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Second order gluon polarization for SU(N) theory in a linear covariant gauge

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The gluon polarization functional is evaluated for a generic linear covariant gauge and for any space dimension in pure Yang-Mills SU(N) theory up to second order in a generalized perturbation theory, where the zeroth order action is freely chosen and can be determined by some variational method. Some numerical data are given for the gluon propagator in the Landau gauge and compared with the Feynman gauge. A comparison is given for several variational methods that can be set up by the knowledge of the second order polarization.

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I. INTRODUCTION

In spite of its phenomenological relevance, the infrared (IR) limit of QCD has not been fully studied yet because of nonperturbative effects that limit the power of standard tools based on perturbation theory. Our current knowledge of the IR limit relies heavily on lattice simulations while, usually, analytical nonperturbative techniques can only describe the phenomenology by insertion of some free parameters that emerge by some unknown sector of the theory like vertex functions [1–6], counterterms [7–16], or renormalization schemes [14]. A mass parameter for the gluon has been shown to capture most of the nonperturbative effects, leading to a reasonable fit of lattice data [17–19], but we still miss a fully consistent and analytical *ab initio* theory without spurious fit parameters.

Quite recently, an optimized perturbation theory has been discussed [20], with zeroth order trial propagators that are optimized by some variational *Ansatz*. Many variational strategies can be set up by the simple knowledge of self-energy and polarization functions, going from the Gaussian effective potential [21–36] up to Stevenson's minimal sensitivity [37] and the novel method of stationary variance [38–40] that has been shown to be viable for pure Yang-Mills SU(3) in the Feynman gauge [41]. At variance with other analytical approaches, these variational methods have the merit of reproducing some lattice features, like the existence of a dynamical mass for the gluon [41], without any free parameter, because the trial quantities are all optimized by the variational *Ansatz*. Of course, the agreement with lattice data is not as good as found in fits, but the approximation can be improved order by order and gives an *ab initio* description that is only based on the original Lagrangian, without spurious parameters or undesired counterterms that would spoil the symmetry of the Lagrangian.

As discussed in Ref. [20], several variational approaches can be implemented if the self-energy (the polarization) is known, order by order in the optimized perturbation expansion, as a functional of the trial propagators. On the other hand, some internal symmetries of the theory could result broken by the truncated expansion and would turn out to be only approximately satisfied so that the actual result would depend on further parameters that have to do with the gauge choice, the renormalization scheme, and even the renormalization group (RG) invariance. All these parameters can also be optimized by a variational *Ansatz* yielding an optimal gauge or an optimal renormalization scheme [42,43]. Thus, it would be desirable to have a general set of explicit expressions for the polarization functionals, holding for any gauge, for any renormalization scheme, and for any trial propagator. Actually, most of these functionals have been reported for a free-particle propagator and in dimensional regularization where many terms vanish. A further proliferation of terms arises from the use of a generic covariant gauge since the trial propagator would be described by two independent functions for the transversal and longitudinal part. Despite many technical problems, the study of a generic linear covariant gauge has attracted some new interest in the last years, and the features of the gluon propagator have been investigated on the lattice [44,45] and in the framework of Dyson-Schwinger equations [13,46]. Moreover, it has been recently shown that even if some IR properties of the gluon propagator, like the dynamical mass, have no effects in the ultraviolet (UV) perturbative regime, they can drive a quark-quark interaction that is equal to that extracted by the ground-state observables [47], thus, enforcing our interest on the gauge dependence of the gluon propagator in the IR.

In this paper we report general integral expressions for the ghost self-energy and the gluon polarization, up to second order in the optimized perturbation theory, as functionals of trial propagators in a generic linear covariant gauge, for pure Yang-Mills SU(N) theory in any space dimension d. The integral expressions hold for any renormalization scheme and have been checked by a comparison with known results in dimensional regularization and in

special gauges like the Feynman and Landau gauges. Then, we use that result for extending to the Landau gauge a previous calculation of the gluon propagator by the method of stationary variance [20,38,39]. In fact, the gluon propagator was studied in the Feynman gauge in Ref. [41], while fixed-gauge lattice data are only available in the Landau gauge. Here, the numerical results of the calculation are compared with lattice data in the same gauge and with the outcome of the same method in the Feynman gauge. It turns out that, after renormalization, the gluon propagator is not very sensitive to the gauge change, in qualitative agreement with Refs. [44,46]. Some different variational methods are discussed and compared, using the same integral expressions for the polarization functionals, but the method of stationary variance emerges as the most reliable among them.

The paper is organized as follows: In Sec. II the generalized perturbation theory is reviewed and described in detail for the special case of pure SU(N) Yang-Mills theory; the first order graphs for the polarization are evaluated in Sec. III; the one-loop second order graphs are reported in Sec. IV, while the two-loop second order graphs are evaluated in Sec. V (the expansion is not loopwise, as it is an expansion in powers of the actual interaction). In Sec. VI the gluon propagator is evaluated in the Landau gauge by the method of stationary variance and compared with lattice data and with previous results in the Feynman gauge; a discussion and comparison of several variational methods follow in Sec. VII; some details on the numerical integration are reported in the Appendix.

II. GENERALIZED PERTURBATION THEORY

Let us consider pure Yang-Mills SU(N) gauge theory without external fermions. The Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{fix}}, \tag{1}$$

where $\mathcal{L}_{\mathrm{YM}}$ is the Yang-Mills term

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\mathrm{Tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}), \tag{2}$$

and $\mathcal{L}_{\mathrm{fix}}$ is a gauge fixing term. In terms of the gauge fields, the tensor operator $\hat{F}_{\mu\nu}$ is given by

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - ig[\hat{A}_\mu, \hat{A}_\nu], \tag{3}$$

where

$$\hat{A}_\mu = \sum_a \hat{X}^a A^a_\mu \tag{4}$$

and the generators of SU(N) satisfy the algebra

$$[\hat{X}^a, \hat{X}^b] = i f_{abc}\hat{X}^c \tag{5}$$

with the structure constants normalized according to

$$f_{abc}f_{dbc} = N\delta_{ad}. \tag{6}$$

A general covariant gauge-fixing term can be written as

$$\mathcal{L}_{\mathrm{fix}} = -\frac{1}{\xi}\mathrm{Tr}[(\partial_\mu \hat{A}^\mu)(\partial_\nu \hat{A}^\nu)], \tag{7}$$

and the quantum effective action $\Gamma[A']$ as a function of an external background field A' reads

$$e^{i\Gamma[A']} = \int_{1PI} \mathcal{D}_A e^{iS[A'+A]} J_{\mathrm{FP}}[A'+A], \tag{8}$$

where $S[A]$ is the action, $J_{\mathrm{FP}}[A]$ is the Faddev-Popov determinant, and the path integral represents a sum over one-particle irreducible $(1PI)$ graphs [48]. Since the gauge symmetry is not broken and we are mainly interested in the propagators, we will limit to the physical vacuum at $A' = 0$, while a more general formalism can be developed for a full study of the vertex functions [34].

The determinant J_{FP} can be expressed as a path integral over ghost fields

$$J_{\mathrm{FP}}[A] = \int \mathcal{D}_{\omega,\omega^\star} e^{iS_{gh}[A,\omega,\omega^\star]}, \tag{9}$$

and the effective action can be written as

$$e^{i\Gamma} = \int_{1PI} \mathcal{D}_{A,\omega,\omega^\star} e^{iS_0[A,\omega,\omega^\star]} e^{iS_I[A,\omega,\omega^\star]}, \tag{10}$$

where the total action in a generic d-dimensional space is

$$S_{\mathrm{tot}} = \int \mathcal{L}_{\mathrm{YM}} \mathrm{d}^d x + \int \mathcal{L}_{\mathrm{fix}} \mathrm{d}^d x + S_{gh}. \tag{11}$$

In a generalized perturbation theory [20,41], we have the freedom to split the action in two parts, a trial *free* action S_0 and the remaining interaction S_I. We *define* the free action S_0 as

$$S_0 = \frac{1}{2}\int A_{a\mu}(x) D^{-1\,\mu\nu}_{ab}(x,y) A_{b\nu}(y) \mathrm{d}^d x \mathrm{d}^d y + \int \omega^\star_a(x) G^{-1}{}_{ab}(x,y)\omega_b(y)\mathrm{d}^d x \mathrm{d}^d y, \tag{12}$$

where $D^{ab}_{\mu\nu}(x,y)$ and $G_{ab}(x,y)$ are unknown trial matrix functions. The interaction is then given by the difference

$$S_I = S_{\mathrm{tot}} - S_0 \tag{13}$$

and can be formally written as the sum of a two-point term and three local terms: the ghost vertex, the three-gluon vertex, and the four-gluon vertex, respectively,

$$S_I = S_2 + \int \mathrm{d}^d x[\mathcal{L}_{gh} + \mathcal{L}_3 + \mathcal{L}_4]. \tag{14}$$

In detail, the two-point interaction can be written as

$$S_2 = \frac{1}{2}\int A_{a\mu}(x)[{D_0}^{-1\mu\nu}_{ab}(x,y) - {D}^{-1\mu\nu}_{ab}(x,y)]A_{b\nu}(y)\mathrm{d}^d x\mathrm{d}^d y + \int \omega^\star_a(x)[{G_0}^{-1}{}_{ab}(x,y) - {G}^{-1}{}_{ab}(x,y)]\omega_b(y)\mathrm{d}^d x\mathrm{d}^d y, \tag{15}$$

where D_0 and G_0 are the standard free-particle propagators for gluons and ghosts and their Fourier transforms read

$$D_{0ab}^{\mu\nu}(p) = -\frac{\delta_{ab}}{p^2}[t^{\mu\nu}(p) + \xi\ell^{\mu\nu}(p)],$$
$$G_{0ab}(p) = \frac{\delta_{ab}}{p^2}. \tag{16}$$

