PHYSICAL REVIEW D **92,** 065011 (2015)

Magnetic and inverse magnetic catalysis in the Bose-Einstein condensation of neutral bound pairs

Bo Feng,[1,2] De-fu Hou,[2] and Hai-cang Ren[3,2]

[1]*School of Physics, Huazhong University of Science and Technology, Wuhan 430074, China*
[2]*Institute of Particle Physics and Key Laboratory of Quark and Lepton Physics (MOE), Central China Normal University, Wuhan 430079, China*
[3]*Physics Department, The Rockefeller University, 1230 York Avenue, New York, New York 10021-6399, USA*

(Received 7 January 2015; published 15 September 2015)

The Bose-Einstein condensation of bound pairs made of oppositely charged fermions in a magnetic field is investigated. We find that the condensation temperature shows the magnetic catalysis effect in weak coupling and the inverse magnetic catalysis effect in strong coupling. The different responses to the magnetic field can be attributed to the competition between the dimensional reduction by Landau orbitals in pairing dynamics and the anisotropy of the kinetic spectrum of fluctuations (bound pairs in the normal phase).

DOI: 10.1103/PhysRevD.92.065011 PACS numbers: 03.75.Hh, 11.10.Wx, 12.38.-t, 74.20.Fg

I. INTRODUCTION

The behavior of a system consisting of charged fermions in a magnetic field attracted considerable interest in recent years, especially in strongly interacting matter, where fundamental constituent quarks exhibit a host of interesting phenomena [1], such as chiral magnetic effect and magnetic catalysis of chiral symmetry breaking. The latter one, which is the main motivation for the present work, involves the dimensional reduction by the Landau orbitals of charged fermions under a magnetic field. We shall investigate another (nonrelativistic) system that shares the same physics, the Bose-Einstein condensation (BEC) of composite bosons—neutral bound pairs made of two oppositely charged fermions in the presence of an external magnetic field.

The underlying theory of strong-interaction quantum chromodynamics(QCD) possesses chiral symmetry for massless quarks, which is spontaneously broken by a long-range order because of the condensation of bound pairs formed by quark and antiquark. As the density of states $D(E) \sim E^2$, with respect to the single quark energy E, vanishes at the Dirac point $E = 0$ (analog of the Fermi surface in a metal), a threshold coupling has to be attained for pairing. The terminology "magnetic catalysis" refers to the fact that chiral symmetry is always spontaneously broken at finite magnetic field regardless of the coupling strength [2,3]. The physical reason for this effect is the dimension reduction in the dynamics of fermion pairing in a magnetic field. The motion of charged particle would be squeezed to a discrete set of Landau orbitals and is one-dimensional within each orbital. The system would thus become $1+1$ dimension when the magnetic field is sufficiently strong than the mass and energy of the fermions, which would be restricted entirely in the lowest Landau level (LLL) only. Consequently, the density of states at the Dirac point becomes a nonzero constant proportional to the magnetic field eB. Such an enhancement would make the chiral condensate happen regardless of the interaction strength, the magnetic field thus plays a role as the catalysis. This is quite similar to the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity, where a nonzero density of states at the Fermi surface supports Cooper pairing with an arbitrarily weak attraction.

It would be natural to expect a higher transition temperature from the chiral broken phase to the chiral symmetric phase due to magnetic catalysis effect. This is indeed the case within mean-field approximations of effective model studies, it was found that the chiral phase transition is significantly delayed by a nonzero magnetic field even including the ρ meson contribution [4–6]. The pseudocritical temperature of chiral restoration was also found to increase linearly with the magnetic field in a quark-meson model using the functional renormalization group equation [7]. The recent Lattice calculations [8], however, provide surprising results that the pseudocritical temperature of chiral restoration drops considerably for an increasing magnetic field. On the other hand, the chiral condensate increases with an increasing magnetic field at low temperature consistent with magnetic catalysis, while it turns out to be monotonously decreasing at high temperature [9], which is in apparent conflict with the magnetic catalysis and termed an "inverse magnetic catalysis," evoking extensive studies [10–17].

While the mean-field approximation gives sensible results in certain circumstances, fluctuations can break it down, especially in the strong coupling domain or in lower dimensions. As was shown in [18] in the absence of a magnetic field, a long-range order cannot survive at a nonzero temperature in the spatial dimensionality two or less because of the fluctuation of its phase. A long wavelength component of the fluctuation variance goes like $1/p^2$ with $\mathbf{p}$ the momentum, which gives rise to infrared divergence of the momentum integration in two and lower dimensions. The anisotropy introduced by a magnetic field $B\hat{\mathbf{z}}$ renders the long wavelength fluctuation $\sim 1/(p_z^2 + \kappa p_\perp^2)$, with κ a positive constant between zero and one. Such a distortion of the bosonic spectrum towards dimensionality one ($\kappa \to 0$), as a consequence of the dimension reduction of the pairing fermions, would enhance the phase fluctuation. A preliminary study of the Ginzburg-Landau theory of the chiral phase transition [19] reveals the same effect and the Ginzburg critical window gets widened in the presence of the magnetic field, indicating the enhancement of the long wavelength fluctuations.

The BEC of bound pairs made of oppositely charged fermions in a magnetic field provides another platform to explore the competition between the enhanced Cooper pairing by Landau orbitals and the enhanced phase fluctuation by the distortion of the bosonic spectrum. We emphasize that our system of BEC has an important difference from the one in the BCS/BEC crossover of cold atoms, where the constituent atoms are neutral and couple to the external magnetic field via different magnetic moment configurations in closed and open channels. The coupling to the magnetic field is, thus, nonminimal. The role of a magnetic field is to tune the interacting strength (or equivalently the scattering length) between the atoms through the Feshbach resonance [20], while the Landau Level effect in the atomic binding is insignificant under a typical laboratory magnetic field. In our model (see (1) in Sec. II), however, the constituent fermions are electrically charged and their coupling to the external magnetic field is minimal [21]. In this regard, the present work is at the stage of a toy model and the conclusions are of theoretical values only. But the physics involved may be relevent to the color-flavor-locked phase or the single flavor planar phase of a dense quark matter in a compact star such as "magnetar" [22,23], where the pairing force stems from the non-perturbative QCD interaction.

We follow the functional integral formulation developed in [24] and calculate the leading (Gaussian) correction to the effective action. A technical simplification in the nonrelativistic BEC is that all summations over Landau orbitals involved can be carried out analytically, resulting in an explicit formula of the critical temperature under an aribitrary magnetic field. We found that the critical temperature for the BEC was dramatically affected by the magnetic field exhibiting magnetic catalysis or inverse magnetic catalysis depending on the coupling strength. In the weak coupling domain, where no bound pairs (composite bosons) exist at the zero magnetic field, the magnetic catalysis induces bound pairs and thereby a BEC. The critical temperature increases with an increasing magnetic field. In the strong coupling domain, where bound pairs exist without a magnetic field, an inverse magnetic catalysis was found. The critical temperature decreases with an increasing magnetic field, signaling the enhanced fluctuation in the magnetic field. Nevertheless, the condensation temperature is always suppressed compared with that of an ideal Bose gas regardless of the coupling strength.

The rest of the paper is organized as follows: in Sec. II we lay out the general formulation and present the mean-field approximation. The fluctuations beyond the mean-field theory, which is necessary for BEC, is calculated under the Gaussian approximation in Sec. III. The magnetic field dependence of the BEC temperature is investigated in Sec. IV. Section V is devoted to the conclusions and outlooks. Some calculation details and useful formulas are presented in the Appendices A, B and C. Throughout the paper, we will work Euclidean signature with the four vector represented by $x^\mu = (i\tau, \mathbf{x})$, $q^\mu = (i\omega_n, \mathbf{q})$ with ω_n the Matsubara frequency for bosons $\omega_n = 2i\pi nT$ and for fermions $\omega_n = (2n+1)i\pi T$.

