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Neutrino pair and gamma beams from circulating excited ions

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We propose a new method of producing a neutrino pair beam that consists of a mixture of neutrinos and antineutrinos of all flavors. The idea is based on a coherent neutrino pair emission from excited ions in circular motion. High energy gamma rays much beyond the kilo-electron-volt range may also be produced by a different choice of excited level.

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I. INTRODUCTION

Synchrotron radiation is a very useful tool of photon emission up to the x-ray energy range, providing a well collimated beam. We examine a similar problem of neutrino pair emission under a circular motion of ions. When excited ions with a high coherence are circulated, emission rates become large with neutrino energies extending to much beyond the kilo-electron-volt (keV) region in the form of a well collimated beam. The produced neutrino beam is a mixture of all pairs of neutrinos, including $\nu_\mu\bar{\nu}_\mu$, $\nu_\tau\bar{\nu}_\tau$. This gives a CP-even neutrino beam, hopefully providing an ideal setting to test fundamental symmetries of particle physics [1], in particular, to measure CP violating phases in the neutrino sector [1–4]. Circulation of highly stripped heavy ions is desirable to achieve the highest neutrino energy in the giga-electron-volt (GeV) region with the largest production rates.

Our method of calculation may be adapted to synchrotron radiation that occurs at an electron machine, giving essentially the same results as in [5], although our method of calculation is different. We shall make it clear how a GeV range intense beam of neutrino pairs is made possible if one uses excited ions instead of ions in the ground state.

One may also produce a high energy gamma ray much beyond the keV range by an appropriate choice of an excited level of a different parity. This may be very useful since the usual electron synchrotron can produce only the keV range photon.

The rest of this paper is organized as follows. In the first two sections we shall explain our semiclassical approximation to treat the ionic motion as given classically and to calculate the probability and its rate of neutrino pair emission in the standard electroweak theory. In Sec. IV we give the core calculation of a phase integral that appears in the rate calculation. We find that with excited ions the phase integral over time contains stationary points of the phase, leading to large neutrino pair emission rates. In Sec. V we compute the differential energy spectrum of neutrino pair production at the synchrotron site. In Sec. VI we discuss a similar problem of photon emission from electric dipole allowed atomic transition. When a good coherence among ions in the excited and the ground levels is prepared and maintained, it might even be possible to have a coherent gamma ray emission much like a laser in the optical region.

In a sequel paper we shall discuss neutrino oscillation experiments that can be done away from the synchrotron.

Throughout this work we use the natural unit of $\hbar = c = 1$.

II. SEMICLASSICAL APPROXIMATION

The total wave function of a composite ion consists of a direct product of the central motion (CM) part of the ion as a whole and its internal part as a consequence of the separation of the Hamiltonian operator into an independent sum of two terms. For the neutrino pair emission process of internal atomic transition, $|e\rangle \to |g\rangle$, the other CM Hamiltonian part never contributes simultaneously. It only contributes to the cases of $|a\rangle \to |a\rangle$, $a = e, g$, and this gives rise to the usual synchrotron emission in much the same way as in the electron machine. For the internal transition, the CM part of wave function Ψ_i gives a weight factor of its probability density $|\Psi_i|^2 = 1/(\gamma V)$ (V is the quantization volume) in the internal part of the Hamiltonian, in accordance to the general rule of the correct property of the lifetime under the Lorentz transformation $\propto \gamma$ [6]. Here γ is the boost factor of the excited ion related to the constant velocity v of circular motion by $v = \sqrt{1 - 1/\gamma^2} \sim 1 - 1/(2\gamma^2)$. Strictly, one needs the instantaneous boost factor $\gamma(t)$ of a time dependent function, but the emission region around the circular orbit is short, and one may replace this by the constant circular

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velocity. For the ion internal state we shall confine ourselves to a two-level system as an approximation, its ionic states being $|e\rangle$ and $|g\rangle$. The metastable state $|e\rangle$ in an upper energy level is assumed to have the same parity as that of the ground state $|g\rangle$ such that a fast electric dipole transition is forbidden, while a magnetic dipole (M1) transition and the neutrino pair emission are both allowed. Relevance of the M1 transition to neutrino pair emission is explained in due course. Another electric dipole case between different parity states is useful for high energy gamma ray emission and is discussed in Sec. VI.

Bilinear forms of wave functions such as the ion current may be described in terms of the density matrix, ρ_{ab}, a, $b = e$, g for its internal part. We assume that the central motion is described by the classical trajectory function $x_A(t)$ (t is the time at observation) of the circular motion. The density matrix for the two-level system is governed by the optical Bloch equation. Its solution may readily be derived in terms of initial values. In particular, the off-diagonal element $\rho_{eg}(t)$ of our system may be described to a good approximation [7] by

$$\rho_{eg}(t) = \rho_{eg}(0)\exp\left[-\left(i\epsilon_{eg} + \frac{1}{T_2}\right)\frac{t}{\gamma}\right], \tag{1}$$

when effects of photon emission are highly suppressed. Throughout this work we use the time in the laboratory system in which measurements of neutrino beam experiments are done. The phase relaxation rate $1/T_2$ is usually larger than its minimum value $1/(2\tau_e)$ (τ_e being the natural lifetime of state $|e\rangle$) that occurs when the phase relaxation is dominated by the spontaneous decay.

In the atomic physics community the quantity ρ_{eg} is called the coherence. For a pure quantum state of a single atom, it is given by a quantum mixture of two states, $|e\rangle$ and $|g\rangle$. Its value is bounded to be less than the value $1/2$ in our normalization convention. Its macroscopic average over a collective body of atoms or ions is usually much less than this maximum value. We shall not discuss the experimental problem of how a large initial coherence given by $\rho_{eg}(0)$ may be prepared.

We can neglect contributions of ionic states that remain in either the excited or the ground state, their rates being proportional to ρ_{ee}^2, ρ_{gg}^2, since they give rise to neutrino pair emission of much smaller rates and much smaller energies, the neutrino-pair analogue of the usual synchrotron radiation. This result originates from the fact that these density matrix elements have no oscillating phase factor as in $\rho_{eg}(t) \propto e^{-i\epsilon_{eg}t/\gamma}$. It is found that both in this case and in the case of electron synchrotron no large neutrino pair production occurs, as is made more evident below.

III. PERTURBATION THEORY OF NEUTRINO PAIR EMISSION

In our semiclassical approximation the Hamiltonian system of an interacting neutrino with atomic electrons is quadratic in neutrino field variables, and one can readily solve the problem of neutrino pair emission, using the perturbation theory of the weak coupling G_F. The four-Fermi interaction of neutrinos and atomic electrons is given by the Hamiltonian (written in terms of neutrino field operators),

$$H_w^{(0)} = \int d^3x \frac{G_F}{\sqrt{2}} \sum_{i,j=1,2,3} (c_{ij}^V V^\beta(x) + c_{ij}^A A^\beta(x))\nu_i^\dagger(x)\sigma_\beta \nu_j(x), \tag{2}$$

