

Analytic investigation of holographic phase transitions influenced by dark matter sector

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We analytically study the phase transitions between s -wave holographic insulator/superconductor and metal/superconductor. The problem is solved by the variational method for the Sturm-Liouville eigenvalue problem in the theory with the dark matter sector of a $U(1)$ -gauge field coupled to the Maxwell field. Additionally in the probe limit we investigate the marginally stable modes of scalar perturbations in the AdS solitonic background, connected with a magnetic field in the dark matter sector. We have found that even with the dark matter sector the superconducting transition temperature T_c is proportional to charge density ρ in power $1/3$. This value seems to be the strong coupling modification of the exponent $2/3$ known from the Bose-Einstein condensation of charged local pair bosons in narrow band superconductors. The holographic droplet solution is affected by the coupling to the dark matter. Interestingly in the probe limit the critical chemical potential increases with the decreasing coupling to dark matter, making it more difficult for the condensation transition to appear.

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I. INTRODUCTION

The gauge/gravity duality in the form of AdS/CFT correspondence [1,2] provides an interesting framework to study strong coupling effects in quantum many-body d -dimensional systems [3] by means of the $(d+1)$ -dimensional spacetime with the negative cosmological constant. In particular, this technique has been widely used to describe phase transitions from the normal or insulating to superconducting state. In the original work on “building the holographic superconductor” [4,5], the single band s -wave superconductor was proposed. The scalar complex field with the appropriate potential has been incorporated into the theory of gravity, and the condensation of its dual operator at finite temperature T , lower than the critical one T_c , was observed. The condensed operator has been identified with the superconducting order parameter. In the bulk, the temperature was introduced as the Hawking’s black hole one.

The aforementioned approach has been extended in many directions, taking into account the relevant aspects of the existing superconducting materials. For instance, the models of d -wave [6] superconductivity have been considered, to shed some light on the strong coupling behavior

of the well-known high temperature superconductors [7], which feature this symmetry of the order parameter. The spin triplet superconducting states of a simple p -wave [8], as well as the chiral $p_x \pm ip_y$ symmetry, has also been elaborated in considerable detail [9]. In Refs. [10], the multiband superconducting systems have been studied in view of many materials in which the coexistence of different orbitals plays a crucial role, e.g., in MgB_2 [11], Sr_2RuO_4 [12], or in heavy fermion superconductors [13]. The description of these materials requires at least two hybridized orbitals, which in the gravity approach translate into two scalar fields.

On the other hand, apart from the aforementioned studies of conductor/superconductor phase transitions, the holographic insulator/superconductor transitions attracted a great deal of attention. Modification of the bulk gravity theory by considering the five-dimensional AdS soliton line element [14] coupled to the Maxwell gauge field and the scalar one allows for building a model of the holographic insulator/superconductor phase transition at zero temperature [15]. In the gauge/gravity duality description, the AdS soliton is dual to a confined field theory with a mass gap, mimicking an insulator phase [16]. It was revealed that in the presence of a chemical potential in the solitonic background, the insulator/metal transition is of the second order. Namely, for the chemical potential greater than some critical value, the considered background turned out to be unstable and nontrivial hair emerged. This fact is interpreted as insulator/superconductor phase transitions.

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In Refs. [17,18] it was shown that the strength of various kinds of matter backreactions could generate new types of phase transitions. Marginally stable modes of scalar/vector perturbations in the AdS solitonic spacetime were studied in [19,20] to reveal the onset of the phase transition as well as to find the magnetic field effects on them. Among others, it was claimed that the magnetic field made it more difficult for the phase transition to occur. Compatibility with earlier investigations [21] was also announced.

Recently, the influence of nonlinear electrodynamics on the holographic insulator/superconductor phase transitions was taken into account [22,23], while the problem of p -wave symmetry of the transitions in question was treated in Refs. [24,25]. On the other hand, the analytical investigations tackling the phase transitions of this type, in Gauss-Bonnet gravity, were discussed in [26].

Moreover, superconducting solutions were examined in which the condensate was confined to a finite region and decayed rapidly outside during conductor/superconductor phase transitions. Both the vortex and the droplet models were constructed for s -wave type superconductors [27]. Non-Abelian droplet solutions emerging during insulator/superconductor phase transition in p -wave and $p + ip$ -wave symmetry were studied in Ref. [28]. It has been shown that in the case of Gauss-Bonnet background, the coupling constant of the theory (like in [26]) affects the transition in question.

Furthermore, the question of the possible matter configurations naturally appears in AdS spacetime. The problem of the strictly stationary Einstein-Maxwell spacetime with negative cosmological constant was treated in [29], while the simply connected Einstein-Maxwell axion-dilaton spacetime with negative cosmological constant and an arbitrary number of $U(1)$ -gauge fields was examined in Ref. [30]. It was revealed that the considered spacetimes could not allow for the existence of nontrivial configurations of complex scalar fields or form fields.

Motivated by the above problems and to provide continuity with our previous studies [31,32], we address here the problem of phase transitions between the insulator/superconductor and metal/superconductor for s -wave holographic superconductors in the theory in which the *dark matter* sector is coupled to the Standard Model. We shall look for the imprints of the dark matter sector in possible holographic experiments.

The importance of examinations of such models goes back to the needed explanation of the 511 keV gamma ray astrophysical observations made by Integral/SPI [33] as well as the experiments showing the electron positron excess in the Galaxy, revealed by ATIC/PAMELA [34,35]. Their energies vary from a few GeV to a few TeV, depending on the experiments. On the other hand, the new physics can explain the 3.6σ discrepancy between the measured value of the muon anomalous magnetic moment and its prediction in the Standard Model [36]. The other

facet concerns the fact that the dark matter model is subject to a key ingredient in the early Universe, where the topological phase transition, giving rise to various topological defects, might have happened.

Our analysis will be addressed to the theory in which, apart from the gravitational action given by

$$S_g = \int \sqrt{-g} d^5x \frac{1}{2\kappa^2} (R - 2\Lambda), \quad (1)$$

where $\kappa^2 = 8\pi G_5$ is the five-dimensional gravitational constant, $\Lambda = -6/L^2$ stands for the cosmological constant, and L is the radius of the AdS spacetime, we shall examine the Abelian-Higgs sector coupled to the second $U(1)$ -gauge field

$$S_m = \int \sqrt{-g} d^5x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - [\nabla_\mu \psi - iq A_\mu \psi]^\dagger \times [\nabla^\mu \psi - iq A^\mu \psi] - V(\psi) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{\alpha}{4} F_{\mu\nu} B^{\mu\nu} \right), \quad (2)$$

where the scalar field potential satisfies $V(\psi) = m^2 |\psi|^2 + \frac{\lambda}{4} |\psi|^4$. $F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]}$ stands for the ordinary Maxwell field strength tensor, while the second $U(1)$ -gauge field $B_{\mu\nu}$ is given by $B_{\mu\nu} = 2\nabla_{[\mu} B_{\nu]}$. Moreover, m , q represent a mass and a charge related to the scalar field ψ . Here α is a coupling constant between $U(1)$ fields. The compatibility with the current observations establishes its order as 10^{-3} .

Within the above model, the backreaction problems of the dark matter sector on an s -wave holographic superconductor were analyzed in Ref. [31]. It was revealed that the greater the dark matter coupling constant is, the smaller the critical temperature. The so-called retrograde condensation takes place for the negative value of the aforementioned constant. In Ref. [32] the nature of the condensate in the external magnetic field and the behavior of the critical field near the transition temperature were examined. The obtained upturn of the critical field constitutes the fingerprint of the strong coupling. In that study α has been found to be limited to positive values.

The organization of the paper is as follows. In Sec. II we start by studying the s -wave holographic zero temperature insulator/superconductor phase transition using the solitonic AdS background. The chemical potential is the control parameter of this phase transition. In Sec. III, the black hole background taken as the gravity configuration allows for the analysis of the transition observed for $\mu > \mu_c$ from holographic metal at high temperatures ($T > T_c$) to a holographic superconductor at temperatures below T_c . The effect of the dark matter sector on the insulator/superconductor transition of the droplet is studied in Sec. IV. We end the paper with summary and discussion of the obtained

results in light of the variety of the existing metal/insulator and metal/superconductor transitions in condensed matter systems. We limit our analysis of the phase transitions to the probe limit.

