PHYSICAL REVIEW D 92, 065022 (2015)

# More on soft theorems: Trees, loops, and strings

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(Received 13 September 2014; published 23 September 2015)

We study soft theorems in a broader context, their universality in effective field theories and string theory, as well as continue the analysis of their fate at loop level. In effective field theories with $F^3$ and $R^3$ interactions, the soft theorems are not modified. However, for gravity theories with $R^2\phi$ interactions, the sub-subleading order soft graviton theorem, which is beyond what is implied by the extended Bondi, van der Burg, Metzner, and Sachs symmetry, requires modifications at tree level for nonsupersymmetric theories and at loop level for $\mathcal{N} \leq 4$ supergravity due to anomalies. For open and closed superstrings at finite $\alpha'$, via explicit calculation for lower-point examples as well as world sheet operator product expansion analysis for arbitrary multiplicity, we show that scattering amplitudes satisfy the same soft theorem as their field-theory counterpart. This is no longer true for closed bosonic or heterotic strings due to the presence of $R^2\phi$ interactions. We also consider loop corrections to gauge theories in the planar limit, where we show that tree-level soft gluon theorems are respected at the integrand level for $1 \leq \mathcal{N} \leq 4$ SYM. Finally, we discuss the fate of soft theorems for finite loop amplitudes in pure Yang-Mills theory and gravity.

DOI: 10.1103/PhysRevD.92.065022 PACS numbers: 12.38.Bx, 11.15.-q, 04.50.-h, 11.10.Kk

## I. INTRODUCTION

It is well known that scattering amplitudes in gauge and gravity theories display universal behavior as one of the external legs becomes soft. Historically, soft theorems at tree level were derived using Feynman diagrams, at leading order [1], and at subleading orders for soft photons [2,3] and for soft gravitons [4]. More recently soft theorems have been revived for gravity [5] and for Yang-Mills theory [6], using BCFW recursion relations [7,8] for tree amplitudes.[1] One of the motivations for studying soft graviton theorems is to understand their relations with the conjectured new infinite dimensional symmetry of gravitational scattering amplitudes [13–19], extending the Bondi, van der Burg, Metzner, and Sachs (BMS) symmetry [20] at null infinity. Given all these different ways of motivating and deriving soft theorems, it is natural to ask if these theorems are respected in more general gauge and gravity theories, including string theory.

Furthermore, the soft behavior of loop-level amplitudes has been studied at leading order [21–23] and more recently at subleading orders [24,25], for both gauge theories and gravity. It is well known that the leading soft graviton theorem is protected from loop corrections [23], but subleading soft graviton theorems and soft gluon theorems both require corrections at loop level. On the other hand, it has been argued in [26] that the distributional nature of the soft limit implies an alternative way of studying soft behaviors at loop level: one should first expand around the soft limit and then perform the loop integrals for the amplitude, which involves an expansion in the regulator. With this prescription, it has been shown in [26] that the subleading soft theorem is not renormalized in the example of one-loop five-point amplitude in $\mathcal{N} = 8$ supergravity. Note that for the purpose of obtaining the correct infrared behavior for scattering amplitudes, it is necessary to abide by the usual procedure of regulating before taking the soft limit [24]. The prescription prescribed by [26] instead serves as constraint one can impose on $D$-dimensional integrands.

In this paper we will continue the investigation of soft theorems along these two directions: their universality in effective field theories and string theory, as well as their fate

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[1]The subleading soft graviton theorem was also proposed in [9]. Both gauge and gravity soft theorems have been proven to hold in arbitrary dimensions [10,11], based on scattering equations [12].

at loop level and its implications. First we consider the question of how universal are they at tree level. Naively one would expect that subleading soft theorems may fail in any effective field theory of gauge or gravity if the three-point interaction is modified. In Sec. II, we will study effective field theories with $F^3$ and $R^3$ interactions, and we will show that soft theorems are not altered in theses cases. A byproduct of our study is a BCFW recursion relation for $F^3$ amplitudes, written in momentum-twistor space in a form very similar to that of Yang-Mills amplitudes. However, for $R^2\phi$ interactions, the sub-subleading soft graviton theorem needs modifications at tree level. Note that while such interactions can be suppressed at tree level via supersymmetry, they are generated in $\mathcal{N} \leq 4$ supergravity due to the presence of $U(1)$ anomalies [27]. This modification does not contradict with that implied by BMS symmetry, since the latter only predicts universality for the subleading soft behavior.

A more interesting aspect of universality is the soft theorems for tree-level string amplitudes. Although $\alpha'$ expansions of string amplitudes are coded in effective field theories, there is *a priori* no Feynman-diagram-like argument for soft theorems at finite $\alpha'$. In Sec. III, we will show, by explicit computations and using four-dimensional kinematics for the cases of four and five points (a six-point computation will be presented in Appendix B), that open superstring amplitudes on the disk satisfy the same soft gluon theorem as the corresponding gauge theory amplitudes. Using KLT relations [28], we will also verify the soft graviton theorem for four- and five-point closed superstring amplitudes. The above result can be understood via BCFW recursion relations for string amplitudes. Combining BCFW recursion relations with the crucial observation that only massless states can contribute to the soft limit, we will argue generally that amplitudes for both bosonic and super open-string theory satisfy the soft theorems. In contrast, while supersymmetric closed-string theory satisfy the soft theorems, the sub-subleading term in soft theorems for bosonic closed-string amplitudes needs corrections.

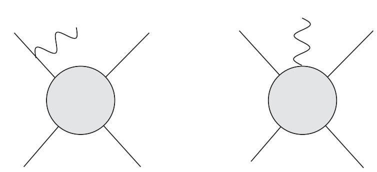
We confirm the above analysis for general multiplicity from a world sheet perspective. We will show that the soft behavior is captured by the operator product expansion (OPE) of the soft vertex operator with adjacent vertex operators in the open-string case and with any hard vertex operator in the closed-string case. BRST symmetry will play a crucial role in the identification of the relevant terms in the OPE and in the choice of the picture for the colliding vertex operators. We will argue that soft theorems hold both in $D = 10$ and in lower dimensions, where the gauge boson and graviton vertex operators simply involve the identity operator of the $\mathrm{CFT}_2$ governing the dynamics of the internal space.

Finally, we will also examine loop-level soft theorems using the prescription of [26]. In Sec. V, we will argue that for gauge theories in the planar limit, loop-level soft gluon theorems can be made manifest already at the integrand level. In particular, we will show that the planar integrands for $\mathcal{N} = 4$ super Yang-Mills (SYM) theory, determined by loop-level BCFW recursion relations [29], satisfy the soft theorem to all loop orders, exactly as for the tree amplitudes. For $1 \leq \mathcal{N} < 4$ SYM, we show explicitly that the same is true for one-loop MHV amplitudes in the CSW representation. In practice, our analysis is simplified significantly by using momentum-twistor variables [30] and choosing to solve momentum conservation in a canonical way for the planar case.

For nonsupersymmetric Yang-Mills theory or theories of gravity, no such representation of the integrands is known; thus, one has to verify the soft theorems in the same way as in [26], i.e. performing the integrals after the soft expansion of the integrand. In Sec. V B, we will carefully examine the integrals from the soft expansion of all-plus one-loop integrands in both Yang-Mills theory and pure gravity theory; we will show that both soft theorems are respected, i.e. the all-plus integrand has the interesting property that taking the soft parameter and IR regulator to zero in different orders commute. This is no longer the case for the single-minus amplitude as observed in [25]. For the latter, we demonstrate that the violation of tree-level soft theorems can be tied to the presence of conformal anomalies at loop level.

### A. Review

We begin with a brief review of soft theorems for tree-level amplitudes in gauge and gravity theories. The $n$-point amplitude involving the emission of a soft photon can be expanded in terms of the soft momentum $s$. The leading and subleading terms in this expansion are given by universal operators acting on the $(n-1)$-point amplitude, a fact that has been well understood since the work of Low [2], who recognized this as a simple consequence of gauge invariance. To see this, separate the Feynman diagrams into two classes:



(a) (b)

Diagram (a) has the soft photon connected to an external line which contributes to the leading divergence in the soft limit, proportional to $\sum_i e_i(\epsilon \cdot k_i)/(s \cdot k_i)$ multiplied by the remaining hard amplitude with one leg slightly off-shell. Subleading terms are distributed between diagrams (a) and (b), where the soft photon is connected to an internal line of the Feynman diagram. Using the fact that the subleading contribution from diagram (a) violates the Ward identity, which is generated by expanding the $(n-1)$-point amplitude near $s=0$, gauge invariance requires the subleading contribution from diagram (b) to be given by differential operators acting on the $(n-1)$-point amplitude.

This observation allowed Low to express the subleading soft limit as a universal soft operator acting on the $(n-1)$-point amplitude. For further extension of Low's result see [3]. Generalizing Low's argument to gravity, Gross and Jackiw [4] obtained soft theorems for gravity accurate up to terms of order $\mathcal{O}(s^2)$, to be compared with $\mathcal{O}(s)$ for gauge theory. Thus the tree-level soft theorems for gravity are universal up to subleading expansion in $s$. For a more recent analysis see [9].

An alternative way to derive the soft theorems is by using BCFW recursion relations for Yang-Mills theory and theories of gravity, as was done in [5,6]. Consider the BCFW representation for tree-level gravity amplitude and choose the soft leg to be one of the shifted lines. If the soft graviton has positive helicity, shift the spinors holomorphically,

$$\lambda_{\hat{s}} = \lambda_s + z\lambda_n, \qquad \tilde{\lambda}_{\hat{n}} = \tilde{\lambda}_n - z\tilde{\lambda}_s, \tag{1.1}$$

and the resulting BCFW representation is given by

$$M_{n+1}(1,2,\ldots,n,s^+) = \sum_{1\le i<n} M_3(\hat{s}^+, i, -\hat{K}_{is})\frac{1}{K_{is}^2} M_n(\hat{K}_{is},\ldots,\hat{n}) + R, \tag{1.2}$$

where $\hat{K}_{is} = k_i + k_{\hat{s}}$, and $R$ represents terms arising from factorization poles $1/(k_s+K)^2$, with $K$ a non-null momentum. The holomorphic soft limit is achieved by scaling $\lambda_s \to \delta\lambda_s$. It was shown explicitly in [5] that the function $R$ is finite under the holomorphic soft limit; thus

$$M_{n+1}(1,2,\ldots,n,\{\delta\lambda_s,\tilde{\lambda}_s\}^+)|_{\text{div}} = \sum_{1\le i<n} M_3(\hat{s}^+, i, -\hat{K}_{is})\frac{1}{K_{is}^2} M_n(\hat{K}_{is},\ldots,\hat{n})|_{\text{div}}, \tag{1.3}$$

where each term on the rhs can be written as

$$M_3(\hat{s}^+, i, -\hat{K}_{is})\frac{1}{K_{is}^2} M_n(\hat{K}_{is},\ldots,\hat{n}) = \mathcal{S}_{si} M_n\left(\left\{\lambda_i, \tilde{\lambda}_i + \delta\frac{\langle sn\rangle}{\langle in\rangle}\tilde{\lambda}_s\right\},\ldots,\left\{\lambda_n, \tilde{\lambda}_n + \delta\frac{\langle si\rangle}{\langle ni\rangle}\tilde{\lambda}_s\right\}\right), \tag{1.4}$$

where "…" indicates unshifted $\{\lambda,\tilde{\lambda}\}$, and $\mathcal{S}_{si}$ is the "inverse soft function" that is independent of the helicity of the $i$th leg,

$$\mathcal{S}_{si} = \frac{1}{\delta^3}\frac{\langle ni\rangle^2[is]}{\langle ns\rangle^2\langle is\rangle}. \tag{1.5}$$

Expanding $M_n(\hat{K}_{is},\ldots,\hat{n})$ in $\delta$, it is straightforward to obtain the divergent part of the holomorphic soft limit,

$$M_{n+1}(1,\ldots,n,\{\delta\lambda_s,\tilde{\lambda}_s\}^+)|_{\text{div}} = \left(\frac{1}{\delta^3}S_{\text{G}}^{(0)} + \frac{1}{\delta^2}S_{\text{G}}^{(1)} + \frac{1}{\delta}S_{\text{G}}^{(2)}\right)M_n, \tag{1.6}$$

where the operator $S_{\text{G}}^{(k)}$ is defined as

$$S_{\text{G}}^{(k)} = \sum_{i=1}^{n-1}\frac{1}{k!}\mathcal{S}_{si}\left(\frac{\langle sn\rangle}{\langle in\rangle}\tilde{\lambda}_s\cdot\frac{\partial}{\partial\tilde{\lambda}_i} + \frac{\langle si\rangle}{\langle ni\rangle}\tilde{\lambda}_s\cdot\frac{\partial}{\partial\tilde{\lambda}_n}\right)^k. \tag{1.7}$$

Note that $M_n$ is here still subject to the $(n+1)$-point amplitude momentum conservation, which is solved by expressing two $\tilde{\lambda}$'s in terms of the remaining $(n-1)$ ones.

Now we turn to the soft gluon theorem. Throughout the paper, we will consider color-ordered, partial amplitudes for gluons (in any gauge theories and open-string theories),

$$\mathbf{A}_n(\{1^{a_1},2^{a_2},\ldots,n^{a_n}\}) = \sum_{\sigma\in S_n/Z_n}\text{Tr}(T^{a_{1_\sigma}}T^{a_{2_\sigma}}\cdots T^{a_{n_\sigma}})A_n(1_\sigma,2_\sigma\ldots n_\sigma), \tag{1.8}$$

where $\mathbf{A}$ denotes the full, color-dressed amplitude and $A$ the corresponding color-ordered amplitude. This is the color decomposition at tree level, but, as we will restrict our analysis to gauge theories in the planar limit wherein $N_c\to\infty$, Eq. (1.8) applies to loop amplitudes as well.

The soft gluon theorem can be derived in a parallel fashion with gravity by using the BCFW representation of tree-level color-ordered amplitudes: the divergent term in the holomorphic soft limit is again isolated into the two-particle channel (only one term, $i=1$, contributes because of the color ordering), and we find

$$A_{n+1}(\{\lambda_1,\tilde{\lambda}_1\},\ldots,\{\lambda_n,\tilde{\lambda}_n\},\{\delta\lambda_s,\tilde{\lambda}_s\}^+)|_{\text{div}} = \sum_{k=0,1}\frac{1}{\delta^{2-k}}S_{\text{YM}}^{(k)}(ns1)A_n(\{\lambda_1,\tilde{\lambda}_1\},\ldots,\{\lambda_n,\tilde{\lambda}_n\}) \tag{1.9}$$

with

$$S_{\text{YM}}^{(k)}(ns1) = \frac{1}{k!}\frac{\langle n1\rangle}{\langle ns\rangle\langle s1\rangle}\left(\frac{\langle sn\rangle}{\langle 1n\rangle}\tilde{\lambda}_s\cdot\frac{\partial}{\partial\tilde{\lambda}_1} + \frac{\langle s1\rangle}{\langle n1\rangle}\tilde{\lambda}_s\cdot\frac{\partial}{\partial\tilde{\lambda}_n}\right)^k. \tag{1.10}$$

Thus, for tree-level amplitudes in Yang-Mills theories, only $S_{\text{YM}}^{(0)}$ and $S_{\text{YM}}^{(1)}$ are universal. Note that if we choose to solve momentum conservation by expressing $\tilde{\lambda}_1,\tilde{\lambda}_n$ in terms of linear combinations of the remaining antiholomorphic spinors, the subleading soft terms actually vanish. This prescription is more natural for planar amplitudes, especially when expressed using momentum twistors, as we will see in Sec. II. In fact, in momentum-twistor representation it is often convenient to consider antiholomorphic soft limits of positive-helicity gluons. The corresponding soft behavior can be straightforwardly obtained from Eqs. (1.9) and (1.10) via little group rescaling,

$$A_{n+1}(\{\lambda_1,\tilde{\lambda}_1\},\ldots,\{\lambda_n,\tilde{\lambda}_n\},\{\lambda_s,\delta\tilde{\lambda}_s\}^+)$$
$$=\frac{\langle n1\rangle}{\langle ns\rangle\langle s1\rangle}A_n(\{\lambda_1,\tilde{\lambda}_1\},\ldots,\{\lambda_n,\tilde{\lambda}_n\})+0\times\delta+\mathcal{O}(\delta^2), \tag{1.11}$$

where the 0 comes from our convention of solving momentum conservation through $\tilde{\lambda}_1, \tilde{\lambda}_n$.

The derivation of the soft theorem from the recursion relation mirrors the work by Low, in that the contribution stems from two-particle channels that involve the soft leg. While in Low's work the subleading contribution also stems from diagrams where the soft leg is attached to an internal line, these contributions are controlled by the leading contribution via Ward identities. Since the representation based on recursion relations uses gauge invariant building blocks, it is not a surprise that only the aforementioned two-particle channels contribute.

## II. SOFT THEOREMS FOR HIGHER-DERIVATIVE INTERACTIONS

In this section, we would like to consider the extent to which the soft theorem is universal for tree-level scattering amplitudes of Yang-Mills and gravity theories coupled matter, or for effective field theories with higher-dimensional operators. The latter can be viewed as posing the same question as tree-level string-theory amplitudes in the $\alpha'$ expansion. Recall that from Low's work the soft gluon/graviton behavior of perturbative scattering amplitudes is determined by the three-point interaction of the theory and gauge invariance; thus, one expects that only higher-dimensional operators that modify the three-point interaction are relevant to the discussion. While such interactions are generically suppressed in the soft limit by the extra power of soft invariants, this does not rule out the possibility of modification in the subleading behaviors.