II. GENERAL FORMULATION AND MEAN FIELD THEORY

We consider a system consisting of nonrelativistic fermions of mass m and chemical potential μ with opposite charge interacting through a short ranged instantaneous attractive interaction. The Hamiltonian density reads

$$\mathcal{H}[\psi,\psi^\dagger] = \sum_{\sigma=\pm}\psi_\sigma^\dagger(x)\left[\frac{(-i\nabla+\sigma e\mathbf{A})^2}{2m}-\mu\right]\psi_\sigma(x) - g\psi_+^\dagger(x)\psi_-^\dagger(x)\psi_-(x)\psi_+(x), \tag{1}$$

where $e > 0$ is the charge magnitude carried by each fermion, $\sigma = \pm$, $g > 0$ and $\mathbf{A}$ is the vector potential underlying an external magnetic field, $\mathbf{B} = \nabla\times\mathbf{A}$. Here the coupling to the magnetic field is minimal. To avoid the Meissner effect, only fermions with opposite charges can pair. For the sake of simplicity, we ignore the spin degrees of freedom. The thermodynamic potential density of the system reads

$$\Omega = -\frac{1}{\beta V}\ln\mathcal{Z}, \tag{2}$$

where $\beta = 1/T$ and V is the volume of the system. The path integral representation of the partition function $\mathcal{Z}$ reads

$$\mathcal{Z} = \int\mathcal{D}\psi_\sigma^\dagger(x)\mathcal{D}\psi_\sigma(x)\exp[\mathcal{S}], \tag{3}$$

with the action $\mathcal{S}$ given by

$$\mathcal{S} = \int d\tau d^3\mathbf{x}\left(-\sum_\sigma \psi_\sigma^\dagger(x)\frac{\partial}{\partial\tau}\psi_\sigma(x) - \mathcal{H}[\psi,\psi^\dagger]\right), \quad (4)$$

where the Grassmann variables ψ and $\psi^\dagger$ are antiperiodic in τ and *independent* of each other. The number density of fermions is given by

$$n = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,B}. \quad (5)$$

Introducing the standard Hubbard-Stratonovich field $\Delta(x)$ coupled to $\psi_+^\dagger\psi_-^\dagger$, the partition function is converted to

$$\mathcal{Z} = \int \mathcal{D}\psi_\sigma^\dagger(x)\mathcal{D}\psi_\sigma(x)\mathcal{D}\Delta^*(x)\mathcal{D}\Delta(x)\exp\left\{\int d\tau d^3\mathbf{x}\left(-\psi_\sigma^\dagger(x)\frac{\partial}{\partial\tau}\psi_\sigma(x) - \psi_\sigma^\dagger(x)\frac{(-i\nabla+\sigma e\mathbf{A})^2}{2m}\psi_\sigma(x)\right.\right.$$
$$\left.\left. + \mu\psi_\sigma^\dagger(x)\psi_\sigma(x) + \Delta(x)\psi_+^\dagger(x)\psi_-^\dagger(x) + \Delta^*(x)\psi_-(x)\psi_+(x) - \frac{|\Delta(x)|^2}{g}\right)\right\}, \quad (6)$$

and becomes bilinear in fermion fields. In terms of the Nambu-Gorkov(NG) spinors,

$$\Psi(x) = \begin{pmatrix}\psi_+(x)\\ \psi_-^\dagger(x)\end{pmatrix}, \qquad \Psi^\dagger(x) = (\psi_+^\dagger(x), \psi_-(x)), \quad (7)$$

the partition function becomes

$$\mathcal{Z} = \mathcal{N}\int \mathcal{D}\Psi^\dagger(x)\mathcal{D}\Psi(x)\mathcal{D}\Delta^*(x)\mathcal{D}\Delta(x)\exp\int d\tau d^3\mathbf{x}$$
$$\times\left[\int d\tau' d^3\mathbf{x}'\Psi^\dagger(x)G^{-1}(x,x')\Psi(x') - \frac{|\Delta(x)|^2}{g}\right], \quad (8)$$

with

$$G^{-1} = \begin{bmatrix} -\frac{\partial}{\partial\tau} - \frac{(-i\nabla+e\mathbf{A})^2}{2m} + \mu & \Delta(x) \\ \Delta^*(x) & -\frac{\partial}{\partial\tau} + \frac{(-i\nabla+e\mathbf{A})^2}{2m} - \mu \end{bmatrix}$$
$$\times\,\delta^4(x-x'), \quad (9)$$

where $\mathcal{N}$ is a constant. Integrating out the fermionic NG fields, we obtain the partition function

$$\mathcal{Z} = \mathcal{N}\int \mathcal{D}\Delta^*(x)\mathcal{D}\Delta(x)\exp(\mathcal{S}[\Delta(x)]), \quad (10)$$

with the action S given by

$$\mathcal{S}[\Delta] = -\int d\tau d^3\mathbf{x}\frac{|\Delta(x)|^2}{g} + \mathrm{Tr}\ln G^{-1}(x,x'), \quad (11)$$

where the trace in (11) is over space, imaginary time and NG indices.

For a uniform magnetic field $\mathbf{B}$ considered in this work, we choose the Landau gauge, in which the vector potential is $A_x = A_z = 0, A_y = Bx$ and the magnetic field is thus along z direction and the system is translationally invariant. To explore the long-range order of the system, we make a Fourier expansion,

$$\Delta(x) = \sqrt{\frac{1}{\beta V}}\sum_{\omega_{n_k},\mathbf{k}} e^{-i\omega_{n_k}\tau + i\mathbf{k}\cdot\mathbf{x}}\Delta(i\omega_{n_k},\mathbf{k}) = \Delta_0 + \Delta'(x), \quad (12)$$

where we have singled out the zero energy-momentum component of the expansion. Carrying out the path integral over $\Delta'(x)$, we end up with

$$\mathcal{Z} = \mathcal{N}\int \mathcal{D}\Delta_0^*\mathcal{D}\Delta_0\exp[-\beta V\Xi(|\Delta_0|)], \quad (13)$$

and the thermodynamic potential density in the infinite volume limit equals to the value of the function $\Xi(|\Delta_0|)$ at its saddle point $\bar{\Delta}_0$ determined by

$$\left(\frac{\partial\Xi}{\partial|\Delta_0|^2}\right)_{T,\mu,B} = 0. \quad (14)$$

A nontrivial saddle point, $\bar{\Delta}_0 \neq 0$, corresponds to a long-range order and the superfluidity phase of the system. $\bar{\Delta}_0$ drops to zero at the transition to the normal phase. Expanding the function Ξ in a power series in $|\Delta_0|^2$,

$$\Xi(|\Delta_0|^2) = \Xi(0) + \alpha(T,\mu,\mathbf{B})|\Delta_0|^2 + \cdots, \quad (15)$$

where $\Xi(0)$, $\alpha(T,\mu,B)$ and the coefficients of higher-order terms of (15) include the contribution from the fluctuation field $\Delta'(x)$ defined in (12). A negative value of the coefficient $\alpha(T,\mu.\mathbf{B})$ signals the instability of the normal phase, $\Delta_0 = 0$, and the critical temperature T_c, and the chemical potential $\bar{\mu}$ for the instability satisfy the condition

$$\alpha(T_c,\bar{\mu},\mathbf{B}) = 0. \quad (16)$$

The critical temperature at a given density is obtained by solving both Eqs. (16) and (5) simultaneously.

The mean-field approximation ignores $\Delta'(x)$ and the eigenvalues of the inverse propagator (9) with $\Delta(x) = \Delta_0$ can be easily found. We obtain

$$\Xi(|\Delta_0|^2) = \frac{1}{g}|\Delta_0|^2 - \frac{1}{\beta V}\sum_n \sum_{k_y,k_z;l} \times \ln[(i\omega_n)^2 - (\varepsilon_{k_z} + l\omega_B - \chi)^2 - |\Delta_0|^2], \quad (17)$$

where $l = 0, 1, 2, \ldots$ are the Landau levels and $\varepsilon_{k_z} = k_z^2/2m$. We have also defined $\chi = \mu - \omega_B/2$ with $\omega_B = eB/m$ the cyclotron frequency. The symbol $V^{-1}\sum_{k_y,k_z;l}$ is the abbreviation of $eB/(2\pi)^2\sum_{l=0}^{\infty} \times \int_{-\infty}^{\infty} dk_z$. The coefficient $\alpha(T, \mu, \mathbf{B})$ under the mean-field approximation can be readily extracted from the Taylor expansion of the rhs of (17) and the condition (16) becomes

$$\frac{1}{g} = \frac{1}{2V}\sum_{k_y,k_z;l}\frac{1}{\varepsilon_{k_z} + l\omega_B - \bar{\chi}}\tanh\frac{\varepsilon_{k_z} + l\omega_B - \chi}{2T_c}. \quad (18)$$

In BCS limit, this equation would be solved to yield the critical temperature with the chemical potential given by that of an ideal Fermi gas at a given density (the limit of Eq. (5) with $\Omega = \Xi$ at $\Delta_0 = 0$, $T = 0$ and $\mathbf{B} = 0$). In the BEC limit, however, the role is reversed [24]. Equation (18) determines the chemical potential. In the latter case, the fluctuation contribution to Ξ has to be restored to determine the critical temperature at a given density through (5).