with $(\sigma_\beta) = (1, -\vec{\sigma})$. We use the neutrino index convention of Roman alphabets, a, b, c, to indicate neutrino flavor states, ν_e, ν_μ, ν_τ, and Roman alphabets, i, j, k, to indicate mass eigenstates ν_1, ν_2, ν_3. The neutrino mass ordering is taken as usual: $m_3 > m_2 > m_1$ for the normal hierarchy case and $m_2 > m_1 > m_3$ for the inverted hierarchy case. Both W- and Z-boson exchange contributions are added, and the Hamiltonian is written in the Fierz-transformed form (charge retention ordered). There are both vector and axial-vector currents, $V(x)$, $A(x)$, with their couplings $c_{ij}^{V,A}$. We may assume the nonrelativistic limit for transitions of internal electron states in the rest frame of the ion, which singles out as the dominant contribution the spatial part of axial 4-vector $c_{ij}^A S^\beta$ in the form of the electron spin current: $S = (0, \vec{S}_e)$, $\vec{S}_e = \sum_a \langle g|\vec{\sigma}_a/2|e\rangle$, where the sum is taken over the valence electrons of ions. Note that the monopole term of the vector part $\propto c_{ij}^V V_0$ vanishes due to the orthogonality of wave functions between $|e\rangle$ and $|g\rangle$. The coefficients of axial-vector parts are

$$C \equiv (c_{ij}^A), \qquad c_{ij}^A = U_{ei}U_{ej}^* - \frac{1}{2}\delta_{ij}, \qquad CC^\dagger = \frac{1}{4}, \tag{3}$$

in the standard electroweak theory.

In the laboratory frame the relevant current becomes [8]

$$(S_\alpha) = \left(\gamma\vec{\beta}\cdot\vec{S}_e, \vec{S}_e + \frac{\gamma^2}{\gamma+1}(\vec{\beta}\cdot\vec{S}_e)\vec{\beta}\right) \sim \gamma(\vec{\beta}\cdot\vec{S}_e, (\vec{\beta}\cdot\vec{S}_e)\vec{\beta}), \tag{4}$$

where $\vec{\beta}$ is the Lorentz boost vector. Averaging over the atomic spin direction to the leading order of large γ gives the squared amplitude summed over neutrino helicities [9],

$$\gamma^2\frac{S_e^2}{3}\left(1 + \frac{1}{3}\frac{\vec{p}_1\cdot\vec{p}_2}{E_1E_2} - \frac{m_1m_2}{2E_1E_2}\delta_M\right), \tag{5}$$

where $\delta_M = 1$ for the Majorana neutrino and $\delta_M = 0$ for the Dirac neutrino. Our experience of calculations for heavy atoms, such as Xe, Yb, etc. [10,11], shows that these matrix elements $\vec{S}_e$ are of order unity or O(0.1) where the intermediate coupling scheme of heavy atoms holds. We assume in the following that the M1 transition matrix element is of this order.

From this consideration it is found that the circulating atomic spin is the source current of neutrino pair emission and the relevant current is given by

$$J^{\alpha}_{eg}(x) = S^{\alpha}\frac{1}{\sqrt{\gamma}}\int dt\rho_{eg}(t)\delta^{(4)}(x - x_A(t)), \tag{6}$$

$$H_w = \int d^3x \frac{G_F}{\sqrt{2}} J^{\beta}_{eg}(x)\cdot \sum_{i,j=1,2,3} C_{ij}\nu_i^{\dagger}(x)\sigma_{\beta}\nu_j(x), \tag{7}$$

where $x_A(t) = (t, \vec{r}_A(t))$ (written in terms of the time in the laboratory frame) is the trajectory function of the excited ion in circular motion given by

$$\vec{r}_A(t) = \rho\left(\sin\frac{vt}{\rho}, 1 - \cos\frac{vt}{\rho}, 0\right), \tag{8}$$

where ρ is the radius of the circular orbit. The factor $1/\sqrt{\gamma}$ in Eq. (6) arises from the overlap of CM wave functions, $\int d^3X|\Psi_i(X)|^2 = 1/\gamma$.

We adopt the interaction picture in which the kinetic and the mass terms of neutrinos are taken as the free part of Hamiltonian H_0, consisting of diagonal terms of $b^{\dagger}b$, $d^{\dagger}d$ where b, d are annihilation operators of neutrino and antineutrino (in the Majorana neutrino case $d = b$) when they are mode decomposed using plane waves of definite helicities. The Fermi interaction (7) due to the circular ion motion gives rise to off-diagonal terms, in particular, terms of the form bd, $b^{\dagger}d^{\dagger}$. In the perturbative picture this means that neutrino pairs may be created at ion synchrotron.

The amplitude $\mathcal{A}_{ij}(p_1h_1, p_2h_2; t)$ of neutrino-pair production of momentum $\vec{p}_i$ (its energy given by $E_i = \sqrt{p_i^2 + m_i^2}$ for the neutrino of mass m_i) and helicity h_i time evolves according to

$$\begin{aligned} i\partial_t\mathcal{A}_{ij}(p_1h_1, p_2h_2; t) &= i\partial_t\langle 0|d_i(p_2h_2; t)b_j(p_1h_1; t)|0\rangle \\ &= \langle 0|[d_i(p_2h_2; t)b_j(p_1h_1; t), H_w]|0\rangle, \end{aligned} \tag{9}$$

reducing the calculation to the commutator between the neutrino bilinear field db and the weak Hamiltonian H_w. The result for neutrino pair emission of a single pair is given by a time integral,

$$\begin{aligned} &\mathcal{A}_{ij}(p_1h_1, p_2h_2; t) \\ &\quad = -i\sqrt{2}G_F\frac{1}{\sqrt{\gamma}}C_{ij}\int_{-\infty}^{t} dt' e^{i(E+E')t'}\tilde{J}_A^{\dagger}(\vec{p}_1 + \vec{p}_2; t')\cdot j_{\nu}, \end{aligned} \tag{10}$$

$$\tilde{J}^{\alpha}_A(\vec{P}; t) = \rho_{eg}(t)S^{\alpha}e^{-i\vec{P}\cdot\vec{r}_A(t)}, \qquad j_{\nu} = u^{\dagger}(p_1h_1)\sigma v(p_2h_2). \tag{11}$$

Here u, v are associated plane-wave solutions of emitted neutrinos.

The basic interaction Hamiltonian (7) indicates a number of striking features of the neutrino pair emission process. Notably, it predicts a coherent (namely, endowed with a definite phase relation among two neutrinos in the pair) mixture of all neutrinos and antineutrinos of three flavors.

The semiclassical approximation in the present work is limited to the neutrino energy region in which ion recoil may be ignored, which allows the GeV region neutrino production since the circulating ion energy is much larger.

IV. PHASE INTEGRAL

The pair emission rate defined by $P_{ij}(t; p_1h_1, p_2h_2) = \partial_t|\mathcal{A}_{ij}(p_1h_1, p_2h_2; t)|^2$ is given by

$$\begin{aligned} &P_{ij}(t_0; p_1h_1, p_2h_2) \\ &\quad = 4G_F^2|\rho_{eg}(0)|^2\frac{1}{\gamma}|C_{ij}|^2\int_{-\infty}^{0} dt S^{\alpha}\Re(\mathcal{N}_{\alpha\beta}(p_1h_1, p_2h_2) \\ &\qquad \times e^{i(\Delta(0)-\Delta(t))})S^{\beta} \\ &\quad = 4G_F^2|\rho_{eg}(0)|^2\frac{1}{\gamma}|C_{ij}|^2\int_{0}^{\infty} dt S^{\alpha}\Re(\mathcal{N}_{\alpha\beta}(p_1h_1, p_2h_2) \\ &\qquad \times e^{i(\Delta(0)-\Delta(-t))})S^{\beta}, \end{aligned} \tag{12}$$

$$\Delta(t) = \left(E_1 + E_2 - \frac{\epsilon_{eg}}{\gamma}\right)t - (\vec{p}_1 + \vec{p}_2)\cdot\vec{r}_A(t), \tag{13}$$

$$\mathcal{N}^{\alpha\beta}(p_1h_1, p_2h_2) = j^{\alpha}_{\nu}(p_1h_1, p_2h_2)(j^{\dagger}_{\nu})^{\beta}(p_1h_1, p_2h_2), \tag{14}$$

by taking an infinite time limit, which effectively means that time contributing to the integral (12) is much larger than a small fraction of the orbital period $2/\pi(\rho/c)$. Explicit forms of $\mathcal{N}_{\alpha\beta}(p_1h_1, p_2h_2)$ may be evaluated by using formulas given in [9].