Here the transverse projector $t_{\mu\nu}(p)$ and the longitudinal projector $\ell_{\mu\nu}(p)$ are defined as

$$t_{\mu\nu}(p) = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2},$$
$$\ell_{\mu\nu}(p) = \frac{p_\mu p_\nu}{p^2}, \tag{17}$$

and $\eta_{\mu\nu}$ is the metric tensor. The three local interaction terms are

$$\mathcal{L}_3 = -gf_{abc}(\partial_\mu A_{a\nu})A^\mu_b A^\nu_c,$$
$$\mathcal{L}_4 = -\frac{1}{4}g^2 f_{abc}f_{ade}A_{b\mu}A_{c\nu}A^\mu_d A^\nu_e,$$
$$\mathcal{L}_{gh} = -gf_{abc}(\partial_\mu\omega^\star_a)\omega_b A^\mu_c. \tag{18}$$

The trial functions G_{ab}, $D^{\mu\nu}_{ab}$ cancel in the total action S_{tot}, which is exact and cannot depend on them. Thus, this formal decomposition holds for any arbitrary choice of the trial functions, and the expansion in powers of the interaction S_I provides a generalized perturbation theory [20,40,41]. Standard Feynman graphs can be drawn for this theory with the trial propagators $D^{\mu\nu}_{ab}$ and G_{ab} as free propagators, and the vertices that can be read from the interaction S_I in Eq. (14). As shown in Fig. 1, we have two-particle vertices for gluons and ghosts that arise from the action term S_2 in Eq. (15), while the local terms in Eq. (18) give rise to three- and four-particle vertices. The effective action Γ can be evaluated by perturbation theory as a sum of Feynman graphs and several variational *Ansätze* can be set up for the best choice of the trial functions [20], mainly relying on stationary conditions that can be easily written in terms of self-energy graphs. Moreover, the propagators can be written in terms of proper self-energy and polarization functions, and their evaluation, up to second order, is the

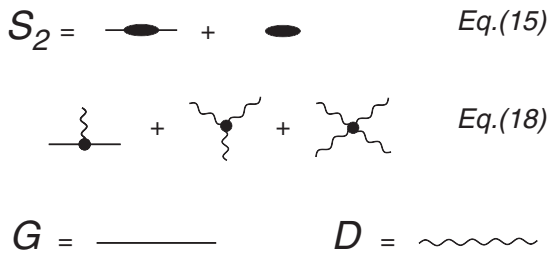

FIG. 1. The two-point vertices in the interaction S_2 of Eq. (15) are shown in the first line. The ghost vertex and the three- and four-gluon vertices of Eqs. (18) are shown in the second line. In the last line, the ghost (straight line) and gluon (wavy line) trial propagators are displayed.

main aim of the present paper. First and second order two-point graphs are shown in Fig. 2.

Since the propagators are gauge dependent, we write the trial function $D^{ab}_{\mu\nu}$ as the most general structure that is allowed by Lorentz invariance, namely,

$$D^{\mu\nu}_{ab}(p) = \delta_{ab}[T(p)t^{\mu\nu}(p) + L(p)\ell^{\mu\nu}(p)], \tag{19}$$

while color symmetry ensures that we can always take

$$G_{ab}(p) = \delta_{ab}G(p) = \delta_{ab}\frac{\chi(p)}{p^2}, \tag{20}$$

where $\chi(p)$ is a trial ghost dressing function. By the same notation, the free-particle propagators in Eq. (16) follow by inserting in Eq. (19) the functions

$$T_0(p) = -\frac{1}{p^2}, \qquad L_0(p) = -\frac{\xi}{p^2}, \qquad G_0(p) = \frac{1}{p^2}. \tag{21}$$

Because of the orthogonality properties of the projectors, the inverse propagator can be trivially written as

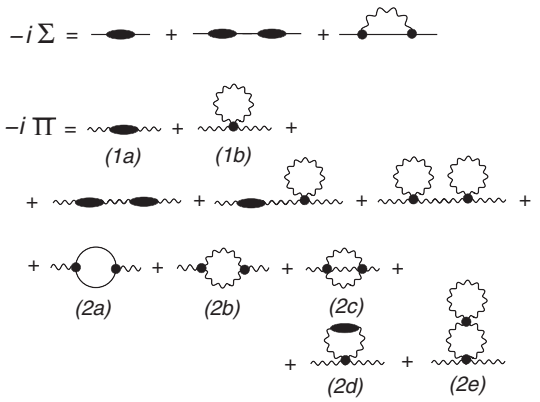

FIG. 2. First and second order two-point graphs contributing to the ghost self-energy and the gluon polarization. Second order terms include nonirreducible graphs.

$$D^{-1}{}^{\mu\nu}_{ab}(p) = \delta_{ab}[T(p)^{-1}t^{\mu\nu}(p) + L(p)^{-1}\ell^{\mu\nu}(p)]. \quad (22)$$

The trial propagator of Ref. [41] is recovered in the Feynman gauge ($\xi = 1$) by taking $T(p) = L(p)$, while in the Landau gauge ($\xi \to 0$), the longitudinal function $L(p)$ vanishes and the propagator is transverse. In both cases, the propagator is described by a single function, but in the general case, two different functions are required.

III. FIRST ORDER

Up to first order, the polarization is given by the sum of graphs $(1a)$ and $(1b)$ in Fig. 2. The tree graph $\Pi_{(1a)}$ is just

$$-i\Pi_{(1a)}{}^{\mu\nu}_{ab} = iD_0{}^{-1}{}^{\mu\nu}_{ab}(p) - iD^{-1}{}^{\mu\nu}_{ab}(p), \quad (23)$$

and in terms of projectors

$$\Pi_{(1a)}{}^{\mu\nu}_{ab}(p) = \delta_{ab}[\Pi^T_{(1a)}(p)t^{\mu\nu}(p) + \Pi^L_{(1a)}(p)\ell^{\mu\nu}(p)], \quad (24)$$

where

$$\Pi^T_{(1a)}(p) = T^{-1}(p) + p^2,$$
$$\Pi^L_{(1a)}(p) = L^{-1}(p) + \frac{p^2}{\xi}. \quad (25)$$

The one-loop term $\Pi_{(1b)}$ follows from the four-point interaction term $\mathcal{L}_4$ in Eq. (18) that gives the bare vertex

$$\Gamma^{\mu\nu\rho\sigma}_{abcd} = -i\frac{g^2}{4!}[T^{\mu\nu\rho\sigma}_{abcd} + T^{\mu\rho\sigma\nu}_{acdb} + T^{\mu\sigma\nu\rho}_{adbc}], \quad (26)$$

where the matrix structure $T^{\mu\nu\rho\sigma}_{abcd}$ is a product of color and Lorentz matrices

$$T^{\mu\nu\rho\sigma}_{abcd} = R_{abcd}S^{\mu\nu\rho\sigma} \quad (27)$$

with

$$R_{abcd} = f_{eab}f_{ecd}, \quad (28)$$

$$S^{\mu\nu\rho\sigma} = \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}. \quad (29)$$

The one-loop graph in Fig. 2 $(1b)$ then reads

$$-i\Pi_{(1b)}{}^{\rho\sigma}_{cd} = \frac{4!}{2}\Gamma^{\mu\nu\rho\sigma}_{abcd}\int\frac{\mathrm{d}^dk}{(2\pi)^d}(i\delta_{ab})[T(k)t_{\mu\nu}(k) + L(k)\ell_{\mu\nu}(k)], \quad (30)$$

and making use of Eq. (6), we can write

$$\Pi_{(1b)}{}^{\mu\nu}_{ab} = \delta_{ab}Ng^2\left\{(d-1)\eta^{\mu\nu}\int\frac{i\mathrm{d}^dk}{(2\pi)^d}T(k) + \int\frac{i\mathrm{d}^dk}{(2\pi)^d}[L(k) - T(k)]t^{\mu\nu}(k)\right\}. \quad (31)$$

Integrating in a d-dimensional Euclidean space, for a generic function $f(k)$ that only depends on k^2, we can use the identity

$$\int\frac{i\mathrm{d}^dk}{(2\pi)^d}\ell_{\mu\nu}(k)f(k) = -\frac{\eta_{\mu\nu}}{d}\int\frac{\mathrm{d}^dk_E}{(2\pi)^d}f(k_E), \quad \text{where } f(k_E) = f(k)|_{k^2=-k_E^2}, \quad (32)$$

and write the polarization in terms of the constant integrals

$$I^T_{n,m} = \int\frac{\mathrm{d}^dk_E}{(2\pi)^d}[T(k_E)]^n(k_E^2)^m,$$
$$I^L_{n,m} = \int\frac{\mathrm{d}^dk_E}{(2\pi)^d}[L(k_E)]^n(k_E^2)^m. \quad (33)$$

We assume that these diverging integrals are made finite by a regulating scheme to be discussed below. The one-loop polarization then reads

$$\Pi_{(1b)}{}^{\mu\nu}_{ab} = -\delta_{ab}\eta^{\mu\nu}M^2, \quad (34)$$

where the first order mass term M^2 is defined as

$$M^2 = \frac{Ng^2(d-1)}{d}[I^L_{1,0} + (d-1)I^T_{1,0}]. \quad (35)$$

It is useful to introduce the transverse and longitudinal massive functions $T_M(p)$, $L_M(p)$,

$$[T_M(p)]^{-1} = [T_0(p)]^{-1} + M^2 = -p^2 + M^2,$$
$$[L_M(p)]^{-1} = [L_0(p)]^{-1} + M^2 = -\frac{p^2}{\xi} + M^2 \quad (36)$$

and the massive propagator

$$D_M{}^{\mu\nu}(p) = T_M(p)t^{\mu\nu}(p) + L_M(p)\ell^{\mu\nu}(p) \quad (37)$$

that describes a free massive particle in a generic covariant gauge. In the special cases of the Feynman gauge ($\xi = 1$) and Landau gauge ($\xi \to 0$), we recover the massive free-particle propagators $D^{\mu\nu}_M(p) = \eta^{\mu\nu}/(-p^2 + M^2)$ and $D^{\mu\nu}_M(p) = t^{\mu\nu}(p)/(-p^2 + M^2)$, respectively. With that notation, the total first order polarization Π_1 can be written in the very simple shape

$$\Pi_1{}^{\mu\nu}_{ab} = \Pi_{(1a)}{}^{\mu\nu}_{ab} + \Pi_{(1b)}{}^{\mu\nu}_{ab} = D^{-1}{}^{\mu\nu}_{ab} - \delta_{ab}D^{-1}_M{}^{\mu\nu}. \quad (38)$$