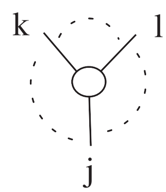
Here we will only consider higher-dimensional operators that involve massless fields. For massive fields, the soft behavior is nontrivial at orders beyond that under discussion for soft theorems. With that in mind, we will consider amplitudes arising from $F^3$, $R^3$, and $R^2\phi$, where the scalar field is a massless dilaton.

### A. Amplitudes from $F^3$

We first consider amplitudes that are generated by the self-dual contribution of a single $F^3$ operator, which have been studied for general multiplicity in [31]. Here, by self-dual we are referring to the part of the $F^3$ that produces an all-minus three-point amplitude.

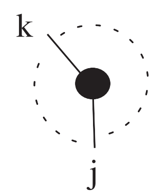
#### 1. CSW representation of $F^3$ Amplitudes

Using a CSW representation, the $n$-point $k$-minus helicity amplitude is given by a single $F^3_{\mathrm{SD}}$ vertex connected with $k-3$ YM MHV vertices [32]. Thus, there are two types of vertices in the CSW rule: (1) A white vertex, representing a $F^3_{\mathrm{SD}}$ vertex, with its associated MHV building block given by



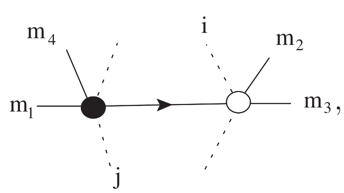

$$: \quad \frac{\langle jk\rangle^2\langle kl\rangle^2\langle lj\rangle^2}{\prod_{i=1}^n\langle ii+1\rangle} \tag{2.1}$$

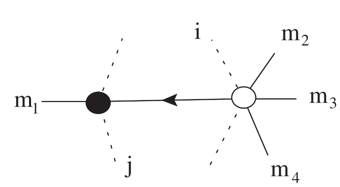
where the lines $j, k, l$ are the negative-helicity legs, and the dots represent positive-helicity legs. (2) A black vertex representing the usual YM MHV vertices,




$$: \quad \frac{\langle jk\rangle^4}{\prod_{i=1}^n\langle ii+1\rangle}\,. \tag{2.2}$$

Here we will consider diagrams with only one white vertex. For example the NMHV amplitude consists of two diagrams (here, $\mathrm{N}^k$MHV refers to $k+3$ minus helicity legs),

(a) :



(b) :



(2.3)

where the arrows on the propagator indicate to which vertex the negative helicity is associated. The dotted lines simply represent the legs that are adjacent to the propagator, and can be one of the minus legs. It is convenient to pull out an overall Parke-Taylor factor, so that the contributions from the above two diagrams are given by

$$
\begin{aligned}
&(a):\ \frac{1}{\prod_{l=1}^{n}\langle l l+1\rangle}\left(\frac{\langle m_1 m_4\rangle^4}{\langle i-1P\rangle\langle Pj\rangle}\frac{\langle i-1i\rangle\langle j-1j\rangle}{P^2}\frac{\langle m_2 m_3\rangle^2\langle m_3 P\rangle^2\langle P m_2\rangle^2}{\langle Pi\rangle\langle j-1P\rangle}\right)\\
&(b):\ \frac{1}{\prod_{l=1}^{n}\langle l l+1\rangle}\left(\frac{\langle m_1 P\rangle^4}{\langle i-1P\rangle\langle Pj\rangle}\frac{\langle i-1i\rangle\langle j-1j\rangle}{P^2}\frac{\langle m_2 m_3\rangle^2\langle m_3 m_4\rangle^2\langle m_4 m_2\rangle^2}{\langle Pi\rangle\langle j-1P\rangle}\right),
\end{aligned}
\tag{2.4}
$$

where $\langle P| = P|\mu]$ for some reference spinor $|\mu]$.

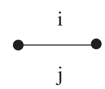
### 2. $F^3$ amplitudes in momentum-twistor space and recursions

To facilitate the analysis, we will now convert the expressions into momentum-twistor space [30]. This will allow us to reveal the fact that amplitudes of the $F^3$ operator with at least one plus-helicity leg respect a BCFW recursion. We write these (super) momentum twistors (with $4|\mathcal{N}$ components) as $\mathcal{Z}_a = (Z_a^I|\eta_a^A) = (\lambda_a^\alpha, \mu_a^{\dot\alpha}|\chi_a^A)$ for $a = 1, \ldots, n$, where for the bosonic part $Z_a^I$, the first two components are the holomorphic spinors $\lambda^\alpha$ and the remaining two components can be used to express the antiholomorphic spinors $\tilde\lambda^{\dot\alpha}$ as follows:

$$
\tilde\lambda_a^{\dot\alpha} = \frac{\mu_{a-1}^{\dot\alpha}}{\langle a a+1\rangle} + \frac{\langle a-1 a+1\rangle\mu_a^{\dot\alpha}}{\langle a-1a\rangle\langle a a+1\rangle} + \frac{\mu_{a+1}^{\dot\alpha}}{\langle a-1 a\rangle}, \tag{2.5}
$$

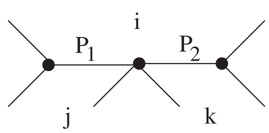
for $a = 1, \ldots, n$ with $a \pm 1$ modulo $n$. The Grassmann variables $\eta^A$ can be written as the same linear combination of the Grassmann part of the twistors $\chi^A$ as $\tilde\lambda^{\dot\alpha}$ of $\mu^{\dot\alpha}$.

The momentum-twistor space CSW prescription for on-shell spinors are as follows. Consider a propagator connecting two vertices defined by two regions $(i, j)$. In momentum-twistor space, they are given by




$$
:\quad \langle aP\rangle \equiv \frac{\langle a[i\rangle\langle i-1]jj-1*\rangle}{\langle ii-1\rangle\langle jj-1\rangle} = -\frac{\langle a[j\rangle\langle j-1]ii-1*\rangle}{\langle ii-1\rangle\langle jj-1\rangle} \tag{2.6}
$$

where the equality holds due to the fact that the reference twistor $Z_* = (0, \mu, 0)$. If two propagators are connected to the same vertex and adjacent, one then has




$$
\begin{aligned}
\langle P_1 P_2\rangle &= \frac{\langle ii-1\,(*jj-1)\cap(*kk-1)\rangle}{\langle ii-1\rangle\langle jj-1\rangle\langle kk-1\rangle} \equiv -\frac{\langle *jj-1[i\rangle\langle i-1]kk-1*\rangle}{\langle ii-1\rangle\langle jj-1\rangle\langle kk-1\rangle}\\
&= \frac{\langle *kk-1\widehat{i-1}\rangle}{\langle ii-1\rangle\langle jj-1\rangle\langle kk-1\rangle}
\end{aligned}
\tag{2.7}
$$

where in the final line $\widehat{i-1} \equiv (ii-1)\cap(*jj-1)$. These will be the fundamental identifications used throughout this paper.

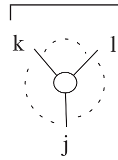
Using these identities, we find that the amplitudes in Eq. (2.4) can be rewritten in the following succinct form:

$$
\begin{aligned}
&(a):\ \frac{1}{\prod_{l=1}^{n}\langle ll+1\rangle}\langle m_1 m_4\rangle^4[ii-1jj-1*]\langle m_2 m_3\rangle^2\langle m_3\widehat{i-1}\rangle^2\langle\widehat{i-1}m_2\rangle^2\\
&(b):\ \frac{1}{\prod_{l=1}^{n}\langle ll+1\rangle}\langle m_1\widehat{i-1}\rangle^4[ii-1jj-1*]\langle m_2 m_3\rangle^2\langle m_3 m_4\rangle^2\langle m_4 m_2\rangle^2,
\end{aligned}
\tag{2.8}
$$

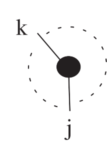
where $[ii-1jj-1*]$ is defined as

$$
[abcde] \equiv \frac{1}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle}. \tag{2.9}
$$

Thus for any CSW diagram, one simply replaces each propagator by a factor of $[*ii-1jj-1]$, while each black or white vertex is dressed with




$$
:\quad \langle jk\rangle^2\langle kl\rangle^2\langle lj\rangle^2,
$$




$$
:\quad \langle jk\rangle^4. \tag{2.10}
$$

Equipped with the momentum-twistor space representation, we will now show that if there is at least one plus-helicity leg, the result from CSW construction satisfies the

BCFW recursion relation similar to that in Yang-Mills theory [(we use $R$ to represent amplitudes with an overall $(\prod_i \langle i i+1\rangle)^{-1}$ stripped off],

$$R^{F^3}_{k,n} = R^{F^3}_{k,n-1} + \sum_j [n-1,n,1,j-1,j] R^{F^3}_{k',j}(1,\ldots,I_j) \times R^{F^2}_{k-1-k',n+2-j}(-I_j,\ldots,\hat{n}_j) + (F^3 \leftrightarrow F^2), \quad (2.11)$$

where $2<j<n$, $I_j = (j-1j)\cap(n-1,n,1)$, $\hat{n}_j = (n-1,n)\cap(1,j-1,j)$, and, similar to above, we have assumed leg $n$ to have positive helicity. Note that in momentum space this corresponds to the $[n-1n\rangle$ shift, for which we have explicitly checked that up to six points, the amplitudes listed in [31] indeed vanish at $z\to\infty$.

The proof proceeds exactly as in $\mathcal{N}=4$ SYM [33], namely, by judiciously choosing the reference twistor, one can show that the difference between the $(n+1)$- and $n$-point CSW representations, $R^{F^3}_{k,n} - R^{F^3}_{k,n-1}$, is given by the last two terms in Eq. (2.11). First, note that as the twistor $Z_n$ is a positive-helicity leg, it generically does not appear in the two expressions, and hence most of the terms cancel immediately. Let us first consider the NMHV tree amplitude, where the mismatch is given simply by

$$R^{F^3}_{k,n} - R^{F^3}_{k,n-1} = \left(\sum_j [*,n-1,n,j-1,j]\bar{X}(n-1,n,j) + \sum_j [*,n,1,j-1,j]\bar{X}(n,1,j) - \sum_j [*,n-1,1,j-1,j]\bar{X}(n-1,1,j)\right), \quad (2.12)$$

where $\bar{X}$ simply denotes the vertex factors for each diagram. Now, if we take $Z_* = Z_1$, the last two terms vanish. To be more precise, while the denominator of $[*,n-1,1,j-1,j]$ contains three zeroes, the factor $\bar{X}$ contains four factors of $\langle a_i P\rangle = \langle a_i [j\rangle\langle j-1]n1*\rangle$, which vanish as $*=1$. Thus the CSW representation for the NMHV tree-level amplitude is simply given as

$$R^{F^3}_{k,n} = R^{F^3}_{k,n-1} + \sum_j [n-1,n,1j-1,j]\bar{X}(n-1,n,j). \quad (2.13)$$

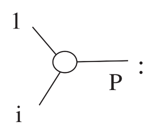
Note that the factor in $\bar{X}$ which involves the propagator leg $|P\rangle$ is evaluated at $(j-1j)\cap(n-1,n,1)$, i.e. it is given by $I_j$. Furthermore, since leg $n$ has positive helicity, it does not appear explicitly in the above representation and we are free to make the identification for $\hat{n}_j$.

For a general N$^k$MHV amplitude, the proof of equivalence again simply follows that of $\mathcal{N}=4$ SYM given in [33]. The classification of all CSW diagrams is given by a collection of $2k$ set of region momenta, separated into $k$ noncrossing pairs. The difference $R^{F^3}_{k,n} - R^{F^3}_{k,n-1}$ is given by CSW diagrams where one of the noncrossing pairs is $(2,i)$. The remaining pairs factorize. Distinct choices of $i$ can then be mapped into distinct helicity distributions in the BCFW recursion. Again the only difference between the $\mathcal{N}=4$ and the present case is the presence of the $\bar{X}$ factors arising from each vertex.

### 3. Soft limits of $F^3$ amplitudes

We now consider the soft limits of $F^3$ amplitudes. Note that the recursion formula derived from above assumes that there is at least one plus-helicity leg, $n$. This is no longer valid for the all-minus amplitude that is also generated by $F^3_{SD}$. We will first treat such amplitudes separately. The limit to analyze is the antiholomorphic soft limit, whose tree-level behavior is simply the complex conjugate of Eq. (1.9), and thus starts at $\delta^{-2}$.

Fortunately, it is straightforward to study the antiholomorphic soft minus gluon limit in the CSW representation, since the only place where antiholomorphic spinors appear in the CSW representation is in the propagators and $\langle P|$. With a generic reference spinor, the only singularities that appear are associated with the soft leg attached to a three-point vertex with another external leg. If the three-point vertex is an $F^3$, then one has (with the propagator included)




$$: \quad \frac{\langle 1P\rangle\langle Pi\rangle}{[i1]} \to \frac{\langle 1i\rangle[i\eta][\eta 1]\langle 1i\rangle}{[i1]}$$

which is finite for the soft leg 1 and thus does not contribute to the leading or subleading soft behavior. This is just a reflection of the fact that $F^3$ operator is higher dimensional and suppresses the soft divergence. If the three-point vertex is the usual MHV vertex, then the soft theorem simply follows from Low's analysis (or by expanding MHV diagrams to the subleading order).

Now consider the recursion in Eq. (2.11), and take the positive-helicity leg $n$ to be soft. We approach the soft limit by deforming

$$Z_n \to \alpha Z_{n-1} + \beta Z_1 + \delta Z_s, \quad (2.14)$$

where $\delta$ is the soft parameter. To see why this corresponds to the soft limit, from Eq. (2.5), observe that the deformation in Eq. (2.14) leads to

$$\tilde{\lambda}_n = \delta \frac{\langle n-11\rangle\mu_s + \langle 1s\rangle\mu_{n-1} + \langle sn-1\rangle\mu_1}{\langle 1n-1\rangle^2\alpha\beta}, \quad (2.15)$$

and thus implies that this corresponds to the antiholomorphic soft limit. Furthermore, since $\tilde{\lambda}_a$ is determined by the twistors $(Z_{a-1}, Z_a, Z_{a+1})$, the deformation in Eq. (2.14) corresponds to deforming $\tilde{\lambda}_{n-1}$ and $\tilde{\lambda}_1$ as well, i.e. the momentum conservation is preserved by having all $a \neq (n-1,1)$ $\tilde{\lambda}_a$'s fixed and solving $\tilde{\lambda}_{n-1}$ and $\tilde{\lambda}_1$ in terms of them. This is precisely the prescription that leads to vanishing subleading soft corrections, as discussed in Sec. I A, which can now be written in momentum-twistor space,

$$\begin{aligned} n\,\text{points:}\ &\{Z_1,\ldots,Z_{n-1},Z_n=\alpha Z_{n-1}+\beta Z_1+\delta Z_s\};\\ (n-1)\quad &\text{points:}\ \{Z_1,\ldots,Z_{n-1}\}. \end{aligned} \tag{2.16}$$

In the soft limit the shifted momentum twistor $I_j$ behaves as $I_j \to \delta(j-1j)\cap(n-1,s,1)$, while all other variables remain unchanged.[2] Let us first look at the factorization terms in Eq. (2.11). The prefactor $[n,1,2,j-1,j]$ behaves as $\delta^{-2}$,

$$\begin{aligned} &[n-1,n,1,j-1,j]\\ &\quad=-\frac{1}{\delta^2\alpha\beta\langle n-11sj-1\rangle\langle n-11sj\rangle\langle n-11j-1j\rangle^3}\\ &\qquad+\mathcal{O}(\delta^{-1}). \end{aligned} \tag{2.17}$$

On the other hand, $I_j$ appears in the tree amplitude on both sides as $\langle I_j x\rangle$ with degree 4 in $\langle I_j|$. Thus the overall result of the factorization terms is of degree $\mathcal{O}(\delta^2)$, and in the anti-holomorphic soft limit, we find

$$R^{F^3}_{k,n}=R^{F^3}_{k,n-1}+\mathcal{O}(\delta^2). \tag{2.18}$$

Putting the stripped Parke-Taylor factor $(\prod_i\langle ii+1\rangle)$ back into the expression, we see the above result is exactly the tree-level Yang-Mills soft theorem in Eq. (1.11).