The important phase factor in the integral is given by

$$\Delta(0)-\Delta(-t)=\left(E_1+E_2-\frac{\epsilon_{eg}}{\gamma}+i\frac{1}{\gamma T_2}\right)t \\ -\rho\left((p_1+p_2)_x\sin\frac{vt}{\rho}+(p_1+p_2)_y \\ \times\left(1-\cos\frac{vt}{\rho}\right)\right). \quad (15)$$

Let us introduce directional angles of emitted neutrinos,

$$\vec{p}_i=p_i\left(\cos\psi_i\cos\left(\theta_i+\frac{vt}{\rho}\right),\cos\psi_i\sin\left(\theta_i+\frac{vt}{\rho}\right),\sin\psi_i\right), \\ -\frac{\pi}{2}\le\psi_i\le\frac{\pi}{2},\quad -\pi\le\theta_i\le\pi. \quad (16)$$

Angles are measured at an observation point away from the circular motion. The forward and the background directions with respect to the ion beam correspond to $|\theta_i|<\pi/2$ and $|\theta_i|>\pi/2$, respectively. See Fig. 1 for this coordinate system.

The phase factor $\Delta(0)-\Delta(-t)$ of Eq. (15) contains three terms: in addition to the main term $\propto E_1+E_2$, one is from the circulating ion proportional to ρ, the radius of the orbit, and the other is proportional to the level spacing ϵ_{eg}. Under the normal condition one may ignore the imaginary component $\propto 1/T_2$, since $\epsilon_{eg}\gg 1/T_2$. The most important observation in the present work is that an input of deexcitation energy ϵ_{eg} may lead to cancellation of three terms and to the existence of stationary phase points in the relevant phase integral along the real axis of time. On the other hand, without the ϵ_{eg} term one can show that the phase is positive definite. As is well known in mathematical physics, a contribution around stationary points does not suffer from large suppression unlike constructive interference contributing with the same sign phase. This was the case without the ϵ_{eg} term such as synchrotron radiation and neutrino pair emission from the ground state ion. The well-known exponential cutoff arising from the constructive interference gives rise to the cutoff energy of the emitted photon $\approx\gamma^3/\rho$ in synchrotron radiation [5]. This cutoff also occurs for neutrino pair emission at the electron synchrotron, restricting available neutrino energies up to a keV range.

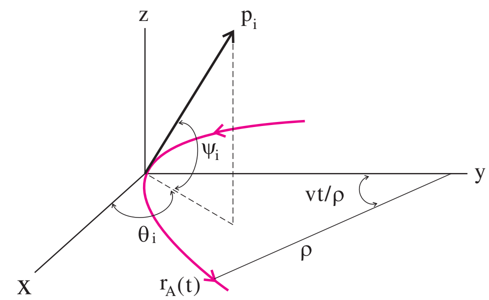

FIG. 1 (color online). Coordinate system for calculation of the phase integral. Observation is made at a point far away on the (positive side of) the x axis. Circular motion of the excited ion in the (x,y) plane is depicted in red. The angle ψ_i is defined to be zero in the ion orbit plane, while θ_i is the angle measured from the tangential direction to the ion beam, with the negative region $\theta_i<0$ being defined toward the inner region of the circular orbit.

The crucial condition for the presence of stationary points is derived by setting the vanishing time derivative of Eq. (15), leading to an equality,

$$E_1+E_2-\frac{\epsilon_{eg}}{\gamma}-v\sum_i p_i\cos\psi_i\cos\left(\theta_i+\frac{vt}{\rho}\right)=0. \quad (17)$$

Infinitely many stationary points exist along the real axis of time t. It turns out that the most important contribution comes from the point nearest to $t=0$, the end point of the integration range $t\ge 0$. Note that without the ϵ_{eg} term there is no stationary point solution on the real axis; hence the possibility of large neutrino pair production at the electron synchrotron is excluded.

Let us consider the in-plane forward direction of $\psi_i=\theta_i=0$ for $E_i\gg\epsilon_{eg}$. One may expand the left hand side function of Eq. (17) in powers of time variable t to solve the stationary point condition. The look-back time of the stationary point $t=t_c$ is then derived as

$$t_c\sim\rho\sqrt{\frac{2\epsilon_{eg}}{\gamma(E_1+E_2)}-\frac{1}{\gamma^2}}, \quad (18)$$

for $E_1+E_2<2\epsilon_{eg}\gamma$. In the following we shall take the kinematical region in which this stationary point exists. To evaluate the phase integral to a good approximation, we use the power series expansion, to derive

$$x=\frac{|A|}{\sqrt{2}\rho}\left(t+\left(1-\frac{1}{\gamma^2}\right)^{-1/2}\frac{p_1\cos\psi_1\sin\theta_1+p_2\cos\psi_2\sin\theta_2}{p_1\cos\psi_1\cos\theta_1+p_2\cos\psi_2\cos\theta_2}\rho\right), \quad (19)$$

$$\Delta(0)-\Delta(-t)\sim\xi\left(-\frac{3}{2}x+\frac{1}{2}x^3\right), \quad (20)$$

$$\xi = \frac{2\sqrt{2}}{3}\frac{\rho}{|A|}\left(\frac{\epsilon_{eg}}{\gamma} - E_1 - E_2 + \left(1 - \frac{1}{\gamma^2}\right)^{1/2}(p_1\cos\psi_1\cos\theta_1 + p_2\cos\psi_2\cos\theta_2) + B\right), \tag{21}$$

$$|A|^2 = \left(1 - \frac{1}{\gamma^2}\right)^{1/2}\left(\frac{p_1\cos\psi_1\cos\theta_1 + p_2\cos\psi_2\cos\theta_2}{\frac{\epsilon_{eg}}{\gamma} - E_1 - E_2 + (1 - \frac{1}{\gamma^2})^{1/2}(p_1\cos\psi_1\cos\theta_1 + p_2\cos\psi_2\cos\theta_2) + B}\right), \tag{22}$$

$$B = \frac{1}{2}\left(1 - \frac{1}{\gamma^2}\right)^{1/2}\frac{(p_1\cos\psi_1\sin\theta_1 + p_2\cos\psi_2\sin\theta_2)^2}{p_1\cos\psi_1\cos\theta_1 + p_2\cos\psi_2\cos\theta_2}. \tag{23}$$

In deriving this equation, we shifted the integration variable t such that $O(t^2)$ terms are eliminated. The power series expansion in terms of time t has been retained up to the third order of t^3, because still higher order terms are suppressed by powers of t_c/ρ.