There is just one first order graph for the ghost self-energy arising from the two-point nonlocal term in Eq. (15) as shown in Fig. 2, so that the first order self-energy can be written as

$$\Sigma_1^{ab}(p) = \delta_{ab}[G^{-1}(p) - G_0^{-1}(p)]. \tag{39}$$

The Gaussian effective potential (GEP) [21–36] can be derived by the requirement that the functional derivative of the first order effective potential with respect to the trial functions D and G is zero, that is equivalent [20] to the self-consistency condition of a vanishing first order self-energy and polarization, $\Pi_1 = 0$ and $\Sigma_1 = 0$. The gap equation that arises was first investigated by Cornwall [49] in 1982 as a simple way to predict a gluon mass. In the present formalism, from Eqs. (38) and (39), the stationary conditions for the GEP that derive from the vanishing of first order self-energy and polarization give a decoupled ghost with $G = G_0$ and a free massive gluon with $D = D_M$, where the mass M follows from the gauge-dependent gap equation (35) that can be formally written by a change of argument in the second integral,

$$M^2 = \frac{N(d-1)^2 g_\xi^2}{d} \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{1}{k_E^2 + M^2}, \tag{40}$$

where the gauge dependence has been absorbed by the effective coupling

$$g_\xi^2 = g^2 \left[1 + \frac{\xi^{d/2}}{d-1}\right]. \tag{41}$$

Accordingly, in the Feynman gauge the larger effective coupling would give a larger mass as compared with the Landau gauge.

IV. SECOND ORDER: ONE LOOP

The generalized perturbation theory that arises from the expansion in powers of the interaction S_I is not a loopwise expansion in powers of the coupling constant, so that one-loop and two-loop graphs coexist in the second order term of the polarization. The standard one-loop graphs, namely, the ghost and gluon one-loop graphs, $\Pi_{(2a)}$ and $\Pi_{(2b)}$ in Fig. 2, are described in this section together with the one-loop ghost self-energy. The other second order terms, namely, the second order one-loop graph $\Pi_{(2d)}$ and the two-loop graphs $\Pi_{(2c)}$ and $\Pi_{(2e)}$ in Fig. 2 will be discussed in the next section.

A. One-loop ghost self-energy

The one-loop ghost self-energy follows from the bare vertex of the ghost-gluon interaction term $\mathcal{L}_{gh}$ in Eq. (18),

$$-i\Sigma_{ab}(p) = g^2 f_{cda} f_{c'bd'} \times \int \frac{\mathrm{d}^d k}{(2\pi)^d} (p_\mu - k_\mu) p_\nu [iD^{\mu\nu}_{cc'}(k) iG_{dd'}(p-k)]. \tag{42}$$

Making use of Eq. (6) and integrating in a d-dimensional Euclidean space, we can split the self-energy in two terms

$$\Sigma_{ab}(p) = \delta_{ab}[\Sigma^T(p) + \Sigma^L(p)], \tag{43}$$

where

$$\Sigma^T(p) = -Ng^2 \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{\chi(k_E - p_E) T(k_E)}{(k_E - p_E)^2} \times \left[p_E^2 - \frac{(k_E \cdot p_E)^2}{k_E^2}\right],$$
$$\Sigma^L(p) = -Ng^2 \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{\chi(k_E - p_E) L(k_E)}{(k_E - p_E)^2} \times \left[\frac{(k_E \cdot p_E)^2}{k_E^2} - (k_E \cdot p_E)\right]. \tag{44}$$

These integrals are functionals of the ghost dressing function χ of Eq. (20) and of the gauge-dependent transverse and longitudinal gluon propagators, respectively.

B. Ghost loop

The one-loop polarization term $\Pi_{(2a)}$, the ghost loop in Fig. 2, also follows from the bare vertex of the ghost-gluon interaction term $\mathcal{L}_{gh}$ in Eq. (18),

$$-i\Pi^{\mu\nu}_{(2a)\,cd}(p) = -g^2 f_{abc} f_{bad} \times \int \frac{\mathrm{d}^d k}{(2\pi)^d} (p+k)^\mu k^\nu iG(p+k) iG(k). \tag{45}$$

Making use of Eq. (6) and integrating in a d-dimensional Euclidean space, with the same notation of Eq. (24), we can write the transverse and longitudinal parts in terms of the trial ghost dressing function χ of Eq. (20),

$$\Pi^T_{(2a)}(p) = -\frac{Ng^2}{(d-1)} \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{\chi(p_E + k_E)\chi(k_E)}{(p_E + k_E)^2} \times \left[1 - \frac{(k_E \cdot p_E)^2}{k_E^2 p_E^2}\right],$$
$$\Pi^L_{(2a)}(p) = -Ng^2 \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{\chi(p_E + k_E)\chi(k_E)}{(p_E + k_E)^2 k_E^2} \times \left[(k_E \cdot p_E) + \frac{(k_E \cdot p_E)^2}{p_E^2}\right], \tag{46}$$

in agreement with the result reported by other authors [11].

C. Gluon loop

The one-loop polarization term $\Pi_{(2b)}$, the gluon loop in Fig. 2, follows from the gluon-gluon interaction term $\mathcal{L}_3$ in Eq. (18) that gives the bare three-particle vertex

$$\Gamma^{\mu\nu\rho}_{abc}(p,q,k) = \frac{gf_{abc}}{3!}\{\eta^{\mu\nu}(p^\rho - q^\rho) + \eta^{\rho\nu}(q^\mu - k^\mu) + \eta^{\mu\rho}(k^\nu - p^\nu)\}. \quad (47)$$

The one-loop graph Fig. 2 $(2b)$ then reads

$$-i\Pi_{(2b)}{}^{\mu\nu}_{aa'}(p) = \frac{3!3!}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} iD_{bb',\rho\rho'}(p+k) iD_{cc',\tau\tau'}(k) \times \Gamma^{\mu\rho\tau}_{abc}(p,-p-k,k) \times \Gamma^{\nu\rho'\tau'}_{a'b'c'}(-p,p+k,-k), \quad (48)$$

and since the trial propagator is defined by two independent functions, there are 36 terms for each of the longitudinal and transverse parts of the polarization. We can write them in a more compact shape by introducing some degree of redundancy in the notation. By Eq. (6), the sum over color indices gives a diagonal matrix so that we can use the same notation of Eq. (24) and drop all color indices. Let us denote by α, β, γ the three momenta in the vertex, $\alpha = -p$, $\beta = p + k$, $\gamma = -k$ so that $\alpha + \beta + \gamma = 0$. Then, we denote by $\hat{A}^{\mu\nu}_a$, $\hat{B}^{\mu\nu}_b$, $\hat{C}^{\mu\nu}_c$ the three projectors

$$\hat{A}^{\mu\nu}_a = P_a{}^{\mu\nu}(\alpha), \quad \hat{B}^{\mu\nu}_b = P_b{}^{\mu\nu}(\beta), \quad \hat{C}^{\mu\nu}_c = P_c{}^{\mu\nu}(\gamma), \quad (49)$$

where a, b, $c = \pm 1$, while $P^{\mu\nu}_\pm$ are the transverse and longitudinal projectors $P^{\mu\nu}_+(k) = t^{\mu\nu}(k)$ and $P_-{}^{\mu\nu}(k) = \ell^{\mu\nu}(k)$ that can also be written as

$$P^{\mu\nu}_a(k) = n_a\eta^{\mu\nu} - a\ell^{\mu\nu}(k), \quad (50)$$

where $n_a = (1+a)/2$.

Moreover, with the same notation of Eq. (24), let us denote by $\mathcal{A}_a$, $\mathcal{B}_b$, $\mathcal{C}_c$ the numbers

$$\mathcal{A}_+ = (d-1)\Pi^T_{(2b)}(\alpha), \quad \mathcal{A}_- = \Pi^L_{(2b)}(\alpha), \quad \mathcal{B}_+ = T(\beta), \quad \mathcal{B}_- = L(\beta), \quad \mathcal{C}_+ = T(\gamma), \quad \mathcal{C}_- = L(\gamma), \quad (51)$$

so that having dropped color indices (not to be confused with the sign indices a, b, c in this section), we can write

$$\mathcal{A}_a = \hat{A}^{\mu\nu}_a \Pi_{(2b)_{\mu\nu}}(p), \quad D^{\mu\nu}(p+k) = \sum_{b=\pm1} \mathcal{B}_b \hat{B}^{\mu\nu}_b, \quad D^{\mu\nu}(k) = \sum_{c=\pm1} \mathcal{C}_c \hat{C}^{\mu\nu}_c. \quad (52)$$