For negative χ with $T \ll |\chi|$, the hyperbolic tangent function in (18) may be approximated by one and we end up with

$$-\frac{m}{4\pi a_s} = \frac{1}{2V}\left[\sum_{k_y,k_z;l}\frac{1}{\varepsilon_{k_z} + l\omega_B - \bar{\chi}} - \sum_{\mathbf{k}}\frac{1}{2\varepsilon_{\mathbf{k}}}\right], \quad (19)$$

where we have introduced a renormalized coupling constant according to

$$\frac{1}{g_R} \equiv \frac{1}{g} - \frac{1}{V}\sum_{\mathbf{k}}\frac{1}{2\varepsilon_{\mathbf{k}}} \equiv -\frac{m}{4\pi a_s} \quad (20)$$

with a_s the s-wave scattering length extracted from the low-energy limit of the two-body scattering in vacuum and in the *absence* of a magnetic field so that the rhs is free from UV divergence. Carrying out the summation explicitly (for details, see Appendix A), we find that

$$-\frac{m}{4\pi a_s} = \frac{\sqrt{\omega_B}m^{3/2}}{4\sqrt{2}\pi}\zeta\left(\frac{1}{2}, \frac{|\bar{\chi}|}{\omega_B}\right). \quad (21)$$

In obtaining this equation, the contributions from all Landau levels have been taken into account and this summation gives rise to the Hurwitz zeta function, which was defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}, \quad (22)$$

for Re$s > 1$ and can be continued to the entire s plane with a pole at $s = 1$ in terms of its integral representation.

Equation (21) sets the chemical potential at the energy of a bound pair of zero center-of-mass momentum in vacuum and this is the condition for the BEC of an ideal Bose gas. The contributions of the bound pairs of nonzero momentum, however, is ignored here. Therefore, the mean-field approximation is not sufficient and the contribution from the bound pairs with nonzero momenta to the density equation (5) has to be restored to determine the transition temperature (the density will be set low enough to justify the approximation $\tanh\frac{\varepsilon_{k_z}+l\omega_B-\bar{\chi}}{2T_c} \simeq 1$.).

In the absence of magnetic field, the rhs of (21) becomes $-m^{3/2}\sqrt{|\bar{\chi}|}/(2\sqrt{2}\pi)$ and we have a solution $\bar{\chi} = -1/(2ma_s^2)$ only for $a_s > 0$, which defines the strong coupling domain. The weak coupling domain, $a_s < 0$, however, entirely resides on the BCS side of the BCS/BEC crossover. When the magnetic field is turned on, the rhs of (21) can take both signs and a solution emerges in the weak coupling domain. This is caused by the dimensional reduction of the Landau orbitals; i.e., the magnetic catalysis and the BEC limit can be approached in both strong and weak coupling domains.

III. GAUSSIAN FLUCTUATION

The Guassian approximation of the fluctuation effect maintains $\Delta'(x)$ to the quadratic order in the path integral (10), while including Δ_0 to all orders. To locate the pairing instability starting from the normal phase, where $\bar{\Delta}_0 = 0$, the Gauss approximation amounts to replace $\mathcal{S}[\Delta]$ of (11) by its expansion to the quadratic order in the entire boson field $\Delta(x)$.

$$\mathcal{S}[\Delta] \simeq \mathcal{S}_{\text{eff}}[\Delta] = \mathcal{S}[0] - \int d\tau d^3\mathbf{x}\frac{|\Delta(x)|^2}{g} - \int d\tau d\tau' d^3\mathbf{x} d^3\mathbf{x}'[G_+(x,x')\Delta(x')G_-(x',x)\Delta^*(x)], \quad (23)$$

with

$$G_\pm(x,x') = \left[-\partial_\tau \mp \left(\frac{(-i\nabla + e\mathbf{A})^2}{2m} - \mu\right)\right]^{-1}\delta^4(x - x'). \quad (24)$$

In terms of the Fourier transformation (12),

$$\mathcal{S}_{\text{eff}}[\Delta] = \mathcal{S}[0] - \sum_{\omega_{n_p},\mathbf{p}}\Gamma^{-1}(i\omega_{n_p}, \mathbf{p})|\Delta(i\omega_{n_p}, \mathbf{p})|^2 \quad (25)$$

where the dependence of the coefficient $\Gamma^{-1}(i\omega_{n_p}, \mathbf{p})$ on T, μ and $\mathbf{B}$ has been suppressed and the thermodynamic potential density reads

$$\Omega = \Omega_0 - \frac{1}{\beta V} \sum_{\omega_{n_p}, \mathbf{p}} \ln \Gamma(i\omega_{n_p}, \mathbf{p}), \tag{26}$$

where $\Omega_0 = -2/(\beta V) \sum_{k_y, k_z; l} \ln [1 + \exp(\varepsilon_{k_z} + l\omega_B - \chi)]$ is the thermodynamic potential of an ideal Fermi gas. It follows that

$$\alpha(T, \mu, \mathbf{B}) = \Gamma^{-1}(0, 0), \tag{27}$$

and

$$n = n_0 + \frac{1}{\beta V} \frac{\partial}{\partial \mu} \sum_{\omega_{n_p}, \mathbf{p}} \ln \Gamma(i\omega_{n_p}, \mathbf{p}), \tag{28}$$

with $n_0 = 2V \sum_{k_y, k_z; l} [\exp(\beta(\varepsilon_{k_z} + l\omega_B - \chi)) + 1]^{-1}$ the fermionic contribution to the density. Continuing $i\omega_{n_p}$ to an arbitrary real frequency ω according to the prescription in [25] and introducing a phase shift defined by $\Gamma(\omega \pm i0, \mathbf{p}) = |\Gamma(\omega, \mathbf{p})| \exp[\pm i\delta(\omega, \mathbf{p})]$, the number equation can also be written as [24]

$$n = n_0 + \frac{1}{V} \sum_{\mathbf{p}} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} n_B(\omega) \frac{\partial \delta}{\partial \mu}(\omega, \mathbf{p}), \tag{29}$$

with $n_B(\omega) = (e^{\beta\omega} - 1)^{-1}$ the Bose-Einstein distribution function. The pair of equations, (18) and (29), at zero magnetic field are widely employed in the context of BCS/BEC crossover in the literature [24,26–28,30,31].

To calculate $\Gamma(\omega, \mathbf{q})$, we write $G_\pm(x, x')$ of (24) in terms of the eigenvalues and eigenfunctions of $G_\pm^{-1}$,

$$G_\pm(x, x') = \sum_K \frac{\psi_K(\tau, \mathbf{x}) \psi_K^*(\tau', \mathbf{x}')}{i\omega_{n_k} \mp (\varepsilon_{k_z} + l\omega_B - \chi)}, \tag{30}$$

with abbreviation $K = (\omega_n; l, k_y, k_z)$ and the notation $\sum_K = (\beta V)^{-1} \sum_{\omega_{n_k}} \sum_{k_y, k_z; l}$. The eigenfunction in the Landau gauge reads

$$\psi_K(\tau, \mathbf{x}) = \frac{1}{\sqrt{L_y L_z}} e^{-i\omega_n \tau + i(k_y y + k_z z)} u_l\left(x - \frac{k_y}{eB}\right), \tag{31}$$

where L_y, L_z are the normalization lengths along y and z axes and the u-function is the wavefunction of a harmonic oscillator given by

$$u_n(z) = \frac{(eB)^{\frac{1}{4}}}{\pi^{1/4} \sqrt{2^n \cdot n!}} e^{-\frac{eBz^2}{2}} H_n(\sqrt{eB} z), \tag{32}$$

with $H_n(z)$ the Hermite polynomial. The u-functions satisfy the orthonormality relation $\int dx u_n(x) u_m(x) = \delta_{nm}$.