The condition that the stationary point is within the integration range $t \geq 0$ gives a limitation of emitted angles. An angular region deep inside the circle of ionic motion gives stationary points in the forbidden region of $t < 0$, and hence does not give large neutrino pair emission rates. This forbidden region is defined by

$$\sqrt{2}(E_1\theta_1 + E_2\theta_2) < -\sqrt{\left(\frac{\epsilon_{eg}}{\gamma} - \frac{1}{2\gamma^2}(E_1 + E_2) - \frac{1}{2}(E_1(\theta_1^2 + \psi_1^2) + E_2(\theta_2^2 + \psi_2^2)\right)(E_1 + E_2)}. \tag{24}$$

The necessary phase integral of the x variable involves a smoothly varying function h of time, and it has a form,

$$\int_0^\infty dx h(x)\cos\xi\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) \sim h(1)\frac{\pi}{3}(J_{1/3}(\xi) + J_{-1/3}(\xi)), \tag{25}$$

where $J_\nu(z)$ is the Bessel function, and $h(x)$ is a smoothly varying function of time given by the squared matrix element of neutrino pair emission. The large radius limit of $\rho \to \infty$, hence the $\xi(\propto \rho) \to \infty$, is important for the calculation of differential rates, since the radius ρ is much larger than any microscopic length scale involved. The limit gives

$$\int_0^\infty dx h(x)\cos\xi\left(\frac{1}{2}x^3 - \frac{3}{2}x\right) \to \sqrt{\frac{2\pi}{3}}\cos\left(\xi - \frac{\pi}{4}\right)\frac{h(1)}{\sqrt{\xi}}, \tag{26}$$

as $\xi \to \infty$. This asymptotic formula may also be derived directly using the steepest descent, or the stationary phase method of mathematical physics. The stationary point appears at $x = 1$, which implies that $t_c = \sqrt{2}\rho/|A|$. In addition to the phase factor given here, there is a constant phase factor arising from the phase at the stationary point, namely $\Delta(0) - \Delta(-t_c)$, which, however, gives a negligible contribution.

The fact that the phase integral proportional to rate, in particular, (26) or the more precise formula without the use of time expansion, can give negative values for some value of ξ might appear odd. But since this is the time derivative of a positive quantity (probability), this may occur without any violation of fundamental principles. Indeed, this also occurs in the usual formula of synchrotron radiation [5]. The quantity ξ is actually a complicated function of neutrino energies, their emission angles, the boost factor γ, and the atomic energy scale ϵ_{eg}. The region of these variables that effectively contributes with a large rate is found to give mostly positive rates. Thus, there is no serious problem of the negative rate. An alternative method used in the case of synchrotron radiation [5] treats the time and one of the angular variables, θ (essentially not measurable), symmetrically in integration by changing integration variables in a clever way. A generalization of this method to the case of neutrino pair emission might be possible with much effort.

For a finite value of T_2, the stationary point moves to a point slightly off the real axis, introducing a small correction to ϵ_{eg} replaced by $\epsilon_{eg} - i/T_2$. The effect of this shift is small.

It would be instructive, before proceeding, to mention the limiting case of $\epsilon_{eg} \to 0$ in our problem. In the limit the stationary point t_c approaches the end point of the time integration range, and the phase space of neutrino momenta shrinks to zero. Thus, the rate due to the mechanism considered vanishes in the limit.

As another extension we would like to mention other contributions than the spin current contribution considered here. Contributions from the excited state and the ground state are proportional to ρ_{ee}, ρ_{gg}, which do not have the ϵ_{eg} factor in the phase; hence, this case too has no stationary point on the real time axis. The result of the phase integral in these cases is given in terms of the modified Bessel function much as in the synchrotron radiation. The neutrino energy spectrum then suffers from the exponential cutoff at energies of order γ^3/ρ, which is typically in the keV region. Since the weak interaction rates scale with energy^5 (the 5th power of energy), rates are negligibly small. We have neglected these contributions.

V. DIFFERENTIAL EMISSION RATE OF A SINGLE PAIR

Let us first write down the squared spin amplitude (4) in our coordinate system,

$$\mathcal{M} = \frac{1}{3}\gamma^2 S_e^2\left(1 + \frac{1}{3}\cos\psi_1\cos\psi_2\cos(\theta_1 - \theta_2) + \frac{1}{3}\sin\psi_1\sin\psi_2\right). \tag{27}$$

For simplicity we took neutrinos to be massless, which is adequate for our purpose. Effects of finite neutrino masses are significant only for $E_i < O(m\gamma)$, which is of order keV for $m = 0.1$ eV, $\gamma = 10^4$. Since neutrino pair production rates are small for this energy range, we shall ignore the effects of finite neutrino masses for the discussion of production rates.

A straightforward calculation using this result gives the differential production rate for a neutrino pair $\nu_i\bar{\nu}_j$ of mass eigenstates. To avoid complication, we shall write down this formula in the leading approximation of a large boost factor,

$$\frac{d^4\Gamma_{ij}}{dE_1 dE_2 d\Omega_1 d\Omega_2} = \frac{4G_F^2}{2^{7/4}\,3\sqrt{3\pi}(2\pi)^6} \times |C_{ij}|^2 S_e^2 N|\rho_{eg}(0)|^2\gamma\sqrt{\rho}E_1^2E_2^2F^{-1/4}, \tag{28}$$

$$F = (E_1 + E_2)\left(\frac{\epsilon_{eg}}{\gamma} - \frac{E_1 + E_2}{2\gamma^2}\right) - \frac{1}{2}(E_1^2\psi_1^2 + E_2^2\psi_2^2) - \frac{E_1E_2}{2}(\theta_1 - \theta_2)^2 - \frac{\epsilon_{eg}}{2\gamma}(E_1\theta_1^2 + E_2\theta_2^2). \tag{29}$$

The spin factor is given by $\mathcal{M} \sim 4\gamma^2S_e^2/9$ in this approximation. The function F is more complicated in the most general case of the boost factor, which may be inferred from Eq. (44) for the (electric dipole) photon emission. There is a constraint on angle factors given by $F \geq 0$. This constraint gives angular restriction worked out for apertures (given for simplicity to the case $E_1 = E_2 = E$),

$$\Delta\psi = O\left(\frac{1}{\gamma}\sqrt{\frac{2(E_m - 2E)}{E}}\right),$$
$$\Delta\theta = O\left(\sqrt{\frac{E_m - 2E}{E_m}}\right), \qquad E_m = 2\gamma\epsilon_{eg}. \tag{30}$$

While $\Delta\theta_i$ is of order unity individually, the opening angle of two neutrinos of the pair is limited by

$$\Delta|\theta_1 - \theta_2| < O\left(\frac{1}{\gamma}\sqrt{\frac{(E_1 + E_2)(E_m - E_1 - E_2)}{E_1E_2}}\right). \tag{31}$$

The suppression by $1/\gamma$ for the opening angle is of great interest from the point of oscillation experiments, since it leaves open for the possibility of a coherent neutrino pair interaction at measurement sites.