Inserting in Eq. (48), the transverse and longitudinal polarizations $\mathcal{A}_a$ can be simply written as

$$\mathcal{A}_a = \frac{Ng^2}{2}\sum_{bc}\int \frac{i\mathrm{d}^d k}{(2\pi)^d}\mathcal{B}_b\mathcal{C}_c\mathcal{F}_{abc}(\alpha,\beta,\gamma), \quad (53)$$

where with the obvious shorthand notation

$$[k\cdot\hat{X}\cdots\hat{Y}\cdot p] = k_\mu \hat{X}^\mu{}_\rho \cdots \hat{Y}^\tau{}_\nu p^\nu, \qquad [\hat{X}\cdot\hat{Y}] = \hat{X}^{\mu\nu}\hat{Y}_{\nu\mu}, \quad (54)$$

the matrix $\mathcal{F}$ reads

$$\mathcal{F}_{abc}(\alpha,\beta,\gamma) = [(\alpha-\beta)\cdot\hat{C}_c\cdot(\alpha-\beta)][\hat{A}_a\cdot\hat{B}_b] + 2[(\alpha-\beta)\cdot\hat{C}_c\cdot\hat{B}_b\cdot\hat{A}_a\cdot(\beta-\gamma)] + \text{cycl.perm.} \quad (55)$$

summed over the three cyclic simultaneous permutations of all the arguments, indices, and projectors, i.e., $\alpha\to\beta\to\gamma\to\alpha$ together with $a\to b\to c\to a$ and $\hat{A}\to\hat{B}\to\hat{C}\to\hat{A}$. A straightforward but tedious calculation yields

$$\mathcal{F}_{abc}(\alpha,\beta,\gamma) = 3(n_a n_b n_c)(d-1)(\alpha^2+\beta^2+\gamma^2) + \left\{(acn_b)\beta^2\left[1 - \frac{(\beta^2-\alpha^2-\gamma^2)^2}{4\alpha^2\gamma^2}\right] + \text{cycl.perm.}\right\} + \left\{(an_bn_c)\left[\frac{\alpha^2}{2} - (\beta^2+\gamma^2) - (2d-3)\frac{(\beta^2-\gamma^2)^2}{2\alpha^2}\right] + \text{cycl.perm.}\right\}. \quad (56)$$

The result holds for any gauge parameter ξ, space dimension d, and for any trial propagator. It has been found in agreement with previous calculations in the Feynman gauge [41] and with older results for a free-particle trial propagator in generic covariant gauge [50].

For instance, in the Feynman gauge, the polarization function $\Pi(p)$ of Ref. [41] is defined as $\Pi(p) = (\mathcal{A}_+ + \mathcal{A}_-)/d$, and the trial propagator is taken to be $T(k) = L(k)$ so that the kernel of the integral in Eq. (53) is just given by

$$\frac{1}{d}\sum_{abc}\mathcal{F}_{abc}(\alpha,\beta,\gamma) = \frac{6(d-1)}{d}(p^2+k^2+p\cdot k), \quad (57)$$

in agreement with Eq. (A11) of Ref. [41] for $d = 4$. In the same work, the function $\Pi'(p)$ is the transverse polarization $\Pi'(p) = \mathcal{A}_+/(d-1)$, and the corresponding kernel in Eq. (53) for $d = 4$ is given by

$$\frac{1}{d-1}\sum_{bc}[\mathcal{F}_{abc}(\alpha,\beta,\gamma)]_{a=1} = 5p^2+2k^2+2p\cdot k+\frac{10}{3}k^2\left[1-\frac{(p\cdot k)^2}{k^2p^2}\right], \quad (58)$$

in agreement with Eq. (A12) of Ref. [41]. In the work of Watson [50], the trial propagator is taken to be the gauge-dependent free-particle function D_0 as defined in Eqs. (16) and (21). The function $\hat{J}_p^{(1)}$ in Eq. (3.3.4) of that work [50] is defined as $(d-1)\hat{J}_p^{(1)} = (\mathcal{A}_+ + \mathcal{A}_-)/p^2$, and the corresponding kernel in Eq. (53) can be written as a polynomial $[w_0 + w_1\xi + w_2\xi^2]$ with coefficients

$$w_0 = \frac{1}{\alpha^2}\sum_a [\mathcal{F}_{abc}(\alpha,\beta,\gamma)]_{b=c=1},$$
$$w_1 = \frac{1}{\alpha^2}\sum_a \left\{ [\mathcal{F}_{abc}(\alpha,\beta,\gamma)]_{\substack{b=+1\\c=-1}} + [\mathcal{F}_{abc}(\alpha,\beta,\gamma)]_{\substack{b=-1\\c=+1}} \right\},$$
$$w_2 = \frac{1}{\alpha^2}\sum_a [\mathcal{F}_{abc}(\alpha,\beta,\gamma)]_{b=c=-1}. \quad (59)$$

The coefficients can be easily evaluated by Eq. (56), and if we drop all terms that vanish under integration (because of symmetry or by dimensional regularization), we obtain

$$w_0 = 3d-\frac{7}{2}+2(2-d)\frac{\alpha^2}{\gamma^2}-\frac{\alpha^4}{4\gamma^2\beta^2},$$
$$w_1 = 1+(2d-5)\frac{\alpha^2}{\gamma^2}+\frac{\alpha^4}{2\gamma^2\beta^2},$$
$$w_2 = \frac{1}{2}+\frac{\alpha^2}{\gamma^2}-\frac{\alpha^4}{4\gamma^2\beta^2}, \quad (60)$$

in agreement with Eq. (3.3.4) of Ref. [50] that was evaluated by a computer routine for algebraic computations. The general result in Eq. (56) holds for any choice of the trial propagator and contains all terms that might not vanish by symmetry or dimensional regularization when a generic massive propagator is considered. Moreover, the result does not depend on a specific regularization scheme and can be used for any kind of calculation.

In the Landau gauge, the propagator is transverse and is defined by only one function $T(p)$, so that the transverse polarization $\mathcal{A}_+$ is obtained by retaining only one term for $b=c=+1$ in Eq. (53), and the corresponding kernel for $d=4$ reads

$$[\mathcal{F}_{abc}(\alpha,\beta,\gamma)]_{a=b=c=+1} = \left[1-\frac{(k\cdot p)^2}{k^2p^2}\right]\left[11(k^2+p^2)+2(k\cdot p)+\frac{p^4+10p^2k^2+k^4)}{(k+p)^2}\right]. \quad (61)$$

V. SECOND ORDER: TWO LOOP

Besides the standard one-loop graphs of the previous section, the second order polarization includes the one-loop and two-loop tadpoles in Fig. 2, $\Pi_{(2d)}$, $\Pi_{(2e)}$, and the two-loop sunset $\Pi_{(2c)}$.

A. Tadpoles

The one-loop and two-loop graphs $\Pi_{(2d)}$ and $\Pi_{(2e)}$ in Fig. 2 follow from the standard tadpole $\Pi_{(1b)}$ by insertion of the total first order polarization in the loop

$$iD^{\mu\nu} \to iD^{\mu\rho}(-i\Pi_{1\rho\sigma})iD^{\sigma\nu}, \quad (62)$$

that is, making use of Eq. (38),

$$D^{\mu\nu} \to D^{\mu\nu} - D^{\mu\rho}D^{-1}_{M\,\rho\sigma}D^{\sigma\nu}. \quad (63)$$

Insertion in Eqs. (34) and (35) yields

$$\Pi_{(2d)}{}^{\mu\nu}_{ab} + \Pi_{(2e)}{}^{\mu\nu}_{ab} = -\delta_{ab}\eta^{\mu\nu}[M^2-\mathcal{M}^2], \quad (64)$$

where the new mass constant reads

$$\mathcal{M}^2 = \frac{Ng^2(d-1)}{d}\left[M^2I^L_{2,0}+\frac{I^L_{2,1}}{\xi}+(d-1)(M^2I^T_{2,0}+I^T_{2,1})\right] \quad (65)$$

The constant integrals $I^T_{n,m}$, $I^L_{n,m}$ were defined in Eq. (33) and are functionals of the transverse and longitudinal trial functions $T(p)$, $L(p)$, respectively.

B. Two-loop sunset

The two-loop graph $(2c)$ in Fig. 2 is the most involved, and even if its contribution is small when the coupling is not too large, it can be relevant in a variational approach when the coupling is allowed to increase enough. The calculation follows from the explicit four-gluon bare vertex in Eq. (26)

$$-i\Pi_{(2c)}{}^{\mu\mu'}_{aa'}(p) = \frac{4!4!}{3!}\Gamma^{\mu\nu\rho\sigma}_{abce}\Gamma^{\mu'\nu'\rho'\sigma'}_{a'b'c'e'}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\int\frac{\mathrm{d}^dq}{(2\pi)^d} \times\int\frac{\mathrm{d}^dt}{(2\pi)^d}[iD_{bb',\nu\nu'}(k)][iD_{cc',\rho\rho'}(q)] \times[iD_{ee',\sigma\sigma'}(t)](2\pi)^d\delta^d(k+q+t+p), \quad (66)$$

and the explicit symmetry by permutation of dummy integration variables ensures that under integration the whole expression is symmetric for the exchange of the corresponding Lorentz and color indices in the matrix factors. Thus, the sum over permutation in Eq. (26) can be replaced by a factor of 3 in one of the vertices, yielding

$$\Pi_{(2c)}{}^{\mu\mu'}_{aa'}(p) = \frac{g^4}{2} T^{\mu\nu\rho\sigma}_{abce}[T^{\mu'\nu'\rho'\sigma'}_{a'bce} + T^{\mu'\rho'\sigma'\nu'}_{a'ceb} + T^{\mu'\sigma'\nu'\rho'}_{a'ebc}] \times \sum_{ijk=\pm1} \int \frac{i\mathrm{d}^d k}{(2\pi)^d} \int \frac{i\mathrm{d}^d q}{(2\pi)^d} \hat{B}_i^{\sigma\sigma'} \hat{C}_j^{\rho\rho'} \hat{E}_k^{\nu\nu'} \mathcal{B}_i \mathcal{C}_j \mathcal{E}_k, \tag{67}$$

with a compact notation that extends that of the previous section: here we define the four vectors $\alpha = p$, $\beta = -(p+q+k)$, $\gamma = q$, $\epsilon = k$ so that $\alpha+\beta+\gamma+\epsilon = 0$ and add a new projector to the set in Eq. (49)

$$\hat{E}_e{}^{\mu\nu} = P_e{}^{\mu\nu}(\epsilon). \tag{68}$$

Moreover, in this section, we denote by $\mathcal{A}_a$, $\mathcal{B}_b$, $\mathcal{C}_c$, $\mathcal{E}_e$ the numbers

$$\begin{aligned} \mathcal{A}_+ &= (d-1)\Pi^T_{(2c)}(\alpha), \qquad \mathcal{A}_- = \Pi^L_{(2c)}(\alpha), \\ \mathcal{B}_+ &= T(\beta), \qquad \mathcal{B}_- = L(\beta), \\ \mathcal{C}_+ &= T(\gamma), \qquad \mathcal{C}_- = L(\gamma), \\ \mathcal{E}_+ &= T(\epsilon), \qquad \mathcal{E}_- = L(\epsilon), \end{aligned} \tag{69}$$

so that dropping color indices, we can write

$$\begin{aligned} \mathcal{A}_a &= \hat{A}_a^{\mu\nu} \Pi_{(2c)_{\mu\nu}}(p), \\ D^{\mu\nu}(p+q+k) &= \sum_{b=\pm1} \mathcal{B}_b \hat{B}_b^{\mu\nu}, \\ D^{\mu\nu}(q) &= \sum_{c=\pm1} \mathcal{C}_c \hat{C}_c^{\mu\nu}, \\ D^{\mu\nu}(k) &= \sum_{e=\pm1} \mathcal{E}_c \hat{E}_e^{\mu\nu}. \end{aligned} \tag{70}$$