In terms of the Fourier components of $\Delta(x)$, the trace term in (23) becomes

$$\begin{aligned} &\mathrm{tr}[G_+(x, y)\Delta(y)G_-(y, x)\Delta^*(x)] \\ &= \sum_{K, l'} \sum_{\omega_{n_p}, \mathbf{p}} \sum_{p_x'} \left[\int dx' e^{ip_x x'} u_l\left(x' - \frac{k_y}{eB}\right) u_{l'}\left(x' - \frac{q_y}{eB}\right)\right] \left[\int dx e^{-ip_x' x} u_l\left(x - \frac{k_y}{eB}\right) u_{l'}\left(x - \frac{q_y}{eB}\right)\right] \\ &\quad \times \frac{\Delta(i\omega_{n_p}, \mathbf{p})}{i\omega_{n_k} - (\varepsilon_{k_z} + l\omega_B - \chi)} \frac{\Delta^*(i\omega_{n_p}, p_x', p_y, p_z)}{i\omega_{n_q} + (\varepsilon_{q_z} + l'\omega_B - \chi)}, \end{aligned} \tag{33}$$

where $q = k + p$, $\omega_{n_q} = \omega_{n_k} + \omega_{n_p}$ and $\mathbf{p} = (p_x, p_y, p_z)$. Upon shifting the integration k_y to $k_y + eBx$, the last two u-functions will no longer be coordinate dependent and the first two u-functions depend only on the relative coordinates. The translational invariance becomes explicit then. It would be convenient to introduce the center of mass coordinate $X = \frac{x' + x}{2}$ and the relative one $r = x' - x$ and we obtain that

$$\begin{aligned} \mathrm{tr}[G_+(x, y)\Delta(y)G_-(y, x)\Delta^*(x)] = &\sum_{\omega_{n_k}, l, k_z; l'} \sum_{\omega_{n_p}, \mathbf{p}} \int dr e^{ip_x r} \left[\int \frac{dk_y}{2\pi} u_l\left(r - \frac{k_y}{eB}\right) u_{l'}\left(r - \frac{q_y}{eB}\right) u_l\left(-\frac{k_y}{eB}\right) u_{-l'}\left(\frac{q_y}{eB}\right)\right] \\ &\times \frac{1}{i\omega_{n_k} - (\varepsilon_{k_z} + l\omega_B - \chi)} \frac{1}{i\omega_{n_q} + (\varepsilon_{q_z} + l'\omega_B - \chi)} |\Delta(i\omega_{n_p}, \mathbf{p})|^2. \end{aligned} \tag{34}$$

Making the variable transformation $s = r - k_y/(eB)$ and $t = -k_y/(eB)$, we have

$$\mathrm{tr}[G_+(x, y)\Delta(y)G_-(y, x)\Delta^*(x)] = \frac{eB}{2\pi} \sum_{\omega_{n_k}, l, k_z; l'} \sum_{\omega_{n_p}, \mathbf{p}} \times \frac{|\Delta(i\omega_{n_p}, \mathbf{p})|^2 |I_{ll'}(p_x, p_y)|^2}{[i\omega_{n_k} - (\varepsilon_{k_z} + l\omega_B - \chi)][i\omega_{n_q} + (\varepsilon_{q_z} + l'\omega_B - \chi)]}, \tag{35}$$

with

$$I_{ll'}(p_x,p_y)=\int_{-\infty}^{\infty}d\xi e^{ip_x\xi}u_l(\xi)e^{-\frac{p_y}{eB}\frac{d}{d\xi}}u_{l'}(\xi), \quad (36)$$

where the identity

$$\exp\left(a\frac{d}{dx}\right)f(x)=f(x+a), \quad (37)$$

with $f(x)$ an arbitrary function is employed. As is shown in Appendix B, the integral (36) can be calculated explicitly with the aid of the raising and lowering operators pertaining to the harmonic oscillator wave function $u_n(\xi)$,

$$\xi=\frac{1}{\sqrt{2eB}}(a+a^\dagger), \quad (38)$$

$$\frac{d}{d\xi}=\sqrt{\frac{eB}{2}}(a-a^\dagger), \quad (39)$$

and we obtain that

$$|I_{ll'}|=\sqrt{\frac{l_<}{l_>}}e^{-\frac{p_\perp^2}{4eB}}\left(\frac{p_\perp^2}{2eB}\right)^{\frac{|l-l'|}{2}}L_{l_<}^{|l-l'|}\left(\frac{p_\perp^2}{2eB}\right), \quad (40)$$

where $l_<=\min(l,l')$, $l_>=\max(l,l')$, $\mathbf{p}_\perp=(p_x,p_y)$ and $L_n^\alpha(z)$ is the generalized Laguerre polynomial. Combining (23), (35) and (40) and carrying out the summation over the Matsubara frequency ω_{n_k} in (35), we end up with

$$\Gamma^{-1}(i\omega_{n_p},\mathbf{p})=\frac{eB}{2\pi}e^{-\frac{p_\perp^2}{2eB}}\sum_{l,l',k_z}\left\{\frac{l_<}{l_>}\left(\frac{p_\perp^2}{2eB}\right)^{|l-l'|}\left[L_{l_<}^{|l-l'|}\left(\frac{p_\perp^2}{2eB}\right)\right]^2\frac{n_F(\varepsilon_{k_z}+l\omega_B-\chi)-n_F(-\varepsilon_{q_z}-l'\omega_B+\chi)}{-i\omega_{n_p}+(\varepsilon_{k_z}+\varepsilon_{q_z})+(l+l')\omega_B-2\chi}\right\}+\frac{1}{g}, \quad (41)$$

with $n_F(z)=(1+e^{\beta z})^{-1}$ the Fermi-Dirac distribution function. The isotropy perpendicular to the magnetic becomes evident in Γ^{-1}. Setting $i\omega_{n_p}=0$ and $\mathbf{p}=0$, we verify the relation $\Gamma^{-1}(0,0)=\alpha(T,\mu,\mathbf{B})$ with $\alpha(T,\mu,\mathbf{B})$ given by the mean-field theory of the previous section and vanishes at $T=T_c$ and $\mu=\bar{\mu}$ according to (18).

For a negative χ with $\beta|\chi|\gg 1$, the case considered in this work, the numerator on the rhs of (41) may be approximated by -1 and the integration over k_z can be carried out analytically. We have

$$\Gamma^{-1}(i\omega_{n_p},\mathbf{p})=-\frac{m^{\frac{1}{2}}eB}{4\pi}e^{-\frac{p_\perp^2}{2eB}}\sum_{l,l'}\frac{\frac{l_<}{l_>}\left(\frac{p_\perp^2}{2eB}\right)^{|l-l'|}\left[L_{l_<}^{|l-l'|}\left(\frac{p_\perp^2}{2eB}\right)\right]^2}{\sqrt{\frac{p_z^2}{4m}-2\chi+(l+l')\omega_B+i\omega_{n_p}}}+\frac{1}{g}. \quad (42)$$

The singularity structure of Γ, with $i\omega_{n_p}$ continued to the entire complex plane, reflects the two-fermion spectrum. There will be an isolated real pole representing the two-body bound pair and a branch cut along the real axis representing the continuum of two-fermion excitations. For sufficiently large $\beta|\chi|$, the contribution to the density is dominated by the bound pair pole. We henceforth consider the expansion of (42) around this pole, which is determined by $\omega=0,\mathbf{p}=0,\mu=\bar{\mu}=\bar{\chi}+\omega_B/2$ with $\bar{\chi}$ the solution to the mean-field equation (21) and $\bar{\chi}<0,\beta|\bar{\chi}|\gg 1$, to the second order in terms of $\mathbf{p}$ and first order in terms of $\omega,\mu-\bar{\mu}$. We obtain that

$$\Gamma^{-1}\simeq a_1\left[-\omega-2(\mu-\bar{\mu})+\frac{p_z^2}{4m}\right]+a_2\frac{p_\perp^2}{4m} \quad (43)$$

with

$$a_1=\frac{m^{3/2}}{16\pi\sqrt{2\omega_B}}\sum_{l=0}^{\infty}\left(l+\frac{|\bar{\chi}|}{\omega_B}\right)^{-\frac{3}{2}}=\frac{m^{3/2}}{16\pi\sqrt{2\omega_B}}\zeta\left(\frac{3}{2},\frac{|\bar{\chi}|}{\omega_B}\right), \quad (44)$$

and

$$\begin{aligned}a_2&=\frac{m^{3/2}}{\pi\sqrt{2\omega_B}}\sum_{l=0}^{\infty}\left(\frac{l+\frac{1}{2}}{\sqrt{l+\frac{|\bar{\chi}|}{\omega_B}}}-\frac{l}{2\sqrt{l-\frac{1}{2}+\frac{|\bar{\chi}|}{\omega_B}}}-\frac{l+1}{2\sqrt{l+\frac{1}{2}+\frac{|\bar{\chi}|}{\omega_B}}}\right)\\&=\frac{m^{3/2}}{\pi\sqrt{2\omega_B}}\left\{\zeta\left(-\frac{1}{2},\frac{|\bar{\chi}|}{\omega_B}\right)-\zeta\left(-\frac{1}{2},\frac{1}{2}+\frac{|\bar{\chi}|}{\omega_B}\right)+\left(\frac{1}{2}-\frac{|\bar{\chi}|}{\omega_B}\right)\left[\zeta\left(\frac{1}{2},\frac{|\bar{\chi}|}{\omega_B}\right)-\zeta\left(\frac{1}{2},\frac{1}{2}+\frac{|\bar{\chi}|}{\omega_B}\right)\right]\right\}, \quad (45)\end{aligned}$$

where the frequency ω is the continuation of the Matsubara frequency $i\omega_{n_p}$ to the neighborhood of the pole. Obviously, the kinetic term becomes anisotropic with respect to the directions along and perpendicular to the magnetic field because of the rotational symmetry breaking by the magnetic field.

