and V^{tt}:

$$\mathcal{M} = (dV^{\phi\phi}, dV^{tt}, dV^{t\phi}).$$

Since the rank of $\mathcal{M}$ is the dimension of the linear space spanned by $dV^{\phi\phi}$, dV^{tt}, $dV^{t\phi}$, it is a geometric object and independent of the choice of coordinates.

If $dV^{t\phi} = 0$, $\mathcal{M}$ might be replaced by the 2×2-matrix $(dV^{\phi\phi}, dV^{tt})$, which will also be denoted by $\mathcal{M}$. Then, instead of the rank of the matrix, the determinant may be used with the obvious correspondences. In case there is an additional linear integral present in a nonflat SAV spacetime (i.e. in addition to p_ϕ and p_t), the rank of $\mathcal{M}$ cannot be full. More precisely:

Lemma 2:

(a) Let (M, g) be in the SAV class.

(1) If there is an additional linear integral (Killing vector field), then the rank of $\mathcal{M}$ is 1, or the spacetime is flat.

(2) Let $\mathrm{rk}(\mathcal{M}) = 1$. Then p_y is a linear integral (Killing vector field) when using Lewis-Papapetrou coordinates with $R = x$.

(b) Let (M, g) be in Weyl's class.

(1) Let $\mathrm{rk}(\mathcal{M}) \leq 1$ be constant. Then there is an additional linear integral on M. In case $\mathrm{rk}(\mathcal{M}) = 1$ this vector field corresponds to p_y in Lewis-Papapetrou coordinates with $R = x$; in case $\mathrm{rk}(\mathcal{M}) = 0$ the spacetime is flat.

(2) If there is an additional linear integral, it is given by p_y in Lewis-Papapetrou coordinates with $R = x$, if M is nonflat.

Proof.—Part (a). For linear integrals we only have two polynomials after taking coefficients with respect to (p_x, p_y) [they are similar to the polynomials of degree 0 and 2 in (10)]. Let us denote the components of the (p_x, p_y)-linear part of the integral as

$$I^{(1)} = b_1 p_x + b_2 p_y, \qquad b = (b_1, b_2). \tag{13}$$

The zeroth order equations read

$$\langle \nabla V^{\phi\phi}, b\rangle = 0, \quad \langle \nabla V^{t\phi}, b\rangle = 0, \quad \langle \nabla V^{tt}, b\rangle = 0. \tag{14}$$

We conclude that the following relations must hold:

$$\begin{aligned} \langle \nabla V^{\phi\phi}, \nabla^\perp V^{t\phi}\rangle &= 0,\\ \langle \nabla V^{\phi\phi}, \nabla^\perp V^{tt}\rangle &= 0,\\ \langle \nabla V^{t\phi}, \nabla^\perp V^{tt}\rangle &= 0. \end{aligned} \tag{15}$$

This means, the potential gradients are pointing all in the same direction, i.e. they are pairwise linearly dependent. Hence, the rank of the potential gradient matrix is 1, provided the spacetime is nonflat. In the flat case we may have Lewis-Papapetrou coordinates with the parameter R constant, making it impossible to choose $R = x$. We therefore exclude the flat case from our considerations. This concludes the proof of claim one and establishes a necessary criterion for the existence of Killing vector fields in everywhere nonflat spacetimes.

Now, since the rank of the potential gradient matrix $\mathcal{M}$ is 1, all rows as well as all columns have to be linearly dependent. This again gives us relations (15), meaning that $\nabla V^{\phi\phi}$, $\nabla V^{t\phi}$ and ∇V^{tt} are pairwise linearly dependent. First, let us assume $\omega \neq 0$. We consider

$$\langle \nabla V^{\phi\phi}, \nabla^\perp V^{t\phi}\rangle = 0.$$

This equation amounts to the requirement

$$x\omega_x U_y - (1 + xU_x)\omega_y = 0$$

or the relation

$$\begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = \kappa \begin{pmatrix} 1 + xU_x \\ xU_y \end{pmatrix}$$

with a scalar function κ to be determined. Inserting this into the requirement

$$\langle \nabla V^{\phi\phi}, \nabla^\perp V^{tt}\rangle = 0$$

yields the relation

$$U_y x^2 e^{4U} = 0$$

and forces U to be a function of x only. Turning back to the relations for ω, we see that $\omega_y = 0$, so ω also is a function of x only.