Under integration, the matrix structure in Eq. (67) simplifies because of the permutation symmetry of dummy integration variables, and the three matrix products can be recast as a single Lorentz matrix $\omega^{\mu\mu'}$ that multiplies three color matrices

$$\begin{aligned} \Pi_{(2c)}{}^{\mu\mu'}_{aa'} &= \omega^{\mu\mu'}[2R_{a'bce}R_{a'bce} - R_{a'bce}(R_{a'ceb} + R_{a'ebc})] \\ &= 3N^2 \delta_{aa'} \omega^{\mu\mu'}, \end{aligned} \tag{71}$$

where the last equality follows by the Jacobi identity. Then, dropping color indices, the transverse and longitudinal polarizations $\mathcal{A}_a$ can be written as

$$\mathcal{A}_a = \frac{3N^2 g^4}{2} \sum_{bce} \int \frac{i\mathrm{d}^d k}{(2\pi)^d} \int \frac{i\mathrm{d}^d q}{(2\pi)^d} \mathcal{B}_b \mathcal{C}_c \mathcal{E}_e \mathcal{F}_{abce}(\alpha,\beta,\gamma,\epsilon), \tag{72}$$

where the kernel $\mathcal{F}$ follows from the projection of $\omega^{\mu\mu'}$ by the projector $\hat{A}$ according to Eq. (70), and with the shorthand notation of Eq. (54), it can be written in terms of traces of projectors

$$\mathcal{F}_{abce}(\alpha,\beta,\gamma,\epsilon) = [\hat{A}_a \cdot \hat{E}_e][\hat{C}_c \cdot \hat{B}_b] - [\hat{A}_a \cdot \hat{E}_b \cdot \hat{C}_a \cdot \hat{B}_b]. \tag{73}$$

Because of the obvious symmetry of the integral in Eq. (72), the result is invariant for any simultaneous permutations of the last three arguments, indices, and projectors, i.e., $\beta \to \gamma \to \epsilon \to \beta$ together with $b \to c \to e \to b$ and $\hat{B} \to \hat{C} \to \hat{E} \to \hat{B}$. Using that symmetry, the kernel $\mathcal{F}$ can be written as

$$\begin{aligned} \mathcal{F}_{abce}(\alpha,\beta,\gamma,\epsilon) &= d(d-1)(n_a n_b n_c n_e) - 3(d-1)(n_a b n_c n_e) - (d-1)(a n_b n_c n_e) + (n_a n_b c e)\left[2 + (d-3)\frac{(\epsilon\cdot\gamma)^2}{\epsilon^2\gamma^2}\right] \\ &+ (a n_b c n_e)\left[2 + (d-3)\frac{(\alpha\cdot\gamma)^2}{\alpha^2\gamma^2}\right] - (n_a b c e)\frac{(\epsilon\cdot\gamma)}{\epsilon^2\gamma^2}\left[(\epsilon\cdot\gamma) - \frac{(\beta\cdot\gamma)(\beta\cdot\epsilon)}{\beta^2}\right] \\ &+ (a n_b c e)\frac{(\epsilon\cdot\gamma)}{\epsilon^2\gamma^2}\left[(\epsilon\cdot\gamma) - \frac{(\alpha\cdot\gamma)(\alpha\cdot\epsilon)}{\alpha^2}\right] \\ &+ 2(a b n_c e)\frac{(\alpha\cdot\beta)}{\alpha^2\beta^2}\left[(\alpha\cdot\beta) - \frac{(\alpha\cdot\epsilon)(\beta\cdot\epsilon)}{\epsilon^2}\right] + (abce)\frac{(\alpha\cdot\beta)(\gamma\cdot\epsilon)}{\alpha^2\beta^2\gamma^2\epsilon^2}[(\alpha\cdot\beta)(\gamma\cdot\epsilon) - (\alpha\cdot\epsilon)(\beta\cdot\gamma)]. \end{aligned} \tag{74}$$

That explicit expression gives the two-loop sunset graph Fig. 2 (2c) by a double integration in Eq. (72) and holds for any covariant gauge, any trial propagator, and any space dimension. For instance, in the Feynman gauge, the polarization function $\Pi(p)$ of Ref. [41] is defined as $\Pi(p) = (\mathcal{A}_+ + \mathcal{A}_-)/d$, and the trial propagator is taken to be $T(k) = L(k)$ so that the kernel of the integral in Eq. (72) is a constant and is just given by

$$\frac{1}{d}\sum_{abce} \mathcal{F}_{abce}(\alpha,\beta,\gamma,\epsilon) = (d-1), \tag{75}$$

in agreement with Eq. (A19) of Ref. [41] for $d = 4$. In the same work, the function $\Pi'(p)$ is the transverse polarization $\Pi'(p) = \mathcal{A}_+/(d-1)$, and the corresponding kernel in Eq. (72) is given by

$$\frac{1}{d-1}\sum_{bce}[\mathcal{F}_{abce}(\alpha,\beta,\gamma,\epsilon)]_{a=1}=(d-1), \tag{76}$$

again in agreement with Eq. (A19) of Ref. [41].

In the Landau gauge, the result is more involved. The propagator is transverse and is defined by one function $T(p)$ so that the transverse polarization $\mathcal{A}_+$ is obtained by retaining only one term for $b=c=e=+1$ in Eq. (72). The corresponding kernel for $d=4$ reads

$$\begin{aligned}[\mathcal{F}_{abce}(\alpha,\beta,\gamma,\epsilon)]_{a=b=c=e=+1} &= 4+\frac{(k\cdot q)}{k^2q^2}\left[\frac{[q\cdot(k+q+p)][k\cdot(k+q+p)]}{(k+q+p)^2}-\frac{(p\cdot q)(p\cdot k)}{p^2}\right]\\ &+\frac{(k\cdot q)^2}{k^2q^2}+\frac{(p\cdot q)^2}{p^2q^2}+\frac{2[p\cdot(p+q+k)]}{p^2(p+q+k)^2}\left[p\cdot(p+q+k)-\frac{(p\cdot k)[k\cdot(k+q+p)]}{k^2}\right]\\ &+\frac{(k\cdot q)[p\cdot(p+q+k)]}{p^2q^2k^2(p+q+k)^2}[[p\cdot(p+q+k)](k\cdot q)-(p\cdot k)[q\cdot(k+q+p)]].\end{aligned} \tag{77}$$

VI. STATIONARY VARIANCE IN LANDAU GAUGE

Explicit expressions for the second order graphs are useful for a direct comparison of variational results in different gauges. For instance, the method of stationary variance [20,38–40] was shown to be viable in the Feynman gauge, and explicit results were reported for the gluon propagator [41]. Here, we explore the outcome of the same method in the Landau gauge where lattice simulations are available [51,52].

The variance is stationary when the trial propagators satisfy the stationary conditions [20]

$$\begin{aligned}\Pi_2&=\Pi_1,\\ \Sigma_2&=\Sigma_1,\end{aligned} \tag{78}$$

where Σ_1 is the first order ghost self-energy in Eq. (39), Σ_2 is the sum of reducible and irreducible second order graphs, Π_1 is the sum of the first order polarization graphs $(1a)$ and $(1b)$ in Fig. 2, and Π_2 is the sum of reducible and irreducible second order polarization graphs. With the notation of Eqs. (38), (39), and (43), the stationary equations read

$$\begin{aligned}&D^{-1\,\mu\nu}_{ab}-\delta_{ab}D_M^{-1\,\mu\nu}\\ &\quad=[D^{-1\,\mu\rho}_{ac}-\delta_{ac}D_M^{-1\,\mu\rho}][D_{ce,\rho\sigma}][D^{-1\,\sigma\nu}_{eb}-\delta_{eb}D_M^{-1\,\sigma\nu}]+\Pi^{\star\,\mu\nu}_{2\,ab}\\ &G^{-1}-G_0^{-1}=[G^{-1}-G_0^{-1}]G[G^{-1}-G_0^{-1}]+\Sigma^T+\Sigma^L,\end{aligned} \tag{79}$$

where the proper polarization $\Pi_2^\star$ is the sum of all the irreducible second order terms

$$\Pi^{\star\,\mu\nu}_{2\,ab}=\Pi_{(2a)^{\mu\nu}_{ab}}+\Pi_{(2b)^{\mu\nu}_{ab}}+\Pi_{(2c)^{\mu\nu}_{ab}}+\Pi_{(2d)^{\mu\nu}_{ab}}+\Pi_{(2e)^{\mu\nu}_{ab}}. \tag{80}$$

The coupled set of integral equations (79) can be written as

$$\begin{aligned}D^{\mu\nu}_{ab}&=\delta_{ab}D_M{}^{\mu\nu}-D_M{}^{\mu\rho}\Pi^\star_{2\,ab,\rho\sigma}D_M{}^{\sigma\nu}\\ G&=G_0-G_0[\Sigma^T+\Sigma^L]G_0\end{aligned} \tag{81}$$

and hold for any gauge. The first of Eqs. (81) shows that the optimal propagator $D^{\mu\nu}_{ab}$ must be diagonal in color indices.