II. PHASE TRANSITION INSULATOR/SUPERCONDUCTOR

In this section we analyze the model of the s -wave holographic insulator/superconductor phase transition in five-dimensional spacetime, where the matter sector is coupled to another $U(1)$ -gauge field, representing the dark matter sector. In the probe limit, we set up the considered model in the AdS soliton background [14]. Its line element is subject to the relation

$$ds^2 = -r^2 dt^2 + L^2 \frac{dr^2}{f(r)} + f(r) d\varphi^2 + r^2 (dx^2 + dy^2), \quad (3)$$

where $f(r) = r^2 - r_0^4/r^2$. The geometry resembles a cigar, if one gets rid of the (r, φ) -coordinates, with a tip located at $r = r_0$. The AdS soliton solution is achieved by making two Wick rotations on a five-dimensional AdS Schwarzschild black hole line element. The asymptotic AdS spacetime tends to $R^{1,2} \times S^1$ topology near the boundary. A conical singularity at r_0 can be removed by the Scherk-Schwarz transformation of the φ -coordinate, i.e., $\varphi \sim \varphi + \pi L/r_0$. Due to the compactification of the φ -direction, the AdS solitonic background allows for a description of a three-dimensional field theory with a mass gap, which echoes an insulator in condensed matter physics. The temperature in the solitonic background is equal to zero.

Without loss of generality we put $L = 1$ and for simplicity we assume that $A_t = \phi(r)$, $B_t = \eta(r)$, and $\psi = \psi(r)$. The underlying system of differential equations for scalar and gauge fields yields

$$\partial_r^2 \psi + \left(\frac{\partial_r f}{f} + \frac{3}{r} \right) \partial_r \psi + \left(\frac{q^2 \phi^2}{r^2 f} - \frac{m^2}{f} \right) \psi = 0, \quad (4)$$

$$\partial_r^2 \phi + \left(\frac{\partial_r f}{f} + \frac{1}{r} \right) \partial_r \phi - \frac{2q^2 \psi^2 \phi}{\tilde{\alpha} f} = 0, \quad (5)$$

$$\partial_r \eta = \frac{c_1}{rf} - \frac{\alpha}{2} \partial_r \phi, \quad (6)$$

where we set $\tilde{\alpha} = 1 - \frac{\alpha^2}{4}$ and c_1 as an integration constant.

Next we impose the boundary conditions on the adequate quantities. Namely, at the tip of the AdS soliton we demand that the solutions will be provided by

$$\psi = \psi_0 + \psi_1(r - r_0) + \psi_2(r - r_0)^2 + \dots, \quad (7)$$

$$\phi = \phi_0 + \phi_1(r - r_0) + \phi_2(r - r_0)^2 + \dots, \quad (8)$$

where ψ_m and ϕ_m , for the range $m = 0, 1, 2, \dots$, are constants. Moreover, in order to achieve the finiteness of the considered quantities, one has to fulfill the Neumann-like boundary conditions ($\psi_1 = 0$ and $\phi_1 = 0$). Contrary to the AdS black hole case, where at the event horizon ϕ is equal to zero, here it can acquire a nonzero value at the tip of the AdS soliton. On the other hand, near $r \rightarrow \infty$, we have the following behaviors:

$$\psi = \frac{\psi^-}{r^{\lambda_-}} + \frac{\psi^+}{r^{\lambda_+}}, \quad \phi = \mu - \frac{\rho}{r^2}, \quad (9)$$

where μ and ρ stand for the chemical potential and charge density in the dual theory, while $\lambda_{\pm} = 2 \pm \sqrt{4 + m^2}$. The coefficients ψ^{\pm} are responsible for the vacuum expectation values of the operators $\langle \mathcal{O}_{\pm} \rangle$ dual to the scalar field. One can impose the condition that either ψ^- or ψ^+ vanish [37]. In what follows we shall assume that ψ^- vanishes and consider $\psi^+ = \langle \mathcal{O}_i \rangle$, with $\langle \mathcal{O}_i \rangle$ denoting the expectation value of the corresponding CFT operator.

It will be convenient to rewrite the above equations in terms of the $z = r_0/r$ variable. They reduce to the forms

$$\psi'' + \left(\frac{f'}{f} - \frac{1}{z} \right) \psi' + \left(\frac{q^2 \phi^2}{z^2 f} - \frac{m^2 r_0^2}{z^4 f} \right) \psi = 0, \quad (10)$$

$$\phi'' + \left(\frac{f'}{f} + \frac{1}{z} \right) \phi' - \frac{2q^2 \psi^2 \phi r_0^2}{\tilde{\alpha} f z^4} = 0, \quad (11)$$

$$\eta' = -\frac{c_1}{zf} - \frac{\alpha}{2} \phi', \quad (12)$$

where the prime denotes the derivation with respect to the z -coordinate.

A. Critical chemical potential

It was revealed in Ref. [15] that when the chemical potential exceeds a critical value, the condensation will set in. This state can be interpreted as a superconductor phase. In the case when $\mu < \mu_c$ the scalar field ψ achieves value close to zero and the phase can be interpreted as the insulator. The system has a mass gap, which is connected with the confinement in $(2 + 1)$ -dimensional gauge theory via performing the Scherk-Schwarz compactification. In light of these facts, the critical value of the chemical potential is the turning point in a superconductor phase transition.

For the chemical potential $\mu = \mu_c$, the scalar field is very small, $\psi \sim 0$, and Eq. (11) for the gauge field ϕ reduces to the form

$$\phi'' + \left(\frac{f'}{f} + \frac{1}{z}\right)\phi' \sim 0. \quad (13)$$

The general solution of this equation can be easily found to read

$$\phi(z) = d_1 + d_2 \log \frac{1 - z^2}{1 + z^2}, \quad (14)$$

where d_1 and d_2 are integration constants.

In order to fulfill the assumed boundary conditions (8) at the tip $z = 1$, we require $d_2 = 0$. Thus ϕ has the constant value μ , when $\psi(z) = 0$. Moreover, from Eq. (9), one obtains that in the considered case $\rho = 0$. These results are in accord with the numerical analysis presented in [15].

By virtue of the above, as $\mu \rightarrow \mu_c$, we have

$$\psi'' + \left(\frac{f'}{f} - \frac{1}{z}\right)\psi' + \left(\frac{q^2\mu^2}{z^2f} - \frac{m^2r_0^2}{z^4f}\right)\psi = 0. \quad (15)$$

By introducing a trial function [37] near the boundary $z = 0$, in the form $\psi(z) = \langle \mathcal{O}_i \rangle z^{\lambda_i} F(z)$, where $i = +$ or $-$, and by imposing the boundary conditions $F(0) = 1$ and $F'(0) = 0$, the underlying equation can be brought to the following form:

$$(p(z)F'(z))' - q(z)F(z) + \mu^2r(z)F(z) = 0, \quad (16)$$

where the various terms in the above relation are provided by

$$p(z) = z^{2\lambda_i - 1}f, \quad (17)$$

$$q(z) = -z^{2\lambda_i - 2} \left(\frac{\lambda_i(\lambda_i - 1)f}{z} + \left(\frac{f'}{f} - \frac{1}{z}\right)\lambda_i f - \frac{m^2r_0^2}{z^3} \right), \quad (18)$$

$$r(z) = q^2 z^{2\lambda_i - 3}. \quad (19)$$

According to the Sturm-Liouville eigenvalue problem, we can specify μ^2 as a spectral parameter and estimate its minimum eigenvalue by varying the following functional:

$$\mu^2 = \frac{\int_0^1 dz [F'(z)^2 p(z) + q(z)F^2(z)]}{\int_0^1 dz r(z)F^2(z)}. \quad (20)$$

The trial function will be set in the form $F(z) = 1 - az^2$. Importantly, the critical value of the chemical potential is unaffected by the dark matter sector parameters. The value of the μ^2 in Eq. (20) depends on the parameter a entering the trial function. Changing a we find the numerically minimal value of μ^2 for $a = a_{\min}$. Both a_{\min} and $\mu_c^2 = \mu^2(a_{\min})$ depend on the parameters of the model—in particular m^2 . In Table I the critical chemical potential has been presented for a few typical values of m^2 fulfilling

TABLE I. Values of the critical chemical potential together with a parameters minimizing the functional (20). In all the above examples we put $q = 1.0$ and $r_0 = 1$. The numerical values of m^2 and λ_+ are chosen for illustration purposes. They obey the physical requirement that the masses m^2 do fulfill the Breitenlohner-Freedman bound $m^2 > -d^2/4 = -16/4$ required for the stability of the AdS $_{d+1}$ spacetime.