Recalling the convention $R = x$, the metric depends on x only, if γ only depends on x, or if it is constant. We infer the secondary Ernst equations,

$$4x^2e^{4U}U_x^2 - \omega_x^2 - 4xe^{4U}\gamma_x = 0,$$
$$\gamma_y xe^{4U} = 0.$$

Consider the latter equation. It means $\gamma_y = 0$, so we are done. Since the metric does not depend on y, p_y must be an integral, and therefore provides a Killing vector field.

Now, assume $\omega = 0$. Then $\langle \nabla V^{\phi\phi}, \nabla^\perp V^{t\phi}\rangle = 0$ trivially and we have to go a slightly different way of reasoning. We consider

$$\langle \nabla V^{\phi\phi}, \nabla^\perp V^{tt}\rangle = 0.$$

It follows that

$$x^2e^{4U}U_y = 0,$$

which means $U_y = 0$ (on the entire neighborhood). Conclude $U = U(x)$, and then $\gamma = \gamma(x)$. Thus, the metric is a function of x only and p_y is a linear integral. This concludes the proof of part (a).

For part (b) let us first remark that if an additional linear integral exists in Weyl's class, it must be a multiple of p_y, or the metric is flat. Two cases need to be checked: firstly, if there is exactly one additional linear integral, it is a multiple of p_y. Secondly, if there are two (independent) additional linear integrals, there are three (say $b^{(k)}, k = 1, 2, 3$). Looking at the equations $\langle \nabla V^{ij}, b^{(k)}\rangle = 0$, this forces all gradients ∇V^{ij} to be zero (or, equivalently, $dV^{ij} = 0$). Hence, V is constant. Thus U and ω are constant, in contradiction to the assumption $R = x$. Therefore, the metric is flat.

With this remark, the first claim of part (b) follows immediately from part (a), keeping in mind that rank 0 corresponds to flat space. The second claim of part (b) follows immediately from the second statement of part (a). ■

III. PROOF OF THE MAIN THEOREM

For the proof we will without loss of generality assume constant rank for the matrix $\mathcal{M}$. If the spacetime M is not of constant rk$\mathcal{M}$, we may still consider the subsets of points in M with constant rank 0, 1, or 2. We may then work with the sets of their inner points ignoring the remaining points of M, which amount only to a null set with respect to the measure induced by the volume form on M. Proving the theorem on a dense set is sufficient because if a degree-3 polynomial integral is identical to a linear combination of products of H, p_ϕ and p_t on an open subset, this is true everywhere.

The proof will be completed in two steps. First we consider spacetimes with rk$\mathcal{M} = 1$, then the case rk$\mathcal{M} = 2$. As we have seen, the rank-1 case is the case when there is one additional Killing vector field. Rank 2 is the case if no additional Killing vector field exists (assuming nonflatness).

Lemma 3: If rk$\mathcal{M} = 1$, then any third-degree integral is reducible by at least one degree.