Integration over four angle factors, with $d\Omega_i = \cos\psi_i d\psi_i d\theta_i$, may be carried out to give

$$\int d\Omega_1\int d\Omega_2 F^{-1/4} \sim \frac{V_4}{14}\left(\frac{\epsilon_{eg}}{\gamma}\right)^{5/4}\frac{(E_1 + E_2)^{5/4}}{(E_1E_2)^{3/2}} \times \left(1 - \frac{E_1 + E_2}{E_m}\right)^{7/4}, \tag{32}$$

where $V_4 = \pi^2/2$ is the volume of the four-dimensional sphere of unit radius. Using this result, the double differential energy spectrum becomes

$$\frac{d^2\Gamma_{ij}}{dE_1dE_2} = \frac{1}{21\times 2^7\times 2^{3/4}\sqrt{3\pi}\pi^4} \times |C_{ij}|^2S_e^2N|\rho_{eg}(0)|^2\sqrt{\rho}\gamma\left(\frac{\epsilon_{eg}}{\gamma}\right)^{5/4} \times G_F^2(E_1E_2)^{1/2}(E_1 + E_2)^{5/4}\left(1 - \frac{E_1 + E_2}{E_m}\right)^{7/4}. \tag{33}$$

The relation $\sum_j|C_{ij}|^2 = 1/4$ was used. Further integration gives the single neutrino energy spectrum and the total pair production rate,

$$\frac{d\Gamma_i}{dE} = \frac{1}{21\times 2^{10}\times\sqrt{6\pi}\pi^4} \times S_e^2N|\rho_{eg}(0)|^2\sqrt{\rho\epsilon_{eg}}G_F^2E_m^4\frac{1}{\gamma}f\left(\frac{E}{E_m}\right), \tag{34}$$

$$f(x) = \sqrt{x}\int_0^{1-x}dy y^{1/2}(y + x)^{5/4}(1 - x - y)^{7/4},$$
$$\int_0^1 dx f(x) \sim 0.00727, \tag{35}$$

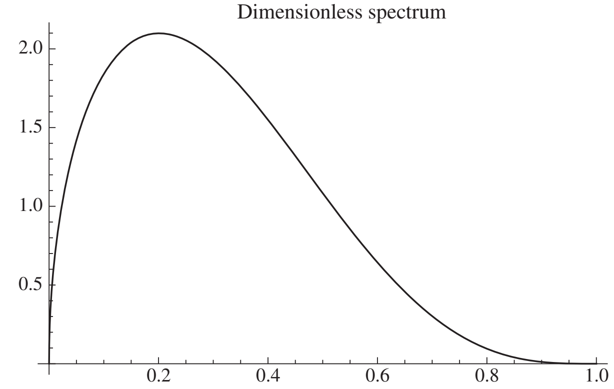

FIG. 2. Normalized universal spectrum function $f(x)$ of Eq. (35) divided by its total integral $\int_0^1 dy f(y)$ in which $x = E/E_m$, $E_m = 2\epsilon_{eg}\gamma$ is the fractional neutrino energy.

$$\Gamma_i \sim 0.0073 \frac{1}{21 \times 2^{10} \times \sqrt{6\pi}\pi^4\gamma} S_e^2 N|\rho_{eg}(0)|^2 \sqrt{\rho\epsilon_{eg}} G_F^2 E_m^5. \quad (36)$$

The production rate does not depend on the mass eigenstate label, since we assumed the massless neutrino for this calculation. The normalized universal spectrum function $f(x)/\int_0^1 dy f(y)$ is plotted in Fig. 2. The average neutrino energy is $\sim 0.30 E_m$. The end point behavior at $x = 1$ gives the threshold behavior $\propto (E_m - E)^{13/4}$, $E_m = 2\epsilon_{eg}\gamma$ at the highest neutrino energy and $\propto \sqrt{E}$ in the infrared region of $E \to 0$.

The dependence of the total rate on parameters ϵ_{eg}, γ taken as independent is $\propto \gamma^4 \epsilon_{eg}^{11/2}$. Along with the relation $E_m = 2\epsilon_{eg}\gamma$, we conclude that it is desirable to choose highly stripped heavy ions in order to achieve both high energy neutrino and large production rates. Rates further depend on $N|\rho_{eg}(0)|^2$ of injected ion, which requires a coherence of large $\rho_{eg}(0)$. We shall derive a constraint on this coherence factor in Sec. VI. A numerical estimate then gives

$$\Gamma = \sum_i \Gamma_i \sim 3.1 \times 10^{21}\ \mathrm{Hz} \left(\frac{\rho}{4\ \mathrm{km}}\right)^{1/2} \frac{S_e^2 N|\rho_{eg}(0)|^2}{10^8} \times \left(\frac{\gamma}{10^4}\right)^4 \left(\frac{\epsilon_{eg}}{50\ \mathrm{keV}}\right)^{11/2}, \quad (37)$$

$$\text{with} \quad E_m = 2\epsilon_{eg}\gamma = 1\ \mathrm{GeV} \frac{\epsilon_{eg}}{50\ \mathrm{keV}} \frac{\gamma}{10^4}. \quad (38)$$

We may offer an interpretation of dependence of the total rate on involved various quantities. Ignoring dimensionless numerical values one has the relation

$$\Gamma \propto \frac{1}{\gamma} N\, |\rho_{eg}(0)|^2\ G_F^2 E_m^5\ \sqrt{\rho\epsilon_{eg}}. \quad (39)$$

Each factor written here has a clear meaning. What this dependence implies is a scaling law with the boosted factor γ in the laboratory frame of the circular motion. Note first that there is a hidden γ factor in the radius of $1/\rho = QeB/(\gamma M_A)$ with M_A the ion mass and Qe the charge of the ion. Except for the first factor $1/\gamma$ which arises from the prolonged lifetime $\propto \gamma$, other factors are dictated by the simple scaling of the basic atomic energy, with $\epsilon_{eg} \to \gamma\epsilon_{eg}$. This scaling law also holds in the photon emission rate discussed in Sec. VI.

For a variety of expected neutrino experiments based on the CP-even neutrino beam, it is important to have a beam of neutrino energy high enough in the GeV region [at minimum, larger than $O(200)$ MeV], since only then can one clearly detect the charged current (CC) interaction of ν_μ, $\bar{\nu}_\mu$. If this requirement is not fulfilled, one only has the CC interaction of ν_e, $\bar{\nu}_e$ and all kinds of neutral current (NC) interaction including ν_τ, $\bar{\nu}_\tau$. The NC process has a lower rate and experiments are harder. The GeV neutrino production requires $2\epsilon_{eg}\gamma \geq 1$ GeV for this combination of the ion parameter and the boost factor. For $\gamma \leq 10^4$, it is necessary to have $\epsilon_{eg} \geq 50$ keV in order to reach 1 GeV neutrino energy. It is then important to excite electrons deeply bound in highly stripped ions in order to reach the keV binding energy of valence electrons in ion. This might be a nontrivial problem, but we assume that this is possible [12]. Judging from Fig. 3 it appears that there is an excellent chance of neutrino experiments in

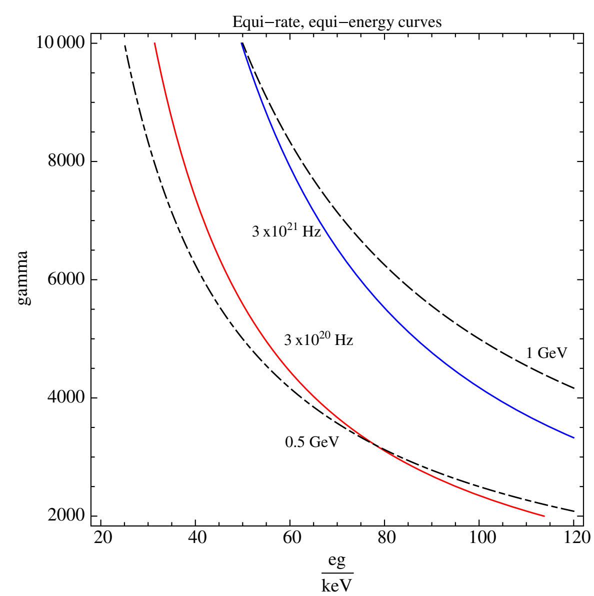

FIG. 3 (color online). Total equirate [given by Eq. (37)] curves for 3×10^{20}, 3×10^{21} Hz in colored solid lines in the $(\epsilon_{eg}/\mathrm{keV}, \gamma)$ plane. Also plotted are the dashed black curve for $E_m = 1$ GeV and the dash-dotted black curve for $E_m = 0.5$ GeV. $N|\rho_{eg}(0)|^2 S_e^2 = 10^8$ is assumed. The normal hierarchical mass pattern of the smallest neutrino mass 0 is taken, with $\epsilon_{eg} = 50$ keV, $\gamma = 10^4$, the synchrotron radius, $\rho = 4$ km.