λ_+	m^2	μ_c	a_{\min}
$\frac{5}{2}$	$-\frac{15}{4}$	1.890	0.330
$\frac{6}{2}$	$-\frac{12}{4}$	2.398	0.371
$\frac{7}{2}$	$-\frac{7}{4}$	2.903	0.407
$\frac{8}{2}$	0	3.406	0.439

the Breitenlohner-Freedman bound $m^2 > -d^2/4 = -16/4$ required for the stability of the AdS $_{d+1}$ spacetime. The same result was analytically obtained in s -wave holographic/superconductor phase transition studies in Einstein-Maxwell scalar theory [24], and is in accord with the numerical examinations provided in Ref. [15]. On the other hand, in Gauss-Bonnet gravity, for the transition in question, one observes the critical potential increase with the growth of curvature corrections, for the same mass of scalar field. For the fixed value of the strength of curvature corrections, with the increase of scalar field mass, the critical potential becomes larger [26].

B. Critical phenomena

In this subsection, we shall concentrate on studies of the critical exponent for the condensation operator as well as on the mutual relations between the charge density ρ and the chemical potential. The question we are asking here is how the order parameter of the superconductor, i.e., $\langle \mathcal{O}_i \rangle$, and the charge density ρ depend on the distance $(\mu - \mu_c)$. Having in mind the form of the scalar field [$\psi(z) = \langle \mathcal{O}_i \rangle z^{\lambda_i} F(z)$, where $F(z)$ is the trial function introduced earlier] near the boundary $z = 0$, when $\mu \rightarrow \mu_c$, the relation for gauge $A_t = \phi(r)$ field can be rewritten as

$$\phi'' + \left(\frac{f'}{f} + \frac{1}{z}\right)\phi' - \frac{2q^2r_0^2}{\tilde{\alpha}f} \langle \mathcal{O}_i \rangle^2 z^{2\lambda_i - 4} F^2(z)\phi = 0. \quad (21)$$

To proceed further, let us recall that for μ slightly above the critical value the condensation scalar operator $\langle \mathcal{O}_i \rangle$ is very small. This enables us to seek the solution in the form

$$\phi(z) \sim \mu_c + \langle \mathcal{O}_i \rangle \chi(z) + \dots \quad (22)$$

In order to recover the previous result $\phi(z) = \mu$, we have to impose boundary condition $\chi(1) = 0$. On the other hand, close to the boundary $z = 0$ one expands the function $\chi(z) = \chi(0) + \chi'(0)z + \frac{1}{2}\chi''(0)z^2 + \dots$ and rewrites the relation (9) as

$$\begin{aligned} \phi(z) \simeq \mu - \rho z^2 \simeq \mu_c + \langle \mathcal{O}_i \rangle \left(\chi(0) + \chi'(0)z \right. \\ \left. + \frac{1}{2} \chi''(0)z^2 + \dots \right). \end{aligned} \quad (23)$$

Comparing the coefficients of the z^0 - and z^1 -terms, in the above equation, one obtains the relations

$$\mu - \mu_c \simeq \langle \mathcal{O}_i \rangle \chi(0) \quad (24)$$

$$\chi'(0) = 0. \quad (25)$$

Inserting the relation (22) into (21), one can easily find that $\chi(z)$ will satisfy the following equation:

$$\begin{aligned} \chi''(z) + \left(\frac{f'}{f} + \frac{1}{z} \right) \chi'(z) - \frac{2q^2 r_0^2}{\tilde{\alpha} f} z^{2\lambda_i - 4} F^2(z) \mu_c \langle \mathcal{O}_i \rangle \\ - \frac{2q^2 r_0^2}{\tilde{\alpha} f} z^{2\lambda_i - 4} F^2(z) \langle \mathcal{O}_i \rangle^2 \chi(z) + \dots = 0. \end{aligned} \quad (26)$$

Close to μ_c the term quadratic in $\langle \mathcal{O}_i \rangle$ is much smaller than the linear one and may be safely neglected, leading to

$$\chi''(z) + \left(\frac{f'}{f} + \frac{1}{z} \right) \chi'(z) = \frac{2q^2 r_0^2}{\tilde{\alpha} f} z^{2\lambda_i - 4} F^2(z) \mu_c \langle \mathcal{O}_i \rangle. \quad (27)$$

In the next step, let us redefine the $\chi(z)$ function by the new one $\xi(z)$ multiplied by the adequate factor

$$\chi(z) = 2 \frac{\langle \mathcal{O}_i \rangle \mu_c}{\tilde{\alpha}} \xi(z). \quad (28)$$

It remains to be checked if the new definition of $\chi(z)$ enables us to get rid of the $\tilde{\alpha}$ in Eq. (27) and to extract the quantity $\langle \mathcal{O}_i \rangle$. It can be inspected that $\xi(z)$ will satisfy the following relation,

$$\xi'' + \left(\frac{f'}{f} + \frac{1}{z} \right) \xi' - \frac{q^2 r_0^2}{f} z^{2\lambda_i - 4} F^2(z) = 0, \quad (29)$$

and consequently the scalar operators $\langle \mathcal{O}_i \rangle$ imply

$$\langle \mathcal{O}_i \rangle = \sqrt{\frac{(\mu - \mu_c) \tilde{\alpha}}{2\mu_c \xi(0)}}. \quad (30)$$

In order to find $\xi(0)$ it will be helpful to rewrite Eq. (29) in the form

$$(fz\xi')' = q^2 r_0^2 z^{2\lambda_i - 3} F^2(z). \quad (31)$$

Then, having in mind the fact that $\xi'(1) = 0$ leads to the conclusion that $\xi(0)$ is provided by

$$\xi(0) = c_1 - \int_0^1 \frac{dz}{fz} \left(c_2 + \int_1^z dy q^2 r_0^2 y^{2\lambda_i - 3} F^2(y) \right). \quad (32)$$

The integration constants c_1, c_2 are determined by the boundary conditions imposed on the $\chi(z)$ -function. We relegate their determination to the Appendix.

On the other hand, the above relations reveal that the operators $\langle \mathcal{O}_i \rangle$ yield

$$\langle \mathcal{O}_i \rangle \simeq \Gamma(\mu - \mu_c)^{\frac{1}{2}}, \quad (33)$$

where the Γ factor contains information of the dependence on the dark matter sector. The bigger the $\tilde{\alpha}$ (the smaller value of the α -coupling constant we take) we consider, the greater factor we obtain.

Moreover, our analytical results show that the holographic s -wave insulator/superconductor phase transition represents the second order phase transition, with the critical exponent of the system attaining the mean-field value $1/2$. The same conclusions were achieved in the case of the ordinary s -wave holographic insulator/superconductor phase transition studies in Refs. [15,20]. On the other hand, the same form of the dependence was also obtained in Gauss-Bonnet theory [26], confirming the previous numerical results [38,39]. The Gauss-Bonnet coupling constant, connected with the influence of curvature corrections, does enter in the multiplier factor of the scalar operators, but the critical value of the exponent takes the mean-field value.