Proof.—By the hypothesis, there is the linear integral p_y in Lewis-Papapetrou coordinates with $R = x$. Consider (10b)

$$\{V^{\phi\phi}, F^{(3)}\} + \{T, F^{(1)}_{\phi\phi}\} = 0$$
$$\{V^{t\phi}, F^{(3)}\} + \{T, F^{(1)}_{t\phi}\} = 0$$
$$\{V^{tt}, F^{(3)}\} + \{T, F^{(1)}_{tt}\} = 0.$$

Each $F^{(1)}$ is a multiple of p_y, so

$$F^{(1)}_{\phi\phi} = h_1p_y, \qquad F^{(1)}_{t\phi} = h_2p_y, \qquad F^{(1)}_{tt} = h_3p_y.$$

This means that the equations in (10b) are of the form

$$\{V^{\phi\phi}, F^{(3)}\} + \{T, h_1\}p_y = 0$$
$$\{V^{t\phi}, F^{(3)}\} + \{T, h_2\}p_y = 0$$
$$\{V^{tt}, F^{(3)}\} + \{T, h_3\}p_y = 0.$$

The leading order term $F^{(3)}$ hence is of the form

$$F^{(3)} = p_x((\ldots)p_y) + fp_y^3 =: Fp_y$$

where the leading p_x is because the potential gradients (or, equivalently, the differentials dV^{ab}) have only p_x-components. The final contribution fp_y^3 accounts for the fact that (10b) only specifies components with at least one p_x.

Now consider (10a),

$$\{T, F^{(3)}\} = \{T, Fp_y\} = \{T, F\}p_y \overset{!}{=} 0.$$

This means $\{T, F\} = 0$, so F is a quadratic integral on the reduced space. It follows that it can be extended to an integral on the initial spacetime, because of the fact that

$$\{V^{\phi\phi}, F^{(3)}\} = \{V^{\phi\phi}, Fp_y\} = \{V^{\phi\phi}, F\}p_y$$

and so on, so we have from (10b) the equations

$$\{V^{\phi\phi}, F\} + \{T, h_1\} = 0$$
$$\{V^{t\phi}, F\} + \{T, h_2\} = 0$$
$$\{V^{tt}, F\} + \{T, h_3\} = 0$$

which makes $\tilde{F} = F + h_1p_\phi^2 + h_2p_\phi p_t + h_3p_t^2$ a quadratic integral on the initial spacetime (see Remark 1 below). Note that $\tilde{F}$ might still be reducible, but can be nonreducible as well. ■

Remark 1: An even-parity quadratic integral $I = I^{(2)} + I^{(0)}_{\phi\phi} p_\phi^2 + I^{(0)}_{t\phi} p_t p_\phi + I^{(0)}_{tt} p_t^2$ satisfies the polynomial equations

$$\{T, I^{(2)}\} = 0 \tag{16a}$$

$$\{V^{\phi\phi}, I^{(2)}\} + \{T, I^{(0)}_{\phi\phi}\} = 0 \tag{16b}$$

$$\{V^{t\phi}, I^{(2)}\} + \{T, I^{(0)}_{t\phi}\} = 0 \tag{16c}$$

$$\{V^{tt}, I^{(2)}\} + \{T, I^{(0)}_{tt}\} = 0. \tag{16d}$$

Proof.—Decompose $\{H, I\} = 0$ by setting each component homogeneous in (p_x, p_y) zero. The first equation is the component of degree 3, the other three equations are components of degree 1. ■

We now turn to the case when there is no additional linear integral in involution with the others. From now on, we will always assume to work in Weyl's class.

Keeping in mind the considerations of the previous sections, we see that this case requires $\mathrm{rk}\mathcal{M} = 2$. Rank 2 requires $\nabla V^{\phi\phi}$ and ∇V^{tt} to be linearly independent. Then, recalling Eq. (11), the scaling functions α_1 and α_2 are equal for Weyl's class. For simplicity we therefore introduce the new function $\alpha = \alpha_1 = \alpha_2$ [step (iii) in the list of Sec. II], so

$$b^{\phi\phi} = \alpha \nabla^\perp V^{\phi\phi}, \qquad b^{tt} = \alpha \nabla^\perp V^{tt}.$$

Lemma 4: Derivatives of α are determined by differential equations of the form

$$\alpha_x = A\alpha$$
$$\alpha_y = B\alpha$$

where A and B are algebraic expressions determined by V^{tt}_x, V^{tt}_y, $V^{\phi\phi}_x$ and $V^{\phi\phi}_y$, which do not contain any higher-than-second derivatives of components of the potential V.

