String bits at finite temperature and the Hagedorn phase

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(Received 21 July 2015; published 22 September 2015)

We study the behavior of a simple string bit model at finite temperature. We use thermal perturbation theory to analyze the high temperature regime. But at low temperatures we rely on the large N limit of the dynamics, for which the exact energy spectrum is known. Since the lowest energy states at infinite N are free closed strings, the $N = \infty$ partition function diverges above a finite temperature β_H^{-1}, the Hagedorn temperature. We argue that in these models at finite N, which then have a finite number of degrees of freedom, there can be neither an ultimate temperature nor any kind of phase transition. We discuss how the discontinuous behavior seen at infinite N can be removed at finite N. In this resolution the fundamental string bit degrees of freedom become more active at temperatures near and above the Hagedorn temperature.

DOI: 10.1103/PhysRevD.92.066007 PACS numbers: 11.25.-w, 12.60.Jv, 11.10.Wx

I. INTRODUCTION

Half a century ago Hagedorn proposed an experimentally successful statistical model of strongly interacting particles in which the density of states grows exponentially with energy $d(E) \sim E^\alpha e^{\beta_H E}$, with the thermodynamic consequence that β_H^{-1} is the ultimate temperature which cannot be exceeded by hadronic matter in thermal equilibrium [1]. The discovery that dual resonance models (also known as string theory) predicted an energy level degeneracy with just this exponential behavior at zero coupling [2] provided unanticipated early support for string theory as a model of strong interactions. Hagedorn's thermal interpretation of an exponential level density has also been exploited to apply string theory to early universe cosmology, first to describe the role of the strong interactions in the hot early universe [3]. Later on, after string theory was promoted from a faulty model of strong interactions to a promising vehicle for unifying quantum gravity with the rest of physics, the Hagedorn model formed the basis for string gas cosmology [4], which provides an alternative to the inflationary universe. Apparently string gas cosmology is still viable after all these years [5].

Thinking about the thermal properties of a system often leads to theoretical insight into puzzling aspects of the system. For example, Atick and Witten, motivated in part by parallels with the temperature dependence of large N QCD [6], interpreted the ultimate temperature of the free string as an artifact of the zero coupling limit. They suggested that, at finite coupling, there should be a phase transition near the Hagedorn temperature to a new phase dominated by the fundamental degrees of freedom underlying string theory [7]. They argued that, much as in QCD where there are many fewer quarks and gluons than mesons and baryons, the true degrees of freedom of string theory are probably much reduced compared to expectations from string field theory.

It has been proposed that string should be regarded as a composite system of fundamental entities [8,9] called "string bits" [10]. It is then of interest to study string bit models at finite temperature and to explore how the string bit degrees of freedom can be exposed at high temperature. In string bit dynamics a string bit is a discrete bit of light cone parametrized [11] string. Then the total $P^+ = (P^0 + P^1)/\sqrt{2}$ of a string is discretized as $\mathcal{M}m$ where $\mathcal{M}$ is the bit number operator. The string itself is simply a long chain of string bits whose nearest neighbor dynamics is implemented by introducing $N \times N$ "color" matrix string bit creation operators, imposing a $U(N)$ color symmetry. Then we identify string perturbation theory as the 't Hooft $1/N$ expansion [12] of string bit dynamics. In early versions of this dynamics [9] the creation operators were fields depending on transverse coordinates $\boldsymbol{x}$, as well as spinor indices in the case of superstring bits [13]. However, each transverse coordinate can effectively emerge from a simple two valued internal flavor degree of freedom [14], so spaceless string bit models (in zero space dimensions) can underlie string theory [15,16] in any space dimension less than or equal to the critical one.

Since the Hagedorn phenomenon is common to all string models, even subcritical ones, we choose to analyze this phenomenon in the simplest stable superstring bit model studied in [15]. Its large N limit describes a noncovariant subcritical light cone string with no transverse coordinates and one Grassmann world sheet field: the string moves in one-dimensional space. The string bit degrees of freedom are specified by bosonic a_α^β, $\bar{a}_\alpha^\beta = (a_\beta^\alpha)^\dagger$ and fermionic b_α^β, $\bar{b}_\alpha^\beta = (b_\beta^\alpha)^\dagger$ $N \times N$ matrix operators satisfying (anti)commutation relations

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$$[a^\beta_\alpha, \bar{a}^\eta_\gamma] = \delta^\eta_\alpha \delta^\beta_\gamma, \qquad \{b^\beta_\alpha, \bar{b}^\eta_\gamma\} = \delta^\eta_\alpha \delta^\beta_\gamma. \tag{1}$$

The bit number $\mathcal{M} = \mathrm{tr}(\bar{a}a + \bar{b}b)$ is identified with $P^+ = m\mathcal{M}$, and the dynamics is given by the Hamiltonian (to be related to P^-)

$$H = \frac{T_0}{2mN} \mathrm{tr}[(\bar{a}^2 - i\bar{b}^2)a^2 - (\bar{b}^2 - i\bar{a}^2)b^2 + (\bar{a}\,\bar{b} + \bar{b}\,\bar{a})ba + (\bar{a}\,\bar{b} - \bar{b}\,\bar{a})ab]. \tag{2}$$