Next, we find the dependence of the charge density ρ on the critical chemical potential. In order to calculate ρ , we use (25), which implies that $\xi'(0) = 0$, together with the previous requirement $\xi(1) = 0$ being subject to the boundary condition. Comparison of the adequate coefficients of z^2 -order in Eq. (23) gives the relation for the charge density in the following form:

$$\rho = -\frac{\langle \mathcal{O}_i \rangle}{2} \chi''(0). \quad (34)$$

In order to find $\chi''(0)$ we rewrite Eq. (27) in the form which implies

$$(fz\chi')' - \frac{2q^2 r_0^2}{\tilde{\alpha}} z^{2\lambda_i - 3} F^2(z) \mu_c \langle \mathcal{O}_i \rangle = 0. \quad (35)$$

Integrating both sides of it and taking into account the aforementioned boundary conditions, one obtains

$$\chi''(0) = \frac{\chi'(z)}{z} \Big|_{z \rightarrow 0} = -2 \langle \mathcal{O}_i \rangle \frac{q^2 \mu_c}{\tilde{\alpha}} \int_0^1 dz z^{2\lambda_i - 3} F^2(z). \quad (36)$$

Then, by virtue of Eqs. (36) and (34), having in mind the relation (30), one arrives at

TABLE II. Values of the prefactors for the condensate and charge density. In all above examples $q = 1.0$ and the trial function was of the form $F = 1 - az^2$.

λ_+	m^2	μ_c	$\xi(0)$	$\frac{\Gamma}{\sqrt{\tilde{\alpha}}}$	\tilde{B}
$-\frac{15}{4}$	1.890	0.081	1.801	1.329	1.620
$-\frac{12}{4}$	2.398	0.062	1.823	1.144	1.520
$-\frac{7}{4}$	2.903	0.049	1.863	1.029	1.420
0	3.406	0.040	1.913	0.948	1.320

$$\rho = (\mu - \mu_c)\tilde{B}, \quad (37)$$

where the quantity \tilde{B} yields

$$\tilde{B} = \frac{q^2}{2\xi(0)} \int_0^1 dz z^{2\lambda_+ - 3} F^2(z). \quad (38)$$

In view of the above relations, the charge density is proportional to the difference $\Gamma_1(\mu - \mu_c)$, where the factor is independent on $\tilde{\alpha}$ characterizing the dark matter sector and accomplishes the ordinary dependence of the form $\rho \sim (\mu - \mu_c)$ achieved analytically in [20] and numerically in Refs. [17,25]. Some typical values of the \tilde{B} factor are presented in Table II. The dependence of the Γ factor on the dark matter sector coupling α is depicted in Fig. 1.

In the case of the Gauss-Bonnet gravity sector, the factor standing in front of $(\mu - \mu_c)$ is a function of mass and the Gauss-Bonnet coupling constant. However, the form of the linear dependence survives [26,38].

III. HOLOGRAPHIC METAL/SUPERCONDUCTOR PHASE TRANSITION

In this section we shall scrutinize the problem of the holographic s -wave metal/superconductor phase transition at low temperatures provided by the black hole background. The problem of the s -wave holographic

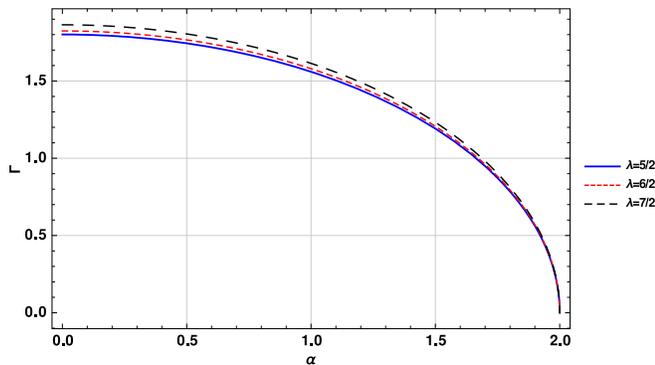


FIG. 1 (color online). The dependence of the prefactor of the condensate function on the coupling constant α . The trial function was chosen as $F = 1 - az^2$ and the charge was set equal to $q = 1$.

superconductor with the dark matter sector in the context of the backreaction of matter fields on the gravitational background was investigated in [31], where the critical temperature was found. On the other hand, for an n -dimensional gravitational background, the expected value of the scalar operator and the influence of the magnetic field on the holographic superconductor was analyzed [32]. For the completeness of the investigations, using quite different methods, we elaborate on the dependence of the critical temperature and the scalar operator on the presence of the dark matter sector.

To commence with, one considers the background of a five-dimensional black hole given by the line element

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + \frac{r^2}{L^2}(dx^2 + dy^2 + dz^2), \quad (39)$$

where $g(r) = r^2/L^2 - r_+^4/r^2L^2$. The Hawking temperature for the black hole has the form $T_{\text{BH}} = r_+/\pi$. In the z -coordinate, the equations of motion imply

$$\psi'' + \left(\frac{g'}{g} - \frac{1}{z}\right)\psi' + \left(\frac{q^2\phi^2}{g^2} - \frac{m^2}{g}\right)\frac{r_+^2}{z^4}\psi = 0, \quad (40)$$

$$\phi'' - \frac{1}{z}\phi' - \frac{2q^2\psi^2\phi r_+^2}{\tilde{\alpha}gz^4} = 0, \quad (41)$$

$$\eta' = -\frac{d_1 z}{r_+^2} - \frac{\alpha}{2}\phi', \quad (42)$$

where the prime denotes the derivation with respect to the z -coordinate. In the following, as in the preceding sections, we set $L = 1$. In order to solve the above equations we need to impose the adequate boundary conditions. At the black hole horizon $z = 1$, it is required that $\phi(1) = 0$ and $\psi(1)$ should be finite. The first requirement is needed for the $U(1)$ -gauge field to have the finite form; the second one exhibits that the black hole has a scalar hair on the event horizon.

When the temperature T tends to the critical value T_c from below, the condensation approaches zero, $\psi \rightarrow 0$. In this limit we write the equation for the ϕ field as

$$\phi'' - \frac{1}{z}\phi' \approx 0. \quad (43)$$

Its general solution is of the form $\phi(z) = c_1 + c_2 z^2$, which together with the aforementioned boundary condition at the horizon leads to $\phi(z) = c_1(1 - z^2)$. Then, having in mind that near the boundary of the bulk the fields behave as

$$\phi \rightarrow \mu - \frac{\rho}{r^2} = \mu - \rho z^2/r_+^2, \quad (44)$$

$$\psi \rightarrow \frac{\psi^-}{r_{\Delta_-}} + \frac{\psi^+}{r_{\Delta_+}}, \quad (45)$$

one arrives at the conclusion that near the critical temperature the A_t gauge field component will behave as $\phi \simeq \lambda r_+(1 - z^2)$, where we have denoted that $\lambda = \rho/r_+^3$. μ and ρ have the same interpretation as in the preceding sections; i.e., they denote the chemical potential and the charge density, respectively. Moreover, Δ_{\pm} have the same values as in Sec. II, but in order to distinguish the nature of the phase transition, we set the new notation for the old quantity. In what follows we concentrate on the $\Delta = \Delta_+$ and the masses m^2 fulfilling the Breitenlohner-Freedman spacetime stability condition. It is important to note that we are looking for the parameter λ which defines the value of both μ and ρ . It depends on the temperature given by $T = T_{\text{BH}} = r_+/\pi$.