### B. Higher-derivative gravitational interactions and their soft limits

From the previous discussion, we have seen via both heuristic arguments and explicit analysis that higher-derivative operators do not modify soft theorems, due to their suppression at small momenta. Extending the argument to gravity, one would reach the same conclusion as gravity operators are further suppressed. However, it is easy to see that this is not always true. Consider the tensoring of two $F^3$ scattering amplitudes via KLT relations [28]. The explicit amplitude up to six points was given in [31]. Take for example

$$\begin{aligned} &M(1^-,2^-,3^-,4^-,5^+)\\ &\quad=is_{12}s_{34}A^{F^3}(1^-,2^-,3^-,4^-,5^+)A^{F^3}(2^-,1^-,4^-,3^-,5^+)\\ &\qquad+\mathcal{P}(2,3). \end{aligned} \tag{2.19}$$

It is straightforward to verify that

$$M(1^-,2^-,3^-,4^-,5^+)|_{\lambda_5\to\delta\lambda_5}=\sum_{i=0}^{2}\frac{1}{\delta^{3-i}}S^{(i)}_{\mathrm{G}}(5)M_4+\mathcal{O}(\delta^0), \tag{2.20}$$

where $M_4=M_4(1^-,2^-,3^-,4^-)$. However, taking the anti-holomorphic soft limit on leg 1, we find

$$\begin{aligned} &M(1^-,2^-,3^-,4^-,5^+)|_{\tilde{\lambda}_1\to\delta\tilde{\lambda}_1}\\ &\quad=\sum_{i=0}^{2}\frac{1}{\delta^{3-i}}S^{(i)}_{\mathrm{G}}(1)M_4+\frac{1}{\delta}\Delta^{(2)}+\mathcal{O}(\delta^0), \end{aligned} \tag{2.21}$$

where, now, $M_4=M_4(2^-,3^-,4^-,5^+)$, and $\Delta^{(2)}$ is an unknown correction to $S^{(2)}_{\mathrm{G}}$. The fact that $S^{(2)}_{\mathrm{G}}$ is violated can be traced back to the presence of a dilaton exchange induced by the higher-dimensional operator $\phi R^2$. Using string theory language the operator $F^3$ is of order $\alpha'$, and thus via KLT one obtains an amplitude that is of order $\alpha'^2$ in the effective field theory. This receives a contribution from $R^3$, which is of order $\alpha'^2$, and two insertions of $\phi R^2$, each of order $\alpha'$. Let us consider the exchange of a dilaton between a $\phi R^2$ vertex and a tree diagram associated with a single $\phi R^2$ operator. In the mostly minus amplitude, the two gravitons on the $\phi R^2$ vertex must be of negative helicity, and the contribution is proportional to

$$\frac{\langle 12\rangle^3}{[12]}\times M_n(\tilde{\phi}), \tag{2.22}$$

where $M_n(\tilde{\phi})$ is a tree-level diagram with the dilaton leg off-shell. As one can see, taking either leg to be soft, one finds a $\frac{1}{\delta}$ contribution proportional to the tree-level amplitude generated by $\phi R^2$. The latter can be easily obtained by KLT tensoring the $F^3$ amplitude with the usual YM $F^2$ amplitude. Indeed, the modification for $S^{(2)}_{\mathrm{G}}$ is precisely given by

$$\Delta^{(2)}=\sum_j -2\frac{\langle 1j\rangle^3}{[1j]}M_n(\phi,i_1^-,i_2^-,\ldots,i_{n-2}^-,n^+), \tag{2.23}$$

where $j$ runs over all remaining minus helicity legs, and $(i_1^-,\ldots,i_{n-2}^-)\neq j$. With this modification we indeed reproduce the correct $\delta^{-1}$ term in Eq. (2.21).[3] Note that

[2] Except for $\hat{n}_j=(n-1,n)\cap(1,j-1j)+\mathcal{O}(\delta)$, but $\hat{n}_j$ never explicitly appears in the expression.

[3] We will find the same conclusion in Sec. III C for bosonic closed-string amplitudes via BCFW recursion relations.

this also explains why the plus-helicity soft limit of the amplitude in Eq. (2.19) does not require modification: for the presence of $\phi R^2$ to appear in the positive-helicity soft channel, there must be at least two positive-helicity legs. Such corrections to the subleading term are very similar to the corrections present in the single-minus amplitude of QCD [25], where the correction term is proportional to a lower-point amplitude with one of the states replaced due to the presence of a new effective vertex.

While the above operators can be ruled out at tree level for supersymmetric theories, such operators can still be generated via anomalies at loop level in supergravity theories. Indeed, the $U(1)$ anomaly in $\mathcal{N}=4$ supergravity is known to generate a term in the effective action that is of the form $(R^+)^2\bar{t}$ [27], where $R^+$ is the anti-self-dual part of the (linearized) Riemann tensor and $\bar{t}$ is the scalar that lies in the same on-shell multiplet as $h^{++}$. Again, amplitudes involving the insertion of $(R^+)^2\bar{t}$ and $(R^-)^2t$ will also encounter the same subleading soft corrections as mentioned before. This would imply, among other things, that the two-loop four-point MHV amplitude will require corrections to $S_{\rm G}^{(2)}$ due to the presence of this term in the effective action, on top of those necessary due to the presence of IR divergences.

## III. SOFT THEOREMS FOR TREE_LEVEL AMPLITUDES IN STRING THEORY

In this section, we will discuss the soft theorem for superstring amplitudes. We will begin with explicit four- and five-point examples in both open- and closed-string theories. After establishing the soft theorem for lower-point amplitudes, we will give a general argument based on BCFW recursion relations of string amplitudes. Furthermore, in Sec. IV, we will present yet another independent analysis of the soft theorems in string amplitudes based on the OPE of world sheet vertex operators.

### A. Soft theorem for open-string amplitudes: Four- and five-point examples

A general $n$-point color-ordered open string gluon amplitude at tree level can be expressed in terms of a basis of $(n-3)!$ functions [34,35],

$$\mathcal{A}(1,2,\ldots,n)=\sum_{\sigma\in S_{n-3}}F^{(2_\sigma,\ldots,(n-2)_\sigma)}\times A_{\rm YM}(1,2_\sigma,\ldots,(n-2)_\sigma,n-1,n), \tag{3.1}$$

where multiple hypergeometric functions are given by

$$F^{(2,\ldots,n-2)}=(-1)^{n-3}\int_{z_i<z_{i+1}}\prod_{j=2}^{n-2}dz_j\left(\prod|z_{il}|^{s_{il}}\right)\left(\prod_{k=2}^{[n/2]}\sum_{m=1}^{k-1}\frac{s_{mk}}{z_{mk}}\right)\left(\prod_{k=[n/2]+1}^{n-2}\sum_{m=k+1}^{n-1}\frac{s_{km}}{z_{km}}\right),$$

where the Mandelstam variables are defined as $s_{ij}\equiv\alpha'(k_i+k_j)^2$. Here we have fixed SL(2) symmetry by choosing $z_1=0, z_{n-1}=1$ and $z_n=\infty$. From the general expression (3.1), we find the four-point amplitude

$$\mathcal{A}(1,2,3,4)=F^{(2)}A_{\rm YM}(1,2,3,4), \tag{3.2}$$

with

$$F^{(2)}=s_{12}\int_0^1 dz_2 z_2^{s_{12}-1}(1-z_2)^{s_{23}}=\frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})}. \tag{3.3}$$

Using the fact that, in soft limit $k_2\to\delta k_2$ with $\delta\to 0$,

$$\frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})}=1+\mathcal{O}(\delta^2), \tag{3.4}$$

it is easy to see that $\mathcal{A}(1,2,3,4)$ satisfies the soft theorem, since $S_{\rm YM}^{(1)}(123)\mathcal{A}(134)=0$.

Let us now move on to the study of the soft limit for the five-point amplitude, which can be written as

$$\mathcal{A}(1,2,3,4,5)=F^{(2,3)}A_{\rm YM}(1,2,3,4,5)+F^{(3,2)}A_{\rm YM}(1,3,2,4,5), \tag{3.5}$$

where

$$\begin{aligned}F^{(2,3)}&=s_{12}s_{34}\int_0^1 dz_2\times\int_{z_2}^1 dz_3 z_2^{s_{12}-1}z_3^{s_{13}}z_{32}^{s_{23}}(1-z_2)^{s_{24}}(1-z_3)^{s_{34}-1},\\ F^{(3,2)}&=s_{13}s_{24}\int_0^1 dz_2\times\int_{z_2}^1 dz_3 z_2^{s_{12}}z_3^{s_{13}-1}z_{32}^{s_{23}}(1-z_2)^{s_{24}-1}(1-z_3)^{s_{34}},\end{aligned} \tag{3.6}$$

with $z_{32}=z_3-z_2$.

In $D=4$, we can take $k_{n-2}=k_3$ to be soft and solve for $\tilde{\lambda}_4$ and $\tilde{\lambda}_5$ using momentum conservation,

$$\tilde{\lambda}_4 = \frac{\langle 5|(1+2)}{\langle 45\rangle} + \delta\frac{\langle 5|3}{\langle 45\rangle}, \qquad \tilde{\lambda}_5 = \frac{\langle 4|(1+2)}{\langle 54\rangle} + \delta\frac{\langle 4|3}{\langle 54\rangle}, \tag{3.7}$$

from which we can conveniently define

$$k'_4 = \frac{|4\rangle\langle 5|(1+2)}{\langle 45\rangle}, \qquad p_4 = \frac{|4\rangle\langle 5|3}{\langle 45\rangle}, \tag{3.8}$$

$$k'_5 = \frac{|5\rangle\langle 4|(1+2)}{\langle 54\rangle}, \qquad p_5 = \frac{|5\rangle\langle 4|3}{\langle 54\rangle}. \tag{3.9}$$

Integrating over $z_3$ and keeping terms up to subleading order, we obtain

$$F_S^{(2,3)} = s_{12}\int_0^1 dz_2 z_2^{s_{12}-1}(1-z_2)^{s_{24'}} \times [1 + \delta(s_{23} + s_{34'} + s_{2p_4})\log(1-z_2)]. \tag{3.10}$$

The leading term simply gives $F(1,2,4',5')$, which appears in the four-point amplitude, and leads to[4]

$$\frac{1}{\delta^2} S^{(0)}_{\mathrm{YM}}(234)\mathcal{A}(1,2,4',5'). \tag{3.11}$$

In contrast, the subleading term, denoted by $F^{(2,3)}_{S^{(1)}}$, reads

$$F^{(2,3)}_{S^{(1)}} = \frac{\langle 34\rangle\langle 51\rangle[31]}{\langle 45\rangle} s_{12} \times \int_0^1 dz_2 z_2^{s_{12}-1}(1-z_2)^{s_{24'}}\log(1-z_2), \tag{3.12}$$

where the identity $s_{23} + s_{34'} + s_{2p_4} = \frac{\langle 34\rangle\langle 51\rangle[31]}{\langle 45\rangle}$ has been used. The above integral can be computed straightforwardly; however, this is not necessary for our purposes, as we will compare its expression with $S^{(1)}_{\mathrm{YM}}(234)\mathcal{A}(1,2,4',5')$ at the level of integrands. Similar consideration applies to $F^{(3,2)}$, which has a subleading contribution only, given by

$$F^{(3,2)}_{S^{(1)}} = -s_{13}s_{24'}\int_0^1 dz_2 z_2^{s_{12}}(1-z_2)^{s_{24'}-1}\log(z_2). \tag{3.13}$$

Combining the two contributions and expanding $\mathcal{A}(1,2,3,4',5')$, we find the subleading term

$$A_{\mathrm{YM}}(1,2,4',5')\frac{1}{\delta}\left(\frac{\langle 24\rangle}{\langle 23\rangle\langle 34\rangle}F^{(2,3)}_{S^{(1)}} + \frac{\langle 12\rangle}{\langle 13\rangle\langle 32\rangle}F^{(3,2)}_{S^{(1)}}\right). \tag{3.14}$$

[4] The leading soft-limit term for $n$-point amplitudes was analyzed in [34].

Now we are ready to compare this with the result of the soft operator acting on the four-point string amplitude,

$$\begin{aligned} S^{(1)}_{\mathrm{YM}}&(234)\mathcal{A}(1,2,4',5') \\ &= \left(\frac{1}{\langle 23\rangle}\tilde{\lambda}_3\cdot\frac{\partial}{\partial\tilde{\lambda}_2} + \frac{1}{\langle 34\rangle}\tilde{\lambda}_3\cdot\frac{\partial}{\partial\tilde{\lambda}_4}\right)\mathcal{A}(1,2,4',5') \\ &= \left(\frac{\langle 24\rangle\langle 51\rangle[31]}{\langle 23\rangle\langle 45\rangle}\frac{\partial}{\partial s_{24'}} + \frac{\langle 12\rangle[13]}{\langle 23\rangle}\frac{\partial}{\partial s_{12}}\right)\mathcal{A}(1,2,4',5'), \end{aligned} \tag{3.15}$$

where it is understood that $\tilde{\lambda}_4$ and $\tilde{\lambda}_5$ are solved by momentum conservation, and thus the result of the action of $\frac{\partial}{\partial\tilde{\lambda}_4}$ on the amplitude vanishes. Now it is straightforward to see that

$$\begin{aligned} \frac{\langle 24\rangle}{\langle 23\rangle\langle 34\rangle}F^{(2,3)}_{S^{(1)}} &= \frac{\langle 24\rangle\langle 51\rangle[31]}{\langle 23\rangle\langle 45\rangle}\frac{\partial}{\partial s_{24'}}F^{(2)}(1,2,4',5') \\ \frac{\langle 12\rangle}{\langle 13\rangle\langle 32\rangle}F^{(3,2)}_{S^{(1)}} &= \frac{\langle 12\rangle[13]}{\langle 23\rangle}\frac{\partial}{\partial s_{12}}F^{(2)}(1,2,4',5'), \end{aligned} \tag{3.16}$$

where $F^{(2)}(1,2,4',5')$ is given in (3.3). In order to check the validity of the second line in the above equation, it is convenient to use

$$F^{(2)}(1,2,4',5') = s_{24'}\int_0^1 dz_2 z_2^{s_{12}}(1-z_2)^{s_{24'}-1}. \tag{3.17}$$

This thus establishes the soft theorem for the five-point open-superstring amplitude. Similar direct analysis can be applied to higher-point amplitudes; we have checked analytically that (3.1) satisfies the soft theorem for six points, see Appendix B.

### B. Soft theorem for closed-string amplitudes: Four- and five-point examples

The tree-level closed-string amplitude can be written in terms of open-string tree amplitudes via KLT relations [28,36],

$$\begin{aligned} \mathcal{M}_n = \pi^{3-n}&\mathcal{A}_n(1,2,\ldots,n) \\ &\times\sum_{\{i\},\{j\}} f(i_1,\ldots,i_{\lfloor\frac{n}{2}\rfloor-1})\bar{f}(j_1,\ldots,j_{\lfloor\frac{n}{2}\rfloor-2})] \\ &\times\mathcal{A}_n(\{i\},1,n-1,\{j\},n) + \mathrm{Perm}(2,\ldots,n-2), \end{aligned} \tag{3.18}$$

where the sum inside the bracket is over $\{i\}\in\mathrm{Perm}(2,\ldots,\lfloor\frac{n}{2}\rfloor)$, $\{j\}\in\mathrm{Perm}(\lfloor\frac{n}{2}\rfloor+1,\ldots,n-2)$, and the functions $f$ and $\bar{f}$ are defined as

$$f(i_1,\ldots,i_m)=\sin(\pi s_{1i_m})\prod_{k=1}^{m-1}\sin\pi\left(s_{1i_k}+\sum_{l=k+1}^{m}g(i_k,i_l)\right),$$
$$\bar f(j_1,\ldots,j_m)=\sin(\pi s_{j_1 n-1})\prod_{k=2}^{m}\sin\pi\left(s_{j_k n-1}+\sum_{l=1}^{k-1}g(j_l,j_k)\right), \tag{3.19}$$

with $g(i,j)=s_{ij}$ for $i>j$ and 0 otherwise. For four points, we have

$$\mathcal{M}_4(\{1,2,3,4\}) = \pi^{-1}\sin(\pi s_{12})\mathcal{A}_4(1,2,3,4)\mathcal{A}_4(2,1,3,4). \tag{3.20}$$

Considering the soft limit $k_2\to\delta k_2, \delta\to 0$, we find

$$\begin{aligned}\mathcal{M}_4(\{1,2,3,4\})|_{\rm div} &= \frac{1}{\delta^3}S^{(0)}_{\rm YM}(123)S^{(0)}_{\rm YM}(421)\\ &\quad\times\mathcal{A}_3^2(1,3,4)(s_{12}-\delta^2\zeta_2 s_{12}^2(s_{12}+s_{23}+s_{24}))\\ &= \frac{1}{\delta^3}s_{12}S^{(0)}_{\rm YM}(123)S^{(0)}_{\rm YM}(421)\mathcal{M}_3(1,3,4),\end{aligned} \tag{3.21}$$

where momentum conservation has been used in the last step. Thanks to

$$s_{12}S^{(0)}_{\rm YM}(123)S^{(0)}_{\rm YM}(421)=S^{(0)}_{\rm G}(2),$$
$$S^{(1)}_{\rm G}(2)\mathcal{M}_3(1,3,4)=S^{(2)}_{\rm G}(2)\mathcal{M}_3(1,3,4)=0,$$

we find that the closed-string four-point amplitude satisfies the soft theorem.

We then study the closed-string amplitude at five points, which again can be expressed via KLT relations

$$\begin{aligned}\mathcal{M}_5(\{1,2,3,4,5\}) &= \pi^{-2}(\mathcal{A}_5(1,2,3,4,5)\mathcal{A}_5(2,1,4,3,5)\\ &\quad\times\sin(\pi s_{12})\sin(\pi s_{34})\\ &\quad+\mathcal{A}_5(1,3,2,4,5)\mathcal{A}_5(3,1,4,2,5)\\ &\quad\times\sin(\pi s_{13})\sin(\pi s_{24})).\end{aligned} \tag{3.22}$$

We will take leg 3 to be soft, and with four-dimensional kinematics we solve $\tilde\lambda_4$ and $\tilde\lambda_5$ using momentum conservation, with $k_4', k_5'$ and $p_4, p_5$ defined as in (3.8).