In the Landau gauge ($\xi\to 0$), according to Eq. (37) the massive propagator $D_M{}^{\mu\nu}$ becomes transversal so that any longitudinal term in the polarization does not play any role in the first of Eqs. (81), yielding a pure transversal solution for the optimal propagator $D^{\mu\nu}_{ab}$. Moreover, in the second of Eqs. (81), the longitudinal term Σ^L is zero according to Eq. (44) and we can drop it, yielding a decoupled set of integral equations for the transversal components. With the same notation of Eq. (24), the transversal component of the proper polarization $\Pi_2^\star$ in Eq. (80) can be written as

$$\Pi^{\star\,T\,\mu\nu}_{2\,ab}(p)=\delta_{ab}t^{\mu\nu}(p)\Pi_2^{\star\,T}(p), \tag{82}$$

where the scalar function $\Pi_2^{\star\,T}(p)$ is the sum of all the transversal second order irreducible polarization graphs $(2a)$–$(2e)$ in Fig. 2. The optimal propagator can be written as

$$D^{\mu\nu}_{ab}(p)=\delta_{ab}t^{\mu\nu}(p)T(p), \tag{83}$$

where the scalar function $T(p)$ satisfies with $G(p)$ the coupled set of stationary equations (81) that in Landau gauge become

$$\begin{aligned}T(p_E)&=\frac{1}{p_E^2+M^2}\left[1-\frac{\Pi_2^{\star\,T}(p_E)}{p_E^2+M^2}\right],\\ G(p_E)&=\frac{1}{-p_E^2}\left[1-\frac{\Sigma^T(p_E)}{-p_E^2}\right],\end{aligned} \tag{84}$$

and we have made use of the explicit expressions of G_0 and D_M in the Euclidean formalism. The last equation can be written in terms of the dressing function Eq. (20) simply as

TABLE I. Mass parameter in the Landau gauge (M_L) and in the Feynman gauge (M_F) for $d = 4$ and $N = 3$.

g	M_L (MeV)	M_F (MeV)
0.5	172	457
1.0	187	573

$$\chi(p_E) = 1 + \frac{\Sigma^T(p_E)}{p_E^2}. \tag{85}$$

The gauge-dependent mass parameter M is given by the one-loop graph $(1b)$ in Fig. 2 and follows from its definition in Eq. (35) which closes the set of equations and must be evaluated by insertion of the actual propagator $T(p)$ instead of the first order massive propagator that was used in Eq. (40) for the GEP. Of course, the mass parameter does depend on the choice of gauge, as it was obvious at first order. Thus, it defines an energy scale that must not be confused with the actual gluon mass of the renormalized propagator. Some typical values of the mass parameter are reported in Table I and compared with the corresponding values in the Feynman gauge as discussed below. In the Landau gauge, the gap equation (35) reads

$$M^2 = \frac{Ng^2(d-1)^2}{d} I^T_{1,0}, \tag{86}$$

where the integral $I^T_{1,0}$ is defined in Eq. (33) and is a functional of the unknown full propagator $T(p)$. The stationary equations (84) together with the gap equation (86) can be made finite by a proper regularization scheme and solved numerically. Details on the numerical calculation for $d = 4$ are reported in the Appendix.

In this work, the integrals are regularized by a finite cutoff Λ in the Euclidean four-dimensional space ($d = 4$) where we take $p_E^2 < \Lambda^2$. The simple choice of a cutoff gives physical results that are directly comparable with the outcome of lattice simulations where a natural cutoff is provided by the lattice spacing. The bare coupling $g = g(\Lambda)$ is supposed to depend on the energy scale Λ, and RG invariance requires that the physical content of the theory should be invariant for a change of scale $\Lambda \to \Lambda'$ followed by a change of coupling $g \to g'$. Then, physical renormalized functions that do not depend on the cutoff can be obtained by scaling. It is important to point out that the present regularization scheme does not need the inclusion of any counterterm in the Lagrangian and especially mass counterterms that are forbidden by the gauge invariance of the Lagrangian. The interaction strength g at a given scale Λ is the only free parameter, while the function $g(\Lambda)$ can be determined by RG invariance. In principle, one could fix the scale by a comparison with experimental data. However, in the present model calculation on pure Yang-Mills theory, we will fix the scale by a comparison with the lattice data. Since the original Lagrangian does not contain any scale, it is useful to take $\Lambda = 1$ and work in units of the cutoff, at a given bare interaction strength g. Thus, the choice of Λ will be equivalent to a choice of the energy units. RG invariance requires that a renormalized propagator $T_R(p)$ can be defined at an arbitrary scale μ by *multiplicative* renormalization, which is equivalent to saying that by scaling, all bare functions at different couplings can be put one on top of the other. Of course, since the approximations and the numerical integration spoil the scaling properties, we will consider the scaling as a test for the accuracy of the whole procedure. We can study the scaling behavior in a log-log plot where the bare functions should go one on top of the other by a simple translation of the axes.

In Fig. 3 the renormalized gluon propagator is shown for $N = 3$, $d = 4$, and for several couplings in the range $g = 0.5$–2.8. Scaling is rather good in the range $g = 0.5$–1.2, but gets spoiled at the rather large coupling $g = 2.8$. That could be a limit of the second order approximation.

For any coupling, the energy scale can be fixed by comparison with the lattice data, yielding a physical renormalized propagator $T(p)$ that is shown in Figs. 4 and 5 for $g = 0.5$ and $g = 1.0$, respectively.

As shown in Table I, when expressed in physical units, the mass parameter M is almost constant with respect to changes of the coupling, while it remains very sensitive to the choice of gauge. However, after scaling, we can define a *physical* mass $m = T(0)^{-1/2} \approx 320$ MeV that does not depend on scale and gauge because the propagator $T(p)$ is barely scaled and made to coincide with the lattice data of Ref. [51].

A direct comparison of results in the Feynman gauge [41] and Landau gauge shows that the renormalized

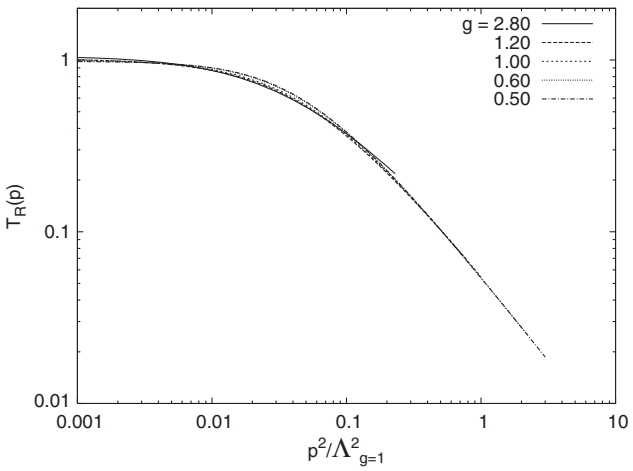

FIG. 3. Log-log plot of the renormalized propagator $T_R(p)$ as obtained by appropriate scaling of the bare propagator for $N = 3$, $d = 4$, and for a bare coupling $g = 0.50, 0.60, 1.00, 1.20, 2.80$. The scale is arbitrary because of scaling: all curves have been scaled in order to fall on top of the $g = 1$ bare propagator. Energy is in units of $\Lambda_{g=1}$ so that $g(1) = 1$ (for $g = 1$, the curve is not rescaled). Some deviations from scaling become more evident at the very large coupling $g = 2.8$.

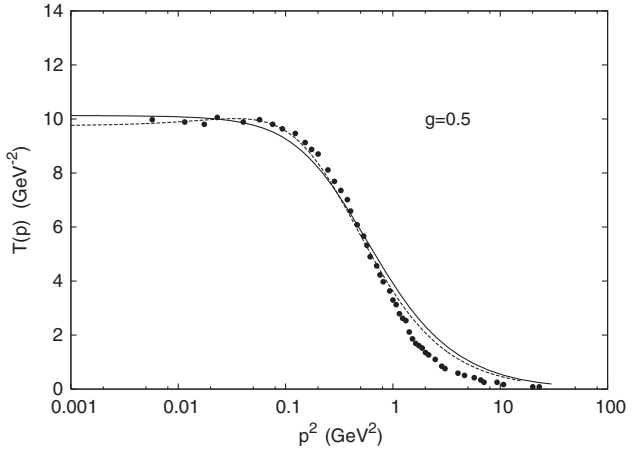

FIG. 4. The renormalized propagator $T(p)$ in the Landau gauge for $N = 3$, $d = 4$, and for a bare coupling $g = 0.5$. The scale has been fixed in order to fit the lattice data of Ref. [51] ($g = 1.02$, L = 96) that are displayed as filled circles. The propagator in the Feynman gauge is shown for comparison as a dotted line.