On the other hand, the scalar field ψ can be cast near the boundary in the form

$$\psi|_{z \rightarrow 0} \simeq \langle \mathcal{C} \rangle \frac{z^{\Delta}}{r_+^{\Delta}} G(z), \quad (46)$$

where we set $G(0) = 1$, $G'(0) = 0$. Inserting this expression into Eq. (40) and using $\phi = \lambda r_+(1 - z^2)$, one finds the scalar field equation of motion, which can be easily rewritten as the Sturm-Liouville eigenvalue problem

$$(\mathcal{P}(z)G'(z))' - \mathcal{Q}(z)G(z) + \lambda^2 \mathcal{R}(z)G(z) = 0, \quad (47)$$

where one has defined the quantities

$$\mathcal{P} = z^{2\Delta-1}g, \quad (48)$$

$$\mathcal{Q} = -\Delta(\Delta-1)gz^{2\Delta-3} - \left(\frac{g}{g} - \frac{1}{z}\right)\Delta z^{2\Delta-2}g + m^2 r_+^2 z^{2\Delta-5}, \quad (49)$$

$$\mathcal{R} = \frac{z^{2\Delta-5}}{g} r_+^4 (1 - z^2)^2 q^2. \quad (50)$$

The minimum value of λ^2 can be found from variation of the functional given by

$$\lambda^2 = \frac{\int_0^1 dz [G'(z)^2 \mathcal{P}(z) + \mathcal{Q}(z)G^2(z)]}{\int_0^1 dz \mathcal{R}(z)G^2(z)}, \quad (51)$$

where the trial function is assumed to be given by $G(z) = 1 - az^2$. Note that in the variable z , the function $g(z) = r_+^2(1/z^2 - z^2)$ and the value of the event horizon radius r_+ factors out of the expression for λ^2 , making it independent on T_{BH} . Because of that and the fact that the analysis is valid close to the transition temperature $T_c \sim T_{\text{BH}}$, one readily finds that

$$T_c = \rho^{1/3} \left(\frac{1}{\pi^3 \lambda_{\text{min}}} \right)^{1/3}. \quad (52)$$

It is important to note that the value λ_{min} results from the variation of Sturm-Liouville functional (51). Holographic superconductors with different values of ρ are characterized by different transition temperatures. This recalls the charge carrier concentration dependence of the local pairing superconductors [40]. However, the charge density dependence of the critical temperature $T_c \propto \rho^{1/3}$ found here markedly differs from the known dependence $T_c \propto n^{2/3}$ for the Bose-Einstein condensation of the low density n superconductors with local pairs of charged hard-core bosons [40]. Whether the difference is the hallmark of the strong coupling behavior remains to be seen. In fact it has been suggested [5] that the condensation transition in holographic models is closer to the Bose-Einstein condensation than the Bardeen-Cooper-Schrieffer (BCS)-like symmetry breaking phase transition. It has to be remembered that the critical temperature does not depend on the carrier concentration n in the standard weak coupling superconductors described by the BCS theory. Instead it depends on the density of states at the Fermi level [41].

It can be observed that the critical temperature does not depend on the dark matter sector. This conclusion is in accord with our previous studies [31], where it was revealed that the backreaction effects introduce the dependence of T_c on the dark matter sector. In the case under consideration we restrict our investigations to the probe limit case; therefore no influence is spotted.

In Table III we have presented the results of the calculation of the superconducting transition temperature assuming the charge density $\rho = 1$. The presented theory is valid for arbitrary allowed values of m^2 , but for illustration we have chosen some exemplary values of it and calculated λ^2 and the critical temperature T_c . Both of these parameters are presented in Table III, together with the value of the parameter a , which minimizes the functional (51).

A. Condensation values

In this subsection our main task will be to find the influence of the dark matter sector field on the condensation

TABLE III. Calculated values of the prefactors of λ^2 minimizing the functional (51), the superconducting transition temperature of the superconductor with charge density $\rho = 1$ and the prefactor \tilde{C} defined in Eq. (60). In all the above examples we assumed $q = 1.0$ and selected the trial function as $G = 1 - az^2$. We also provide the parameter a minimizing the functional (51).

Δ	m^2	λ^2	a_{min}	$T_c (\rho = 1)$	\tilde{C}
$\frac{5}{2}$	$-\frac{15}{4}$	9.586	0.619	0.218	55.67
$\frac{6}{2}$	$-\frac{12}{4}$	18,22	0.721	0.196	137.8
$\frac{7}{2}$	$-\frac{7}{4}$	30.50	0.797	0.180	326.7
$\frac{8}{2}$	0	46.89	0.853	0.168	748.3

operator. Near the critical temperature, the equation for the gauge field can be rewritten as

$$\phi'' - \frac{\phi'}{z} = \frac{2q^2 r_+^2 \phi}{\tilde{\alpha} g} z^{2\Delta-4} G^2(z) \mathcal{A}, \quad (53)$$

where we have denoted $\mathcal{A} = \langle \mathcal{C} \rangle^2 / r_+^{2\Delta}$. Having in mind the fact that near the critical temperature the \mathcal{A} quantity is small, one expands ϕ near $z \rightarrow 0$. On account of this, we can write the following:

$$\frac{\phi}{r_+} = \lambda(1 - z^2) + \mathcal{A}\chi(z) + \dots \quad (54)$$

In the next step, comparing the coefficients in z^2 -order terms, we reveal that

$$\frac{\rho}{r_+^3} = \lambda - \frac{\mathcal{A}}{2} \chi''(0). \quad (55)$$

On the other hand, considering the relation (54) and the equation of motion for the ϕ field, we get

$$\chi'' - \frac{\chi'}{z} = \frac{2\lambda q^2 r_+^2}{\tilde{\alpha} f} (1 - z^2) z^{2\Delta-4} G^2(z). \quad (56)$$

Consequently, proceeding as in the last section, it can be verified that the following is satisfied:

$$\chi''(0) = \left. \frac{\chi'(z)}{z} \right|_{z=0} = -2\lambda \int_0^1 dz \frac{q^2 r_+^2}{\tilde{\alpha} f} (1 - z^2) z^{2\Delta-5} G^2(z). \quad (57)$$

On evaluating the expression for \mathcal{A} , in the case when $T \rightarrow T_c$, the condensation operator in question is provided by

$$\langle \mathcal{C} \rangle = \sqrt{\frac{2\tilde{\alpha}}{\mathcal{B}}} (\pi T_c)^\Delta \sqrt{1 - \frac{T}{T_c}} = \sqrt{\tilde{\alpha}} \langle \mathcal{C} \rangle_{\text{no dark sect}}, \quad (58)$$

where by $\langle \mathcal{C} \rangle_{\text{no dark sect}}$ we denoted the value in the theory without the dark matter sector. The term \mathcal{B} is given by the relation

$$\mathcal{B} = 2 \int_0^1 dz \frac{q^2 r_+^2}{g} (1 - z^2) z^{2\Delta-5} G^2(z). \quad (59)$$

One can see that the condensation operator depends on the α constant coupling of the *dark matter* sector. The greater the α -coupling is, the easier it is for condensation to form. From the point of view of the AdS/CFT correspondence, the operator in question can be interpreted as the operator for the pairing mechanism. The greater the expectation value it achieves, the more difficult it is for condensation to

occur. In order to better understand the dependence of the condensate on the dark matter sector, we factor out its dependence on the rest of the parameters in the following way:

$$\langle \mathcal{C} \rangle_{\text{norm}} = \frac{\langle \mathcal{C} \rangle}{T_c^\Delta} \equiv \sqrt{\tilde{\alpha}} \tilde{C} \sqrt{1 - \frac{T}{T_c}}, \quad (60)$$

where $\tilde{C} \equiv \sqrt{\frac{2}{\mathcal{B}}} \pi^\Delta$ and $\langle \mathcal{C} \rangle_{\text{norm}}$ represent renormalized values of the condensate. The factor \tilde{C} depends on all the remaining parameters (except temperature) and its typical values are presented in Table III. These facts can potentially constitute the way of determining the dark matter sector in future “possible” superconductor experiments.

We remark that, in the case of Gauss-Bonnet theory [42], the value of $\langle \mathcal{C} \rangle$ is dependent on the higher curvature term corrections. When it grows, the value of the operator also increases. This conclusion is in agreement with the previous studies (see, e.g., [38] and references therein).

IV. HOLOGRAPHIC DROPLET IN S-WAVE INSULATOR/SUPERCONDUCTOR PHASE TRANSITION

The term superconducting droplet refers to the solutions that are confined in space and rapidly decay at large distances. This happens if the studied superconductor is exposed to a strong external magnetic field. The size of the confining region diminishes with the increase of the magnetic field. In this section we shall investigate the onset of the transition by studying the marginally stable modes of scalar perturbations. It has been shown earlier [19,20,43] that in the holographic approach the marginally stable modes signal the appearance of the insulator-superconductor transition.

Here we are interested in the insulator-superconductor phase transition in the AdS solitonic background coupled to the dark matter sector. The presence of the magnetic field introduced via dark matter potential enables us to examine the droplet solution via the aforementioned technique.