Proof.—We use the relations (12), i.e. we use the six equations from (10b) and combine them in a straightforward way to find expressions for the coefficients a_0 through to a_3 of $I_T = \sum_i a_i p_x^{d-i} p_y^i$. In this way, we find two different expressions for a_1, and two for a_2, corresponding to the above identities. The expressions are polynomials in derivatives of the potential V, i.e. they are determined by V^{tt}_x, V^{tt}_y, $V^{\phi\phi}_x$ and $V^{\phi\phi}_y$ and do not contain derivatives of order higher than 2. The coefficients of the a_i are just integer multiples of $\nu = \langle \nabla V^{tt}, \nabla^\perp V^{\phi\phi} \rangle$, which is nonzero because we required ∇V^{tt} and $\nabla V^{\phi\phi}$ to be linearly independent.

We can then eliminate a_1 and a_2 and deduce two equations which have the following form:

$$\langle \nabla V^{tt}, \nabla^\perp V^{\phi\phi} \rangle \alpha_x = (\ldots)\alpha$$
$$\langle \nabla V^{tt}, \nabla^\perp V^{\phi\phi} \rangle \alpha_y = (\ldots)\alpha.$$

The expressions abbreviated by $(\ldots)$ are polynomials in derivatives of V of at most second order. Dividing by the nonzero coefficient of the α-derivatives yields the required result. ■

The integrability condition for α is a necessary requirement for the existence of nonreducible Killing tensor fields [step (iv) in the list of Sec. II]:

Lemma 5: Let $\mathrm{rk}\mathcal{M} = 2$ and $\omega = 0$, but $\nabla V^{\phi\phi}, \nabla V^{tt} \neq 0$. If there is a Killing tensor field of valence 3, then either $\alpha = 0$ or $A_y - B_x = 0$.

We note that in case $\alpha = 0$ the integral $F_3 = F^{(3)} + F^{(1)} = 0$, so the lemma actually provides a necessary criterion for the existence of nontrivial Killing tensor fields of valence 3.

Proof of Lemma 5.—Compute

$$\begin{aligned}(\alpha_x)_y - (\alpha_y)_x &= A_y\alpha + A\alpha_y - B_x\alpha - B\alpha_x \\ &= (A_y - B_x)\alpha + (AB - BA)\alpha \\ &= (A_y - B_x)\alpha.\end{aligned}$$

■

We give an example where this idea provides information on the reducibility of linear integrals:

Example 1: The Zipoy-Voorhees family of metrics is a family in Weyl's class that is parametrized by a nonnegative number δ.

We can use the method as described above, but we take H in a modified form, namely

$$H = \frac{p_x^2}{2\Omega_1} + \frac{p_x^2}{2\Omega_1} + V^{\phi\phi} p_\phi^2 + V^{tt} p_t^2.$$

The Zipoy-Voorhees metric satisfies, in prolate spheroidal coordinates:

$$\begin{aligned}\Omega_1 &= \frac{1}{2}\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}\left(\frac{x+1}{x-1}\right)^{\delta}\frac{x^2-y^2}{x^2-1} \\ \Omega_2 &= \frac{1}{2}\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}\left(\frac{x+1}{x-1}\right)^{\delta}\frac{x^2-y^2}{1-y^2} \\ V^{\phi} &= \left(\left(\frac{x+1}{x-1}\right)^{\delta}(x^2-1)(1-y^2)\right)^{-1} \\ V^{t} &= -\left(\frac{x+1}{x-1}\right)^{\delta}.\end{aligned}$$