the $O(0.5-1\ \mathrm{GeV})$ energy range, which roughly gives a large rate of order, $10^{20}-10^{21}$ Hz of the neutrino pair emission. A large value of ϵ_{eg} and a large boost factor γ are required to accomplish this goal.

Very important as a caveat, a new scheme of continuous injection or generation of coherent excited ion beam should be invented, because an excited ion, once it produces the neutrino pair, is not expected to be reusable for another source. The design and realization of this scheme might be challenging, and one may have to tolerate a sizable and unavoidable reduction of an effectively usable ion number.

A possible problem of highly stripped heavy ions is their large magnetic dipole (M1) transition rate [13]. For concreteness let us take an example of the He-like ion, Pb^{80+}. A good candidate for the initial ionic level is $|e\rangle = ((2s)(1s))^3_{J=1}$ (a spin triplet state described in the jj coupling scheme) of level spacing $\epsilon_{eg}\sim 70$ keV. The beam loss rate due to M1 photon emission is

$$\Gamma_\gamma = \gamma_{M1} N \rho_{ee}(0), \tag{40}$$

where the M1 decay rate γ_{M1} is $\sim 3.4\times 10^{13}$ Hz according to [13]. By requiring that this loss rate is smaller than the neutrino pair emission rate, one derives

$$|\rho_{eg}(0)|^2 > O(0.1)\rho_{ee}(0)\left(\frac{\gamma}{10^4}\right)^{-4}. \tag{41}$$

Assuming the relation $|\rho_{eg}(0)|^2 = \rho_{ee}(0)\rho_{gg}(0)$ that holds for a quantum mixture of pure states, one may further derive a lower bound for $\rho_{gg}(0)$ of order $0.1(\gamma/10^4)^{-4}$. This condition in the general case of He-like ions, with the inequality $\rho_{gg}(0) < 1$, further gives constraint on a relation of the boost factor and the level spacing. Details on these constraints should be worked out after more detailed R and D investigation on candidate heavy ions is made.

VI. HIGH ENERGY GAMMA RAY BEAM

For completeness we shall present the main results for high energy gamma emission from the circulating excited ion. The high energy gamma ray emission occurs between different parity states among which the $E1$ transition is allowed. The formalism in the main text is readily adapted to this case, and we shall be brief in presenting results.

The basic Hamiltonian operator of $E1$ photon emission is (using a similar notation as in the previous case)

$$H_\gamma = \int d^3x \frac{e}{m_e}\vec{A}(x)\cdot\vec{J}_\gamma(x),$$
$$\vec{J}_\gamma(x) = \frac{1}{\sqrt{\gamma}}\gamma\int dt\rho_{eg}(t)\vec{p}_{eg}\delta(x-x_A(t)), \tag{42}$$

where $\vec{A}(x) = e^{i\vec{k}\cdot\vec{x}}\vec{e}_{\vec{k}}/\sqrt{2\omega V}$ is the vector potential of the emitted plane-wave photon ($\vec{e}_{\vec{k}}$ being the polarization of the photon). The other contribution arising from the center of mass (CM) motion part of the ion as a whole $\propto \vec{P}_A$ (atomic momentum) has been omitted, because it does not contribute to the internal atomic transition of $|e\rangle\to|g\rangle$. The CM part gives a contribution similar to the usual electron's synchrotron radiation and gives rates much smaller than the rest of the contribution. See below on more of this point.

The matrix element of the internal part $\propto \vec{p}_{eg}$ leading to Eq. (42) has been worked out as follows. The relativistic form of the interaction Hamiltonian density after the Lorentz boost of the γ factor is given by

$$e\gamma\int d^3x\langle g|(\vec{A}\cdot\psi^\dagger\vec{\alpha}\psi + \vec{A}\cdot\vec{\beta}\psi^\dagger\psi)|e\rangle, \tag{43}$$

using the radiation gauge in the atomic rest frame, where $\vec{\alpha}$ is the Dirac 4×4 matrix, $\vec{\alpha}=\gamma_0\vec{\gamma}$, and $\vec{\beta}$ is the Lorentz boost vector. The orthogonality of (nonrelativistic) wave functions of $|e\rangle$ and $|g\rangle$ gives the vanishing second contribution $\propto \psi^\dagger\psi$ to the leading first order to v/c (v being the velocity of the atomic electron in its rest frame). The first contribution gives the internal contribution of Eq. (42) when the matrix element $\int d^3x\langle g|e^{i\vec{k}\cdot\vec{x}}\psi^\dagger\vec{\alpha}\psi|e\rangle$ is written in the atomic rest frame, taking the long wavelength approximation of $\vec{k}\to 0$ valid comparing with a larger inverse atomic length scale. The atomic matrix element may further be recast into the usual dipole form, using the equation of motion: $\vec{p}_{eg}/m_e = -i\epsilon_{eg}\vec{r}_{eg}$ with $e\vec{r}_{eg}$ the dipole matrix element for $E1$ transition.

Calculation of the phase integral involves the energy-momentum $(\omega,\vec{k})$, $|\vec{k}|=\omega$ of a single photon. Stationary phase points appear due to the presence of the energy ϵ_{eg} in the phase integral. Straightforward calculations using the same approximation as in the previous case lead to photon emission rates. It would be instructive to start from a detailed discussion of the angular distribution. The double differential emission rate is given, to the best accuracy we know of, by

$$\frac{d^2\Gamma}{d\omega d\Omega} = \frac{1}{2^{1/4}16\pi^3}N|\rho_{eg}(0)|^2\gamma\sqrt{\rho}\frac{\gamma_{eg}}{\epsilon_{eg}}\omega^{3/4}$$
$$\times\left(\cos\theta\cos\psi\left(\frac{\epsilon_{eg}}{\gamma}-\omega+\frac{1}{2}\left(1-\frac{1}{\gamma^2}\right)^{1/2}\right.\right.$$
$$\left.\left.\times\,\omega\frac{1+\cos^2\theta}{\cos\theta}\cos\psi\right)\right)^{-1/4}, \tag{44}$$

where the squared dipole moment $\vec{r}_{eg}^{\,2}$ was replaced by the related decay rate γ_{eg} (Einstein's A coefficient). The bracketed quantity in the argument of the negative fractional power $-1/4$ must be positive definite. To the leading order of the boost factor γ, this quantity is approximated near the forward direction by

$$\frac{\epsilon_{eg}}{\gamma} - \frac{\omega}{2\gamma^2} - \frac{\omega}{2}\psi^2 - \frac{\epsilon_{eg}}{2\gamma}\theta^2. \quad (45)$$