It turned out that the quasinormal modes (QNMs) technique was established as a method of examining stability of a spacetime background [44]. In the case when the imaginary part of the QNMs is negative, the modes decrease in time and result in the disappearance of perturbations (the background is stable against perturbations). On the other hand, when the imaginary part is positive, the background is unstable against the perturbations in question. The marginally stable modes are the modes in which frequencies go to zero ($\omega = 0$) near the critical point of the phase transition. Their appearance thus signals the phase transitions [21,45].

To commence, we consider an AdS soliton metric, taking into account the symmetry of the problem in question, and

rewrite the line element used in Sec. II in the coordinates $(t, r, \tilde{\rho}, \varphi, \theta)$ as

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{f(r)} + f(r) d\varphi^2 + r^2 (d\tilde{\rho}^2 + \tilde{\rho}^2 d\theta^2). \quad (61)$$

Further, we assume the existence, in addition to a constant chemical potential bounded with an A_t gauge field component, of the B_θ potential which corresponds to the dark matter gauge field and is proportional to the constant value of magnetic field B :

$$A_t = \mu, \quad B_\theta = \frac{1}{2} B \tilde{\rho}^2. \quad (62)$$

The above ansatz stems from the fact that one considers the gauge sector close to the critical point of the phase transition, i.e., $\mu \sim \mu_c$ and $\psi \sim 0$, as well as the implementation of the polar coordinates, in order to envisage the symmetry of the problem in question [20].

Having in mind the exact form of B_θ , given by the above relation, we calculate from the equation of motion,

$$\nabla_\mu B^{\mu\nu} + \frac{\alpha}{2} \nabla_\mu F^{\mu\nu} = 0, \quad (63)$$

the A_θ component, which yields

$$A_\theta = \frac{D_1 r \tilde{\rho}^2}{\alpha} - \frac{B}{\alpha} \tilde{\rho}^2 + D_2. \quad (64)$$

D_1 and D_2 are integration constants. To proceed further, we assume A_θ to be the function of $\tilde{\rho}$ only, which implies that D_1 and D_2 have to be equal to zero. Our studies are devoted to the probe limit; i.e., the $U(1)$ -gauge fields and the scalar one do not backreact on the AdS soliton background metric. Without the condensate (i.e., for $\psi = 0$) the solution of the equations of motion for the A gauge field components are given by $A_\theta = -\frac{B}{\alpha} \tilde{\rho}^2$ and $A_t = \mu$. We are interested in finding the solution of the ψ equation close to the critical chemical potential $\mu \sim \mu_c$, where the value of the scalar field reaches nearly zero. It implies that one can treat the ψ field as a probe into the background consisting of an AdS Schwarzschild soliton with constant electric and magnetic field. The equation describing the ψ field yields

$$\nabla_\mu \nabla^\mu \psi - q^2 A_\mu A^\mu \psi - m^2 \psi = 0. \quad (65)$$

By virtue of the above, the explicit form of the equation for the ψ field may be written as

$$\begin{aligned} \partial_r^2 \psi + \left(\frac{\partial_r f}{f} + \frac{3}{r} \right) \partial_r \psi + \frac{1}{f^2} \partial_\varphi^2 \psi - \frac{1}{r^2 f} \partial_t^2 \psi + \frac{1}{r^2 f \tilde{\rho}} \partial_{\tilde{\rho}} (\tilde{\rho} \partial_{\tilde{\rho}} \psi) \\ + \frac{1}{r^2 f} \left(q^2 \mu^2 - m^2 r^2 - \frac{q^2 B^2 \tilde{\rho}^2}{\alpha^2} \right) \psi = 0. \end{aligned} \quad (66)$$

In order to solve the above equation we choose an ansatz for the ψ field:

$$\psi = F(r, t) H(\varphi) U(\tilde{\rho}). \quad (67)$$

This form enables us to separate variables. After a simple algebra we arrive at the following set of equations:

$$\begin{aligned} \partial_r^2 F + \left(\frac{3}{r} + \frac{\partial_r f}{f} \right) \partial_r F - \frac{1}{r^2 f} \partial_t^2 F \\ + \frac{1}{r^2 f} \left(q^2 \mu^2 - m^2 r^2 - \frac{\lambda^2 r^2}{f} - k^2 \right) F = 0, \end{aligned} \quad (68)$$

$$\frac{\partial_\varphi^2 H}{H} = -\lambda^2, \quad (69)$$

$$\frac{1}{\tilde{\rho}} \partial_{\tilde{\rho}} (\tilde{\rho} \partial_{\tilde{\rho}} U) - \frac{q^2 B^2 \tilde{\rho}^2}{\alpha^2} U = -k^2 U, \quad (70)$$

From periodicity property $H(\varphi) = H(\varphi + \pi L/r_0)$ of $H(\varphi)$ we identify that $\lambda = 2r_0 n/L$, where $n \in \mathbb{Z}$. In what follows without loss of the generality we set $r_0 = 1$ and $L = 1$, which leads to $\lambda = 2n$. We expect that the lowest mode will be first to condense and result in the most stable solution.

The equation for $U(\tilde{\rho})$ is a two-dimensional harmonic oscillator one. In order to solve it we recall that the function $U(\tilde{\rho})$ should satisfy the boundary conditions $U(\tilde{\rho} \rightarrow \infty) = 0$. It is possible to investigate this kind of differential equation by the Frobenius method. The customary procedure is first to factor out the behavior of the relevant solutions at infinity by setting

$$U(\tilde{\rho}) = e^{-\Lambda \tilde{\rho}^2/2} D(\tilde{\rho}), \quad (71)$$

which results in a Hermite type of equation. Inserting (71) into the underlying equation, we obtain the expected type of the differential solution if the condition

$$\Lambda^2 = \frac{q^2 B^2}{\alpha^2} \quad (72)$$

is satisfied. Then, the resulting equation yields

$$\frac{1}{\tilde{\rho}} \partial_{\tilde{\rho}} (\tilde{\rho} \partial_{\tilde{\rho}} D) - 2\Lambda \tilde{\rho} \partial_{\tilde{\rho}} D + (k^2 - 2\Lambda) D = 0. \quad (73)$$

In order to find the exact form of $D(\tilde{\rho})$, one sets $D(\tilde{\rho}) = \sum_{k=0}^{\infty} a_k \tilde{\rho}^{k+1}$ (see, e.g., [46]).

Note that $D(\tilde{\rho}) = \text{const}$ is the well-known lowest energy solution of the harmonic oscillator. It leads to the condition

$$k^2 = 2\Lambda = \left| \frac{2qB}{\alpha} \right|. \quad (74)$$

Equation (71) shows that in the presence of a magnetic field the superconducting region is confined in space. For the chosen solution $D(\tilde{\rho}) = \text{const}$ it forms a droplet [47] of radius

$$\langle \tilde{\rho} \rangle = \frac{\int d\tilde{\rho} \tilde{\rho} U(\tilde{\rho})}{\int d\tilde{\rho} U(\tilde{\rho})} = \frac{1}{\sqrt{2\pi\Lambda}} = \sqrt{\frac{\alpha}{4\pi qB}}. \quad (75)$$

In the field theory it corresponds to the condensation in the lowest Landau level [48]. As far as the time dependence of the $F(r, t)$ is concerned, we substitute it in the form $F(r, t) = e^{-i\omega t} R(r)$. The requirement concerning marginally stable modes leads to the condition $\omega = 0$. Redefining the coordinates as $z = r_+/r$ enables us to arrive at the equation given by

$$\partial_z^2 R(z) + \left(\frac{\partial_z f}{f} - \frac{1}{z} \right) \partial_z R(z) + \frac{1}{z^2 f} \left(q^2 \mu_c^2 - \frac{2Bq}{\alpha} - \frac{m^2}{z^2} - \frac{4n^2}{z^2 f} \right) R(z) = 0. \quad (76)$$