H has been chosen to commute with the supercharge $Q = \mathrm{tr}[\bar{a}be^{i\pi/4} + \bar{b}ae^{-i\pi/4}]$, which satisfies $Q^2 = \mathcal{M}$, and so respects a supersymmetry. We have written the coefficient as T_0/m with T_0 the rest tension of the emergent string for which m will disappear as a parameter. Later, when we study thermal perturbation theory, we will take $g = T_0/(2m\sqrt{2})$ as the expansion parameter. Here it is important that m and g are independent parameters at the level of string bits. We shall work directly with this Hamiltonian in analyzing the high temperature behavior of the system, which is best described in terms of the fundamental string bits.

The eigenstates of H in the color singlet sector were obtained in [15] in the limit $N \to \infty$. These $N = \infty$ eigenstates can be pictured as containing several noninteracting (discretized) closed chains of bits. A single closed chain state is a linear combination of single trace states with fixed bit number M. The ground single chain state has energy

$$E_G = -\frac{1}{2}\sum_n \omega_n = -\frac{T_0}{m}\cot\frac{\pi}{2M} = -\frac{2T_0 M}{m\pi} + \frac{\pi T_0}{6Mm} + \mathcal{O}(M^{-3}), \tag{3}$$

where $\omega_n = (2T_0/m)\sin(n\pi/M)$, and in the last form we have taken M large to show the limit in which a chain becomes a continuous string. If we identify

$$P^- \equiv \frac{2T_0}{m\pi}\mathcal{M} + H, \tag{4}$$

the dispersion relation $P^-(P^+)$ is Lorentz invariant in $1+1$ dimensional spacetime, in the limit $M \to \infty$: $2P^+P^-_G = \pi T_0/3$.

In this simplest string bit model the excited states of a single closed chain are those of left and right moving statistics waves [17,18] described in the emergent string theory by a Grassmann world sheet field. A wave in the nth normal mode adds energy ω_n to the ground state. If mode number $n < M/2$ is left moving then mode number $M - n$ is right moving. These two modes have the same frequency ω_n. The mode number n takes on the values $0, 1, 2, \ldots, M-1$. The zero mode $n = 0$ is a fermionic operator, whose square is unity, which converts a state satisfying Bose-Einstein statistics to one satisfying Fermi-Dirac statistics or vice versa. There is a cyclic constraint on the occupied modes $\{n_i\}$, which is that $\sum_i n_i$ is a multiple of M if M is odd, but it is an odd multiple of $M/2$ if M is even. This mismatch of cyclic constraints for M even and odd is due to the fact that the number of fermionic bits b is odd (1 in this model). In the limit of continuous string the cyclic constraint reduces to the familiar $L_0 = \tilde{L}_0$ constraint of closed string theory.

The energy of states with several closed strings is simply the sum of the energies of the individual closed strings, reflecting the absence of interactions between them when $N = \infty$. All of these multistring states are color singlets and all have finite P^- in the limit $m \to 0$ with $P^+ = mM$ fixed. As noted in [15] the color nonsinglet states have P^- of order T_0/m. Thus one has true color confinement in the limit $m \to 0$ since in that limit the only finite energy states are the color singlets with $M = \infty$. If m is kept finite but small, then the color nonsinglets have energy much greater than $\sqrt{T_0}$, and we can say we have effective confinement.

At zero temperature the large N limit is given by summing all planar Feynman graphs. At finite temperature, the limit is given by summing the planar graphs of thermal perturbation theory, reviewed for the string bit system in the Appendix. The canonical partition function is given by[1] $Z = \mathrm{Tr}e^{-\beta P^0}$ where

$$P^0 = \frac{P^+ + P^-}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left[\left(m + \frac{2T_0}{m\pi}\right)\mathcal{M} + H\right] \equiv \omega\mathcal{M} + \frac{H}{\sqrt{2}}. \tag{5}$$

The vertices of the thermal graphs are determined by the terms in H. The sum of connected graphs calculates $\ln Z$. This sum is easily shown to have the structure

$$\ln Z = N^2 f_0(\beta) + f_1(\beta) + \frac{1}{N^2} f_2(\beta) + \cdots, \tag{6}$$

where the leading term f_0 is found by calculating the sum of planar diagrams. The presence of the N^2 term is due to the fact that the operators a, b are $N \times N$ matrices with N^2 elements. As we shall find, the calculation of $\ln Z$ using the known energy spectrum at $N = \infty$, at low temperature (β large) gives a contribution to $f_1(\beta)$ (because color singlet states with large M dominate). This contribution blows up when $\beta < \beta_H$, predicting the ultimate temperature β_H^{-1}. In string models the singularity in $f_1(\beta)$ is of the form $(\beta - \beta_H)^p$ where the power $p = -\alpha - 1$ depends on the details of the model. For the simple model studied here we find $\alpha = -3/2$ implying $p = 1/2$. The corresponding contribution to f_0 comes from color adjoint states and is suppressed by factors of $e^{-\beta T_0/m}$. In the limit of absolute confinement $m \to 0$, this implies that $f_0(\beta) = 0$ at low temperature. On the other hand, calculating with the