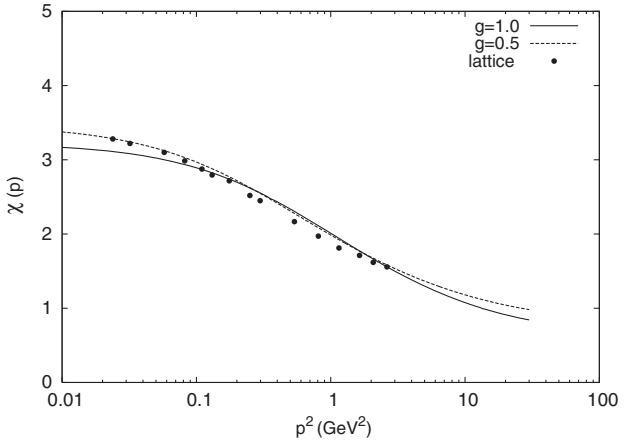

FIG. 6. The subtracted dressing function in the Landau gauge for $N = 3$, $d = 4$, and for a bare coupling $g = 0.5$ (broken line) and $g = 1.0$ (solid line). The energy scale is the same as in Figs. 4 and 5. The filled circles are the lattice data of Ref. [51] ($g = 1.02$, L = 96).

propagator is not very sensitive to the choice of gauge. The effects of a different mass scale, more than double in the Feynman gauge, are absorbed by renormalization. As shown in Figs. 4 and 5, where the propagator in the Feynman gauge is reported for comparison, we can say that the difference between the lattice data and the calculated curves cannot be ascribed to gauge differences but is probably a consequence of the finite order (second order) of the approximation. In fact, even if optimized by a variational method, the nature of the calculation is perturbative and can be improved by inclusion of higher orders. Actually, we do not expect that a perfect agreement could be reached in the UV limit of standard perturbation theory because of the simple renormalization scheme of the present calculation that is based on an energy cutoff. In that scheme, the spurious quadratic divergence would spoil the result for the UV limit. The problem of canceling that divergence without affecting the IR limit has been discussed by several authors and recently reviewed in Ref. [14]. It is a major problem that has not found a satisfactory solution yet. The ghost dressing function is reported in Fig. 6 for $g = 1$ and $g = 0.5$. Again, the functions are scaled and subtracted along the lines discussed in Ref. [41].

Finally, it is instructive to compare the weight of the single graphs in the second order polarization for the studied case $N = 3$, $d = 4$. At a rather strong coupling $g = 1$ ($\beta = 6$), the total polarization is reported in Fig. 7

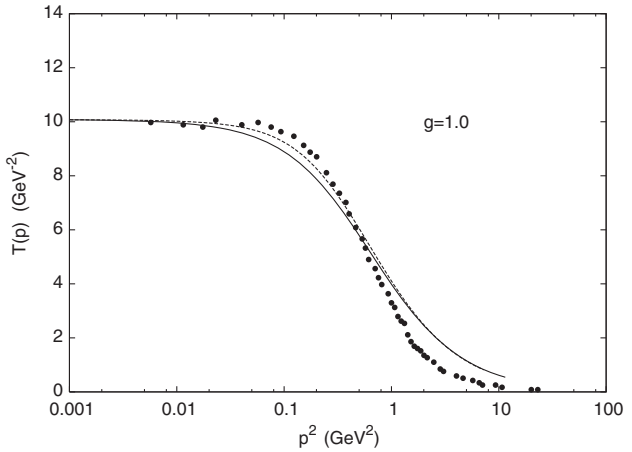

FIG. 5. The renormalized propagator $T(p)$ in the Landau gauge for $N = 3$, $d = 4$, and for a bare coupling $g = 1.0$. The scale has been fixed in order to fit the lattice data of Ref. [51] ($g = 1.02$, L = 96) that are displayed as filled circles. The propagator in the Feynman gauge is shown for comparison as a dotted line.

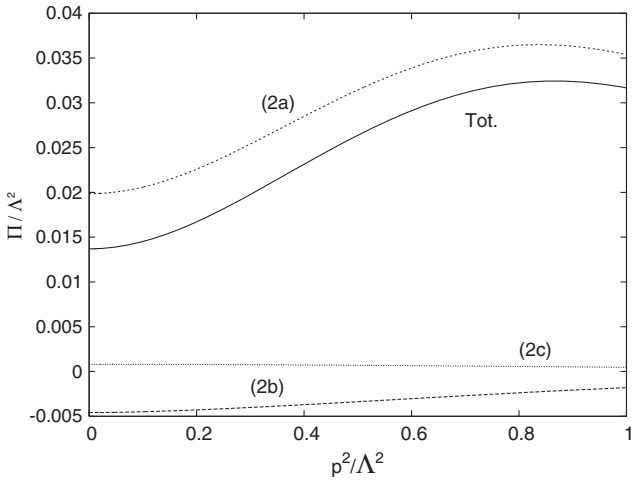

FIG. 7. The total polarization is displayed together with the second order contributions of the one-loop graphs (2a), (2b), and of the two-loop graph (2c) for $N = 3$, $d = 4$. The coupling is $g = 1$ ($\beta = 6$). The total polarization includes the constant contributions $\Pi_{(2d)} = -5.25 \times 10^{-3}$ and $\Pi_{(2e)} = 2.87 \times 10^{-3}$ in units of the cutoff.

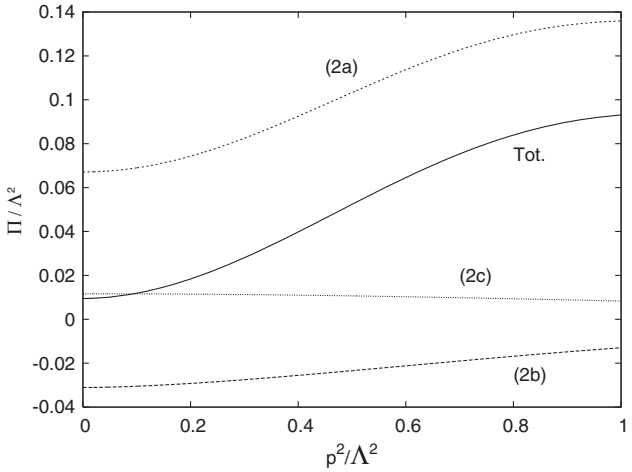

FIG. 8. Same as Fig. 7 but for a strong coupling $g = 2.8$. The constant terms are $\Pi_{(2d)} = -7.30 \times 10^{-2}$ and $\Pi_{(2e)} = 3.48 \times 10^{-2}$ in units of the cutoff.

together with the contributions of the one-loop graphs (2a), (2b), and of the two-loop graph (2c). The total polarization includes the constant contributions, $\Pi_{(2d)} = -5.25 \times 10^{-3}$ and $\Pi_{(2e)} = 2.87 \times 10^{-3}$ in units of the cutoff that are not negligible compared to the one-loop graphs. We observe that the two-loop term (2c) is very small and rather constant so that it could be neglected without serious consequences. On the other hand, at the very strong coupling $g = 2.8$, Fig. 8 shows that the two-loop graph is still rather constant but becomes quite important in the low energy limit where it is as large as the total polarization. The constant terms are also rather large and amount to $\Pi_{(2d)} = -7.30 \times 10^{-2}$ and $\Pi_{(2e)} = 3.48 \times 10^{-2}$ in units of the cutoff.

VII. DISCUSSION

By the explicit knowledge of the second order polarization, several variational strategies can be set up for the optimization of the perturbation expansion. The method of stationary variance [20,38–41] has been discussed in the previous section. The generalized perturbation theory can also be optimized by other methods like minimal sensitivity [37] or by the self-consistent requirement of a vanishing self-energy. Here, we give a brief description and comparison of some different methods.

As discussed in a recent paper [20], the stationary conditions for Stevenson's method of minimal sensitivity [37] can be written simply as

$$\Pi_2 = 0, \qquad \Sigma_2 = 0 \tag{87}$$

to be compared with Eqs. (78) for the stationary variance. Equations (87) are equivalent to the requirement that the second order effective potential is stationary for any variation of the trial propagators. We explored this method in the Landau gauge, but Eqs. (87) have no physical solution for $N = 3$ and $d = 4$. In fact, it is well known that sometimes that method does not show any range of parameters where the effective potential is stationary. A second derivative would be required for imposing that the sensitivity is minimal.

Another simple approach would be based on a first order optimization of the expansion followed by a second order evaluation of polarization and propagator [53]. That would be equivalent to taking the trial propagator equal to the massive propagator $D_M = (-p^2 + M^2)^{-1}$. Even if the mass would not be dynamical in the trial propagator, a second order propagator D_2 can be defined as usual by Dyson equations

$$D_2^{-1}(p) = D^{-1}(p) - \Pi_1(p) - \Pi_2^\star(p), \tag{88}$$

where $D(p)$ is the trial propagator and $D = D_M$ in the actual case. Since the first order optimization is self-consistent, it requires that $\Pi_1 = 0$, and in the Landau gauge by Eqs. (82) and (83) we can define a transversal second order propagator T_2 that reads

$$T_2(p) = [-p^2 + M^2 - \Pi_2^{\star T}(p)]^{-1}. \tag{89}$$

Thus, the second order propagator would acquire a dynamical mass that in the low energy limit tends to $m^2 = M^2 - \Pi_2^{\star T}(0)$. The advantage of this basic approach is that the trial propagator is simple, and the theory can be renormalized by standard dimensional regularization. A similar massive model has been recently studied [17–19] and shown to be in close agreement with the lattice data. However, the mass was regarded as a free parameter rather than a variational parameter. It would be interesting to see how close the result would be for the variational approach that does not contain free parameters at all.

The simple first order optimization would not be self-consistent at second order, but one can attempt and extend it by a self-consistent approach. Equation (88) is quite general and can be made self-consistent by the simple requirement that the total proper polarization vanishes exactly

$$\Pi_1(p) + \Pi_2^\star(p) = 0. \tag{90}$$

That would generalize the first order stationary condition $\Pi_1 = 0$ which holds for the GEP. If the polarization were not truncated at the second order, Eq. (90) would be the exact condition that the trial propagator must satisfy in order to be the exact one. The method would be equivalent to Dyson-Schwinger equations. Of course, truncation spoils it, and the approximation depends on the accuracy of the polarization function that can be evaluated up to second order in any gauge by the explicit expressions of the present paper.