To solve it close to μ_c when the field $\psi \approx 0$ we introduce a correction function $\Theta(z)$ in the form

$$R(z)|_{z \rightarrow 0} \sim \langle \mathcal{O}_i \rangle z^{\lambda_i} \Theta(z), \quad (77)$$

with the boundary conditions $\Theta(0) = 1$ and $\Theta'(0) = 0$. After some algebra, the resulting equation can be converted into the standard Sturm-Liouville eigenvalue equation, which can be rewritten as

$$\partial_z(a(z)\Theta') - b(z)\Theta + \delta^2 c(z)\Theta = 0, \quad (78)$$

where $\delta^2 = q^2 \mu_c^2 - 2qB/\alpha$ and the remaining quantities are defined by the relations

$$a(z) = f z^{2\lambda_i - 1}, \quad (79)$$

$$b(z) = -f z^{2\lambda_i - 1} \left(\frac{\lambda_i(\lambda_i - 1)}{z^2} + \left(\frac{\partial_z f}{f} - \frac{1}{z} \right) \frac{\lambda_i}{z} - \frac{1}{z^4 f} \left(m^2 + \frac{4n^2}{f} \right) \right), \quad (80)$$

$$c(z) = z^{2\lambda_i - 3}. \quad (81)$$

The eigenvalues of δ^2 can be found by the method of minimizing the functional

$$\delta^2 = q^2 \mu_c^2 - \frac{2Bq}{\alpha} = \frac{\int_0^1 dz (\Theta'(z))^2 a(z) + b(z)\Theta(z)^2}{\int_0^1 dz (c(z)\Theta^2(z))}. \quad (82)$$

In order to estimate δ^2 , we choose function $\Theta(z) = 1 - az^2$. Minimization of the functional provides an estimation of the value of δ which depends on m^2 and n , resulting from the periodicity property of $H(\varphi)$. The above relation between δ^2 and the critical chemical potential μ_c can be rewritten as

$$\mu_c = \frac{\sqrt{\delta^2 + \frac{2qB}{\alpha}}}{q}. \quad (83)$$

It follows that μ_c depends on the coupling to the dark matter sector α . Interestingly, for a constant magnetic field B the critical chemical potential diverges for $\alpha \rightarrow 0$. It means that at the constant magnetic field, it is more difficult for the condensation to occur for smaller values of α . However, one has to remember that the zero value of α is not allowed, as simultaneously one has to take $B = 0$, so without the dark matter field the standard relation, valid for zero magnetic field [28] $\delta^2 = q^2 \mu_c^2$, is recovered. The increase of the magnetic field causes the increase of μ_c , which in turn results in the more difficult condensation. The aforementioned behavior is depicted in Fig. 2 for two values of the magnetic field. The discussed increase of μ_c for small α is clearly visible both for $B = 0.1$ and for $B = 1$.

V. DISCUSSION AND CONCLUSION

The main aim of our paper is to find the quantitative or at least qualitative imprints of the dark matter sector on the properties of s -wave holographic superconductor phase transitions. The nonordinary features might constitute possible hints for future experiments testing the considered model of dark matter. In the model in question, apart from the electromagnetic matter field we have taken into account the dark matter sector described by another $U(1)$ -gauge field, bounded with the Maxwell field by the coupling constant α .

The models where dark matter is a part of a larger sector that interacts with visible matter were successfully implemented as possible explanations of various astrophysical anomalous observations like the excess of electrons in the Galaxy having energies of a few GeV and TeV, or gamma rays of 511 keV [49]. There have also been efforts to find new physics explaining the anomalous muon magnetic moment, possible implications for parity violation, rare meson decays [50], and some implications of boson and dark boson mixing for high energy experiments [51]. This problem is of a great importance, especially in light of the latest claim of nongravitational interactions of dark matter in colliding galaxy clusters [52], which can disfavor some extensions of the Standard Model.

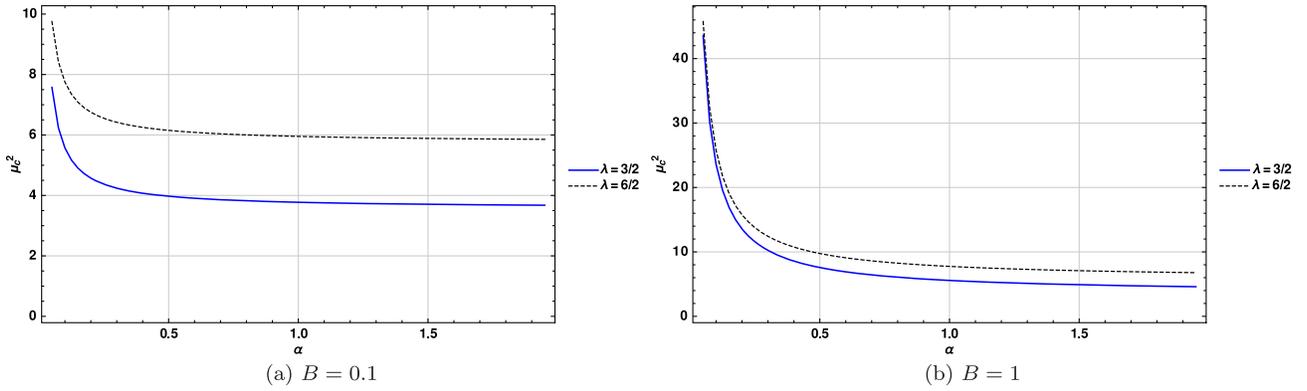


FIG. 2 (color online). The dependence of the critical chemical potential on α in the droplet case, for the fixed value of the magnetic field. We set $B = 0.1$ (for the left panel) and $B = 1$ (for the right panel), the rest of the parameters are equal to $q = 1.0$, $n = 0$, $m^2 = -\frac{15}{4}$ ($\lambda = \frac{5}{2}$), $m^2 = -3$ ($\lambda = 3$).

In the paper we have discussed analytically various phase transitions toward the s -wave holographic superconductor in the probe limit. Our results for $\alpha = 0$ agree with the previous numerical and analytical studies [20,38] of holographic superconductor transitions. The coupling between ordinary and dark matter changes the values of the parameters at the transition, however, only quantitatively for s -wave superconductors.

To make contact between transitions studied here and those known in the condensed matter systems, let us recall some basic facts from the latter field of research. In the condensed matter systems there exist a number of metal/insulator transitions. They differ by the role played by the interactions between carriers and the lack of periodicity of the underlying crystal lattice. The metal/insulator transition may appear when the carriers strongly interact with each other. On physical grounds it can be argued that in such a strongly correlated system the electron movement is hindered by the repulsive interactions and as a result, the insulator may form. The literature on the Mott-Hubbard metal/insulator transition [53,54] is vast and the transition itself is still not fully understood [55]. The other interesting transition appears in the system that is not translationally invariant. The transition is driven by disorder. Strong disorder makes some states localized and thus unable to carry the electric current. This is called Anderson metal/insulator transition [56]. In real materials one usually finds transitions in which both interactions and disorder do play an essential role [57].

Our study is restricted to the system that is periodic on the boundary, and the insulator/metal transition we study should be related to the Mott-Hubbard one. The sequels to the transitions we are discussing, i.e., insulator/superconductor at zero temperature and metal/superconductor at higher temperatures, are realized in high temperature cuprate and iron superconductors with an increase of the carrier doping. The $T = 0$ insulator/superconductor transition is an analog of the Hawking-Page-like soliton-black

hole transition. In real superconductors, an increase of charge density beyond the upper limit induces (at low temperatures) a reverse superconductor-to-metal transition, which seemingly has not been hitherto found in holographic analogy.

Treating μ^2 as a spectral parameter we analyzed the behavior of it in the s -wave insulator/superconductor phase transition. We did not observe the influence of the dark matter sector on this quantity. The charge density is proportional to the difference $(\mu - \mu_c)$ and also does not depend on the dark matter sector in the probe limit. In the case of the scalar operator, it was revealed that it is proportional to $(\mu - \mu_c)^{1/2}$ and represents the second order phase transition. The critical exponent of the considered system has the mean-field value, while the proportionality factor is subject to dark matter coupling constant dependence. The smaller the value of α one considers, the greater the factor one obtains; i.e., the more difficult it is for the condensation to happen (it will also be the case in the s -wave holographic metal/superconductor phase transition).

The same conclusions were drawn studying the s -wave type of transition in question in Gauss-Bonnet theory. The Gauss-Bonnet coupling which envisages the influence of higher curvature corrections does influence the considered factor, and the form of linear dependence survives.