[1] We have used tr to denote the trace over matrix indices; here we use Tr to denote the thermal trace.

graphical expansion shows no problem with arbitrarily high temperatures which are dominated by the string bit description suggesting there is no limiting temperature. For these two facts to be compatible, the ultimate temperature must be an artifact of the large N limit. At finite N the string bit system has a finite number of degrees of freedom, and hence $\ln Z$ should be a smooth function of β for the whole range $0 < \beta < \infty$.

In this paper we shall calculate, from the $N = \infty$ eigenvalues of H, the value of β_H and also determine the power $p = 1/2$. Then $f_1(\beta) = -K_1(\beta - \beta_H)^{1/2} + K_2$ for β slightly larger than β_H. It then follows that $f_1'' \sim (K_1/4)(\beta - \beta_H)^{-3/2}$. Since $\partial^2 \ln Z/\partial\beta^2 > 0$, we must have $K_1 > 0$. It is natural to guess that at finite N, this function is made smooth by the substitution $(\beta - \beta_H)^{-3/2} \to ((\beta - \beta_H)^2 + \eta(N))^{-3/4}$, where $\eta(N)$ is some function of N which vanishes as $N \to \infty$. With this ansatz one can then integrate back to determine that $\ln Z$ for β near β_H would be proportional to the function[2]

$$g(\beta, N) = \frac{N^2}{\sqrt{\gamma}}(\beta_H - \beta)\frac{\Gamma(1/4)\sqrt{\pi}}{\Gamma(3/4)} + \int_0^\beta dt(\beta - t)\left[(t - \beta_H)^2 + \frac{\gamma^2}{N^8}\right]^{-3/4} \tag{7}$$

$$\sim \begin{cases} -4(\beta - \beta_H)^{1/2} + 4\beta_H^{1/2} - 2\beta\beta_H^{-1/2} & \beta > \beta_H \\ \frac{N^2}{\sqrt{\gamma}}(\beta_H - \beta)\frac{\Gamma(1/4)\sqrt{\pi}}{\Gamma(3/4)} - 4(\beta_H - \beta)^{1/2} + 4\beta_H^{1/2} - 2\beta\beta_H^{-1/2} & \beta < \beta_H, \end{cases} \tag{8}$$

where the last lines show the large N behavior. The determination that $\eta(N) = \gamma^2/N^8$ is made by requiring that the divergence as $N \to \infty$ be precisely proportional to N^2 as dictated by the rules of the $1/N$ expansion. We have fixed the integration constants $A\beta + B$ so that the N^2 term is absent when $\beta > \beta_H$. We have not yet learned enough about $1/N$ corrections to confirm the validity of this ansatz, but if it is valid, the physical interpretation of the N^2 term, which is present only for $\beta < \beta_H$, is that it signals the liberation of the fundamental string bit degrees of freedom.

In the following sections we discuss our results in detail. Section II gives a brief review of the Hagedorn phenomenon for a single free string as it is described in light cone parametrization. Section III then extends the discussion to the string discretized as a chain of string bits. We obtain the Hagedorn temperature as a function of the discretization unit m. Section IV concludes the paper. An Appendix which reviews thermal perturbation theory, needed in the high temperature analysis of Sec. III, in the context of string bit models is included at the end.

II. THE FREE LIGHT CONE STRING AT FINITE TEMPERATURE

A. A general ideal gas

Consider a system of bosons of various species b and fermions of various species f. Here b and f can include momentum as well as internal state labels. In the absence of interactions the canonical partition function is

$$Z = \prod_b \frac{1}{1 - e^{-\beta\epsilon_b}} \prod_f (1 + e^{-\beta\epsilon_f}) \tag{9}$$

$$\ln Z = \sum_f \ln(1 + e^{-\beta\epsilon_f}) - \sum_b \ln(1 - e^{-\beta\epsilon_b}) \tag{10}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left[\sum_b e^{-n\beta\epsilon_b} + (-)^{n-1}\sum_f e^{-n\beta\epsilon_f}\right]. \tag{11}$$

We see that the gas partition function can be expressed in terms of the partition functions for a single particle immersed in heat baths of temperatures $\beta^{-1}, (2\beta)^{-1}, \ldots, (n\beta)^{-1}), \ldots$. More specifically the nth term involves either the single particle partition function

$$z(n\beta) = \sum_b e^{-n\beta\epsilon_b} + \sum_f e^{-n\beta\epsilon_f} = \sum_k e^{-n\beta\epsilon_k}, \quad \text{for } n \text{ odd} \tag{12}$$

or the single superparticle partition function

$$z^S(n\beta) = \sum_b e^{-n\beta\epsilon_b} - \sum_f e^{-n\beta\epsilon_f}, \quad \text{for } n \text{ even.} \tag{13}$$

When the particle spectrum is supersymmetric, as in the model studied here, $z^S = 0$.