Taking an approach similar to Lemma 5, we first check that $\det\mathcal{M} \neq 0$. We find the following:

$$\det \mathcal{M} = 0$$
$$\Leftrightarrow \delta^2 y^2 \frac{x^8 - 4x^6y^2 + 6y^4x^4 - 4y^6x^2 + y^8}{(x-1)^2(x^2-1)^4(-1+y^2)^6(x+1)^2} = 0$$

which obviously is not true for generic x, y, if $\delta \neq 0$. Then we compute the necessary criterion as in Lemma 5. Using computer algebra (Maple 18), we find

$$A_y - B_x \overset{!}{=} 0$$
$$\Leftrightarrow \frac{(x-y)^4(x+y)^4(-3\delta x^2 + 4\delta^2 x + 2x - 3\delta)\delta^2 y}{(x^2-1)^8(y^2-1)^6} \overset{!}{=} 0$$

which is not true for generic x, y since $\delta > 0$. We therefore must conclude $\alpha = 0$, which means the integral of degree 3 is zero.

We now turn to step (v) in the list at the beginning of Sec, II:

Lemma 6: A polynomial equation of degree $N > 0$ for a function $f(x, y)$ with coefficients that depend on x only is independent of y, so $f = f(x)$.

Proof.—Denote the equation by $\sum_{n=0}^{N} a_n(x) f^n(x,y) = 0$. If we can factor out $f(x, y)$, then $f = 0$ and is independent of both x and y. Otherwise, we take one derivative with respect to y and obtain $\sum_{n=1}^{N} a_n(x) n f^{n-1} f_y = 0$. Then either $f_y = 0$ or we divide by f_y and proceed similarly. At some point, we either get $f_y = 0$ or we end up with $a_N = 0$, which contradicts the hypothesis that the polynomial equation be of degree N. Thus we have $f_y = 0$ and f is a function of x only. ■

Lemma 7: Let $U_x = U_x(x)$ be a function of x only. Let the StAV spacetime have a nonreducible third-degree integral. Then $U_y = 0$.

Proof.—The proof has two parts: (1) show that U_y has to be constant. (2) Show that the constant is zero. For the first part, consider the $p_1^3 p_2$-component of (10a). Use the Ernst equation to substitute derivatives U_{yy}. In this way, obtain the equation:

$$10x^3U_y^4 + 156x^2{}_x(1 + xU_x)U_y^2 + 36x^2U_xU_{xx} - 126x^3U_x^4$$
$$- 126xU_x^2 + 18xU_{xx} - 18U_x - 252x^2U_x^3 = 0. \qquad (17)$$

This is a polynomial equation of degree 4 for U_y, and all coefficients are functions of x only. By Lemma 6, this means $U_{yy} = 0$, so $U_y = \text{const} =: c$.

For the second part of the proof, we insert this result into the p_1^4-component of (10a). If we substitute U_{xx} with the help of the Ernst equation, we find

$$6U_x(1 + xU_x)(1 + 2xU_x) = 0.$$

Hence, there are 3 cases: $U_x = 0$, $U_x = -\frac{1}{x}$ and $U_x = -\frac{1}{2x}$. We treat them separately:

(i) If $U_x = 0$, use again the $p_1^3 p_2$-component which reads
$$10x^3c^6 = 0,$$
so $c = 0$.

(ii) For $U_x = -\frac{1}{x}$ we have the same equation, so again $c = 0$.

(iii) The case $U_x = -\frac{1}{2x}$ is slightly more involved. The $p_1^3 p_2$-component reads
$$\frac{c^2(9 - 312x^2c^2 + 80x^4c^4)}{8x} = 0.$$
Now, either $c = 0$ directly, or $9–312x^2c^2 + 80x^4c^4$. In the latter case, $xc = \text{const}$ and hence $c = 0$. ■

Lemma 8: Using Lewis-Papapetrou coordinates with $R = x$, assume the potential function U to be

$$e^{2U} = k_U \frac{2y + c + \sqrt{4x^2 + 4y^2 + 4cy + c^2}}{x^2},$$
$$\text{with} \quad k_U, c \in \mathbb{R}.$$

This provides a parametrization of flat space.