At the leading order, we have $\sin(\pi s_{3i})=\pi s_{3i}+\mathcal{O}(\delta^3)$, and using the leading soft theorem for open-string amplitudes we have (if we take the holomorphic limit)

$$\begin{aligned}\mathcal{M}_5 &= \delta^{-3}s_{34'}S^{(0)}_{\rm YM}(2,3,4')S^{(0)}_{\rm YM}(4',3,5')[\pi^{-1}\sin(\pi s_{12})\mathcal{A}_4(1,2,4',5')\mathcal{A}_4(2,1,4',5')]\\ &\quad+\delta^{-3}s_{13}S^{(0)}_{\rm YM}(1,3,2)S^{(0)}_{\rm YM}(5',3,1)[\pi^{-1}\sin(\pi s_{24'})\mathcal{A}_4(1,2,4',5')\mathcal{A}_4(4',2,5',1)]+\mathcal{O}(\delta^{-2}),\end{aligned} \tag{3.23}$$

where we recognize that the two combinations inside square brackets are two KLT representations of the same four-point amplitude, $\mathcal{M}_4(\{1,2,4',5'\})$, and the prefactors combine to the leading gravity soft factor

$$S^{(0)}_{\rm G}(3)=\sum_{i=1}^{5}\frac{[3i]}{\langle 3i\rangle}\frac{\langle xi\rangle\langle yi\rangle}{\langle x3\rangle\langle y3\rangle}=\sum_{i=1,4}s_{3i}\frac{\langle i2\rangle\langle i5\rangle}{\langle i3\rangle^2\langle 32\rangle\langle 35\rangle}, \tag{3.24}$$

where we have used the four-dimensional form of $S^{(0)}_G$ and for the gauge choice we choose $x=2, y=5$.

The subleading order of Eq. (3.22) receives contribution from the subleading order of $\mathcal{A}_5$'s: for the first term, we have $\frac{\partial}{\partial\tilde\lambda_2}$ in $S^{(1)}_{\rm YM}(2,3,4')\mathcal{A}_4(1,2,4',5')$, and for the second term, $\frac{\partial}{\partial\tilde\lambda_{1,2}}$ in $S^{(1)}_{\rm YM}(1,3,2)\mathcal{A}_4(1,2,4',5')$ and $\frac{\partial}{\partial\tilde\lambda_1}$ in $S^{(1)}_{\rm YM}(5',3,1)\mathcal{A}_4(4',2,5',1)$. Combining these terms and the subleading term from $\sin(\pi s_{24})=\sin(\pi s_{24'})+\delta\pi\cos(\pi s_{24'})s_{2p_4}$, we find

$$\begin{aligned}\mathcal{M}_5|_{\mathcal{O}(\delta^{-2})} &= \pi^{-1}\frac{1}{\langle 23\rangle}\tilde\lambda_3\cdot\frac{\partial\mathcal{A}_4(1,2,4',5')}{\partial\tilde\lambda_2}\left(\sin(\pi s_{12})\frac{[34']\langle 4'5'\rangle}{\langle 35'\rangle}\mathcal{A}_4(2,1,4',5')-\sin(\pi s_{24'})\frac{[13]\langle 5'1\rangle}{\langle 5'3\rangle}\mathcal{A}_4(4',2,5',1)\right)\\ &\quad+\pi^{-1}\sin(\pi s_{24'})\frac{1}{\langle 13\rangle}\tilde\lambda_3\cdot\frac{\partial\mathcal{A}_4(1,2,4',5')}{\partial\tilde\lambda_1}\frac{[13]\langle 5'1\rangle}{\langle 5'3\rangle}\mathcal{A}_4(4',2,5',1)\\ &\quad+\pi^{-1}\sin(\pi s_{24'})\frac{1}{\langle 13\rangle}\tilde\lambda_3\cdot\frac{\partial\mathcal{A}_4(4',2,5',1)}{\partial\tilde\lambda_1}\frac{[1,3]\langle 21\rangle}{\langle 23\rangle}\mathcal{A}_4(1,2,4',5')\\ &\quad-\cos(\pi s_{24'})\tilde\lambda_3\cdot\frac{\partial s_{2,4'}}{\partial\tilde\lambda_1}\frac{[13]\langle 12\rangle}{\langle 13\rangle\langle 32\rangle}\mathcal{A}_4(1,2,4',5')\mathcal{A}_4(4',2,5',1),\end{aligned} \tag{3.25}$$

where on the last line we have rewritten $s_{2p_4}S^{(0)}_{\rm YM}(132)S^{(0)}_{\rm YM}(5'31)$ as a derivative operator acting on $s_{24'}$.

Now we compare this with $S_G^{(1)}(3)M_4$, which is given by

$$\frac{1}{2}\sum_{i=1,i\neq 3}^{5}\frac{[3i]}{\langle 3i\rangle}\left(\frac{\langle xi\rangle}{\langle x3\rangle}+\frac{\langle yi\rangle}{\langle y3\rangle}\right)\tilde{\lambda}_3^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\alpha}}}M_4. \tag{3.26}$$

The crucial step in dealing with the big bracket in (3.25) is the use of the monodromy relation

$$\sin(\pi s_{12})\mathcal{A}_4(2,1,4',5') = \sin(\pi s_{24'})\mathcal{A}_4(4',2,5',1), \tag{3.27}$$

in order to simplify it to $\frac{[32]\langle 25'\rangle}{\langle 5'3\rangle}\sin(\pi s_{24'})\mathcal{A}_4(4',2,5',1)$. This in turn can be combined with the third line to produce $(S_{\rm G}^{(1)}(3)\mathcal{A}_4(1,2,4',5'))\sin(\pi s_{1,2})\mathcal{A}_5(4',2,5',1)$ with the gauge choice $x=y=5$. Since $S_{\rm G}^{(1)}(3)$ is gauge invariant, we can make a different choice $x=y=2$; in this form the result is simply $S_{\rm G}^{(1)}(3)$ acting on the second KLT representation of $\mathcal{M}_4$ in Eq. (3.23),

$$\begin{aligned}\mathcal{M}_5|_{\mathcal{O}(\delta^{-2})} &= \frac{[13]\langle 12\rangle}{\langle 13\rangle\langle 32\rangle}\tilde{\lambda}_3\cdot\frac{\partial}{\partial\tilde{\lambda}_1}\\ &\quad\times[\pi^{-1}\sin(\pi s_{24'})\mathcal{A}_4(1,2,4',5')\mathcal{A}_4(4',2,5',1)]\\ &= S_{\rm G}^{(1)}(3)\mathcal{M}_4(\{1,2,4',5'\}).\end{aligned} \tag{3.28}$$

Finally, we move to the order $\mathcal{O}(\delta^{-1})$, where one needs to consider the product of subleading contributions from the $\mathcal{A}_5$'s, the sub-subleading contribution from the sin factors, and the sub-subleading contribution from either of the $\mathcal{A}_5$'s. We have worked out all contributions analytically (the details can be found in Appendix C); checked numerically against $S_{\rm G}^{(2)}(3)\mathcal{M}_4(\{1,2,4',5'\})$, we found perfect agreement.

Two comments regarding closed-string soft theorems are in order. First, we believe that the pattern we observed in the proof for $S_G^{(0)}$ and $S_G^{(1)}$ at five points can be generalized to higher points. It would be desirable to explicitly check these first two orders of the soft graviton theorem by KLT relations and the repeated use of monodromy relations.

In addition, we want to stress that the agreement at sub-subleading order, unlike the first two orders, is not a direct consequence of KLT and monodromy relations. In particular, in the KLT representation the agreement involves nonuniversal sub-subleading soft behavior of open-string amplitudes, and it would be interesting to better understand how they combine nicely into the universal $S_G^{(2)}$ acting on the lower-point amplitude.

### C. Soft theorems of string amplitudes from BCFW recursion relations

In this section we will give a general argument for the soft theorems in string theories based on BCFW recursion relations. BCFW recursion relations for scattering amplitudes in field theories [7,8] have been generalized to open- and closed-string amplitudes [37,38].[5] For instance, for the color-ordered open-string amplitudes, one has

$$\begin{aligned}\mathcal{A}(1,2,\ldots,n-1,n) = \sum_i\sum_{\text{states I}}\mathcal{A}_L(\hat{1},2,\ldots,i,I)\frac{1}{k_I^2+m_I^2}\\ \times\mathcal{A}_R(-I,i+1,\ldots,\hat{n}).\end{aligned} \tag{3.29}$$

In practice, since the sum runs over an infinite number of states, the recursion may not be so useful for computing scattering amplitudes in string theories. (See Refs. [39,40] for recent developments on the application of BCFW recursion relations in string amplitudes.) However, the above recursion relation is very useful for our purpose of proving the soft theorems. Here we take holomorphic soft limit on leg 1. First, the terms with $i>2$ in the recursion relation (3.29) are regular, just as the recursion relations for field theories. As for the case when $i=2$, the crucial observation is that only massless states can contribute to the soft limit, since the singularity arises from $\frac{1}{k_I^2+m_I^2}$. Thanks to the recursion relation, in the soft limit, the divergent part of an open-superstring amplitude reduces to

$$\mathcal{A}(1,2,\ldots,n-1,n)|_{\rm div} = \mathcal{A}_3(\hat{1},2,I)\frac{1}{k_I^2}\mathcal{A}_{n-1}(-I,3,\ldots,\hat{n}). \tag{3.30}$$

Note that the internal state is a massless gluon now. Since the three-point open-superstring amplitude is identical to the one in SYM, we see that the result of this particular BCFW channel takes the same form as for Yang-Mills amplitudes, i.e. Eq. (1.9),

$$\begin{aligned}\mathcal{A}(1,2,\ldots,n-1,n)|_{\rm div} = \left(\frac{1}{\delta^2}S_{\rm YM}^{(0)}(n12)+\frac{1}{\delta}S_{\rm YM}^{(1)}(n12)\right)\\ \times\mathcal{A}_{n-1}(2,3,\ldots,n)\end{aligned} \tag{3.31}$$

The same argument applies to closed-superstring amplitudes.

The BCFW argument can also apply to bosonic string amplitudes. For the case of open strings, the conclusion is the same since there is no other massless state, except for the gluon. In contrast, for bosonic closed strings as well as heterotic strings, in addition to the graviton we have also the massless dilaton (the Kalb-Ramond field does not contribute since there is no three-point amplitude with two gravitons and a Kalb-Ramond field), which could contribute to $\mathcal{M}_n|_{\rm div}$. The contribution of the dilaton $\phi$ is of order $\mathcal{O}(\delta^{-1})$, and spoils the $S_{\rm G}^{(2)}\mathcal{M}_{n-1}$ term by a factor of

[5] We are aware that the recursion relation has only been explicitly checked to be correct for a few examples.

$$\mathcal{M}^{\phi}(1^+,2,\ldots,n-1,n)|_{\rm div} = \sum_i \mathcal{M}_3(\hat{1}^+,i^+,I)\frac{1}{k_I^2} \times \mathcal{M}_{n-1}(-I,3,\ldots,\hat{n}) = -\frac{2}{\delta}\sum_i \frac{[1i]^3}{\langle 1i\rangle}\mathcal{M}_{n-1}(\phi,3,\ldots,\hat{n}),$$

where we have emphasized the fact that only the amplitude with helicity $(h^{++},h^{++},\phi)$ (or its conjugate) is nonvanishing by making helicity dependence explicit.

## IV. SOFT LIMIT OF SUPERSTRING AMPLITUDES: WORLD SHEET ANALYSIS

We here discuss how to derive soft theorems for string amplitudes from the perspective of world sheet OPE in the Neveu-Schwarz Ramond (NS-R) approach. The analysis can be systematized and, in principle, one can even derive further subleading terms and investigate their universality.

### A. Preliminaries

The Euclidean world sheet is parameterized by the coordinates $z=e^w$, $w=\tau+i\sigma$, where for open strings $\sigma\in[0,\pi]$, $\tau\in(-\infty,+\infty)$, and for closed strings we have $\sigma\in[0,2\pi]$, $\tau\in(-\infty,+\infty)$. For convenience, we will use units such as $2\alpha'=1$ for open strings and $\alpha'=2$ for closed strings [41].

We will analyze both the bosonic string and the superstring. For the open bosonic string, the vertex operator for a massless vector boson is

$$V_A = (\epsilon\cdot\partial X)e^{ikX}, \tag{4.1}$$

where $k^2=\epsilon\cdot k=0$. Similarly, for the closed bosonic string, the graviton vertex operator is

$$V_G = E_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu e^{ikX}, \tag{4.2}$$

where $E_{\mu\nu}=E_{\nu\mu}$, $k^2=k^\mu E_{\mu\nu}=g^{\mu\nu}E_{\nu\mu}=0$. In explicit computations, it is often convenient to set $E_{\mu\nu}=\epsilon_\mu\epsilon_\nu$ and factorize the vertex into two chiral parts.

In the Neveu-Schwarz (NS) sector of the superstring, the vertex operator for a gauge boson in the (-1) superghost picture is

$$V_A^{(-1)} = (\epsilon\cdot\psi)e^{-\varphi}e^{ikX}, \tag{4.3}$$

where $\varphi$ is the boson for the superghosts. For the graviton, one has

$$V_G^{(-1,-1)} = E_{\mu\nu}\psi^\mu\tilde{\psi}^\nu e^{-\varphi}e^{-\tilde{\varphi}}e^{ikX}. \tag{4.4}$$

The vertex operators in the (0) picture are

$$V_A^{(0)} = (i\epsilon\cdot\partial X + k\cdot\psi\epsilon\cdot\psi)e^{ikX} \tag{4.5}$$

and

$$V_G^{(0,0)} = E_{\mu\nu}(i\partial X^\mu + k\cdot\psi\psi^\mu)(i\bar{\partial}X^\mu + k\cdot\tilde{\psi}\tilde{\psi}^\mu)e^{ikX}. \tag{4.6}$$

We will use the following normalization for the correlators:

$$\langle X^\mu(z_1)X^\nu(z_2)\rangle = -\alpha' g^{\mu\nu}\ln|z_1-z_2|^2, \qquad \langle \psi^\mu(z_1)\psi^\nu(z_2)\rangle = \frac{g^{\mu\nu}}{z_1-z_2}. \tag{4.7}$$

In the following, we will need the generators of the Lorentz group. In the open bosonic strings they are

$$J^{\mu\nu} = \frac{1}{\pi}\int_0^\pi d\sigma[X^\mu\partial_\tau X^\nu - X^\nu\partial_\tau X^\mu], \tag{4.8}$$

while for the open superstring in the $q=0$ superghost picture, we have

$$J^{\mu\nu}_{(0)} = \frac{1}{\pi}\int_0^\pi d\sigma[X^\mu\partial_\tau X^\nu - X^\nu\partial_\tau X^\mu + \psi^\mu\psi^\nu]. \tag{4.9}$$

The commutator of $J^{\mu\nu}$ with the gauge boson vertex operator takes the form

$$[J_{\mu\nu},V_A(k)] = \left(\epsilon_{[\mu}\frac{\partial}{\partial\epsilon^{\nu]}} + k_{[\mu}\frac{\partial}{\partial k^{\nu]}}\right)V_A(k). \tag{4.10}$$

This analysis extends directly to the open superstring (or the other open fermionic strings) and to the closed bosonic and super- (or fermionic) strings. In the latter cases one should keep in mind that there is a single conserved center of mass momentum $P^\mu=p_0^\mu$ and a single conserved angular momentum

$$J^{\mu\nu}_{cl} = x_0^\mu p_0^\nu - x_0^\nu p_0^\mu + \hat{J}^{\mu\nu}_L + \hat{J}^{\mu\nu}_R, \tag{4.11}$$

where $\hat{J}^{\mu\nu}_{L,R}$ denotes the contribution of the oscillators including fermionic zero modes $\psi_0^\mu\psi_0^\nu$ or $\bar{\psi}_0^\mu\bar{\psi}_0^\nu$ (the Ramond sector of the superstring). With some effort one can check that

$$[J_{cl\mu\nu},V_G(k)] = \left(2\epsilon_{[\mu}\frac{\partial}{\partial\epsilon^{\nu]}} + k_{[\mu}\frac{\partial}{\partial k^{\nu]}}\right)V_G(k) \tag{4.12}$$

for the graviton with $E_{\mu\nu}=\epsilon_\mu\epsilon_\nu$. An important property that will be relevant to our discussion is that $J^{\mu\nu}$ is BRST invariant, and thus the commutator of $V$ and $J$ remains BRST invariant. Note also that the leading term in the gluon vertex operator contains the world sheet current $\mathcal{J}_P^\mu = \partial_z X^\mu = \partial_\tau X^\mu = \Pi^\mu$ (momentum conjugate to $X^\mu$) for the space-time momentum operator $P^\mu$, while the subleading term contains the world sheet current $\mathcal{J}_J^{\mu\nu} = X^\mu\partial_z X^\nu - X^\mu\partial_z X^\mu + \psi^\mu\psi^\nu$ for angular momentum $J^{\mu\nu}$.

This is in line with the fact that the on-shell vertex operator for a massless vector at $k=0$, i.e. with a constant field strength, is precisely $V_F = F^{\mu\nu}\int dz[X_\mu\partial X_\nu - X_\nu\partial X_\mu + \psi_\mu\psi_\nu]$. Indeed, when $V_F$ is inserted in the action it changes the boundary conditions from $X_\mu\partial_\sigma X^\mu|_{\sigma=0,\pi}=0$ to $X_\mu\partial_\sigma X^\mu|_{\sigma=0,\pi} = X_\mu F^\mu{}_\nu\partial_\tau X^\nu|_{\sigma=0,\pi}$ and similarly for fermions (when present).