The partition function (10) under the Gaussian approximation of fluctuations may be written as

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}\phi^* \mathcal{D}\phi \exp\left\{ \sum_{\omega_{n_p},\mathbf{p}} \phi_p^*(\omega - \omega_b + 2\mu)\phi_p \right\}, \tag{46}$$

where ϕ is the rescaled field of the fluctuation Δ and $\omega_b = -E_B + \omega_B + p_z{}^2/(4m) + \kappa p_\perp^2/(4m)$ is the bosonic dispersion relation with $E_B = -2\bar{\chi}$ the binding energy that is measured from the lowest Landau level. We have also the explicit expression of the anisotropy factor

$$\kappa \equiv a_2/a_1 = 16 \frac{\zeta\left(-\frac{1}{2}, \frac{|\bar{\chi}|}{\omega_B}\right) - \zeta\left(-\frac{1}{2}, \frac{1}{2} + \frac{|\bar{\chi}|}{\omega_B}\right) + \left(\frac{1}{2} - \frac{|\bar{\chi}|}{\omega_B}\right)\left[\zeta\left(\frac{1}{2}, \frac{|\bar{\chi}|}{\omega_B}\right) - \zeta\left(\frac{1}{2}, \frac{1}{2} + \frac{|\bar{\chi}|}{\omega_B}\right)\right]}{\zeta\left(\frac{3}{2}, \frac{|\bar{\chi}|}{\omega_B}\right)}. \tag{47}$$

As is shown in Appendix C, $\kappa \leq 1$ for an arbitrary value of the ratio $|\bar{\chi}|/\omega_B$ and is a monotonically increasing function of this ratio.

The partition function (46) is nothing but an ideal Bose gas with anisotropy in kinetic term and $\Gamma(\omega, \mathbf{p})$ is proportional to the boson propagator. The condensation temperature is determined by setting the chemical potential in (29) at the solution of the mean-field equation (21), i.e. $\mu = \bar{\mu} = \omega_B/2 + \bar{\chi}$, and the phase shift there reads

$$\delta(\omega, \mathbf{p}) = \pi\theta(\omega - \omega_b + 2\bar{\mu}), \tag{48}$$

where $\theta(x)$ is the Heaviside step function with $\theta(x \geq 0) = 1$ and otherwise zero. It follows then that

$$n = 2 \int \frac{d^3p}{(2\pi)^3} \left[\exp\left(\frac{p_z^2 + \kappa p_\perp^2}{4mT_c}\right) - 1\right]^{-1}, \tag{49}$$

where the n_0 term of Eq. (29) is ignored with $T_c \ll |\bar{\chi}|$. Consequently, the BEC temperature is given by

$$T_c = \kappa^{\frac{2}{3}} T_c^0, \tag{50}$$

where

$$T_c^0 = \left[\frac{n}{2\zeta(3/2)}\right]^{2/3} \frac{\pi}{m} \tag{51}$$

is the condensation temperature of an ideal Bose gas of the same density at zero magnetic field.

Beyond the Gaussian approximation, we have also calculated the quartic term, $\mathcal{S}_{\text{quartic}}[\Delta]$, of the effective action (11) in the limit of low energy and momentum of $\Delta(x)$ and obtained a term

$$-\frac{3m^{\frac{3}{2}}\omega_B^{-\frac{3}{2}}}{64\sqrt{2}\pi} \zeta\left(\frac{5}{2}, \frac{\bar{\chi}}{\omega_B}\right) \sum_{\omega_{n_p},\mathbf{p}} |\Delta(i\omega_{n_p}, \mathbf{p})|^4 \tag{52}$$

to be added to Eq. (25). This term gives rise to a repulsive interaction between the bound pairs.

IV. BOSE-EINSTEIN CONDENSATION IN A MAGNETIC FIELD

In this section, we shall explore the magnetic field dependence of the BEC temperature (50) for both strong coupling, $a_s > 0$ and weak coupling, $a_s < 0$.

As the mean-field equation (21) and the formula (47) depend on the ratio $r \equiv |\bar{\chi}|/\omega_B$ through the Hurwitz zeta function, we shall begin with an examination of the two asymptotic behaviors $r \gg 1$ and $r \ll 1$ of the Hurwitz zeta function $\zeta(s, r)$.

The large r expansion follows from the Hermite formula

$$\zeta(s, r) = \frac{r^{-s}}{2} + \frac{r^{1-s}}{s-1} + 2\int_0^\infty \frac{(r^2 + y^2)^{-s/2} \sin s\theta}{e^{2\pi y} - 1} dy, \tag{53}$$

with $\theta = \arctan(y/r)$, and reads

$$\zeta(s, r) \simeq \frac{r^{-s+1}}{s-1} + \frac{r^{-s}}{2} + \frac{sr^{-s-1}}{12} - \frac{s(s+1)(s+2)r^{-s-3}}{720} + O(r^{-s-5}). \tag{54}$$

The negative value of $\zeta(1/2, r)$ in this limit leads us to the strong coupling domain via the mean-field equation (21)

$$\frac{1}{a_s} \simeq \sqrt{2m|\bar{\chi}|} - \frac{1}{2}\sqrt{\frac{m}{2|\bar{\chi}|}}\omega_B > 0. \tag{55}$$

If follows that the approximate binding energy,

$$|\bar{\chi}| \simeq \frac{1}{2ma_s^2}(1 + eBa_s^2), \tag{56}$$

with $a_s \ll \frac{1}{\sqrt{eB}}$. The anisotropy factor (47) reads

$$\kappa \simeq 1 - \frac{1}{16}\left(\frac{\omega_B}{|\bar{\chi}|}\right)^2 \simeq 1 - \frac{1}{4}(eB)^2 a_s^4 \tag{57}$$

and gives rise to a slight suppression of the condensation temperature according to (50), corresponding to an inverse magnetic catalysis.

The small r behavior follows from the relation

$$\zeta(s,r) = r^{-s} + \zeta(s, 1+r) \simeq r^{-s} + \zeta(s), \tag{58}$$

which, for $s > 0$, is dominated by the first term on the rhs and corresponds to the lowest Landau level approximation in our problem. The positivity of $\zeta(\frac{1}{2}, \frac{|\bar{\chi}|}{\omega_B})$ in this case, i.e. $|\bar{\chi}| \ll \omega_B$, together with the mean-field equation (21) implies a negative a_s and thereby the weak coupling domain, i.e.

$$\frac{1}{a_s} \simeq -\sqrt{\frac{m}{2|\bar{\chi}|}}\omega_B < 0. \tag{59}$$

It follows that the binding energy,

$$|\bar{\chi}| \simeq \frac{1}{2} m\omega_B^2 a_s^2, \tag{60}$$

is entirely induced by the magnetic field, as a consequence of the magnetic catalysis. In terms of the solution (60), the inequality $|\bar{\chi}| \ll \omega_B$ implies $|a_s| \ll \frac{1}{\sqrt{eB}}$. The anisotropy factor,

$$\kappa \simeq 8\frac{|\bar{\chi}|}{\omega_B} \simeq 4eBa_s^2 \ll 1, \tag{61}$$

in this case and maximizes the suppression of the condensation temperature.

Since $\zeta(1/2, r)$ is a monotonically decreasing function of r and is negative (positive) for a large (small) r, its zero, r_c, serves a demarcation between the strong coupling domain, where $a_s > 0$ and $|\bar{\chi}|/\omega_B > r_c$, and the weak coupling domain, where $a_s < 0$ and $|\bar{\chi}|/\omega_B < r_c$. The value of r_c as well as the solution of the mean-field equation (21) and the condensation temperature for $|\bar{\chi}|/\omega_B = O(1)$ can only be calculated numerically. We find $r_c \simeq 0.303$,

$$|\bar{\chi}| \simeq r_c\omega_B \simeq 0.303\omega_B, \tag{62}$$

and $\kappa \simeq 0.792$ as $B \to \infty$.