B. Hagedorn temperature for the light cone string

The Hagedorn temperature is by definition the lowest temperature above which the partition function of the system diverges. Assuming the divergence does not come from the sum over n in (11), we see that the Hagedorn temperature satisfies $z(\beta_H - \epsilon) = \infty$, because then all the $z((2n+1)\beta_H - \epsilon), z^S(2n\beta_H - \epsilon)$ for $n > 0$ are finite. Thus to determine the Hagedorn temperature, it suffices to

[2]More generally one could construct a family $g_k(\beta, N)$, each with a different γ_k. Then a linear combination of this family would remove the discontinuities of the $N = \infty$ limit in the same way.

examine the partition function for a single particle in a heat bath.

In the light cone description the energy of a closed string is expressed as

$$P^0 = \frac{1}{\sqrt{2}}(P^+ + P^-), \qquad P^- = \frac{4\pi T_0(L_0 + \tilde{L}_0 + 1/12)}{2P^+}, \tag{14}$$

where L_0 ($\tilde{L}_0$) is the transverse string mode number operator for left (right) moving waves. They depend in detail on the string model of interest. Here we assume the simplest possible transverse dynamics, namely a single fermion field on a closed string world sheet (hence the $1/12$ in P^- above). For more elaborate string models one simply adds more world sheet fields. For example, for a closed string world sheet system of s left-right pairs of periodic Grassmann fields and d bosonic fields, the c-number $1/12$ is replaced by $(s - d)/12$. Here we have chosen $s = 1$ and $d = 0$.

For all models, the physical states of a closed string satisfy the constraint $(L_0 - \tilde{L}_0)|\psi\rangle = 0$. We must therefore insert the projection operator

$$\mathcal{P}_{\text{Phys}} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(L_0 - \tilde{L}_0)} \tag{15}$$

in the thermal trace. The canonical partition function for a single string in a heat bath at temperature β^{-1} is then

$$\begin{aligned} z(\beta) &= \int_0^\infty dP^+ \int_0^{2\pi} \frac{d\theta}{2\pi} \mathrm{Tr} e^{-\beta P^0} e^{i\theta(L_0 - \tilde{L}_0)} \\ &= \int_0^\infty dP^+ \int_0^{2\pi} \frac{d\theta}{2\pi} \mathrm{Tr} e^{-(\beta/\sqrt{2})(P^+ + P^-)} e^{i\theta(L_0 - \tilde{L}_0)} \\ &= \int_0^\infty dP^+ e^{-(\beta/\sqrt{2})[P^+ + \pi T_0/(6P^+)]} \\ &\quad \times \int_0^{2\pi} \frac{d\theta}{2\pi} \prod_{n=1}^\infty |1 + e^{-\beta\pi T_0 n\sqrt{2}/P^+ + in\theta}|^2. \end{aligned} \tag{16}$$

The Hagedorn temperature is the temperature above which the integral over P^+ diverges. Putting $z = e^{-\beta\pi T_0\sqrt{2}/P^+ + i\theta}$, we see that $z \to e^{i\theta}$ for $P^+ \to \infty$ and the product $\prod_n (1 + z^n)(1 + z^{*n})$ will be maximized in this limit for $\theta = 0$. To find β_H in this simple model we need the $z \to 1$ behavior of the product $\prod_n |1 + z^n|^2$. One can either change variables via the Jacobi imaginary transformation, or for the leading behavior as $z \to 1$, it is enough to write

$$\begin{aligned} \ln \prod_n (1 + z^n) &= \sum_{n=1}^\infty \ln(1 + z^n) = \sum_{k=1}^\infty \frac{(-)^{k-1}}{k} \frac{z^k}{1 - z^k} \\ &\sim \frac{1}{1 - z} \sum_{k=1}^\infty \frac{(-)^{k-1}}{k^2} = \frac{\pi^2}{12} \frac{1}{1 - z} \to \frac{\pi^2}{12} \frac{P^+}{\beta\pi T_0 \sqrt{2}}, \end{aligned} \tag{17}$$

where the last form inserted the value of z at $\theta = 0$ and large P^+ for our model. It is evident that the large P^+ behavior of the P^+ integrand is of the form $(P^+)^\alpha e^{-h(\beta)P^+}$ where the power α is determined to be $\alpha = -3/2$ by integrating θ in the neighborhood of zero. Thus the Hagedorn temperature is determined by

$$0 = h(\beta_H) = \frac{\beta_H}{\sqrt{2}} - \frac{\pi}{12} \frac{\sqrt{2}}{\beta_H T_0}, \qquad \beta_H = \sqrt{\frac{\pi}{6T_0}}. \tag{18}$$

Because $\alpha = -3/2 < -1$, $z(\beta_H)$ is actually finite for this model, the Hagedorn singularity being a square root branch point ($p = 1/2$).