### B. Open superstring amplitudes on the disk

Color-ordered disk amplitudes are given by

$$\mathcal{A}(1,2,\ldots,n) = ig_s^{n-2}\int_{0\le z_2\le\ldots z_{n-2}\le 1} dz_2\ldots dz_{n-2}\langle cV(1) \times V(2)\ldots cV(n-1)cV(n)\rangle, \tag{4.13}$$

where $V$ denotes the vertex operators and $c$ the conformal ghost. In order to saturate the superghost charge one needs $\sum_i q_i = -2$. This can be satisfied by taking two vertices in the $q=-1$ picture and the remaining $n-2$ in the $q=0$ picture. In order to make the analysis of the soft limit transparent, it is convenient to take the vertex that goes "soft" in the $q=0$ picture and the two neighboring ones in the $q=-1$ picture. We will follow our previous convention where the soft leg is in the last position labeled by $n+1$.

We now consider the OPE between the soft vertex $V_A^{(0)}$ and its adjacent vertices $V_A^{(-1)}$ at $z_1$ and $z_n$,

$$\begin{aligned} V_A^{(0)}(z_s)V_A^{(-1)}(z_n) \approx{}& |z_s-z_n|^{k_s\cdot k_n-1}e^{-\varphi(z_n)}e^{i(k_s+k_n)X(z_n)} \\ &\times(\epsilon_s\cdot k_n\epsilon_n\cdot\psi - \epsilon_n\cdot k_s\epsilon_s\cdot\psi \\ &+\epsilon_n\cdot\epsilon_s k_s\cdot\psi)(z_n)+\cdots, \end{aligned}\tag{4.14}$$

where … indicate terms subleading in $|z_s - z_n|$. The integral over $z_s$ can be done using the identity[6]

$$\int_0^\epsilon x^{s-1}f(x) = \frac{f(0)}{s} + \mathcal{O}(s^0); \tag{4.15}$$

thus the leading term in the expansion of $k_s$ is simply $(\epsilon_s\cdot k_n/k_s\cdot k_n)V_A^{(-1)}(n)$.

At the next order, from the terms appearing in Eq. (4.14) we obtain

$$\frac{2}{k_s\cdot k_n}e^{-\varphi(z_n)}e^{ik_nX(z_n)}(i\epsilon_s\cdot k_n\epsilon_n\cdot\psi k_s\cdot X + \epsilon_n\cdot k_s\epsilon_s\cdot\psi - \epsilon_n\cdot\epsilon_s k_s\cdot\psi)(z_n). \tag{4.16}$$

The term proportional to $k_s\cdot X$ is responsible for the logarithms that appear in the explicit expansion of the amplitudes in the soft limit [see e.g. (3.10)] and can be decomposed into a symmetric and antisymmetric piece under the exchange $k_s \leftrightarrow \epsilon_s$. The symmetric piece is BRST exact. To see this, note that the term we are interested in, $\epsilon_s\cdot k_n k_s\cdot X + k_s\cdot k_n\epsilon_s\cdot X$, can be written as

$$\begin{aligned} \epsilon_{s\mu}k_{s\nu}X^{(\mu}k_n^{\nu)} &= \frac{\epsilon_{s\mu}k_{s\nu}}{\pi}\int_0^\pi d\sigma\partial_\tau X^{(\mu}X^{\nu)} \\ &= \frac{\epsilon_{s\mu}k_{s\nu}}{\pi}\int_0^\pi d\sigma\{Q_{\mathrm{BRST}}, bX^\mu X^\nu\}, \end{aligned}\tag{4.17}$$

where $b$ is the antighost. Thus only the antisymmetric piece is in the BRST cohomology. Putting everything together, we find that the subleading soft term is given by

$$\begin{aligned} &\frac{(F_s)_{\mu\nu}}{k_s\cdot k_n}(ik_n^\mu X^\nu\epsilon_n\cdot\psi + \epsilon_n^\mu\psi^\nu)e^{-\varphi}e^{i(k_n)X}(z_n) \\ &\quad = \frac{(F_s)_{\mu\nu}}{k_s\cdot k_n}\left(k_n^\mu\frac{\partial}{\partial k_{n\nu}} + \epsilon_n^\mu\frac{\partial}{\partial\epsilon_{n\nu}}\right)V_A^{(-1)}(z_n), \end{aligned}$$

where $F_s \equiv k_{s[\mu}\epsilon_{s\nu]}$. In other words, the two terms combined neatly produce

$$\frac{(F_s)_{\mu\nu}}{k_s\cdot k_n}[J^{\mu\nu}, V_A^{(-1)}(z_n)], \tag{4.18}$$

where $J_{\mu\nu}$ is the total angular momentum, defined before, that acts on both polarization (spin) and momentum (orbital). Thus we find that in the soft limit, the subleading contribution is given by the commutator of a BRST invariant operator with its adjacent vertex operators,

$$\begin{aligned} &\frac{(F_s)_{\mu\nu}}{k_s\cdot k_n}\langle[J^{\mu\nu}, V_A^{(-1)}(z_n)]V_A^{(-1)}(z_1)\cdots\rangle \\ &\quad - \frac{(F_s)_{\mu\nu}}{k_s\cdot k_1}\langle V_A^{(-1)}(z_n)[J^{\mu\nu}, V_A^{(-1)}(z_1)]\cdots\rangle. \end{aligned}\tag{4.19}$$

Let us stress that the final results, derived with a specific choice of superghost pictures and position of the soft gluon, are very general and do not depend on these choices at all. In particular, had we chosen one of the "hard" vertices to be in the $q=0$ picture or the "soft" vertex to be in the $q=-1$ picture, the leading singularity in the OPE would have contained terms like

$$|z_s - z|^{k_s\cdot k-2}\epsilon_s\cdot\epsilon e^{i(k_s+k)X}\times(1 \quad \text{or} \quad e^{-2\varphi}) \tag{4.20}$$

that would have not contributed to the leading term in the soft limit, since it would have produced a "pole" $1/(k_s\cdot k-1)$ upon integration over $z_s$ around $z$. The subleading terms in the OPE such as

$$|z_s - z|^{k_s\cdot k-1}e^{i(k_s+k)X}[\epsilon_s\cdot\epsilon(k_s-k)\cdot\partial X + \epsilon_s\cdot\psi\epsilon\cdot\psi] \times(1 \quad \text{or} \quad e^{-2\varphi}) \tag{4.21}$$

[6]This is a consequence of $\delta(x) = \lim_{s\to 0} s x^{s-1}$.

would have then produced the desired pole $1/k_s \cdot k$ in the soft limit. With some effort, one could check that the leading and subleading terms in the soft expansion are the same as in our analysis.

Moreover, our analysis applies to superstring gluon amplitudes at tree level in any dimension $D \leq 10$. Indeed, even after compactification the vertex operator for a massless gluon remains unchanged. One should simply restrict momentum and polarization to have nonzero components only along the noncompact directions. In other words the vertex operator involves the "identity" operator of the $\mathrm{CFT}_2$ governing the dynamics of the internal space. In particular, in $D = 4$ there are only two physical polarizations, and one can conveniently switch to the spinor helicity basis, wherein a generic massless vector polarization is the sum of plus and minus helicities.

### C. Closed superstring amplitudes on the sphere

In order to derive the behavior of graviton (in fact any NS-NS massless state) amplitudes for closed superstrings on the sphere, we start from the standard definition

$$\mathcal{M}(1,2,\ldots,n) = ig_s^{2(n-2)} \int_{S^2} dz_2 \ldots dz_{n-2} \langle c\bar{c}V(1) \times V(2)\ldots c\bar{c}V(n-1)c\bar{c}V(n)\rangle, \tag{4.22}$$

where $V = V_L V_R$ denotes closed-string vertex operators and $c$ the conformal ghost.

As in the open superstring case, in order to saturate the superghost charge on the sphere one needs $\sum_i q_i = -2$ both for left and right movers. The simplest way to satisfy this condition is to take two vertices in the $q = -1$ picture and the remaining $n-2$ in the $q = 0$ picture. In order to make the analysis of the soft limit transparent, it is convenient to take the closed-string vertex that becomes soft in the $q = 0$ picture.

In the soft limit, $k \to 0$, $V_G^{(0,0)}(z_s)$ becomes a total derivative and the integral over $z_s$ only receives a contribution from the boundary points $z_s = z_i$, where the soft vertex in the $q = 0$ picture collides with nonsoft ones. If the latter is in the $q = -1$ picture, the result is completely determined by the OPE,

$$V_G^{(0,0)}(z_s)V_G^{(-1,-1)}(z_i) \approx |z_s - z_i|^{2k_s\cdot k_i - 2} e^{-\varphi(z_i)-\tilde{\varphi}(\bar{z}_i)} e^{i(k_s+k_i)X(z_i,\bar{z}_i)} \times (\tilde{\epsilon}_s \cdot k_i\tilde{\epsilon}_i \cdot \psi - \tilde{\epsilon}_i \cdot \tilde{F}_s \cdot \tilde{\psi})(\bar{z}_i)(\epsilon_s \cdot k_i\epsilon_i \cdot \psi - \epsilon_i \cdot F_s \cdot \psi)(z_i) + \cdots$$

Integration over $z_s$ produces a pole $\pi/k_s \cdot k_i$ from the most singular term in the OPE and, up to an overall operator $e^{-\varphi(z_i)-\tilde{\varphi}(\bar{z}_i)} e^{ik_iX(z_i,\bar{z}_i)}$, the numerator can be expanded in $k_s$ as

$$\begin{aligned}
&\mathcal{O}(k_s^0)\colon\ (\tilde{\epsilon}_s \cdot k_i\tilde{\epsilon}_i \cdot \tilde{\psi})(\epsilon_s \cdot k_i\epsilon_i \cdot \psi)\\
&\mathcal{O}(k_s^1)\colon\ \{i(k_s \cdot X)(\tilde{\epsilon}_s \cdot k_i\tilde{\epsilon}_i \cdot \tilde{\psi})(\epsilon_s \cdot k_i\epsilon_i \cdot \psi) - \epsilon_s \cdot k_i\epsilon_i \cdot \psi(\epsilon_i \cdot \tilde{F}_s \cdot \tilde{\psi})\\
&\qquad - \tilde{\epsilon}_s \cdot k_i\tilde{\epsilon}_i \cdot \tilde{\psi}(\epsilon_i \cdot F_s \cdot \psi)\}\\
&\mathcal{O}(k_s^2)\colon\ \{i(k_s \cdot X)[(\tilde{\epsilon}_s \cdot k_i\tilde{\epsilon}_i \cdot \tilde{\psi})\epsilon_i \cdot F_s \cdot \psi + (\epsilon_s \cdot k_i\epsilon_i \cdot \psi)\tilde{\epsilon}_i \cdot \tilde{F}_s \cdot \tilde{\psi}]\\
&\qquad - (k_s \cdot X)^2(\tilde{\epsilon}_s \cdot k_i\tilde{\epsilon}_i \cdot \tilde{\psi})(\epsilon_s \cdot k_i\epsilon_i \cdot \psi)/2 + \tilde{\epsilon}_i \cdot \tilde{F}_s \cdot \tilde{\psi}\epsilon_i \cdot F_s \cdot \psi\}.
\end{aligned} \tag{4.23}$$

At $\mathcal{O}(k_s^0)$, this gives the leading soft behavior as

$$\mathcal{O}(k_s^{-1})\colon\ \pi\frac{(\tilde{\epsilon}_s \cdot k_i)(\epsilon_s \cdot k_i)}{k_s \cdot k_i} V_G^{(-1,-1)}(z_i). \tag{4.24}$$

From the open-string analysis, we have seen that it is convenient to rewrite the relevant terms in the form

$$\epsilon_i \cdot F_s \cdot \psi = F_s^{\mu\nu}\epsilon_{i\mu}\frac{\partial}{\partial\epsilon_i^\nu}\epsilon_i \cdot \psi, \qquad i(X \cdot [k_s)(\tilde{\epsilon}_s] \cdot k_i\tilde{\epsilon}_i \cdot \tilde{\psi})e^{ik_i\cdot X} = \tilde{F}_s^{\mu\nu}k_{i\nu}\frac{\partial}{\partial k_i^\mu}\tilde{\epsilon}_i \cdot \tilde{\psi}e^{ik_i\cdot X}. \tag{4.25}$$

Using these identifications and taking into account the symmetrization of the polarization vectors on leg $s$, for the subleading term we find

$$\begin{aligned}
\mathcal{O}(k_s^0)\colon\ &\frac{1}{2k_s \cdot k_i}[(\epsilon_s \cdot k_i\tilde{\epsilon}_s \cdot k_i)k_s^\mu\frac{\partial}{2\partial k_i^\mu} - (\epsilon_s \cdot k_ik_s \cdot k_i)\tilde{\epsilon}_s^\mu\frac{\partial}{2\partial k_i^\mu} - \epsilon_s \cdot k_i\tilde{F}_s^{\mu\nu}(\tilde{\epsilon}_{i\mu} \cdot \partial_{\tilde{\epsilon}_i^\nu})\\
&+ (\epsilon \leftrightarrow \tilde{\epsilon})]V_G^{(-1,-1)}(z_i) = \pi\frac{k_{i\mu}E_s^{\mu\rho}}{k_s \cdot k_i}[k_s^\nu J_{\rho\nu}^{\mathrm{total}}, V_G^{(-1,-1)}(z_i)],
\end{aligned} \tag{4.26}$$

where $J^{\mathrm{total}} = J + \tilde{J}$ and $E_s^{\mu\nu} = \epsilon^{(\mu}\tilde{\epsilon}^{\nu)}/2$. Similar analysis for the sub-subleading order contribution yields

$$\mathcal{O}(k_s^1):\ \pi \frac{E_s^{\mu\nu}}{2k_s \cdot k_i}[k_s \cdot J_\mu^{\mathrm{total}} k_s \cdot J_\nu^{\mathrm{total}}, V_G^{(-1,-1)}(z_i)]. \quad (4.27)$$

Thus we see that by soft expanding the result of the OPE between the soft and hard vertex operators, we recover the field theory soft theorem, written in BRST invariant operator language.

For closed bosonic and heterotic strings, the presence at tree level of the higher derivative $\phi R^2$ coupling spoils the universality of the sub-subleading terms. The higher derivative $R^3$, present in the bosonic string but not in the heterotic string, does not affect the soft theorem, as already observed earlier.

Finally, notice that if one replaces the soft graviton with a soft dilaton or a soft Kalb-Ramond B-field, the leading term vanishes. It is well known that the soft-dilaton limit of the $(n+1)$-point amplitude gives the derivative of the amplitude with respect to the string tension, since the zero-momentum dilaton vertex operator is essentially the world sheet action [42,43].

In general, the dilaton in $D = 10$ and the other moduli fields in lower dimensions are governed by a nonlinear $\sigma$ model and decouple at zero momentum like soft pions. An $(n+1)$-point amplitude with a soft modulus field is finite and given by the sum of $n$ contributions that represent the derivative with respect to the constant Vaccum Expectation Value (VEV) of the modulus field of the $n$-point amplitude without modulus field. Following this line of argument, many threshold corrections to (higher-derivative) terms in the effective superstring actions have been computed. See e.g. [41] for a pedagogical presentation and references therein.

A slightly different story can be told for the insertion of a soft dilaton in the bulk of a disk with open-string insertions on the boundary. The soft dilaton tadpole captures the divergence of the loop amplitude on a cylinder in the limit where it becomes infinitely long and thin. This divergence studied in detail in the early days of "dual" models [44] is absent in any consistent superstring background since it is related by supersymmetry to tadpoles in the R-R sector which, in turn, cancel in anomaly-free theories [45].

## V. THE SOFT THEOREMS FOR LOOP INTEGRANDS

As discussed in the Introduction, the loop-level soft theorem can be formulated in two distinct prescriptions: (1) taking $\epsilon \to 0$ before expanding in the soft parameter $\delta$, or (2) first expanding the integrand in the soft parameter $\delta$, and then performing the integration with the regularization. For general integrands the two limits do not commute, as was pointed out in [26]. That this is the case can be simply understood from the fact that soft expansion of the integrand assumes that the loop momentum is hard compared to the soft external momenta. This assumption becomes untenable in the region where the loop momentum itself is soft, which is precisely the region to be regulated by $\epsilon$. For the purpose of obtaining the correct infrared physics, one should take prescription (1), as discussed in detail in [24].

On the other hand, it is still interesting to ask whether or not the soft behavior is modified in the context of prescription (2), as it may yield nontrivial constraints for the integrand of the theory. As we will see, the planar integrand of $\mathcal{N} \leq 4$ SYM manifestly respects the tree-level soft theorems prior to integration. For all-plus YM and gravity amplitudes, we will show that the soft behavior of one-loop amplitudes is nonrenormalized in both prescriptions; in other words, the relevant integrands enjoy the property that the two limits commute. The tree-level soft theorem is known to be violated for the single-minus one-loop amplitudes [24,25].

### A. Soft limits of the planar integrand of SYM

In this subsection, we consider supersymmetric Yang-Mills theories in the planar limit. The advantage of working in the planar limit is that the four-dimensional integrand is well defined. We will argue that the Yang-Mills soft theorem works directly at the level of the integrand, and can be derived in essentially the same way as the BCFW derivation at tree level.