In the strong coupling domain, $a_s > 0$, bound pairs exist in the absence of magnetic field with the binding energy $E_b = 1/(ma_s^2)$ and condense at the temperature T_c^0. The mean-field equation (21) and the condensation temperature (50) in a magnetic field can be expressed in terms of dimensionless quantities, i.e.

$$b^{-\frac{1}{2}} = -\frac{1}{2}\zeta\left(\frac{1}{2}, \frac{v}{b}\right) \tag{63}$$

and

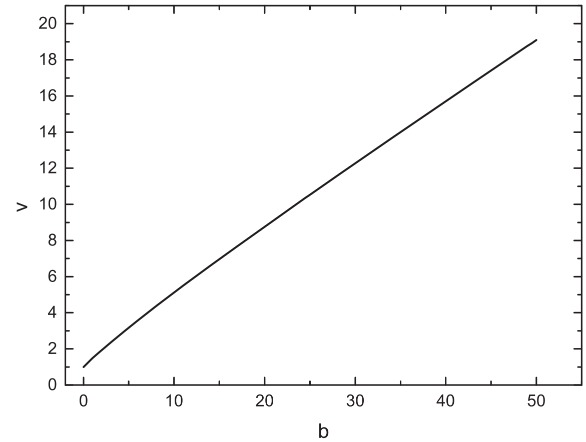

FIG. 1. The scaled binding energy v versus the dimensionless magnetic field b in strong coupling domain.

$$t_c = \kappa^{\frac{2}{3}}\left(\frac{v}{b}\right), \tag{64}$$

where $b \equiv \frac{\omega_B}{E_b}$, $v \equiv \frac{|\bar{\chi}|}{E_b}$ and $t_c \equiv \frac{T_c}{T_c^0}$. The solution of (63) for v and t_c versus the dimensionless magnetic field b are plotted in Fig. 1 and Fig. 2. We find that the binding energy starts with a nonzero value at $b = 0$, indicating the existence of the bound pairs without magnetic field, and grows linearly for large b, consistent with the asymptotic behavior (62). The condensation temperature, however, deceases as the magnetic field increases, consistent with the large r limit. The physical reason for this inverse magnetic catalysis is the enhanced fluctuations by the anisotropic distortion of the bosonic spectrum, $\kappa < 1$, in the magnetic field. The effect is, however, rather mild with κ decreasing from one at $b = 0$ to about 0.9 at $b = 50$ because the ratio r never drops to a level to warrant the LLL approximation within the strong coupling domain.

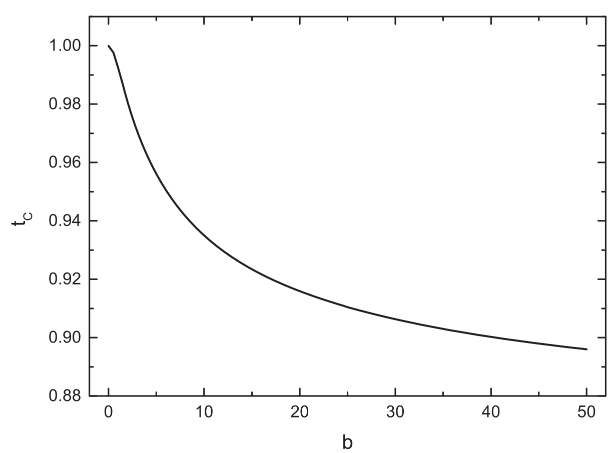

FIG. 2. The ratio of BEC temperature t_c versus the dimensionless magnetic field b in strong coupling domain.

In the weak coupling domain, $a_s < 0$, bound pairs are formed through the mechanism of magnetic catalysis. The mean-field equation becomes

$$b^{-\frac{1}{2}} = \frac{1}{2}\zeta\left(\frac{1}{2}, \frac{v}{b}\right), \tag{65}$$

with the sign on the rhs opposite to that of (63). The formula for the condensation temperature, (64), remains unchanged. Here the denominator of b and t_c, $|E_b|$ and T_c^0, do not carry direct physical meaning other than reference scales because the bound pairs do not exist in the absence of magnetic field. The solution of the mean-field equation for v and t_c versus b in this case are plotted in Fig. 3 and Fig. 4. The strong field limit of the binding energy also follows (62). The difference, however, from the case in the strong coupling domain is that the binding energy at zero magentic field vanishes. The bound pairs exist only at nonzero magnetic field, suggesting a BCS/BEC crossover induced by magnetic field. The condensation temperature in Fig. 4 increases as magentic field increases, which is consistent with the analysis in small r limit. The LLL approximation works in the limit $r \to 0$, where the anisotropy of the bosonic spectrum is maximized. An increasing magnetic field raises the ratio r and promotes the contribution from higher LLs, and thereby increases the condensation temperature.

Notice, however, that the condensation temperature is always suppressed compared with that of an ideal Bose gas of mass $2m$ regardless of the coupling strength because of the inequality $\kappa < 1$ for all real a_s.

Before concluding this section, we would like to comment on the validity of the Gauss approximation of fluctuations in the context of the BEC limit, which ignored the quartic and higher powers on $\Delta(x)$ in (11). These terms represents the interactions among the Cooper pairs, which becomes significant when their wave functions overlap. Therefore the approximation may deteriorate at the density at which the inter-particle distance $n^{-1/3}$ becomes comparable to the size of the bound pairs.

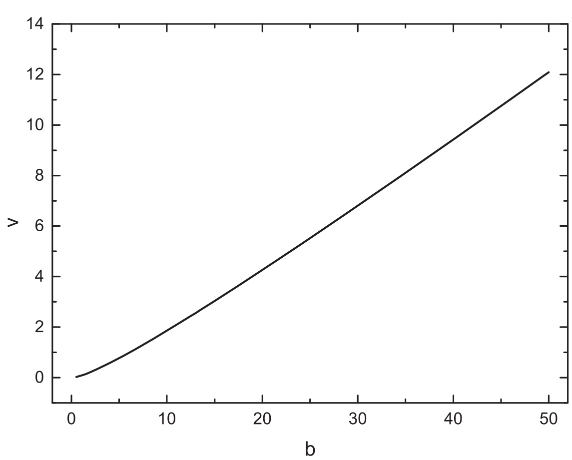

FIG. 3. The scaled binding energy v versus the dimensionless magnetic field b in weak coupling domain.

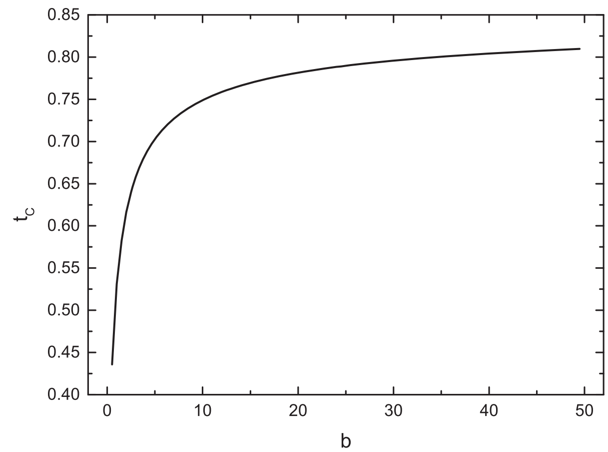

FIG. 4. The ratio of BEC temperature t_c versus the dimensionless magnetic field b in weak coupling domain.

V. SUMMARY AND OUTLOOK

We have investigated a system of nonrelativistic bound pairs made of oppositely charged fermions in the presence of an external magnetic field. We found that the variation of the BEC temperature with respect to the magnetic field depends on the coupling strength of pairing. In the strong coupling domain where the bound pairs (composite bosons) exist already without magnetic field, we found the inverse magnetic catalysis that the condensation temperature decreases as increasing magnetic field. In the weak coupling domain where the bound pairs are induced by magnetic field, the transition temperature exhibits the usual magnetic catalysis effect. In either domain, the condensation temperature is lower than that of an ideal Bose gas of the same mass, $2m$, in the absence of magnetic field. The suppression effect is maximized when the lowest Landau level approximation works which requires the ratio of binding energy relative to the lowest Landau level over the spacing between adjacent Landau levels, $r = |\chi|/\omega_B \ll 1$. This condition is realized in the weak coupling domain under a weak magnetic field. Otherwise, the ratio is order $O(1)$ and the suppression effect is less pronounced. In particular, the binding energy diverges like $|\chi| \simeq 0.303\omega_B$ in the strong field limit, for both strong and weak couplings, making the ratio 0.303 with the suppression factor $\kappa \simeq 0.792$. Of course, the BEC approximation requires the fermion density of the system to be sufficiently low such that the bound pairs do not overlap. With increasing density, individual bound pairs lose their identities and BCS condensation emerges. The fluctuations beyond the Gaussian approximation may also come into play then.