The positivity requires the inside region of an ellipsoid in the (θ, ψ) plane,

$$\psi^2 + \frac{\epsilon_{eg}}{\gamma\omega}\theta^2 \leq \frac{\omega_m - \omega}{\gamma^2\omega}, \qquad \omega_m = 2\gamma\epsilon_{eg}, \quad (46)$$

along with $\omega \leq \omega_m$. Thus, there exists an interesting angular asymmetry: the cylindrical symmetry around the tangential direction is broken. With this approximation, the double differential rate becomes

$$\frac{d^2\Gamma}{d\omega d\Omega} \sim \frac{1}{2^{1/4} \times 16\pi^3} N|\rho_{eg}(0)|^2 \gamma\sqrt{\rho}\frac{\gamma_{eg}}{\epsilon_{eg}}\omega^{3/4} \times \left(\frac{\epsilon_{eg}}{\gamma} - \frac{\omega}{2\gamma^2} - \frac{\omega}{2}\psi^2 - \frac{\epsilon_{eg}}{2\gamma}\theta^2\right)^{-1/4}. \quad (47)$$

The small angle approximation here is valid only for a small value of $2(\omega_m - \omega)/(\gamma^2\omega)$. The approximation clearly breaks down at the infrared limit $\omega \to 0$.

Further angular integration is straightforward if one uses the small angle approximation, leading to the photon energy spectrum and finally the total emission rate,

$$\frac{d\Gamma}{d\omega} = \frac{1}{24\pi^2} N|\rho_{eg}(0)|^2 \gamma \frac{\gamma_{eg}}{\epsilon_{eg}} \sqrt{\rho\epsilon_{eg}} \left(\frac{\omega}{\omega_m}\right)^{1/4} \left(1 - \frac{\omega}{\omega_m}\right)^{3/4}, \quad (48)$$

$$\Gamma = \frac{I}{12\pi^2} N|\rho_{eg}(0)|^2 \gamma^2 \gamma_{eg} \sqrt{\rho\epsilon_{eg}}, \qquad I = \int_0^1 dy y^{1/4}(1-y)^{3/4} \sim 0.4165. \quad (49)$$

The dimensionless spectrum function $x^{1/4}(1-x)^{3/4}/I$, $x = \omega/\omega_m$ is plotted in Fig. 4 after renormalization. Its end point is at $\omega_m = 2\gamma\epsilon_{eg}$, and the averaged energy value is $0.42\omega_m$. A typical value of the total photon emission rate is

$$\Gamma \sim 1.1 \times 10^{29}\ \text{Hz} \frac{\gamma_{eg}}{100\ \text{MHz}} \sqrt{\frac{\rho}{4\ \text{km}}} \left(\frac{\epsilon_{eg}}{50\ \text{keV}}\right)^{1/2} \left(\frac{\gamma}{10^4}\right)^2 \times \frac{N|\rho_{eg}(0)|^2}{10^8}, \quad (50)$$

for the $\omega_m = 1$ GeV case.

In a similar fashion as for the neutrino pair emission, the formula of the total $E1$ photon emission rate may be interpreted using the γ-scaling law. Let us ignore dimensionless numerical factors for this purpose. The total rate then has dependence on various quantities,

$$\Gamma \propto \frac{1}{\gamma}\ N|\rho_{eg}(0)|^2\ e^2 \vec{r}_{eg}^{\,2} \omega_m^3\ \sqrt{\rho\epsilon_{eg}}, \qquad \omega_m = 2\gamma\epsilon_{eg}. \quad (51)$$

When one regards the atomic dipole $e\vec{r}_{eg}$ as an invariant and intrinsic quantity to the atom, the other energy factor scales as $\propto \gamma$ under the Lorentz transformation, along with the prolonged lifetime factor $1/\gamma$ in front. This law explains γ and ϵ_{eg} dependence of the photon emission rate $\propto \gamma^2\epsilon_{eg}^{7/2}$, as well as that of neutrino pair emission $\propto \gamma^4\epsilon_{eg}^{11/2}$ (ρ regarded as γ independent).

Comparison with the usual synchrotron radiation may be instructive and also interesting. We can work out photon emission caused by ion circular motion in the ground state (or kept in the excited state) or simply an electron's circular motion, using the same calculation technique as above. The basic Hamiltonian arises from the omitted $\propto \vec{P}_A$ term, and the calculation is purely classical unlike the semiclassical approximation in the case of photon emission from an excited level. There is no lifetime related factor $\propto 1/\gamma$ in this case, because the synchrotron emission is not a decay process. The result differs in an essential way from the case of the excited ion, in that there is no stationary point of the time integral. The phase integral in this case takes the form

$$\int_0^\infty dx h(x) \cos\xi\left(\frac{1}{2}x^3 + \frac{3}{2}x\right) \to \sqrt{\frac{\pi}{6}} e^{-\xi} \frac{h(0)}{\sqrt{\xi}}. \quad (52)$$

There is no phase cancellation unlike in the case of the excited ion. Instead, the exponential cutoff emerges for large ξ.

We can finally derive in the large radius (ρ) limit a compact result for the energy spectrum and the total rate,

$$\frac{d\Gamma}{d\omega} = N\sqrt{\frac{2\pi}{3}} \frac{Q^2\alpha}{4\pi} \frac{1}{\gamma^2} \int_{\omega/\omega_c}^{\eta\sqrt{\eta}\omega/\omega_c} d\xi \frac{e^{-\xi}}{\sqrt{\xi}} \left(\left(\frac{\omega_c \xi}{\omega}\right)^{2/3} - 1\right), \qquad \omega_c = \frac{3}{2\rho}\gamma^3 \sim 75\ \text{eV}\left(\frac{\gamma}{10^4}\right)^3 \frac{4\ \text{km}}{\rho}, \quad (53)$$

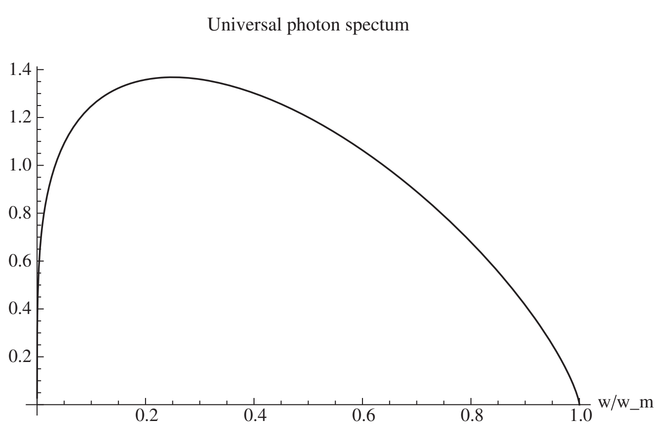

FIG. 4. Normalized universal energy spectrum of photons emitted from excited ions.