In the case of the holographic metal/superconductor phase transition, the critical temperature does not depend on the dark matter sector. The conclusion is in accord with our previous studies [31], where it was shown that the backreaction effects introduced the dependence of T_c on the dark matter sector. On the contrary, the condensation operator reveals a linear dependence on the α -coupling constant. The greater the dark matter coupling constant one considers, the easier it is for condensation to form.

By examining the s -wave droplet insulator/superconductor phase transition, it was found that the chemical potential and magnetic field were bounded with the linear dependence. Not only does the magnetic field

influence the condensation but also the coupling constant of the dark matter sector does this. The increase of the magnetic field causes the increase of μ_c , which eventuates in the harder condensation. For the fixed value of the magnetic field, it happens that the smaller the value of the dark matter sector coupling constant one chooses, the harder it is for condensation to take place. A very similar behavior was envisaged in the holographic droplet in the p -wave insulator/superconductor phase transition case.

To conclude, we remark that there are some points which are in contrast to the ordinary behavior (without the dark matter sector) during the aforementioned phase transitions, which may constitute indicators for the future experiments for detecting dark matter and elucidating its nature. Testing s -wave holographic superconductors with the dark matter sector is only the tip of the iceberg, and some more complicated models like p -wave or $p_x + ip_y$ should be taken into account. We hope to investigate these problems elsewhere. Our preliminary calculations show stronger modifications of the p -wave superconductor characteristics by the dark matter coupling α .

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APPENDIX: INTEGRATION CONSTANTS IN EQ. (32)

Equation (32) provides formal solution of the differential equation (29) and contains two integration constants. The value of $\xi(0)$ enters the prefactors Γ and \tilde{B} of the dependence of the condensation operator $\langle O_i \rangle$ and the density ρ on the chemical potential.

To find numerical values of the coefficients Γ and \tilde{B} we note that the integrals entering Eq. (32) still contain terms that are singular at $z = 1$ and that have to be eliminated by the proper choice of constants. To this end we take $F(y) = 1 - ay^2$ and for numerical evaluation of the constants use the values of $a = a_{\min}$ that minimize the functional (20) for μ^2 . Evaluating the (nonsingular) integral over y in Eq. (32) leads to

$$\begin{aligned} q^2 r_0^2 \int_1^z dy y^{2\lambda_i-3} F^2(y) &= q^2 r_0^2 \left[y^{2\lambda_i} \left(-\frac{a}{\lambda_i} + \frac{1}{2(\lambda_j-1)y^2} + \frac{a^2 y^2}{2(\lambda_i+1)} \right) \right]_1^z \\ &= q^2 r_0^2 \left(\left[z^{2\lambda_i} \left(-\frac{a}{\lambda_i} + \frac{1}{2(\lambda_j-1)z^2} + \frac{a^2 z^2}{2(\lambda_i+1)} \right) \right] - \left[\left(-\frac{a}{\lambda_i} + \frac{1}{2(\lambda_j-1)} + \frac{a^2}{2(\lambda_i+1)} \right) \right] \right). \end{aligned} \quad (\text{A1})$$

Inserting the above result denoted as $R(z)$ into (32) and performing the integral over z one gets

$$\xi(0) = c_1 - \int_0^1 dz \frac{z}{1-z^4} [c_2 + R(z)] = c_1 - \left[\frac{c_2}{4} (\log(1+z^2) - \log(1-z^2)) + q^2 r_0^2 W(z) \right]_0^1, \quad (\text{A2})$$

where we have denoted

$$\begin{aligned} W(z) &= \left[2\lambda_i(1+a^2(\lambda_i-1) + \lambda_i) z^{2\lambda_i+4} {}_2F_1 \left(1, 1 + \frac{\lambda_i}{2}; 2 + \frac{\lambda_i}{2}, z^4 \right) - (2 + \lambda_i)(-2z^{2\lambda_i}(\lambda_i + 1) \right. \\ &\quad + 4a(\lambda_i - 1)z^{2\lambda_i+2} {}_2F_1 \left(1, \frac{1 + \lambda_i}{2}; \frac{3 + \lambda_i}{2}, z^4 \right) + (a^2\lambda_i(\lambda_i - 1) - \lambda_i(\lambda_i + 1) - 2a(\lambda_i^2 - 1)) \log(1 - z^2) \\ &\quad \left. + (-2a + \lambda_i + a^2\lambda_i + \lambda_i^2 + 2a\lambda_i^2 - a^2\lambda_i^2) \log(1 + z^2) \right] / [8(\lambda_i - 1)\lambda_i(\lambda_i + 1)(\lambda_i + 2)]. \end{aligned} \quad (\text{A3})$$

Here ${}_2F_1(a, b; c, z)$ is the hypergeometric function [58]. For the special values of parameters with $c = a + b$, as in the above expression, it diverges for $z \rightarrow 1$ and takes the form [58]

$$\lim_{z \rightarrow 1} {}_2F_1(a, b; a + b, z) = -\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \log(1-z). \quad (\text{A4})$$

It is easy to check that $W(0) = 0$. On the other hand the boundary condition $\xi(1) = 0$ requires

$$c_2 = -\frac{q^2 r_0^2}{\lambda_i - 1}, \quad (\text{A5})$$

and

$$c_1 = -q^2 r_0^2 \left[\frac{\log 2}{4(\lambda_i - 1)} + \frac{1}{4\lambda_i(\lambda_i - 1)} + \left(\frac{a}{4\lambda_i(\lambda_i + 2)} - \frac{a^2}{4(\lambda_i + 1)(\lambda_i + 2)} - \frac{1}{8(\lambda_i - 1)(\lambda_i + 2)} \right) \log 2 \right]. \quad (\text{A6})$$

Introducing the obtained results into Eq. (25), one gets the required formula

$$\langle \mathcal{O}_i \rangle = \Gamma \sqrt{\mu - \mu_c}, \quad (\text{A7})$$

where the prefactor $\Gamma = \sqrt{\frac{\tilde{\alpha}}{\mu_c q^2}} \sqrt{\frac{4(\lambda_i - 1)}{(a-1)^2 \log 2}}$. On the other hand, the charge density yields

$$\rho = \frac{4(\lambda_i - 1) \int_0^1 y^{2\lambda_i - 3} F^2(y) dy}{(a-1)^2 \log 2} (\mu - \mu_c). \quad (\text{A8})$$

Another way to find the required value $\xi(0)$ is by direct solution of Eq. (29). For a general value of the parameter λ_i , it is given in terms of the hypergeometric ${}_2F_1(a, b; c, z)$ functions [58] as

$$\begin{aligned} \xi(z) = C_2 + \frac{q^2 r_0^2}{4(\lambda_i + 2)(\lambda_i + 1)\lambda_i(\lambda_i - 1)} & \left[\lambda_i (a^2(\lambda_i - 1) + \lambda_i + 1) z^{2\lambda_i + 4} {}_2F_1 \left(1, \frac{\lambda_i}{2} + 1; \frac{\lambda_i}{2} + 2, z^4 \right) \right. \\ & \left. - (\lambda_i + 2) \left(2a(\lambda_i - 1) z^{2\lambda_i + 2} {}_2F_1 \left(1, \frac{\lambda_i + 1}{2}; \frac{\lambda_i + 3}{2}, z^4 \right) \right) + (\lambda_i + 1) \left(-z^{2\lambda_i} + \lambda_i(\lambda_i - 1) C_1 \log \frac{1 - z^2}{1 + z^2} \right) \right]. \quad (\text{A9}) \end{aligned}$$

Remembering that [58] $\lim_{z \rightarrow 1} {}_2F_1(1, \frac{\lambda_i}{2} + 1; \frac{\lambda_i}{2} + 2, z^4) = -(1 + \frac{\lambda_i}{2}) \log(1 - z^2)$ and $\lim_{z \rightarrow 0} {}_2F_1(1, \frac{\lambda_i}{2} + 1; \frac{\lambda_i}{2} + 2, z^4) = 1$ one chooses the constants C_1 and C_2 in such a way that the solution is finite with $\chi(1) = 0$ and finds $\xi(0) = C_2$ with C_2 given by the formula (A6) above.

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