To highlight the Landau level effect in the minimal electromagnetic coupling, we ignored the spin degrees of freedom of the fermions as they do not contribute to the pairing dynamics. Our qualitative conclusion, however, can be carried over to the case with spins and their coupling (nonminimal) to the magnetic field. As an example, we consider the following Hamiltonian:

$$\mathcal{H}[\psi,\psi^\dagger] = \sum_{\sigma=\pm}\psi_\sigma^\dagger(x)\left[\frac{(-i\nabla+\sigma e\mathbf{A})^2}{2m} - \mu - \sigma\omega_B\sigma_3\right]\psi_\sigma(x) - g\psi_+^\dagger(x)\sigma_2\tilde{\psi}_-^\dagger(x)\tilde{\psi}_-(x)\sigma_2\psi_+(x), \tag{66}$$

with $\psi_\pm$ two component spinors and σ_2 and σ_3 Pauli matrices, which pairs fermions of opposite charges in the spin singlet channel, with

$$\tilde{\psi}_-(x)\sigma_2\psi_+(x) = i[-\psi_{-\uparrow}(x)\psi_{+\downarrow}(x) + \psi_{-\downarrow}(x)\psi_{+\uparrow}(x)]. \tag{67}$$

The one-loop contribution to $\Gamma^{-1}(\omega,\mathbf{p})$ consists of two branches of the two-fermion spectrum now,

$$\epsilon_{k_{z1}} = \frac{k_z^2}{2m} + \frac{k_z'^2}{2m} + (l+l')\omega_B - 2\mu \tag{68}$$

and

$$\epsilon_{k_{z2}} = \frac{k_z^2}{2m} + \frac{k_z'^2}{2m} + (l+l'+2)\omega_B - 2\mu, \tag{69}$$

in contrast to the single branch of the two-fermion spectrum in the spinless case,

$$\frac{k_z^2}{2m} + \frac{k_z'^2}{2m} + (l+l'+1)\omega_B - 2\mu. \tag{70}$$

Consequently, the rhs of the mean-field equation (21) becomes a sum of two terms of the same form, one with χ replaced by μ and the other with χ replaced by $\mu - \omega_B$, i.e.

$$-\frac{m}{4\pi a_s} = \frac{\sqrt{\omega_B}m^{3/2}}{4\sqrt{2}\pi}\left[\zeta\left(\frac{1}{2},\frac{|\bar{\mu}|}{\omega_B}\right) + \zeta\left(\frac{1}{2},\frac{|\bar{\mu}|}{\omega_B}+1\right)\right]. \tag{71}$$

Likewise, the coefficients a_1 and a_2 of (44) and (45) as well as the quartic term (52) each becomes the sum of two terms of the same form, one with $|\chi|/\omega_B$ replaced by $|\mu|/\omega$ and the other by $|\mu|/\omega_B + 1$. This change does not modify our statement that the anisotropy factor (47) $\kappa < 1$ and is a monotonically increasing function of the ratio $|\mu|/\omega_B$ and that the lowest Landau level dominates the weak coupling limit.

The Hamiltonian (66) is the nonrelativistic limit of the relativistic Hamiltonian,

$$\mathcal{H}[\psi,\psi^\dagger] = \sum_{\sigma=\pm}\Psi_\sigma^\dagger(x)[\gamma_4\boldsymbol{\gamma}\cdot(\nabla + i\sigma e\mathbf{A}) + m\gamma_4 - \mu]\Psi_\sigma(x) + g\bar{\Psi}_+(x)\gamma_5\Psi_-^c(x)\bar{\Psi}_-^c(x)\gamma_5\Psi_+(x), \tag{72}$$

which we are currently exploring, where $\Psi_\pm(x)$ are Dirac spinors and the charge conjugation $\Psi_\pm^c(x) = \gamma_2\tilde{\Psi}_\pm^\dagger(x)$ with γ's gamma matrices (Hermitian). Beyond the nonrelativistic approximation, the one-loop diagram underlying $\Gamma^{-1}(\omega,\mathbf{p})$ will be quadratically divergent. The leading divergence can be removed by the coupling constant renormalization like (20), but the logarithmic divergence remains, which requires an explicit UV cutoff $\Lambda \gg m$ of the pairing force. Consequently, the solution of the mean-field equation (21) as well as the anisotropy factor will carry an explicit dependence on $\ln\Lambda$. In addition to the weak coupling domain where the pairing dynamics is dominated by the lowest Landau level and the nonrelativistic approximation works, the lowest Landau level also dominates under an ultrastrong magnetic field, $eB \gg \Lambda^2$. The bosonic spectrum is expected to be highly anisotropic for $eB \gg \Lambda^2$ with the critical temperature of BEC strongly suppressed by the fluctuations [29]. The relativistic BCS/BEC crossover in a magnetic field was reported in [30] in the context of a boson-fermion model, where the boson is represented by an independent field with isotropic spectrum to the zeroth order of coupling. The anisotropic distortion can only occur in higher orders there.

ACKNOWLEDGMENTS

The authors would like to thank I. Shovkovy, E. J. Ferrer and V. de la Incera for discussions and valuable comments. This research is supported by the Ministry of Science and Technology of China (MSTC) under the “973” Project No. 2015CB856904. It is also supported by National Natural Science Foundation of China (NSFC) under Grants No. 11305067, No. 11375070, No. 11221504, No. 11135011 and No. 11535005 and by Huazhong University of Science and Technology (HUST) under the Fundamental Research Funds No. 2013QN015 and No. 2015TS016.

APPENDIX A: THE REGULARIZATION IN THE MEAN-FIELD EQUATION (19)

To regularize the rhs of the mean-field equation (19), the summation over Landau orbitals is restricted to $l \le N$ and, correspondingly, the transverse kinetic energy in the second term (the sum over $\mathbf{k}$) is restricted below $N\omega_B$ i.e. $\frac{1}{2}(k_x^2 + k_y^2) \le N\omega_B$. The limit $N \to \infty$ will be taken in the end. Carrying out the momentum integral of the second term and the integration over (k_y, k_z) of the first term, we find that

$$-\frac{m}{4\pi a_s}=\frac{m^{\frac{3}{2}}\sqrt{\omega_B}}{4\sqrt{2}\pi}\lim_{N\to\infty}\left(\sum_{l=0}^{N}\frac{1}{\sqrt{l+\frac{\chi}{\omega_B}}}-2\sqrt{N}\right). \tag{A1}$$

To evaluate the limit, we introduce a sequence of analytic functions

$$f_N(s)\equiv\sum_{l=0}^{N}\left(l+\frac{|\chi|}{\omega_B}\right)^{-s}-\frac{N^{1-s}}{1-s}, \tag{A2}$$

for positive integers N's. The sequence converges uniformly in any closed domain with $\mathrm{Re}s>0$ and $s\neq 1$, and the limit,

$$f(s)=\lim_{N\to\infty}f_N(s), \tag{A3}$$

is therefore an analytic function within the same domain. For $\mathrm{Re}s>1$, the limit of $N^{1-s}/(1-s)$ vanishes and we have

$$f(s)=\zeta\left(s,\frac{|\chi|}{\omega_B}\right). \tag{A4}$$

Following the principle of analytic continuation, we end up with

$$\lim_{N\to\infty}\left(\sum_{l=0}^{N}\frac{1}{\sqrt{l+\frac{\chi}{\omega_B}}}-2\sqrt{N}\right)=f\left(\frac{1}{2}\right)=\zeta\left(\frac{1}{2},\frac{|\chi|}{\omega_B}\right) \tag{A5}$$

and Eq. (21) follows.

APPENDIX B: CALCULATION OF THE INTEGRAL (36)

In this appendix, we show the details of the explicit calculation of the integral (36). In terms of the raising and lowering operators (39), the integral can be written as

$$I_{ll'}=\langle l|e^{i\frac{p_x}{\sqrt{2eB}}(a+a^\dagger)}e^{-\frac{p_y}{\sqrt{2eB}}(a-a^\dagger)}|l'\rangle, \tag{B1}$$

with $u_l(\xi)=\langle\xi|l\rangle$. Using the operator relation

$$e^{A+B}=e^Ae^Be^{-\frac{1}{2}[A,B]}, \tag{B2}$$

with $[A,B]$ commuting with both A and B twice, we find that

$$I_{ll'}=e^{i\frac{p_xp_y}{2eB}-\frac{1}{2}|w|^2}\langle l|e^{iw^*a^\dagger}e^{iwa}|l'\rangle, \tag{B3}$$

with

$$w=\frac{1}{\sqrt{2eB}}(p_x+ip_y). \tag{B4}$$

Expanding the exponential functions in a and $a^\dagger$ and using the relation $a|n\rangle=\sqrt{n}|n-1\rangle$, we find

$$\begin{aligned}&\langle l|e^{iw^*a^\dagger}e^{iwa}|l'\rangle\\&=\sum_{n,n'\leq l'}\frac{1}{n!n'!}\sqrt{\frac{l!l'!}{(l-n)!(l'-n')!}}i^{n+n'}w^{*n}w^{n'}\langle l-n|l'-n'\rangle\\&=(iw^*)^{l-l'}\sum_{n'=0}^{l'}\frac{\sqrt{l!l'!}}{n'!(l-l'+n')!(l'-n')!}(-)^{n'}|w|^{2n'}\\&=\sqrt{\frac{l'!}{l!}}(iw^*)^{l-l'}L_{l'}^{l-l'}(|w|^2),\end{aligned} \tag{B5}$$

for $l\geq l'$. For $l<l'$, we find

$$\begin{aligned}\langle l|e^{iw^*a^\dagger}e^{iwa}|l'\rangle&=\langle l'|e^{-iw^*a^\dagger}e^{-iwa}|l\rangle^*\\&=\sqrt{\frac{l!}{l'!}}(iw)^{l'-l}L_l^{l'-l}(|w|^2).\end{aligned} \tag{B6}$$

Combining (B3), (B4), (B5) and (B6), we derive (40).