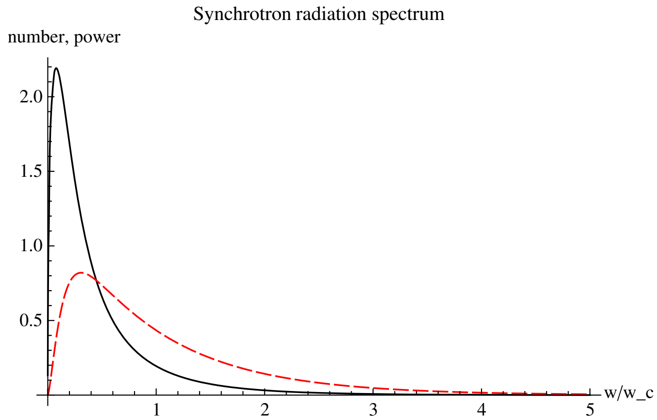

FIG. 5 (color online). Normalized synchrotron spectrum shapes. Number spectrum in solid black curve and power spectrum in dashed red curve.

$$\Gamma = \sqrt{\frac{3}{2}}\frac{Q^2\alpha}{8}\frac{1}{\rho}N\gamma \sim 8.4\times 10^{24}\ \mathrm{Hz}Q^2\frac{4\ \mathrm{km}}{\rho}\frac{N}{10^{19}}\frac{\gamma}{10^4}, \tag{54}$$

where Qe is the charge of the ion. The value of the total rate given here corresponds to $1C$ ions equivalent to $\sim 10^{19}$ ion numbers. The value η in the upper bound of the ξ integral is estimated around 5 from the available angular area of 4π. This result is in a fair agreement with the standard results given in textbooks such as [8], considering the crudeness of the matrix element estimate given here. We show the spectrum in Fig. 5 for the reader's reference.

It is interpreted that for electron synchrotron radiation the Zeeman splitting energy $eB/(\gamma m_e) = 1/\rho$ is extended by the γ^3 factor. Use of the internal atomic energy in our problem has two important effects: (1) larger energy spacing than the Zeeman splitting, and (2) kinematical power law cutoff at $2\epsilon_{eg}\gamma$ rather than the exponential cutoff $eB\gamma^2/m_e = \gamma^3/\rho \sim 2\gamma^3\ \mathrm{neV}(100\ \mathrm{m}/\rho)$ for synchrotron radiation.

One may work out a requirement on the coherence $\rho_{eg}(0)$ by demanding that the synchrotron radiation is not an obstacle against the neutrino pair emission. It is imposed that the number of emitted neutrino pair per revolution of circular motion $\Gamma\times 2\pi\rho/c$ (equivalent to the number of deexcited ions caused by neutrino pair emission) is much larger than the number of emitted synchrotron photons per revolution. This condition gives a constraint on the coherence,

$$|\rho_{eg}(0)| \gg 1\times 10^{-4}Q\left(\frac{\gamma}{10^4}\right)^{-3/2}\left(\frac{\rho}{4\ \mathrm{km}}\right)^{-3/4} \times\left(\frac{\epsilon_{eg}}{50\ \mathrm{keV}}\right)^{-11/4}, \tag{55}$$

taking the spin factor to be unity, $S_e^2 = 1$. If this condition is violated in the case of a large Q, one may have to think of compensating the loss of excited ions by irradiation of the laser each time of revolution [12].

We have several comments based on results of the gamma emission from excited ions. First, a different kind of coherence effect over a larger volume may further enhance photon emission rates by the superradiance mechanism of Dicke [14]. In the case of two-photon emission mentioned above the macrocoherent paired superradiance (PSR) may further enlarge the coherent region [10] not restricted to an area of the photon wavelength in the Dicke case. It might even be possible to produce the coherent gamma ray "laser," with the help of macrocoherence. As an example, the $2s\to 1s$ two-photon transition of the H^- ion may be an excellent source of coherent two-photon emission due to its long lifetime of the $2s$ excited ion. The achievable energy is not large, however, of order 200 keV $\gamma/10^4$ for the hydrogen ion. The molecular vibrational transition $v = 1\to 0$ of pH_2^+ may be better due to their easiness of Raman excitation. Recently, the macrocoherent PSR of the vibrational transition of neutral pH_2 was observed [15], in which we achieved a macrocoherence of approximately several percent over a target of 15 cm long. Rates of two-photon emission from circulating excited ions may be worked out as in the rate calculation of neutrino pair emission. Our γ-scaling law suggests that two-photon emission rates are large despite their effective, weaker coupling.

Even as a technical strategy toward a high intensity neutrino beam, it would be wise to first study the basic experimental feasibility of heavy ion excitation aiming at high energy photon emission, since it would be easier to detect and study the mechanism of photon emission in detail.

In summary, a new method of producing a CP-even coherent neutrino beam from circulating excited ions was proposed. Large production rates of neutrino energies extending to much beyond the keV region were derived. When ions are excited to a different $E1$ allowed level, they may provide a high intensity gamma ray beam much beyond the keV range. Evidently, many R and D works, both theoretical and experimental, are needed to determine a realistic design using a specific ion.

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Notes added in proof.—The phase build-up time must be shorter than the neutrino pair emission time given by $1/\Gamma_{2\nu}$ ($\Gamma_{2\nu}$ simply denoted by Γ in the text). The phase build-up time is estimated by analyzing the time integral of the phase factor, which is an Airy-type of integral:

$$\int_0^\infty dt \cos\Phi(t), \quad \Phi(t) = \frac{\omega + E_1 + E_2}{2\rho\gamma}\sqrt{D}\left(t - \frac{\rho}{\gamma}D\right)^2,$$

$$D = 1 - \frac{2\epsilon_{eg}\gamma + \gamma^2(m_1^2/E_1 + m^2/E_2)}{\omega + E_1 + E_2} - (\text{quadratic function of angles}).$$

This gives rise to the development time t_r and the width Δt_r of resonance in the time domain:

$$\text{resonance in time domain:}\ t_r = \frac{\rho}{\gamma}D,$$

$$\text{width;} \quad \Delta t_r = \rho\sqrt{\frac{2}{(\omega + E_1 + E_2)t_r}} \ll t_r$$

$$\int_0^\infty dt \cos\Phi(t) \sim \int_0^\infty dt \cos\frac{(t - t_r)^2}{(\Delta t_r)^2} \sim \sqrt{\frac{2\pi}{3}}\Delta t_r = \sqrt{\frac{\pi}{3}}\left(\frac{\rho\gamma}{\omega + E_1 + E_2}\right)^{1/2} D^{-1/4},$$

for the GeV neutrinos. The dimensionless function D is in the range, $0 \sim 1$.

For a single excited ion of $S_e^2 N|\rho_{eg}(0)|^2 = 1$,

$$\Gamma_{2\nu} \sim 3.1 \times 10^{13}\ \text{Hz}\left(\frac{\rho}{4\ \text{km}}\right)^{1/2}\left(\frac{\gamma}{10^4}\right)^4\left(\frac{\epsilon_{eg}}{50\ \text{keV}}\right)^{11/2}.$$

The inequality $\Delta t_r < 1/\Gamma_{2\nu}$ gives a constraint which is satisfied in major parts of the contour plot of Fig. (3).

A more serious constraint arises for the gamma ray emission, since its rates are larger. Using the gamma emission rate Γ_γ given in the text, the inequality $\Delta t_r < 1/\Gamma_\gamma$ gives a constraint on the boost factor,

$$\gamma < 0.9\left(\frac{\gamma_{eg}}{100\ \text{MHz}}\right)^{-1/2}\left(\frac{\rho}{4\ \text{km}}\right)^{-1/4}\left(\frac{\epsilon_{eg}}{50\ \text{keV}}\right)^{-1/4}.$$

There may be two solutions conceivable:

(1) Go to a lower energy hard X-ray region with a smaller ρ and smaller ϵ_{eg}.

(2) Construct equilibrium solution among emitted photons and excited ions in the beam, which requires a new theoretical work to find its resolution.

We appreciate K. Yokoya for raising a question that led to the constraint given here.

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