For color-ordered amplitudes in the planar limit, we find it convenient to choose the momenta adjacent to the soft particle for solving momentum conservation, in which case the soft theorem states that the subleading term should vanish. We will show that this is indeed the case for loop integrands of amplitudes in planar SYM theories with $\mathcal{N}$ supercharges. For convenience, let us strip off an overall MHV prefactor

$$A_0 \equiv \frac{\delta^{4|2\mathcal{N}}(\sum_{a=1}^n \lambda_a^\alpha (\tilde{\lambda}_a^{\dot{\alpha}}|\eta_a^A))}{\langle 12\rangle \ldots \langle n-1n\rangle\langle n1\rangle}, \quad (5.1)$$

with $\alpha = 1, 2$, $\dot{\alpha} = \dot{1}, \dot{2}$ Lorentz indices, and $A = 1, \ldots, \mathcal{N}$ the $SU(\mathcal{N})$ R-symmetry index. Note that, by definition, MHV tree amplitudes are given by $\langle a, b\rangle^{4-\mathcal{N}}$ where $a, b$ are the two negative-helicity particles (for $\mathcal{N} = 4$ it is simply unity).

For the $n$-point, N$^k$MHV amplitude at $L$ loops, $A_{n,k}^{(L)}$, let us denote the integrand (after stripping off $A_0$) by $R_{n,k}^{(L)}$,

$$A_{n,k}^{(L)} = A_0 \times \int d^D\ell_1 \cdots d^D\ell_L R_{n,k}^{(L)}(1, \ldots, n; \ell_1, \ldots, \ell_L), \quad (5.2)$$

where $\ell_1, \ldots \ell_L$ denotes the loop variables, and $D = 4 - 2\epsilon$ with $\epsilon$ as the dimensional regulator.

We will again consider the loop integrand in momentum-twistor space. The new ingredient is that the loop variables are given by $L$ bitwistors $\ell_i = (A_i, B_i)$ for $i = 1, \dots, L$. In terms of these variables, $R^{(L)}_{n,k}$ is a degree-$(4k-8)$ polynomial of $\chi^A$'s and a rational function of the totally antisymmetric contractions $\langle abcd\rangle \equiv \epsilon_{IJKL} Z^I_a Z^J_b Z^K_c Z^L_d$ of external and loop (bosonic) twistors. Note that the two-bracket of holomorphic spinors is given by $\langle ab\rangle \equiv \langle abI\rangle$ where $I$ is the infinity (bi)twistor projecting any twistor to its first two components.

We would now like to show that the subleading soft expansion of momentum-twistor space integrand begins at $\mathcal{O}(\delta^0)$ for a negative-helicity soft leg, and at $\mathcal{O}(\delta^2)$ for a positive-helicity soft leg.[7] It suffices to focus on the case of a positive-helicity particle, i.e. the $k$-preserving soft limit, in which case we will take Eq. (2.14) supersymmetrically. Note that the MHV prefactor absorbs the leading soft factor $S^{(0)}_{\mathrm{YM}}$, thus making the stripped amplitude behave trivially at leading order. We claim that the following soft theorem holds for the planar integrand of SYM to any loop order:

$$R^{(L)}_{n,k}(\mathcal{Z}_1, \dots, \mathcal{Z}_n) = R^{(L)}_{n-1,k}(\mathcal{Z}_1, \dots, \mathcal{Z}_{n-1}) + 0 \times \delta + \mathcal{O}(\delta^2). \tag{5.3}$$

### *1. All-loop integrand of $\mathcal{N} = 4$ SYM*

We first consider the $\mathcal{N} = 4$ integrand, which satisfies a BCFW-like recursion relation most compactly written in momentum-twistor space [29],

$$R^{(L)}_{n,k} = R^{(L)}_{n-1,k} + \sum_{L',k',i} R^{(L')}_{i,k'}(1, \dots, i-1, I_i)[1, i-1, i, n-1, n] R^{(L-L')}_{n+2-i,k-1-k'}(I_i, i, \dots, \hat{n}_i) + \int_{\mathrm{GL}(2)} [1, A, B, n-1, n] R^{(L-1)}_{n+2,k+1}(1, \dots, \hat{n}, A, \hat{B}), \tag{5.4}$$

where we suppress the sum over distributions of loop variables $\ell_1, \dots, \ell_L$ on both factorization and forward-limit terms, and where for the latter one needs to perform fermionic and GL(2) integrals. In addition, $\hat{n}_i = (n-1n) \cap (1i-1i)$, $I_i = (i-1i) \cap (1n-1n)$, $\hat{n} = (n-1n) \cap (1AB)$, $\hat{B} = (AB) \cap (1n-1n)$ with the intersection defined as $(ab) \cap (ijk) \equiv \mathcal{Z}_a\langle bijk\rangle - \mathcal{Z}_b\langle aijk\rangle$, and the R-invariant of five (super)twistors is defined as

$$[a, b, c, d, e] \equiv \frac{\delta^{0|4}(\chi_a\langle bcde\rangle + \mathrm{cyc})}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle}. \tag{5.5}$$

It is not a coincidence that we choose to shift the momentum twistor $\mathcal{Z}_n$ of the soft particle *à la* BCFW. For this shift, the first term in the recursion corresponds to the special BCFW factorization term: the $(n-1)$-point, $k$-preserving amplitude, multiplied by three-point anti-MHV amplitude; we will show that it is the only term that contributes to the first two orders of the soft expansion, which is a fact we are familiar with at tree level. This turns out to be a direct generalization of the BCFW argument for soft theorem at tree level.

Let us first see how the soft-theorem is derived from Eq. (5.4) for tree amplitudes, $L = 0$, where only the first line contributes. In the soft limit, $I_i = \delta(i-1i) \cap (1n-1s) \equiv \delta I'_i$, $\mathcal{Z}_{\hat{n}_i} = \mathcal{Z}_1 + \mathcal{O}(\delta)$; thus, the two subamplitudes are both nonsingular as we take $\delta \to 0$. The R-invariant, $[1, i-1, i, n-1, n]$, however, becomes of order $\delta^2$,

$$\frac{\delta^2}{\alpha\beta} \times \frac{\delta^{0|4}(\chi_{[i-1}\langle i]n-1s1\rangle + \chi_s\langle 1i-1in\rangle)}{\langle 1i-1in-1\rangle^3\langle n-1s1i-1\rangle\langle n-1s1i\rangle} + \mathcal{O}(\delta^3), \tag{5.6}$$

where in the numerator we have used the fact that terms involving $\chi_{n-1}$ and $\chi_1$ cancel with each other, and $[i-1, i]$ means antisymmetrization with respect to the two labels. Thus we recovered the soft gluon theorem at tree level,

$$R^{(0)}_{n,k} = R^{(0)}_{n-1,k} + \mathcal{O}(\delta^2). \tag{5.7}$$

Now it becomes clear that the first two orders in the soft expansion of the loop integrand are identical to those of tree amplitudes. The factorization part works exactly as before, except that now we need to use the fact that subamplitudes are nonsingular at the loop integrand level. For the forward-limit term, the R-invariant, $[1, A, B, n-1, n]$, behaves exactly as that in the factorization term,

$$[1, A, B, n-1, n] = \frac{\delta^2}{\alpha\beta} \times \frac{\delta^{0|4}(\chi_{[A}\langle B]n-1s1\rangle + \chi_s\langle 1ABn\rangle)}{\langle AB1n-1\rangle^3\langle 1An-1s\rangle\langle 1Bn-1s\rangle} + \mathcal{O}(\delta^3). \tag{5.8}$$

In addition, the lower-loop integrand is again nonsingular, with $\mathcal{Z}_{\hat{n}} = \mathcal{Z}_1 + \mathcal{O}(\delta)$ and $\hat{B} = \delta(AB) \cap (1n-1s) \equiv \delta\hat{B}'$.

[7] It is $\mathcal{O}(\delta^2)$ for the positive-helicity leg because we need to rescale the holomorphic soft behavior by $\delta^2$ to see the anti-holomorphic soft behavior.

After performing the fermionic and GL(2) integrals we find that the entire forward-limit term goes like $\mathcal{O}(\delta^2)$ in the limit; thus, we conclude that the soft theorem holds for all-loop integrand in $\mathcal{N}=4$ SYM,

$$R^{(L)}_{n,k} = R^{(L)}_{n-1,k} + \mathcal{O}(\delta^2). \tag{5.9}$$

Note that although the sub-subleading [$\mathcal{O}(\delta^2)$] order is no longer universal, it takes a relatively simple form: it is given by factorization and forward-limit terms with Eqs. (5.6), (5.8), where the dependence on the parameters is always through the prefactor $\delta^2/(\alpha\beta)$.

Before ending the discussion for $\mathcal{N}=4$ SYM, let us look at the soft behavior of forward-limit terms even more explicitly for the one-loop integrand. One can easily see that indeed each forward-limit term at one loop goes like $\mathcal{O}(\delta^2)$ when we take the BCFW-shifted particle, $n$, to be soft. For example, forward-limit terms for the one-loop MHV integrand, $K_{i,n}$ with $2<i<n$, are given by [29],

$$\begin{aligned} K_{i,n} &= -\frac{\langle AB(1i-1i)\cap(1n-1n)\rangle^2}{\langle AB1i-1\rangle\langle AB1i\rangle\langle ABi-1i\rangle\langle AB1n-1\rangle\langle AB1n\rangle\langle ABn-1n\rangle} \\ &= \frac{\delta^2}{\alpha\beta}\times\frac{\langle AB(1i-1i)\cap(1n-1s)\rangle^2}{\langle AB1i-1\rangle\langle AB1i\rangle\langle ABi-1i\rangle\langle AB1n-1\rangle^3} + \mathcal{O}(\delta^3). \end{aligned} \tag{5.10}$$

### 2. Integrands for $\mathcal{N}<4$ SYM

Now we turn to the soft theorem for $\mathcal{N}<4$ SYM theories. It is illuminating to first write down BCFW recursion relations for tree amplitudes in any $\mathcal{N}<4$ gauge theories in terms of momentum-twistor variables [46], from which again the soft gluon theorem follows immediately.

When taking the BCFW shift of $\mathcal{Z}_n$, without loss of generality we assume the helicity of particle $n$ to be positive; then, the recursion relation is almost identical to the $\mathcal{N}=4$ case,

$$\begin{aligned} R^{(0)}_{n,k} &= R^{(0)}_{n-1,k} + \sum_{k',i} R^{(0)}_{i,k'}(1,\ldots,i-1,I_i)[a,b,c,d,e]_{\mathcal{N}} \\ &\quad\times R^{(0)}_{n+2-i,k-1-k'}(-I_i,i,\ldots,\hat{n}^+), \end{aligned} \tag{5.11}$$

where the shifted twistors are the same as above, the helicity of $I_i$ depends on $k'$ and $i$ [46], and the general $\mathcal{N}<4$ five-bracket is defined as

$$[a,b,c,d,e]_{\mathcal{N}} \equiv \frac{\delta^{\mathcal{N}}(\eta_a\langle bcde\rangle + \text{cyclic})}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle}. \tag{5.12}$$

To see the soft theorem at work, note that although the R-invariant behaves like $\delta^{\mathcal{N}-2}$ in the soft limit, the two subamplitudes will provide the additional powers of $\delta$. This is because, unlike $\mathcal{N}=4$ amplitudes in momentum-twistor space, $\mathcal{N}<4$ amplitudes carry nonzero weights for negative-helicity particles, which is the case for one of the $I_i$'s in the subamplitudes. Since $I_i\equiv\delta I'_i$, we have

$$\begin{aligned} &R^{(0)}_{i,k'}(1,\ldots,i-1,I_i)R^{(0)}_{n+2-i,k-1-k'}(-I_i,i,\ldots,\hat{n}^+) \\ &\quad\sim\mathcal{O}(\delta^{4-\mathcal{N}}), \end{aligned} \tag{5.13}$$

thus rendering these factorization terms again vanishing as $\delta^2$.

At loop level, integrands in $\mathcal{N}<4$ SYM can also be obtained from e.g. CSW diagrams [32,47]. For $\mathcal{N}=4$ SYM, McLoughlin and one of the authors [33] proved that the integrand obtained from CSW diagrams is identical to the one from BCFW recursion relations above (see Sec. II A for a generalization to $F^3$ amplitude). Given the similarity of the structures of integrands in $\mathcal{N}=4$ and $\mathcal{N}<4$, we conjecture that the soft theorem again holds at the integrand level.

We now study one-loop amplitudes explicitly, as an example which provides strong evidence for our conjecture. The integrand for $\mathcal{N}<4$ SYM amplitudes at one loop can be written in terms of the one in $\mathcal{N}=4$ and including a part from $\mathcal{N}=1$ chiral multiplets, and it is sufficient to look at the soft behavior of the latter. A compact formula for the $\mathcal{N}=1$ chiral part of the integrand has been written in momentum-twistor space using CSW diagrams [46]: with $a,b$ as the negative-helicity particles, the $\mathcal{N}=1$ chiral part of the integrand, $R^{(1)\text{chiral}}_{n,2}$, is given by

$$R^{(1),\text{chiral}}_{n,2} - R^{(1),\text{chiral}}_{n-1,2} = -\frac{\langle a\hat{B}\rangle\langle b\hat{B}\rangle}{\langle AB\rangle^2\langle AB1n-1\rangle\langle ABn-1n\rangle\langle AB1n\rangle}\sum_{a<i\leq b}\frac{\langle aI_i\rangle\langle bI_i\rangle}{\langle AB1i-1\rangle\langle ABi-1i\rangle\langle AB1i\rangle}, \tag{5.14}$$

where the hallmark of an $\mathcal{N}=1$ chiral integrand is the appearance of the prefactor $1/\langle AB\rangle^2 = 1/\langle ABI\rangle^2$.

The soft behavior of $R^{(1),\mathrm{chiral}}_{n,2}$ is given by $R^{(1),\mathrm{chiral}}_{n-1,2}$, plus that of the rhs of Eq. (5.14). Recall that $I_i = \delta I'_i$ and $\hat{B} = \delta \hat{B}'$; we see that the soft behavior is identical to the $\mathcal{N}=4$ case in Eq. (5.10),

$$-\frac{\langle a\hat{B}\rangle\langle b\hat{B}\rangle\langle aI_i\rangle\langle bI_i\rangle}{\langle AB\rangle^2\langle AB1n-1\rangle\langle ABn-1n\rangle\langle AB1n\rangle\langle AB1i-1\rangle\langle ABi-1i\rangle\langle AB1i\rangle}$$
$$=\frac{\delta^2}{\alpha\beta}\times\frac{\langle a\hat{B}'\rangle\langle b\hat{B}'\rangle\langle aI'_i\rangle\langle bI'_i\rangle}{\langle AB\rangle^2\langle AB1n-1\rangle^3\langle AB1i-1\rangle\langle ABi-1i\rangle\langle AB1i\rangle}. \tag{5.15}$$

Thus, the soft theorem holds for the one-loop MHV integrand in $\mathcal{N}<4$ SYM. In addition, to obtain the $\mathcal{N}=1$ chiral part for non-MHV amplitudes, one only needs to dress the above formula with two tree subamplitudes, so we conclude that the soft theorem, Eq. (5.3), holds for all one-loop amplitudes in $1\leq\mathcal{N}<4$ SYM.

Note that although the soft theorem is quite transparent using the BCFW-like recursion (when we shift the soft particle), it can be very nontrivial to see in terms of other representations of the same integrand, such as the local form based on leading singularities [48]. For example, in that representation, the subleading terms cancel between different terms in a nontrivial way even for the one-loop integrand.

More importantly, the soft theorem is generally not manifest at the integrand level for other representations, such as the form in [49] and [50]) for the one-loop five-point amplitude in $\mathcal{N}=4$ SYM, which is given by scalar boxes and pentagons related to Eq. (5.10) by integral reduction. The soft theorem is expected to hold only when we perform the integrals after the soft expansion.

We have not discussed loop integrands in pure Yang-Mills theory, $\mathcal{N}=0$, because it is not clear to us how to write down a four-dimensional integrand that manifests the soft theorem. It is also unclear how to apply our argument to cases where the definition of an integrand may be ambiguous, e.g. nonplanar theories such as gravity. In Sec. V B, we will discuss the soft theorem with the integrals performed for the case of all-plus amplitudes in both YM theory and gravity.

### B. Soft theorems for finite loop amplitudes

We now consider an interesting example where the integrand does not manifestly satisfy the tree-level soft theorem, but does so only after integration. These are the finite rational terms of all-plus Yang-Mills and gravity one-loop amplitudes.

#### *1. All-plus Yang-Mills amplitude*

The $D$-dimensional all-plus integrand can be obtained straightforwardly from the $\mathcal{N}=4$ SYM integrand by simply multiplying it by extra powers of the regulator mass $(\mu^2)^2$ [51]. Naively, since we have already shown that the planar integrand vanishes for the subleading term in the kinematic configuration of Eq. (5.3), multiplying by an overall factor would not change this result. However, as one converts the momentum-twistor integrand into momentum space, the nonuniqueness of the identification of $\ell$ obscures this property and integration is necessary to show the vanishing of the subleading terms.