In the Landau gauge, Eq. (90) can be solved numerically as an integral equation for the trial propagator $T(p)$,

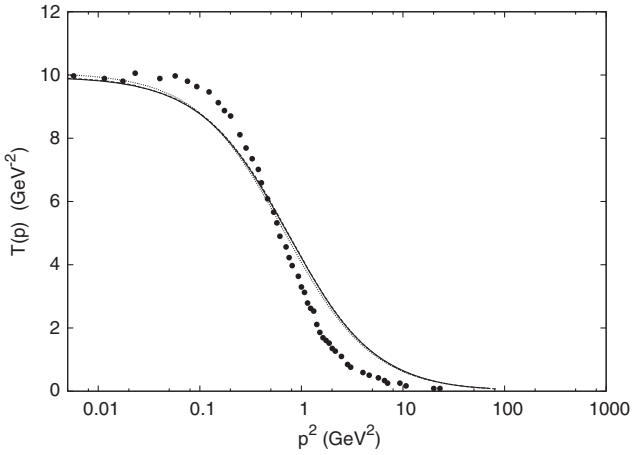

FIG. 9. The self-consistent renormalized propagator $T(p)$ in the Landau gauge for $N = 3$, $d = 4$. The scale has been fixed in order to give a rough fit of the lattice data of Ref. [51] ($g = 1.02$, $L = 96$) that are displayed as filled circles. The second order propagator of Ref. [41] in the Feynman gauge is shown for comparison as a dotted line.

yielding a self-consistent second order propagator that satisfies $T_2(p) = T(p)$ because of Eq. (88). The propagator satisfies a perfect scaling and can be fitted with good accuracy by the simple expression

$$T(p_E) \approx \frac{Z}{p_E^2 + m^2}, \tag{91}$$

where Z and m are real parameters. In fact, that explains the perfect scaling, as any form like that, with just two free parameters, can be scaled on top of each other by a change of units. But, on the other hand, it means that the approximation is very poor, since it is well known that the lattice data cannot be described by a simple Yukawa propagator. As shown in Fig. 9, after renormalization the self-consistent propagator can be put on the lattice data, but the agreement is worse than found by the method of stationary variance in Figs. 4 and 5 of the previous section.

In summary, while the method of stationary variance seems to be more reliable than other variational approaches, other attempts can be devised by the knowledge of the explicit expressions for the polarization up to second order. Once optimized, the perturbation theory does not show divergences in the infrared, while the ultraviolet ones can be cured by standard regularization techniques. The explicit expressions of the present paper hold for any regularization scheme and any choice of the gauge parameter. That would suggest a further way to optimize the expansion, with the gauge parameter and the renormalization scheme that can be regarded as variational parameters [42,43]. A comparison between the present Landau gauge calculation and previous results in the Feynman gauge [41] shows that the gluon propagator is not too sensitive to gauge changes in qualitative agreement with some recent results by Dyson-Schwinger equations [46] and lattice simulation [44].

ACKNOWLEDGMENTS

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APPENDIX: DETAILS ON THE NUMERICAL INTEGRATION

For $d = 4$, all numerical integrations have been calculated as successive one-dimensional integrations by the standard Simpson method in the Euclidean space and with an energy cutoff $p_E^2 < \Lambda^2$. Four-dimensional integrals of simple functions of k_E^2 are reduced to simple one-dimensional integrals before numerical integration, according to

$$\int_\Lambda \frac{\mathrm{d}^4 k_E}{(2\pi)^4} A(k_E^2) = \frac{1}{8\pi^2} \int_0^\Lambda A(k^2) k^3 \mathrm{d}k. \tag{A1}$$

Four-dimensional integrals of functions of the two variables $(k_E \cdot p_E)$ and k_E^2 are reduced to two-dimensional integrals according to

$$\int_\Lambda \frac{\mathrm{d}^4 k_E}{(2\pi)^4} A[(k_E \cdot p_E), k_E^2] = \int_0^\Lambda \frac{y^2 \mathrm{d}y}{4\pi^3} \int_{-\sqrt{\Lambda^2 - y^2}}^{\sqrt{\Lambda^2 - y^2}} A[(x p_E), (x^2 + y^2)] \mathrm{d}x. \tag{A2}$$

The eight-dimensional integral of the two-loop sunset graph $(2c)$ Eq. (72) can be written as a four-dimensional integral by exact integration of some variables. We notice that each single term contributing in Eq. (72) can be written as

$$\int_\Lambda \frac{\mathrm{d}^4 k_E}{(2\pi)^4} \int_\Lambda \frac{\mathrm{d}^4 q_E}{(2\pi)^4} f_i(p_E^2, k_E^2, q_E^2, q_E \cdot (k_E + p_E), p_E \cdot k_E) \times g_i(p_E^2, k_E^2, q_E^2, q_E \cdot k_E, q_E \cdot p_E, p_E \cdot k_E), \tag{A3}$$

where the function f_i has one argument less than the function g_i, and p is the external momentum. Let us introduce the vector $V = k_E + p_E$ and split the four-vector q as the sum of two orthogonal two-dimensional vectors (q_1, q_2) and (q_x, q_y) that are orthogonal and parallel to the k-p plane, respectively. Moreover, we take the direction q_y to be parallel to the direction of V. Omitting the variables p_E^2, k_E^2, V^2 that are constant in the internal integration, the integral in Eq. (A3) reads

$$\int_\Lambda \frac{\mathrm{d}^4 k_E}{(2\pi)^4} \frac{1}{2(2\pi)^4} \int_0^{\Lambda^2} \mathrm{d}q^2 \int_0^{2\pi} \mathrm{d}\phi \int_{-q}^{q} \mathrm{d}q_y f_i(q^2, q_y) \times \int_{-\sqrt{q^2 - q_y^2}}^{\sqrt{q^2 - q_y^2}} \mathrm{d}q_x g_i(q^2, q_x, q_y), \tag{A4}$$

where $\tan\phi = q_2/q_1$ and $q^2 = q_E^2$. The angle ϕ can be integrated exactly yielding a factor of 2π and denoting by $\tilde{g}_i(q^2, q_y)$ the integrated function

$$\tilde{g}_i(q^2, q_y) = \int_{-\sqrt{q^2-q_y^2}}^{\sqrt{q^2-q_y^2}} \mathrm{d}q_x g_i(q^2, q_x, q_y), \tag{A5}$$

in the Landau gauge by Eq. (77) we can write the transverse polarization term $\Pi^T_{(2c)}$ as

$$\Pi^T_{(2c)}(p) = \frac{(Ng^2)^2}{32\pi^3}\int_\Lambda \frac{\mathrm{d}^4k_E}{(2\pi)^4}\int_0^{\Lambda^2}\mathrm{d}q^2\int_{-q}^{q}\mathrm{d}q_y h(q^2, q_y) \times \sum_{i=1}^{11} f_i(q^2, q_y)\tilde{g}_i(q^2, q_y), \tag{A6}$$

where the function h is

$$h(q^2, q_y) = T(q_E)T(k_E)T(k_E + q_E + p_E) \tag{A7}$$

that only depends on $q_E \cdot V \sim q_y$ and $q^2 = q_E^2$ in the internal integration. The 11 functions g_i turn out to be polynomials, and the integrated functions $\tilde{g}_i$ can be evaluated exactly so that we are left with a two-dimensional internal integration, and the result can only depend on k_E^2 and $k_E \cdot p$. Then, the external integration follows by Eq. (A2), yielding a total four-dimensional integration to be evaluated numerically. The 11 terms f_i, g_i, $\tilde{g}_i$ follow from inspection of Eq. (77). Dropping the E in the Euclidean vectors, denoting by $s = \sqrt{q^2 - q_y^2}$ and by k_x, k_y, p_x, p_y the components of k_E, p_E that are parallel to q_x and q_y, respectively, we can write

$$\begin{aligned}
f_1 &= 4, & \tilde{g}_1 &= 2s,\\
f_2 &= \frac{1}{q^2k^2}, & \tilde{g}_2 &= I_2(k,k),\\
f_3 &= \frac{1}{q^2p^2}, & \tilde{g}_3 &= I_2(p,p),\\
f_4 &= \frac{q^2 + Vq_y}{k^2q^2(k+q+p)^2}, & \tilde{g}_4 &= I_2(k,k) + (V\cdot k)I_1(k),\\
f_5 &= \frac{-(p\cdot k)}{k^2q^2p^2}, & \tilde{g}_5 &= I_2(p,k),\\
f_6 &= \frac{2}{p^2(k+p+q)^2}, & \tilde{g}_6 &= I_2(p,p),\\
f_7 &= \frac{4(p\cdot V)}{p^2(k+p+q)^2}, & \tilde{g}_7 &= I_1(p),\\
f_8 &= \frac{2(p\cdot V)^2}{p^2(k+p+q)^2}, & \tilde{g}_8 &= 2s,\\
f_9 &= \frac{-2(p\cdot k)}{p^2k^2(k+p+q)^2}, & \tilde{g}_9 &= I_2(k,p) + (k\cdot V)I_1(p) + (p\cdot V)I_1(k) + (2s)(p\cdot V)(k\cdot V),\\
f_{10} &= \frac{1}{p^2k^2q^2(k+p+q)^2}, & \tilde{g}_{10} &= (p\cdot V)^2I_2(k,k) + I_4(k,p) + 2(p\cdot V)I_3(k,k,p),\\
f_{11} &= \frac{-(k\cdot p)(q^2+Vq_y)}{p^2k^2q^2(k+p+q)^2}, & \tilde{g}_{11} &= I_2(k,p) + (p\cdot V)I_1(k),
\end{aligned} \tag{A8}$$

where the functions I_n are defined as

$$\begin{aligned}
I_1(k) &= (2s)q_yk_y,\\
I_2(k,p) &= (2s)\left[q_y^2k_yp_y + \frac{1}{3}s^2k_xp_x\right],\\
I_3(k,k,p) &= \frac{2}{3}s^3q_y(k_x^2p_y + 2k_xk_yp_x) + (2s)q_y^3(k_y^2p_y),\\
I_4(k,p) &= \frac{2}{5}s^5(k_x^2p_x^2) + \frac{2}{3}s^3q_y^2(k_x^2p_y^2 + k_y^2p_x^2 + 4k_xk_yp_xp_y) + (2s)q_y^4(k_y^2p_y^2).
\end{aligned} \tag{A9}$$

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