III. SUPERSTRING BIT MODEL AT FINITE TEMPERATURE

A. Low temperature behavior at $N = \infty$

The color singlet eigenstates of the Hamiltonian (2) at $N = \infty$ were obtained in [15]. They are states of multi-closed-chains with $P^+ = Mm$ each with fermion world sheet fields for which the normal mode frequencies are $\omega_n = (2T_0/m)\sin(n\pi/M)$. The cyclic constraint can be imposed through the projection operator

$$\mathcal{P} = \frac{1}{M} \sum_{k=0}^{M-1} (-)^{k(M-1)} e^{2\pi i k \mathcal{N}/M}, \tag{19}$$

where $\mathcal{N}$ is the mode number operator with values $\sum_i n_l$ on a state with modes n_i occupied. Then the partition function for a single chain in a heat bath at temperature β^{-1} is

$$\begin{aligned} z(\beta) &= \sum_{M=1}^\infty e^{-\beta E_G} \frac{1}{M} \sum_{k=0}^{M-1} (-)^{k(M-1)} \\ &\quad \times \prod_{n=1}^{M-1} (1 + e^{-(\sqrt{2}\beta T_0/m)\sin(n\pi/M) + 2i\pi nk/M}) \end{aligned} \tag{20}$$

$$E_G \equiv \frac{mM + P_G^-}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(mM + \frac{2MT_0}{m\pi} - \frac{T_0}{m} \cot \frac{\pi}{2M} \right). \tag{21}$$

Multichain states can then be included as usual by including the $z((2n + 1)\beta)$ and $z^S(2n\beta)$ terms of the ideal gas formula for $\ln Z$.

The summand is maximized by the $k = 0$ term, so the Hagedorn temperature for this string bit model is given by the condition

$$0 = \frac{\beta_H m}{\sqrt{2}} - \lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{M-1} \ln(1 + e^{-(\sqrt{2}\beta_H T_0/m)\sin(n\pi/M)}). \tag{22}$$

To analyze this condition, it is convenient to define the variable $\xi = \sqrt{2}\beta_H T_0/m$ and then rewrite the equation as a formula for the discretization unit m as a function of ξ:

$$\begin{aligned}\frac{m^2}{2T_0} &= \lim_{M\to\infty}\frac{1}{M\xi}\sum_{n=1}^{M}\ln\left(1+e^{-\xi\sin(n\pi/M)}\right)\\ &= \frac{1}{\xi}\int_0^1 dx\ln\left(1+e^{-\xi\sin(x\pi)}\right)\\ &= \frac{2}{\xi}\int_0^{1/2} dx\ln\left(1+e^{-\xi\sin(x\pi)}\right).\end{aligned} \tag{23}$$

Then one should choose ξ so that $m^2/T_0 \ll 1$ to compare to the continuous string. Evidently $m \to 0$ when $\xi \to \infty$. So to recover the continuous string result we need the large ξ behavior of the integral on the right side,

$$\begin{aligned}2\int_0^{1/2} dx\ln\left(1+e^{-\xi\sin(x\pi/2)}\right) &= \frac{2}{\pi}\int_0^1\frac{du}{\sqrt{1-u^2}}\ln(1+e^{-\xi u})\\ &= \frac{2}{\pi}\frac{1}{\xi}\sum_{k=1}^{\infty}\frac{(-)^{k-1}}{k^2}+O(\xi^{-3})\\ &= \frac{\pi}{6\xi}+O(\xi^{-3}),\end{aligned} \tag{24}$$

so that

$$\frac{m^2}{2T_0}\sim\frac{\pi}{6\xi^2},\qquad \xi\to\infty \tag{25}$$

$$\beta_H = \frac{m\xi\sqrt{2}}{2T_0}\sim\frac{\xi\sqrt{2}}{2T_0}\sqrt{\frac{2\pi T_0}{6\xi^2}} = \sqrt{\frac{\pi}{6T_0}} \tag{26}$$

in agreement with the direct continuum calculation.

To assess the consequences of discreteness one can simply take ξ finite. If ξ is large m will be small and the effects of discreteness will be small. For example put $\xi = 10$ and $M = 50000$ and calculate:

$$\begin{aligned}\frac{m}{\sqrt{T_0}} &= 0.1029839985,\\ \beta_H\sqrt{T_0} &= 0.7282068418 = 1.006364824\sqrt{\frac{\pi}{6}}.\end{aligned} \tag{27}$$

For $\xi = 2$ and $M = 50000$ the numbers become

$$\begin{aligned}\frac{m}{\sqrt{T_0}} &= 0.52988823849,\\ \beta_H\sqrt{T_0} &= 0.7493668536 = 1.035607454\sqrt{\frac{\pi}{6}}.\end{aligned} \tag{28}$$

It appears that the value of the Hagedorn temperature is rather insensitive to the discreteness parameter m.