Let us first consider the one-loop five-point all-plus amplitude. The $D$-dimensional integrand is given as [51]

$$A_5^{+,+,+,+,+}=\frac{2}{\prod_{i=1}^5\langle ii+1\rangle}\left(-\frac{1}{2}\left[\frac{\mu^4 s_{12}s_{23}}{d_1d_2d_3d_5}+\text{cyclic}\right]+\frac{4i\mu^6\epsilon(1234)}{d_1d_2d_3d_4d_5}\right), \tag{5.16}$$

where $d_i=\ell_i^2$ and $\ell_i=\ell+\sum_{j=1}^i k_i$, and thus $\ell$ is positioned between 5 and 1. In the soft limit, the numerators of the above integrand behave as

$$\begin{gathered}
s_{12}s_{23}=s_{1'2}s_{23}+\delta s_{23}s_{2p_1},\qquad s_{23}s_{34}=s_{23}s_{34'}+\delta s_{23}s_{3p_4},\\
s_{34}s_{45}=\delta s_{34'}s_{54'}+\mathcal{O}(\delta^2),\qquad s_{45}s_{51}=\mathcal{O}(\delta^2),\qquad s_{51}s_{12}=\delta s_{1'2}s_{51'}+\mathcal{O}(\delta^2)\\
\epsilon(1,2,3,4)=\delta\epsilon(p_1,k_2,k_3,k_{4'})+\delta\epsilon(k_{1'},k_2,k_3,p_4)+\mathcal{O}(\delta^2),
\end{gathered} \tag{5.17}$$

where $s_{ip_j}=(k_i+p_j)^2$ and we have used the following notation:

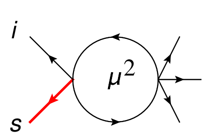
$$k_1 = k'_1 + \delta p_1, \qquad k'_1 = -|1\rangle \sum_{i=2,3} \frac{\langle 4i\rangle}{\langle 41\rangle}[i|, \qquad p_1 = -|1\rangle \frac{\langle 45\rangle}{\langle 41\rangle}[5|$$
$$k_4 = k'_4 + \delta p_4, \qquad k'_4 = -|4\rangle \sum_{i=2,3} \frac{\langle 1i\rangle}{\langle 14\rangle}[i|, \qquad p_4 = -|4\rangle \frac{\langle 15\rangle}{\langle 14\rangle}[5|. \tag{5.18}$$

Note that $k_{1'} + k_2 + k_3 + k_{4'} = 0$. Since the Parke-Taylor prefactor behaves as $1/\delta^2$, the leading soft contribution comes from the first two terms in the square bracket in Eq. (5.16), which indeed is $S^{(0)}A_4^{+,+,+,+,+}$ at the integrand level. For the subleading term, again only for the first two terms in the square bracket does one need to soft expand the integrand. Note that since the integrand integrates to a constant, there is no subleading contribution if one follows prescription (1). On the other hand, since

$$I_m[\mu^{2r}] = -\epsilon(1-\epsilon)\cdots(r-1-\epsilon)(4\pi)^r I_m^{D=4+2r-2\epsilon}, \tag{5.19}$$

the fact that the preexpanded integral is a constant implies that the $I_m^{D=4+2r-2\epsilon}$ in the above is logarithmic divergent. The soft expansion then introduces an additional propagator which would render $I_m^{D=4+2r-2\epsilon}$ finite, leading to a vanishing result as well. Thus to order $\epsilon$, the two prescriptions agree and the soft theorem is nonrenormalized in both cases.

The same analysis applies to general $n$. As $\mathcal{N}=4$ SYM contains no triangles or bubbles, the dimension-shifting formula tells us that the all-plus integrand can be simply expressed in terms of scalar boxes and pentagons multiplied by $(\mu^2)^2$. The subleading soft expansion of these integrals vanishes, in agreement with the expansion of the integrated results. Note that if the integrand includes scalar triangle and bubbles, $I_3[\mu^{2r}]$ and $I_2[\mu^{2r}]$, the two limits may no longer commute. This is due to the fact that the soft expansion can introduce scale-free integrals which strictly integrate to zero in dimension regularization, but are of order $\delta$ if one expands the integrated result. A trivial example would be the following bubble integral:




which integrates to $k_{si}^2$ in prescription (1), and thus becomes of order $\delta$, while in prescription (2) it integrates to $\delta \times 0$, since in the soft limit the integrand becomes a massless bubble integral. Similarly for $I_3[\mu^4]$, if the soft leg is on a massless corner the soft expansion is of order $\delta$ in prescription (1), while it vanishes in prescription (2). The possible disagreement of soft theorems between prescriptions (1) and (2) for the single-minus amplitude can be traced to the presence of these integrals in the final answer. Indeed, at four points $A_4(-,+,+,+)$ already contains the bubble integrals mentioned above [52].

### 2. All-plus gravity amplitude

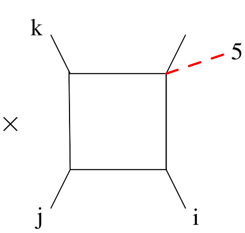
We now consider all-plus gravity amplitudes. The integrand is given by dimension-shifting formulas from the one-loop MHV amplitude in $\mathcal{N}=8$ supergravity [23]. Note that due to higher powers of $\mu^2$, the fact that the two limits commute is rather nontrivial. Consider

$$M_5 = \beta^{123(45)} I^{123}[(\mu^2)^4] + \gamma^{12345} I^{12345}[(\mu^2)^{10}] + \text{Perm}, \tag{5.20}$$

where

$$\beta^{123} = -\frac{[12]^2[23]^2[45]}{\langle 14\rangle\langle 15\rangle\langle 34\rangle\langle 35\rangle\langle 45\rangle},$$
$$\gamma^{12345} = -2\frac{[12][23][34][45][51]}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \tag{5.21}$$

and one sums over 30 inequivalent box integrals and 12 pentagons. First let us consider to which order in $\delta$ one should expand the integrals in the above representation. For the pentagon, since the prefactor begins at order $\delta^{-2}$, for the first subleading behavior of the integrand, we do not need to expand the pentagon integrand. For the box integrals, there are three distinct types to consider in the soft limit: (I) If the soft leg is on the massive corner, there are 12 such diagrams. (II) If the soft leg is on the massless corner adjacent to the massive corner, there are again 12 such diagrams. (III) If the soft leg is diagonal to the massive corner, there are six diagrams. The coefficient for the last case (III) behaves as $\mathcal{O}(\delta^0)$ in the soft limit and thus will not participate in the discussion. The prefactor for case (II) behaves as $\mathcal{O}(\delta^2)$ and thus there is no need to expand the integrand. Finally, case (I) is of order $\frac{1}{\delta^3}$, and thus we need the result of the integral expanded to order $\delta$. Denoting the integrand by its three massless legs $I_4(i,j,k)$,

$I_4(i,j,k) = \quad 420 \times$  . (5.22)

We list the order $\mathcal{O}(\delta)$ contribution in the following table:

| | $\mathcal{O}(\delta^1)$ |
|---|---|
| $I_4(1', 2, 3)$ | $-2u(p_1 \cdot k_4{}') - (4s + t)(p_1 \cdot k_2)$ |
| $I_4(3, 1', 2)$ | $-(6s + 3u)(p_1 \cdot k_2) - (6u + 3s)(p_1 \cdot k_3)$ |
| $I_4(3, 4', 1')$ | $-(4s + t)(p_4 \cdot k_3) - (5s + 12t)(p_4 \cdot k_1{}') - (7s + 14t)(p_1 \cdot k_4{}') - 2[u(p_5 \cdot k_3) + (2s + 4t)(p_5 \cdot k_4{}') + (s + 3t)(p_5 \cdot k_1{}')]$ |
| $I_4(1', 3, 4')$ | $-(6u + 3s)(p_1 \cdot k_3) + 2t(p_1 \cdot k_4{}') + 2t(p_4 \cdot k_1{}') - (3u + 6s)(p_4 \cdot k_3)$ |

while all others are related by symmetry. It is straightforward to check that the above result is the same as $\mathcal{O}(\delta^1)$ of

$$I_4(1,2,3) = -\frac{2s_{12}^2 + 2s_{23}^2 + 2(K^2)^2 + s_{12}s_{23} + 2s_{12}K^2 + 2s_{23}K^2}{2}, \tag{5.23}$$

where $K$ is the momenta on the massive leg. Thus we see for the subleading soft contribution that the two prescriptions again commute and the soft theorem is unrenormalized in both descriptions.

The above analysis should come as no surprise given the fact that the integrals involved remain finite, whether or not the soft expansion is done before or after the integration, and thus the limits should commute. Again, for bubble and triangle integrals, the two limits no longer commute; thus, the fact that the soft theorem for the all-plus gravity amplitude agrees in both prescriptions can be associated with the fact that the dimension-shifting formula allows only box and pentagon integrals in the representation.

An alternative way of understanding why the tree-level soft theorems are not corrected for all-plus amplitudes, and fail for single-minus amplitudes, is to use symmetry principles. As discussed in [53], given the leading soft function the subleading soft operator is determined by the conformal symmetry of the tree-level amplitude. Thus the tree-level soft functions can be viewed as the homogenous solutions to the differential equation implied by the conformal boost generators. The all-plus one-loop amplitudes are generated by the self-dual Yang-Mills theory [54], and hence preserve conformal invariance at loop level. The same is no longer true for the single minus. We leave a detailed discussion to Appendix D.

## VI. CONCLUSIONS

In this paper we addressed two questions regarding soft gluon and graviton theorems. (1) Are the tree-level soft theorems protected and unmodified, for effective theories with higher-dimensional operators or string theory at finite $\alpha'$? (2) How are tree-level soft theorems modified (or not) for loop-level integrands and integrated amplitudes? For (1), we found that soft theorems are respected in a wide range of effective field theories, even for those with $F^3$ or $R^3$ interaction vertices; more importantly, they hold for open- and closed-superstring tree-level amplitudes, as verified by explicit computations as well as by general analysis based on BCFW recursion relations and world sheet OPE. However, the sub-subleading soft graviton theorem is modified at tree level for theories with a $R^2\phi$ vertex, and for closed bosonic as well as heterotic string theory. Note that while $R^2\phi$ interaction terms appear at tree level in heterotic strings, and they can be generated at one loop in type II superstrings. For $\mathcal{N} \leq 4$ supergravity theories the $R^2\phi$ arises as a consequence of U(1) anomalies. Concerning (2), we have found that for planar $\mathcal{N} = 4$ SYM, the momentum-twistor representation derived from loop-level BCFW recursion indeed manifests the soft behavior dictated by the unrenormalized (tree) soft theorem. A similar conclusion can be arrived at for one-loop amplitudes for $\mathcal{N} < 4$ SYM in the CSW representation.

It is highly desirable to generalize our investigations to string amplitudes with higher genus. In this respect, it is quite remarkable that the BCFW-like recursion relation (5.4) derived in [29] closely resembles the three boundary contributions (pinching limits) of the world sheet moduli space of a string amplitude at higher genus. The first corresponds to the collision of two external vertices, the second to the factorization into two lower-genus amplitudes (separating tube), and the third to the degeneration of a tube/strip (pinching cycle). This analogy strongly suggests that, at least in the maximally supersymmetric case, superstring loop amplitudes should satisfy the same soft theorems as at tree level. It would also be interesting to further investigate the role of soft dilaton limits in the renormalization of the string tension and coupling constant.

Regarding (2), one interesting further direction would be turning integrand soft theorems into a constructive way of constraining the form of loop integrands in more general theories. We have seen that only those exact integrands in planar SYM exhibit manifest soft behavior identical to that of tree-level amplitudes. As discussed in [26], it could be worthwhile to interpret not only the soft limit but also collinear and factorization limits for loop amplitudes as kinematic limits to be taken before expanding in regulators. In this way loop integrands behave very similar to tree-level amplitudes, as we can see from the BCFW-like recursion in $\mathcal{N} = 4$. It would be fascinating to explore other formulations of loop integrands resembling those at tree level (e.g. twistor-string [55] or scattering-equation [12] formulas), in

$\mathcal{N}=4$ and beyond, based on their behavior in such kinematic limits.

For integrated soft theorems, we have shown that loop corrections can be easily understood via the presence of symmetry anomalies, in particular conformal anomalies. Note that we have only used the conformal anomaly associated with generic kinematics, whose analytic form is not well known. On the other hand, the conformal anomaly associated with collinear kinematics is well studied, and thus it will be interesting to work out what constraints these collinear anomalies do impose. Finally, the fact that gluon soft theorems for all-plus amplitude are not renormalized can be associated with conformal symmetry being unbroken at loop level for self-dual Yang-Mills. Similarly, the all-plus amplitude for gravity is also unrenormalized. Might there be some hidden symmetry for tree-level gravity amplitudes that is respected at loop level for self-dual gravity, such that the soft theorems are protected?.

## ACKNOWLEDGMENTS

It is a pleasure to thank Zvi Bern, Henrik Johansson (for private communications), and Andreas Brandhuber, Radu Roiban, Rodolfo Russo, Gabriele Travaglini, and Brian Wecht (for helpful discussions). The work of S. H. is supported by Zurich Financial Services Membership and the Ambrose Monell Foundation. The work of Y.-t. H. is supported by the National Science Foundation Grant No. PHY-1314311 and MOST Grant No. 103-2112-M-002-025-MY3. The work of C. W. is supported by the Science and Technology Facilities Council Consolidated Grant No. ST/J000469/1, "String theory, gauge theory and duality." The work of M. B. is partially supported by the ERC Advanced Grant No. 226455, "Superfields," and was initiated while M. B. was at QMUL holding a Leverhulme Visiting Professorship. M. B. would like to thank I. Antoniadis for discussions on this point.

*Note added.*—Recently, the work by Schwab [56] has appeared, which has some overlap with the results in Sec. III A and Appendix B.

## APPENDIX A: SYMMETRY CONSTRAINTS ON SOFT FUNCTIONS

Here, we will derive the supersoft functions using the special SUSY generator $\mathfrak{S}_{Aa}=\sum_i \frac{\partial^2}{\partial\lambda_i^a\partial\eta_i^A}$, which holds classically for super Yang-Mills theory. Again we impose

$$\left(\mathfrak{S}_0+\frac{1}{\delta}\mathfrak{S}_s\right)\left(\frac{1}{\delta^2}\mathcal{S}^{(0)}A_n+\frac{1}{\delta}\mathcal{S}^{(1)}A_n\right)=0. \tag{A1}$$

We will begin with the well-known result that $\mathcal{S}^{(0)}=S^{(0)}$; then, order $\delta^{-3}$ is trivially satisfied. For $\delta^{-2}$ we have the following constraint:

$$\mathfrak{S}_0 S^{(0)}A_n+\mathfrak{S}_s\mathcal{S}^{(1)}A_n=-\left(\frac{\lambda_n}{\langle ns\rangle^2}\frac{\partial}{\partial\eta_n}+\frac{\lambda_1}{\langle 1s\rangle^2}\frac{\partial}{\partial\eta_1}\right)A_n + \mathfrak{S}_s\mathcal{S}^{(1)}A_n=0. \tag{A2}$$

Now applying $\mathfrak{S}_s$ on the bosonic part of $\mathcal{S}^{(1)}$ gives 0; thus, in order for the above equation to hold, one must include a fermionic term. Again going through the same analysis, one finds that the requisite fermionic piece is given by

$$\frac{\eta_s}{\langle s1\rangle}\frac{\partial}{\partial\eta_1}+\frac{\eta_s}{\langle sn\rangle}\frac{\partial}{\partial\eta_n}. \tag{A3}$$

Thus we see that the supersymmetrized soft function is given by

$$\mathcal{S}^{(1)}=\mathcal{S}^{(0)}\left[\frac{\langle sn\rangle}{\langle 1n\rangle}\left(\tilde\lambda_s\cdot\frac{\partial}{\partial\tilde\lambda_1}+\eta_s\cdot\frac{\partial}{\partial\eta_1}\right) + \frac{\langle s1\rangle}{\langle n1\rangle}\left(\tilde\lambda_s\cdot\frac{\partial}{\partial\tilde\lambda_n}+\eta_s\cdot\frac{\partial}{\partial\eta_n}\right)\right], \tag{A4}$$

which is exactly what was found in [25] via recursion relations.