Proof.—Determine the function γ from the secondary Ernst equations and find

$$e^{2\gamma} = 2k_\gamma \sqrt{4x^2 + 4y^2 + 4cy + c^2}\, e^{2U}, \qquad k_\gamma \in \mathbb{R}.$$

Then compute the Riemann curvature tensor for the metric

$$g = e^{2U}(e^{-2\gamma}(dx^2 + dy^2) + x^2 d\phi^2) - e^{-2U} dt^2$$

and find that it vanishes identically. Thus, the potential function U defines a flat metric, which of course is StAV. ■

Lemma 9: Let $\mathrm{rk}\mathcal{M} = 2$ and $\omega = 0$, but $\nabla V^{\phi\phi}, \nabla V^{tt} \neq 0$. Assume $\alpha \neq 0$. Then there is no nontrivial Killing tensor of valence 3.

Proof.—We assume there was such a Killing tensor. Then, by the necessary criterion (Lemma 5), $A_y - B_x = 0$. In addition, consider (10a), and the Ernst equations. Since we chose Lewis-Papapetrou coordinates with $R = x$, we have $U_y \neq 0$.

Consider (10a) in combination with the necessary criterion from Lemma 5, plus the Ernst equations. The Ernst equations are to be invoked mainly in order to substitute $\frac{d^2U}{dy^2}$. We take derivatives with respect to x and y of (10a). Then, we have 18 equations [(10a) plus the necessary criterion $A_y - B_x = 0$ from Lemma 5]. Using the Ernst equations, we have only the following unknown functions:

$$U_{xxxx}, U_{xxxy}, U_{xxx}, U_{xxy}, U_{xx}, U_{xy}, U_x, U_y, U, \gamma.$$

Use the x-derivative of the $p_1^3 p_2$-component to substitute U_{xxxy}, and the x-derivative of the $p_1^2 p_2^2$-component to substitute U_{xxxx} in terms of lower order derivatives. The quantity U_{xxy} can be substituted for via the x-derivative of the integrability criterion, but only if

$$(1+2xU_x)(xU_x^2-3xU_y^2+U_x)\neq 0. \qquad (18)$$

In this case, we can proceed as follows: substitute U_{xxx} by the x-derivative of the p_1^4-component, and use this component to substitute U_{xx}. Finally, substitute U_{xy} using the integrability condition.

With all these substitutions at hand, we now have only equations in the unknowns U_x, and U_y left. For instance, the derivative with respect to y of the p_1^4-component of (10a) reads

$$xU_x^2(1+2xU_x)(1+xU_x)^2(xU_x^2+U_x+xU_y^2)^3=0.$$

Therefore, either $U_x=0$ or $U_x=-\frac{1}{x}$ or $U_x=-\frac{1}{2x}$, or $xU_x^2+U_x+xU_y^2=0$. The three cases mentioned first are covered by Lemma 7, and obviously in contradiction to the hypothesis $U_y\neq 0$.

We are left with the forth case. Solve the equation and obtain

$$U_y^2=-\frac{1}{x}U_x(1+xU_x). \qquad (19)$$

Then substitute this into the p_1^4-component of (10a) and obtain an expression for U_{xx}, and from the integrability condition we obtain an expression for U_{xy}.[1] The other components of (10a) are then satisfied trivially. At this point, it is a good idea to go back to the expressions for U_y^2 and U_{xy}. Combining both, we find the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}U_y=-4xU_y^3$$

which is an ODE for U_y and can be solved in a straightforward way. The solution is

$$U_y=\frac{1}{\sqrt{(4x^2-f_1(y))}}.$$

Use this to replace U_y^2 in (19):