We should stress that these results depend on $N = \infty$. We have also restricted the partition sum to color singlet states. Nonsinglet states would add a positive amount and certainly could not remove the singularity. But the energies of the nonsinglet states are of order T_0/m and so are highly suppressed (when $m^2 \ll T_0$) at low temperatures.

In our study of thermal perturbation theory, the expansion parameter $g = T_0/(2m\sqrt{2})$ will be small in the opposite limit $m^2 \gg T_0$. In that case we need to put $\xi \ll 1$, for which

$$\frac{m^2}{2T_0}\sim\frac{\ln 2}{\xi},\qquad \xi\ll 1. \tag{29}$$

Then we find the Hagedorn temperature $\beta_H \sim \sqrt{\xi\ln 2}/\sqrt{T_0}$. In this limit $\xi \sim (g^2/T_0)16\ln 2$. Thus the Hagedorn temperature goes to ∞ in the limit of $g = 0$:

$$\beta_H \sim \frac{4g\ln 2}{\sqrt{T_0}}. \tag{30}$$

B. High temperature

In the high temperature limit, we expect the fundamental constituents to play an active visible role. We shall use thermal perturbation theory, reviewed in the Appendix, to analyze this limit. Write the energy of the string bit system as

$$P^0 = \frac{1}{\sqrt{2}}\left[\left(m+\frac{2T_0}{m\pi}\right)\mathcal{M}+H\right] \tag{31}$$

so that in the notation of the Appendix $\omega = (m + 2T_0/(m\pi))/\sqrt{2}$ and $g = T_0/(2m\sqrt{2})$ is the expansion parameter. The small expansion parameter condition $g \ll \omega$ translates to $m^2 \gg T_0$ which is the opposite of the continuous string limit we are eventually interested in, but we hope some of our qualitative insights will retain some validity, at least at high temperature.

Even if g is assumed small, the high temperature limit requires at least partial summation to all orders, because the bare boson propagators blow up as $(\beta\omega)^{-1}$ as $\beta \to 0$ making successive terms in perturbation theory blow up more and more severely. In the Appendix we show that the solution to a one loop Dyson equation for the boson propagator reduces the divergence to $(2\beta g)^{-1/2}$ which is sufficient to make the remaining terms in the expansion finite as $\beta \to 0$. The upshot is that the zeroth order high temperature behavior of the partition function is modified from $(\beta\omega/2)^{-N^2}$ to $(\beta g)^{-N^2/2}$. Some support for believing this result captures the qualitative small β behavior is that the power of β (though not the constant) agrees with the known exact solution of the $N = 1$ case.

IV. CONCLUDING REMARKS

In this paper we have analyzed the Hagedorn phenomenon in string bit models. We started with a derivation of the Hagedorn temperature of the continuous string from the point of view of light cone quantization, which is perhaps less familiar than other treatments. Then we basically repeated this derivation for the simplest string bit model. In this case the Hagedorn temperature depends on m the discrete unit of P^+. It ranges from $\sqrt{\pi T_0/6}$ for $m/\sqrt{T_0} \to 0$ to ∞ for $m/\sqrt{T_0} \to \infty$. We also analyzed the high temperature behavior of the system employing thermal perturbation theory. In string bit models the Hagedorn phenomenon cannot reflect a phase transition at finite N. We presented a possible hypothesis (8) for the behavior of $\ln Z$ near the Hagedorn temperature, which illustrates how a perfectly smooth function of temperature at finite N induces the Hagedorn phenomenon when $N \to \infty$. Namely, in that limit the leading term $N^2 f_0$ is present only at temperatures above the Hagedorn temperature. The string bit degrees of freedom start to become more active, but in a smooth way, above and near the Hagedorn temperature. So far (8) is merely an educated guess—it needs to be tested by studying higher orders in the $1/N$ expansion.

ACKNOWLEDGMENTS

This research was supported in part by the Department of Energy under Grant No. DE-FG02-97ER-41029.

APPENDIX: PERTURBATION THEORY

Here we give a brief review of thermal perturbation theory taking advantage of the special features of string bit models. We concentrate on the simplest model defined by (31):

$$P^0 = \omega \mathcal{M} + \frac{g}{N}\mathrm{tr}\mathcal{V}, \tag{A1}$$

where $\mathcal{V} = \bar{a}\bar{a}aa + \cdots$ are the quartic operators within the square brackets of (2). We develop the perturbation expansion as a power series in g:

$$Z = \mathrm{Tr}e^{-\beta P^0} = \sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Tr}e^{-\beta\omega\mathcal{M}}\left(\frac{g}{N}\mathrm{tr}\mathcal{V}\right)^n \tag{A2}$$