## APPENDIX B: SOFT THEOREM FOR SIX-POINT OPEN-STRING AMPLITUDE

The six-point open-superstring amplitude can be expressed in terms of $(6-3)!=6$ YM amplitudes and as many multiple hypergeometric functions, which only depend on the momenta. We will separate its contributions into two classes according to the color ordering of Yang-Mills amplitudes. Each class contains three terms. The first class includes terms with color ordering $\{1,2,3,4,5,6\}$, $\{1,2,4,3,5,6\}$, and $\{1,4,2,3,5,6\}$, whereas the second class includes terms with color ordering $\{1,3,2,4,5,6\}$, $\{1,3,4,2,5,6\}$, and $\{1,4,3,2,5,6\}$. We will prove that in the soft limit $k_4\to 0$, the sum of terms in the first class reduces to the soft factors multiplying $A_{\mathrm{YM}}(1,2,3,5,6)F^{(2,3)}$, appearing in the five-point amplitude, and the sum of the terms in the second class reduces to the soft factors multiplying $A_{\mathrm{YM}}(1,3,2,5,6)F^{(3,2)}$. It is convenient to solve for $\tilde\lambda_5$ and $\tilde\lambda_6$ using momentum conservation, and define

$$k_5'=\frac{|5\rangle\langle 6|(1+2+3)}{\langle 56\rangle},\qquad p_5=\frac{|5\rangle\langle 6|4}{\langle 56\rangle}, \tag{B1}$$

$$k_6'=\frac{|6\rangle\langle 5|(1+2+3)}{\langle 65\rangle},\qquad p_6=\frac{|6\rangle\langle 5|4}{\langle 65\rangle}. \tag{B2}$$

Let us start with the terms in the first class. From the term with color ordering $\{1,2,3,4,5,6\}$, we have

$$F^{(234)} = -\int dz_2 dz_3 dz_4 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{s_{12}}{z_{12}} \left(\frac{s_{34}}{z_{34}} + \frac{s_{35}}{z_{35}}\right) \frac{s_{45}}{z_{45}}, \tag{B3}$$

where we use SL(2) to fix $z_1 = 0, z_5 = 1$, and $z_6 = \infty$. It is straightforward to see that this term produces a leading term given by

$$\frac{1}{\delta^2} S^{(0)}_{\mathrm{YM}}(345) F^{(2,3)}(1,2,3,5',6') A_{\mathrm{YM}}(1,2,3,5',6'). \tag{B4}$$

Focusing on the subleading part, we find

$$\int dz_2 dz_3 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \frac{s_{12}}{z_{12}} F^{(234)}_\delta, \tag{B5}$$

where the Koba-Nielsen factor $\prod_{i<l} |z_{il}|^{s_{il}}$ is for five-point kinematics $\{k_1, k_2, k_3, k'_5, k'_6\}$ and $F^{(234)}_\delta$, of order $\mathcal{O}(\delta)$, is given by

$$F^{(234)}_\delta = \frac{\delta}{z_{35}} [(s_{4'5'} + s_{34} + s_{3p_5})[1 + s_{35'} \log(1 - z_3)] + (s_{24} + s_{2p_5}) s_{35'} \log(1 - z_2)].$$

Similarly, from the terms with color ordering $\{1,2,4,3,5,6\}$ and $\{1,4,2,3,5,6\}$, we find that the corresponding $F^{(243)}_\delta$ and $F^{(423)}_\delta$ are given by

$$F^{(243)}_\delta = \delta \frac{s_{35'}}{z_{35}} [s_{14} \log(z_3) + s_{24} \log(z_{23}) - s_{24} \log(1 - z_2)]$$
$$F^{(423)}_\delta = -\delta \frac{s_{35'}}{z_{35}} s_{14} \log(z_3). \tag{B6}$$

Combining all the terms and putting back $\delta$-independent terms, we obtain

$$\frac{1}{\delta^2} A_{\mathrm{YM}}(1,2,3,5,6) \int dz_2 dz_3 \frac{s_{12}}{z_{12}} \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \times \left(\frac{\langle 35 \rangle}{\langle 34 \rangle \langle 45 \rangle} F^{(234)}_\delta + \frac{\langle 23 \rangle}{\langle 24 \rangle \langle 43 \rangle} F^{(243)}_\delta + \frac{\langle 12 \rangle}{\langle 14 \rangle \langle 42 \rangle} F^{(423)}_\delta\right),$$

which we find to agree with

$$\frac{1}{\delta} S^{(1)}_{\mathrm{YM}}(345) F^{(2,3)}(1,2,3,5',6') A_{\mathrm{YM}}(1,2,3,5',6'), \tag{B7}$$

at the level of the integrand.

We then consider the expansion of terms in the second class. First we observe that color orderings $\{1,3,2,4,5,6\}$ and $\{1,3,4,2,5,6\}$ both contain leading terms, and they combine to produce

$$\frac{1}{\delta^2} S^{(0)}_{\mathrm{YM}}(345) F^{(3,2)}(1,3,2,5',6') A_{\mathrm{YM}}(1,3,2,5',6'). \tag{B8}$$

Now consider the subleading terms. From color ordering $\{1,3,2,4,5,6\}$, we get

$$\int dz_2 dz_3 \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) F^{(324)}_\delta \frac{s_{13}}{z_{13}}, \tag{B9}$$

with the subleading term $F^{(324)}_\delta$ given by

$$F^{(324)}_\delta = \frac{s_{25'}}{z_{25}} [(s_{45'} + s_{34} + s_{3p_5}) \log(1 - z_3) + (s_{24} + s_{2p_5}) \log(1 - z_2)] + \frac{1}{z_{25}} (s_{24} + s_{2p_5}).$$

Finally, from terms with color ordering $\{1,3,4,2,5,6\}$ and $\{1,4,3,2,5,6\}$, we find

$$F^{(342)}_\delta = \frac{\delta}{z_{25}} [s_{25'} (s_{24} \log(z_{32}) + k_2 \cdot p_5 \log(1 - z_2) + (s_{34} + s_{45'} + s_{3p_5}) \log(1 - z_3)) + s_{2p_5}]$$
$$F^{(432)}_\delta = -\frac{\delta}{z_{25}} s_{14} [s_{25'} \log(z_3) + 1]. \tag{B10}$$

Combining all the relevant terms, we find

$$\frac{1}{\delta^2} A_{\mathrm{YM}}(1,3,2,5',6') \int dz_2 dz_3 \frac{s_{13}}{z_{13}} \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \times \left(\frac{\langle 25 \rangle}{\langle 24 \rangle \langle 45 \rangle} F^{(324)}_\delta + \frac{\langle 32 \rangle}{\langle 34 \rangle \langle 42 \rangle} F^{(342)}_\delta + \frac{\langle 13 \rangle}{\langle 14 \rangle \langle 43 \rangle} F^{(432)}_\delta\right),$$

which can be checked to agree with

$$\frac{1}{\delta} S^{(1)}_{\mathrm{YM}}(345) F^{(3,2)}(1,2,3,5',6') A_{\mathrm{YM}}(1,3,2,5',6'). \tag{B11}$$

This ends the proof of the soft theorem for six-point open-superstring amplitudes.

## APPENDIX C: SOFT THEOREM FOR FIVE-POINT CLOSED-STRING AMPLITUDES

In this appendix we will check the validity of the soft theorem, especially $S^{(2)}_{\mathrm{G}}$, for closed-superstring amplitudes at five points. As we discussed in Sec. III B, in order to use the KLT formula, we need to expand five-point open-superstring amplitudes to sub-subleading order. Here we will again solve for $\tilde\lambda_4$ and $\tilde\lambda_5$, and take $k_3$ to be the soft leg. Expanding up to order $\mathcal{O}(\delta^2)$, we obtain the five-point disk integral for open superstring amplitudes

$$F^{(2,3)} = \frac{\Gamma(1+s_{24'})\Gamma(1+s_{12})}{\Gamma(1+s_{24'}+s_{12})}[1+\delta f_1^{(2,3)}+\delta^2 f_2^{(2,3)}]+\mathcal{O}(\delta^3), \tag{C1}$$

where the subleading and sub-subleading terms are given by

$$\begin{aligned}
f_1^{(2,3)} &= (s_{2p_4}+s_{23}+s_{34'})[H(s_{24'})-H(s_{24'}+s_{12})],\\
f_2^{(2,3)} &= s_{13}s_{34'}[(\psi^{(0)}(s_{12})-\psi^{(0)}(1+s_{24'}+s_{12}))(\psi^{(0)}(1+s_{24'})-\psi^{(0)}(1+s_{24'}+s_{12}))\\
&\quad -\psi^{(1)}(1+s_{24'}+s_{12})+\frac{s_{12}}{(1+s_{24'}+s_{12})}F(\{1,1,1,1+s_{12}\},\{2,2,2+s_{24'}+s_{12}\},1)]\\
&\quad +\frac{1}{2}(s_{2p_4}+s_{23}+s_{34'})^2[(\psi^{(0)}(1+s_{24'}+s_{12})-\psi^{(0)}(1+s_{24'}))^2+\psi^{(1)}(1+s_{24'})\\
&\quad -\psi^{(1)}(1+s_{24'}+s_{12})]-s_{34'}(s_{13}+s_{23})\zeta_2,
\end{aligned} \tag{C2}$$

where $H$ is the harmonic number, $F$ is the generalized hypergeometric function, and finally $\psi^{(m)}(z)=\frac{d^{m+1}}{dz^{m+1}}\log(\Gamma(z))$ is the polygamma function of order $m$. Similarly, we find the result of expanding $F^{(3,2)}$, which now starts from subleading order, as

$$F^{(3,2)} = s_{13}\frac{\Gamma(1+s_{24'})\Gamma(1+s_{12})}{\Gamma(1+s_{24'}+s_{12})}[\delta f_1^{(3,2)}+\delta^2 f_2^{(3,2)}]+\mathcal{O}(\delta^3), \tag{C3}$$

and for $f_1^{(3,2)}, f_2^{(3,2)}$, which are given by

$$\begin{aligned}
f_1^{(3,2)} &= H(s_{24'}+s_{12})-H(s_{12}),\\
f_2^{(3,2)} &= s_{2p_4}[(\psi^{(0)}(1+s_{24'}+s_{12})-\psi^{(0)}(s_{24'}))(\psi^{(0)}(1+s_{12})-\psi^{(0)}(1+s_{24'}+s_{12}))\\
&\quad +\psi^{(1)}(1+s_{24'}+s_{12})-\frac{1}{s_{24'}}[\psi^{(0)}(1+s_{12})-\psi^{(0)}(1+s_{24'}+s_{12})]]\\
&\quad +\frac{(1+s_{12})(s_{23}+s_{34'})}{1+s_{24'}+s_{12}}F(\{1,1,1,2+s_{12}\},\{2,2,2+s_{24'}+s_{12}\},1)\\
&\quad -\frac{1}{2}(s_{13}+s_{23})[(\psi^{(0)}(1+s_{12})-\psi^{(0)}(1+s_{24'}+s_{12}))^2+\psi^{(1)}(1+s_{12})\\
&\quad -\psi^{(1)}(1+s_{24'}+s_{12})]-(s_{23}+s_{34'})\zeta_2.
\end{aligned} \tag{C4}$$

We thus obtain the expansion of the five-point open-string amplitude up to sub-subleading order by substituting the above expansions into the expression for $\mathcal{A}_5(1,2,3,4,5)$,

$$\mathcal{A}_5(1,2,3,4,5) = F^{(2,3)}A_{\mathrm{YM}}(1,2,3,4,5)+F^{(3,2)}A_{\mathrm{YM}}(1,3,2,4,5).$$

Similarly, one can work out other open-superstring amplitudes entering the KLT relation for the five-point closed-superstring amplitude,

$$\begin{aligned}
\mathcal{M}_5(\{1,2,3,4,5\}) = \pi^{-2}(&\mathcal{A}_5(1,2,3,4,5)\mathcal{A}_5(1,4,3,5,2)\sin(\pi s_{12})\sin(\pi s_{34})\\
&+\mathcal{A}_5(5,1,3,2,4)\mathcal{A}_5(2,5,3,1,4)\sin(\pi s_{13})\sin(\pi s_{24})).
\end{aligned} \tag{C5}$$

With the above results up to the necessary order, we find that $\mathcal{M}_5(\{1,2,3,4,5\})$ satisfies the soft theorem by numerically comparing it with

$$\left(\frac{1}{\delta^3}S_{\mathrm{G}}^{(0)}(3)+\frac{1}{\delta^2}S_{\mathrm{G}}^{(1)}(3)+\frac{1}{\delta}S_{\mathrm{G}}^{(2)}(3)\right)\mathcal{M}_4(\{1,2,4',5'\}). \tag{C6}$$

This explicit numerical test of the soft theorem is consistent with the argument based on BCFW recursion relations and the world sheet OPE analysis presented in Secs. III and IV.

## APPENDIX D: CONFORMAL ANOMALY AND THE INTEGRATED SOFT THEOREMS

An alternative way to understand why the integrated soft theorems for the all-plus amplitude is not corrected, while the single-minus are, is via symmetries. Indeed, it was demonstrated in [53] that given the leading soft function, with suitable assumptions, the subleading soft operator is determined by the conformal symmetry of tree-level amplitude. Thus the tree-level soft functions can be viewed as the homogenous solutions to the differential equation implied by the symmetry constraints. From this point of view, the loop-level corrections can be attributed to the fact that this symmetry becomes anomalous at loop level. In particular, since the all-plus amplitude is generated by the self-dual sector of Yang-Mills theory, it is protected and conformal symmetry is preserved, implying that the soft function is not corrected. For the single-minus amplitude, this is no longer the case and potential correction terms arise, as verified in [24,25].

To see this, note that conformal symmetry of the $(n+1)$-point amplitude implies[8]

$$\left(\mathfrak{K}_0+\frac{1}{\delta}\mathfrak{K}_s\right)\left(\frac{1}{\delta^2}S_{\rm YM}^{\rm tree(0)}A_n+\frac{1}{\delta}S_{\rm YM}^{\rm tree(1)}A_n\right)=0, \quad \text{(D1)}$$

where we have separated the conformal boost generator into

$$\mathfrak{K}_0=\sum_{i=1}^{n}\frac{\partial}{\partial\lambda_i}\frac{\partial}{\partial\tilde{\lambda}_i}, \qquad \mathfrak{K}_s=\frac{\partial}{\partial\lambda_s}\frac{\partial}{\partial\tilde{\lambda}_s}, \quad \text{(D2)}$$

where we have suppressed the Lorentz indices $\alpha,\dot{\alpha}$. Now, starting with $S^{(0)}=\frac{\langle n1\rangle}{\langle ns\rangle\langle s1\rangle}$, at order $\mathcal{O}(\delta^{-3})$ Eq. (D1) is trivially satisfied, while at $\mathcal{O}(\delta^{-2})$ we have the following constraint:

$$\mathfrak{K}_0S_{\rm YM}^{\rm tree(0)}A_n+\mathfrak{K}_sS_{\rm YM}^{\rm tree(1)}A_n=-\left(\frac{\lambda_n}{\langle ns\rangle^2}\frac{\partial}{\partial\tilde{\lambda}_n}+\frac{\lambda_1}{\langle 1s\rangle^2}\frac{\partial}{\partial\tilde{\lambda}_1}\right)A_n +(\mathfrak{K}_sS_{\rm YM}^{\rm tree(1)})A_n=0. \quad \text{(D3)}$$

One can check that the tree-level soft function $S_{\rm YM}^{\rm tree(1)}$ is the homogenous solution to the above conformal boost equation. The same analysis applies to the supersoft functions, as we show in Appendix A.

A consequence of this analysis is that if conformal symmetry becomes anomalous, as one expects at loop level, then the soft function has to be modified. Let us consider the conformal boost equations in the presence of anomalies,

$$\left(\mathfrak{K}_0+\frac{1}{\delta}\mathfrak{K}_s\right)A_{n+1}(\{\lambda_i,\tilde{\lambda}_i\},\delta\lambda_s,\tilde{\lambda}_s)=\sum_i a_{n+1}^{(i)}\delta^i, \quad \text{(D4)}$$

where the $a_i$'s are the conformal anomaly expanded in the soft parameter. We begin with the following ansatz for the soft expansion of $A_{n+1}$:

$$\sum_{i=0}^{1}\frac{1}{\delta^{i+1}}S_{\rm YM}^{\rm tree(i)}A_n+\Delta^{(i)}+\mathcal{O}(\delta^0). \quad \text{(D5)}$$

From Eq. (D4), we have the following constraints on the unknown function $\Delta^{(i)}$:

$$\begin{aligned} &\mathcal{O}(\delta^{-3}) && \mathfrak{K}_s(S_{\rm YM}^{\rm tree(0)}A_n+\Delta^{(0)})=a_{n+1}^{(-3)},\\ &\mathcal{O}(\delta^{-2}) && \mathfrak{K}_0(S_{\rm YM}^{\rm tree(0)}A_n+\Delta^{(0)})+\mathfrak{K}_s(S_{\rm YM}^{\rm tree(1)}A_n+\Delta^{(1)})=a_{n+1}^{(-2)}. \end{aligned} \quad \text{(D6)}$$

Now, as the all-plus amplitude is associated with the self-dual sector of YM theory, which is exact, this implies that its amplitude is conformally invariant. Thus we expect no correction to the soft functions, i.e. $\Delta^{(i)}=0$. For the single-minus amplitude, this is no longer true and potential correction terms may arise. It is straightforward to verify that in the soft limit, if the soft leg is minus helicity, then the anomaly is finite; thus, Eq. (D6) reduces to zero on the rhs, leading to the conclusion that one only has the tree-level soft theorem.

For the negative-helicity leg the anomaly begins at $\delta^{-2}$. The absence of $a_{n+1}^{(-3)}$ implies $\Delta^{(0)}=0$, and thus $\Delta^{(1)}$ must satisfy

$$\mathfrak{K}_s(\Delta^{(1)})=a_{n+1}^{(-2)}-a_n^{(0)}. \quad \text{(D7)}$$

The explicit correction term for the single-minus amplitude is given in [25],

$$\Delta^{(1)}=-\frac{\langle n1\rangle^4}{\prod_{i=1}^{n}\langle ii+1\rangle}\frac{\langle n-1s\rangle[nn+1]}{\langle n-1n\rangle\langle ns\rangle^2}. \quad \text{(D8)}$$

We have explicitly verified that the above expression indeed satisfies Eq. (D7).

[8]Unlike other sections, here we put a superscript "tree" on $S_{\rm YM}^{\rm tree(i)}$ to emphasize they are tree-level results, and we will consider corresponding loop corrections.

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