$$f_1(y)=-\frac{x(1+4xU_x+4x^2U_x^2)}{U_x(1+xU_x)}.$$

Solve this for U_x. There are two branches of possible solutions:

$$U_x=\frac{1}{2}\frac{-4x^2-f_1\pm\sqrt{4x^2f_1+f_1^2}}{x(4x^2+f_1)}.$$

We can use the integrability criterion to find an explicit form for f_1. First, obtain two differential equations:

$$(f_1)_y\pm 4\sqrt{f_1}=0.$$

Up to the sign of the integration constant, both have the same solution

$$f_1=(2y+c)^2=4y^2+4cy+c^2.$$

Using again the equation for U_y^2 in terms of U_x and integrating, one finds

$$U=\frac{1}{2}\ln(2y+c+\sqrt{4x^2+4y^2+4yc+c^2})+f_2(x)$$

with f_2 first being an unspecified integration "constant." Checking if this solution is compatible with the expression for U_x found above, $(f_2)_x$ can take two possible values: either $(f_2)_x=-\frac{1}{x}$, or

$$\frac{df_2}{dx}=(f_2)_x(x)$$
$$=\frac{-4x}{(2y+c+\sqrt{4x^2+(2y+c)^2})\sqrt{4x^2+(2y+c)^2}}.$$

Now, consider the Ernst equation $U_x+xU_{yy}+xU_{xx}=0$. For the first solution for $(f_2)_x$, this implies $x=0$, so this is no valid solution. Therefore, conclude

$$U=\frac{1}{2}\ln(2y+c+\sqrt{4x^2+4y^2+4yc+c^2})-\ln(x)+c_2$$

with an additional integration constant $c_2\in\mathbb{R}$. By Lemma 8, the metric is flat and therefore all Killing tensor fields are reducible. This concludes step (vi) in the list of Sec. II.

To complete the proof, we still have to account for the case when (18) is not satisfied. In this case, either $U_x=-\frac{1}{x}$ (this is covered by Lemma 7) or

$$U_x(1+xU_x)-3xU_y^2=0.$$

We solve for U_y^2:

$$U_y^2=\frac{U_x(1+xU_x)}{3x}.$$

From the p_1^4-component and the integrability criterion, we can also get another expression for U_y^2:

[1] One might want to check that both expressions are compatible, which is true.

$$U_y^2 = \frac{3U_x(1+xU_x)}{x}.$$

The only way to allow both solutions to be true is if $U_x = 0$ or $U_x = -\frac{1}{x}$. Both cases are covered by Lemma 7. ■

We have considered odd-parity third-degree integrals in a StAV spacetime. Let us now summarize the results and merge them into one theorem:

Proof of theorem 1.—If M is flat on a neighborhood, then it is totally reducible there [21]. Thus assume that M is nonflat. First, consider only odd-parity integrals:

Claim: Let M be nonflat with $\omega = 0$, $\nabla V^{\phi\phi} \neq 0$, and $\nabla V^{tt} \neq 0$. Let F be a third-degree integral of odd parity in M. Then F is reducible by at least one degree.

Proof of the claim.—First, let us consider the case when there is an additional Killing vector field. As we have seen in Lemma 3, this means that the odd-parity third-degree integral is reducible by the (linear) integral p_y. So, the assertion is proven in this case.

Second, if there is no additional Killing vector field, proposition 9 tells us (provided $\alpha \neq 0$) that there is no odd-parity third-degree integral. In the case $\alpha = 0$, we have $F = 0$. Thus, the assertion is proven. ■

Now, consider the even-parity contributions. The quadratic contributions $F_\phi^{(2)}$ and $F_t^{(2)}$ must obey the equation

$$\{T, F_k^{(2)}\} = 0$$

as well as equations of the form

$$\{T, F_{abk}^{(0)}\} + \{V^{ab}, F_k^{(2)}\} = 0,$$

where $a, b, k \in \{\phi, t\}$.