$$= \left(\frac{1+e^{-\beta\omega}}{1-e^{-\beta\omega}}\right)^{N^2}\sum_{n=0}^{\infty}\frac{1}{n!}\left\langle\left(\frac{-\beta g}{N}\mathrm{tr}\mathcal{V}\right)^n\right\rangle \tag{A3}$$

$$\langle\Omega\rangle \equiv \mathrm{Tr}e^{-\beta\omega\mathcal{M}}\Omega, \tag{A4}$$

where tr denotes the matrix trace and Tr denotes the thermal trace. The average is computed by applying Wick's theorem with the contraction rules,

$$\langle\bar{a}^\beta_\alpha a^\kappa_\gamma\rangle = \delta^\kappa_\alpha\delta^\beta_\gamma\frac{1}{e^{\beta\omega}-1}, \quad \langle a^\kappa_\gamma\bar{a}^\beta_\alpha\rangle = \delta^\kappa_\alpha\delta^\beta_\gamma\frac{e^{\beta\omega}}{e^{\beta\omega}-1} \tag{A5}$$

$$\langle\bar{b}^\beta_\alpha b^\kappa_\gamma\rangle = \delta^\kappa_\alpha\delta^\beta_\gamma\frac{1}{e^{\beta\omega}+1}, \quad \langle b^\kappa_\gamma\bar{b}^\beta_\alpha\rangle = \delta^\kappa_\alpha\delta^\beta_\gamma\frac{e^{\beta\omega}}{e^{\beta\omega}+1}. \tag{A6}$$

A graphical representation of Wick's theorem is constructed using 't Hooft's double line notation for the matrix operators. The propagator and vertex are shown in Fig. 1.

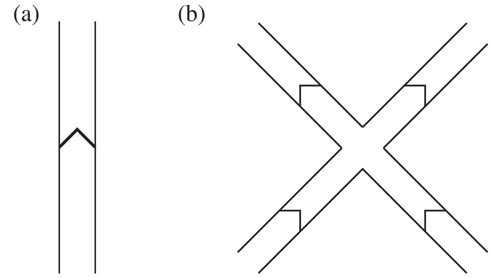

FIG. 1. Graphical rules for thermal perturbation theory. The arrow points from an $\bar{a}$ to an a. Operators ordered left to right are represented by graph elements ordered top to bottom. Thus the propagator (a) contributes a factor $(e^{\beta\omega}-1)^{-1}$ if its arrow points down and contributes the factor $e^{\beta\omega}(e^{\beta\omega}-1)^{-1}$ if its arrow points up. In a graph contributing to nth order the top to bottom order of the vertices (b) coincides with the left to right ordering of the n perturbation operators.

First and second order examples of the application of the graphical rules are shown in Fig. 2.

To define the $1/N$ expansion, we calculate the sum of connected graphs which gives the perturbation expansion for $\ln Z$. Then restricting the sum to planar diagrams gives the leading order as $N \to \infty$, which is of order N^2. Both diagrams shown in Fig. 2 are planar and contribute to leading order. The structure of the $1/N$ expansion in string bit models is that shown in (6).

Since the boson propagators blow up at high temperature ($\beta \to 0$), this limit is not amenable to straight perturbation theory. At least some form of (partial) summation to all orders must be attempted. A relatively simple partial summation is to set up a Dyson equation for the "self-energy" Π defined as the one particle irreducible two point function, in terms of which the fully corrected propagator Δ is, at high temperature,

$$\Delta = \frac{1}{\beta\omega}\sum_{n=0}^{\infty}\frac{\Pi^n}{(\beta\omega)^n} = \frac{1}{\beta\omega - \Pi} \tag{A7}$$

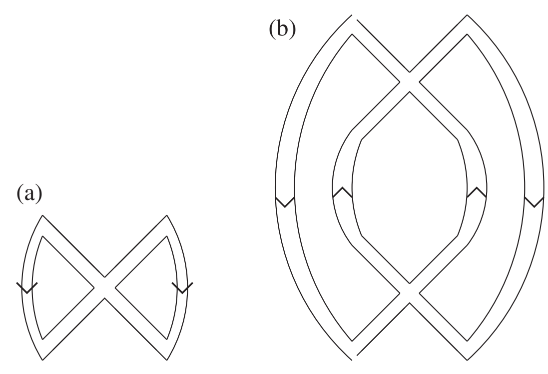

FIG. 2. Example of graphs contributing to the perturbation expansion (a) at first order and (b) at second order of the partition function. Its logarithm $\ln Z$ is calculated by restricting to connected graphs. These particular graphs, being planar contribute to leading order in the large N limit.

At finite temperature the formalism is complicated by the distinction that must be kept between $\langle \bar{a}a\rangle \neq \langle a\bar{a}\rangle$. At high temperature, this distinction disappears and both propagators are approximately $(\beta\omega)^{-1}$. Here we limit the discussion to this simplifying limit. In general both Δ and Π are matrices in internal color space. There are two cases where they can be treated as numbers: namely $N=1$ when they *are* numbers, and $N=\infty$ when the indices are simply spectators, which factor out of both sides of the Dyson equation.

In the latter case ($N=\infty$) the one loop Dyson equation for the boson propagator reads (for $N=1$ change the 2 to a 4)

$$\Pi = -2\beta g \frac{1}{\beta\omega - \Pi} \tag{A8}$$

which is a quadratic equation for Π:

$$0 = \Pi^2 - \Pi(\beta\omega) - 2\beta g \tag{A9}$$

$$\Pi = \frac{1}{2}\left[\beta\omega - \sqrt{\beta^2\omega^2 + 8\beta g}\right], \tag{A10}$$

where the branch of the square root is chosen so that $\Pi \to 0$ at $g=0$. We see that in the high temperature limit $\Pi \sim -\sqrt{2\beta g}$. In this approximation, the high temperature behavior of $\ln Z$ is

$$\begin{aligned}\ln Z &\sim -N^2 \ln(1-e^{-\beta\omega}) + N^2 \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\Pi}{\beta\omega}\right)^n \\ &\sim -N^2\ln(\beta\omega) - N^2 \ln\left(1 - \frac{\Pi}{\beta\omega}\right) \\ &= -N^2 \ln \frac{\beta\omega + \sqrt{\beta^2\omega^2 + 8\beta g}}{2} \\ &\sim -\frac{N^2}{2}\ln(2\beta g).\end{aligned} \tag{A11}$$

The effect of the interactions in this approximation is to soften the singularity in Z at $\beta \to 0$ but not to remove it entirely.

Of course this only takes into account a subset of the terms in the perturbation expansion, after which the terms in perturbation expansion are at least finite as $\beta \to 0$. But it would be helpful to test how it does in a context where the exact answer is known. The special case $N=1$ is an instructive example. For simplicity we drop the fermionic operators. In that case the Hamiltonian is a function of the number operator $\mathcal{M} = \bar{a}a$:

$$H = \mathcal{M}\omega + g\bar{a}\bar{a}aa = \mathcal{M}\omega + g(\mathcal{M}^2 - \mathcal{M}) \tag{A12}$$

and hence

$$Z = \sum_{n=0}^{\infty} e^{-n\beta\omega - \beta g(n^2-n)}. \tag{A13}$$

Taking the limit $\beta \to 0$ the sum can be approximated by an integral over a variable $x = n\sqrt{\beta}$:

$$\begin{aligned}Z &\sim \frac{1}{\sqrt{\beta}}\int_0^\infty dx e^{-x\sqrt{\beta}\omega - g(x^2 - x\sqrt{\beta})} \\ &\approx \frac{1}{\sqrt{\beta}}\int_0^\infty dx e^{-gx^2} = \frac{\sqrt{\pi}}{2\sqrt{\beta g}}.\end{aligned} \tag{A14}$$

We see that the one loop approximation gets the power of β right but not the constant prefactor.

It is also encouraging that this last result can be obtained by analyzing the perturbation expansion for Z. At order n, one can count the number of Wick contraction schemes of $\langle(\bar{a}^2a^2)^n\rangle$ and find that there are exactly $(2n)!$. Since at $N=1$ and high temperature all graphs at each order are equal, we can conclude that

$$Z = \frac{1}{1-e^{-\beta\omega}} \sum_{n=0}^{\infty}\left(\frac{-\beta g}{\beta^2\omega^2}\right)^n \frac{(2n)!}{n!} \tag{A15}$$

a sum that has zero radius of convergence. However we can use the Borel summation trick

$$(2n)! = \int_0^\infty dt t^{2n} e^{-t} \tag{A16}$$

to interpret the sum as

$$\begin{aligned}Z &= \frac{1}{1-e^{-\beta\omega}}\int_0^\infty dt e^{-t-gt^2/(\beta\omega^2)} \\ &= \frac{1}{1-e^{-\beta\omega}}\sqrt{\frac{\beta\omega^2}{g}}\int_0^\infty dt e^{-t\omega\sqrt{\beta/g}-t^2}\end{aligned} \tag{A17}$$

$$\sim \frac{1}{2}\sqrt{\frac{\pi}{\beta g}}, \quad \text{as } \beta \to 0. \tag{A18}$$

Thus a complete analysis of perturbation theory in this simple case gets the power of β *and* the prefactor right. If we could get an accurate count of the number of planar connected vacuum diagrams at each order n, we could make a similar statement about high temperature in the large N limit. Our one loop Dyson equation evaluation provides support for the high temperature behavior $\ln Z \to (N^2/2)\ln(\beta g) + C$ in the $N \to \infty$ limit but gives no reliable nonperturbative information about the constant C.

Because fermion propagators are perfectly finite as $\beta \to 0$, the presence of fermion lines in the graphical rules does not affect the singular high temperature behavior of the partition function, which is entirely determined by the bosonic degrees of freedom. However, they will certainly contribute to the subleading behavior and in particular can be expected to contribute to the constant C.

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