These, however, are precisely equations (16) for quadratic integrals with leading term $F_\phi^{(2)}$ or $F_t^{(2)}$, respectively. This means that

$$p_a(F_a^{(2)} + F_{a\phi\phi}^{(0)} p_\phi^2 + F_{at\phi}^{(0)} p_t p_\phi + F_{att}^{(0)} p_t^2),$$

$a = \phi, t$, are quadratic integrals and therefore, the even-parity contributions to the degree-3 integral F are reducible by p_ϕ and p_t, respectively. Hence, also the entire integral F (consisting of the parts with degree from 3 down to 0) is reducible by one degree. ■

A. Zipoy-Voorhees

Consider again the Zipoy-Voorhees class. We already considered third-degree odd-parity integrals in such spacetimes. Let us now consider even-parity components and assume without loss of generality $\delta \neq 0$. We use for the Hamiltonian H the representation

$$H = \Omega_1 p_x^2 + \Omega_2 p_y^2 + V_\phi p_\phi^2 + V_t p_t^2 \qquad (20)$$

and denote the integral by

$$F = a_0 p_x^2 + a_1 p_x p_y + a_2 p_y^2 + b_0 p_\phi^2 + b_1 p_\phi p_t + b_2 p_t^2.$$

From each polynomial of degree 1 after split with respect to p_x, p_y (cf. Remark 1), we obtain integrability conditions for b_0 and b_2. Automatically, $b_1 = \text{const}$ is no longer of interest.

Combining the Bertrand-Darboux relations and equations obtained from the degree-3 polynomial after splitting with respect to (p_x, p_y), we can solve for derivatives of a_0, a_1 and a_2, and derive integrability conditions for them. From the integrability conditions, we can deduce that $a_1 = 0$ and that (at least if $\delta \neq 1$)

$$(y^2 - 1)a_2 + (x^2 - 1)a_0 = 0.$$

From the Bertrand-Darboux equations for b_0, b_2, we can now deduce $d(a_0)$ in terms of a_0 and solve the system of differential equations, obtaining

$$a_0 = c_1 (y^2 - x^2)^{1-\delta^2} (x+1)^{\delta^2+\delta-1} (x-1)^{\delta^2-\delta-1}.$$

Then, we can immediately compute a_2:

$$a_2 = -c_1 \left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2} \left(\frac{x+1}{x-1}\right)^{\delta} \frac{x^2-y^2}{y^2-1}.$$

Finally, from the equations obtained from the degree-1 polynomial after split with respect to p_x, p_y, we obtain the derivatives of b_0, b_2, and by integration

$$b_0 = -c_1 (y^2-1)(x^2-1)\left(\frac{x+1}{x-1}\right)^{\delta} + c_2$$

$$b_2 = -c_1 \left(\frac{x-1}{x+1}\right)^{\delta} + c_3.$$

Comparing this result to the Hamiltonian shows that

$$F = c_1 H + c_2 p_\phi^2 + c_3 p_\phi p_t + c_4 p_t^2.$$

This means that every quadratic integral is reducible, provided $\delta \neq 1$ (in case $\delta = 1$, we obtain the Schwarzschild metric, which is integrable with the additional integral in involution being given by a quadratic integral. This quadratic integral, however, is reducible by linear integrals that are not in involution). Together with Theorem 1, this proves the assertion. ■

IV. CONCLUSION

In this paper we gave a proof for the reducibility of valence-3 Killing tensor fields in static and axially symmetric vacuum spacetimes (Weyl's class). We saw that using prolongation-projection is an efficient way to decide on the existence of integrals in SAV metrics even if the metric is not given specifically. We plan to extend the result for degree 3 to the fully stationary SAV case. Though computationally more challenging, we are not aware of major conceptual problems with this. As for generalizations beyond the SAV context, the line of reasoning made here is in principle not specific to the SAV class of spacetimes, and an analogous approach might work for other classes of spacetimes, too.

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