APPENDIX C: THE PROPERTIES OF THE ANISOTROPY FACTOR κ

To explore the properties of the anisotropy factor κ as a function of $r=\frac{|\chi^*|}{\omega_B}$, we write

$$\kappa(r)=\frac{f(r)}{g(r)}, \tag{C1}$$

with

$$f(r)=16\left\{\zeta\left(-\frac{1}{2},r\right)-\zeta\left(-\frac{1}{2},\frac{1}{2}+r\right)+\left(\frac{1}{2}-r\right)\left[\zeta\left(\frac{1}{2},r\right)-\zeta\left(\frac{1}{2},\frac{1}{2}+r\right)\right]\right\}, \tag{C2}$$

and

$$g(r)=\zeta\left(\frac{3}{2},r\right). \tag{C3}$$

It turns out that the Hermite formula is not convenient for this purpose and we start with the series representations in (44) and (45). We have

$$f(r) = 16\sum_{l=0}^{\infty}\left(\frac{l+\frac{1}{2}}{\sqrt{l+r}} - \frac{l}{2\sqrt{l-\frac{1}{2}+r}} - \frac{l+1}{2\sqrt{l+\frac{1}{2}+r}}\right)$$
$$= \frac{16}{\sqrt{\pi}}\sum_{l=0}^{\infty}\int_0^\infty dx x^{-\frac{1}{2}}\left[\left(l+\frac{1}{2}\right)e^{-(l+r)x} - \frac{l}{2}e^{-(l-\frac{1}{2}+r)x} - \frac{l+1}{2}e^{-(l+\frac{1}{2}+r)x}\right]$$
$$= \frac{8}{\sqrt{\pi}}\int_0^\infty dx x^{-\frac{1}{2}}\frac{e^{-rx}}{(1+e^{-\frac{x}{2}})^2} \geq 0, \tag{C4}$$

where we have interchanged the order of integration and summation and have carried out the summation explicitly. Likewise,

$$g(r) = \frac{2}{\sqrt{\pi}}\int_0^\infty dx x^{\frac{1}{2}}\frac{e^{-rx}}{1-e^{-x}} \geq 0. \tag{C5}$$

It follows that

$$f(r) - g(r) = \frac{8}{\sqrt{\pi}}\int_0^\infty dx x^{-\frac{1}{2}}\frac{e^{-rx}}{1-e^{-x}}\left(\tanh\frac{x}{4} - \frac{x}{4}\right) \leq 0. \tag{C6}$$

Therefore, $f(r) \leq g(r)$ and $\kappa(r) \leq 1$. Taking the derivatives with respect to r, we find

$$\frac{df}{dr} - \frac{dg}{dr} = -\frac{8}{\sqrt{\pi}}\int_0^\infty dx x^{\frac{1}{2}}\frac{e^{-rx}}{1-e^{-x}}\left(\tanh\frac{x}{4} - \frac{x}{4}\right) \geq 0, \tag{C7}$$

and then $\frac{df}{dr} \geq \frac{dg}{dr}$. Finally,

$$\frac{d}{dr}\ln\kappa(r) = \frac{1}{f}\frac{df}{dr} - \frac{1}{g}\frac{dg}{dr} \geq 0, \tag{C8}$$

and $\frac{d\kappa}{dr} \geq 0$. The statements on κ following (47) are proved.

[1] D. E. Kharzeev, K. Landsteiner, A. Schmitt, and H.-u. Yee, Lect. Notes Phys. **871**, 1 (2013) and references therein.
[2] K. Klimenko, Theor. Math. Phys. **89**, 1161 (1991); Z. Phys. C **54**, 323 (1992).
[3] V. Gusynin, V. Miransky, and I. Shovkovy, Phys. Rev. Lett. **73**, 3499 (1994); Phys. Rev. D **52**, 4718 (1995); Phys. Lett. B **349**, 477 (1995); Nucl. Phys. **B462**, 249 (1996).
[4] K. Fukushima, M. Ruggieri, and R. Gatto, Phys. Rev. D **81**, 114031 (2010).
[5] A. J. Mizher, M. Chernodub, and E. S. Fraga, Phys. Rev. D **82**, 105016 (2010).
[6] V. Skoko, Phys. Rev. D **85**, 034026 (2012).
[7] K. Kamikado and T. Kanazawa, J. High Energy Phys. 03 (2014) 009.
[8] G. Bali, F. Bruchmann, G. Endrődi, Z. Fodor, S. D. Katz, S. Krieg, A. Schäfer, and K. K. Szabó, J. High Energy Phys. 02 (2012) 044.
[9] G. Bali, F. Bruchmann, G. Endrődi, Z. Fodor, S. D. Katz, S. Krieg, and A. Schäfer, Phys. Rev. D **86**, 071502 (2012).
[10] F. Preis, A. Rebhan, and A. Schmitt, J. High Energy Phys. 03 (2011) 033.
[11] K. Fukushima and J. M. Pawlowski, Phys. Rev. D **86**, 076013 (2012).
[12] K. Fukushima and Y. Hidaka, Phys. Rev. Lett. **110**, 031601 (2013).
[13] T. Kojo and N. Su, Phys. Lett. B **720**, 192 (2013).
[14] F. Bruckmann, G. Endrodi, and T. G. Kovacs, J. High Energy Phys. 04 (2013) 112.
[15] J. Chao, P. Chu, and M. Huang, Phys. Rev. D **88**, 054009 (2013); L. Yu, H. Liu, and M. Huang, Phys. Rev. D **90**, 074009 (2014); L. Yu, J. Doorsselaere, and M. Huang, Phys. Rev. D **91**, 074011 (2015).
[16] E. J. Ferrer, V. de la Incera, and X. Wen, Phys. Rev. D **91**, 054006 (2015).
[17] R. Farias, K. Gomes, G. Krein, and M. Pinto, Phys. Rev. C **90**, 025203 (2014).
[18] N. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966); P. Hohenberg, Phys. Rev. **158**, 383 (1967); S. Coleman, Commun. Math. Phys. **31**, 259 (1973).
[19] B. Feng, D.-f. Hou, and H.-c. Ren (to be published).
[20] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Rev. Mod. Phys. **82**, 1225 (2010).
[21] Despite the different coupling to the external magnetic field, our system corresponds to the BEC limit of the BCS/BEC crossover, which has been studied extensively in the absence of a magnetic field for nonrelativistic fermions [24,26] and relativistic ones [27,28,30,31].
[22] E. J. Ferrer, V. de la Incera, and C. Manuel, Phys. Rev. Lett. **95**, 152002 (2005); Nucl. Phys. **B747**, 88 (2006); B. Feng, E. J. Ferrer, and V. de la Incera, Nucl. Phys. **B853**, 213 (2011).
[23] A. Schmitt, Phys. Rev. D **71**, 054016 (2005); A. Schmitt, Q. Wang, and D. H. Rischke, Phys. Rev. Lett. **91**, 242301 (2003); B. Feng, D-f. Hou, H-c. Ren, and P-p. Wu, Phys. Rev. Lett. **105**, 042001 (2010).
[24] C. Sá de Melo, M. Randeria, and J. R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
[25] G. Baym and N. D. Mermin, J. Math. Phys. **2**, 232 (1961).
[26] P. Noziéres and S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985).
[27] Y. Nishida and H. Abuki, Phys. Rev. D **72**, 096004 (2005); H. Abuki, Nucl. Phys. **A791**, 117 (2007); H. Abuki, G. Baym, T. Hatsuda, and Y. Yamamoto, Phys. Rev. D **81**, 125010 (2010).

[28] J. Deng, A. Schmitt, and Q. Wang, Phys. Rev. D **76**, 034013 (2007).
[29] B. Feng, D.-f. Hou, and H.-c. Ren (to be published).
[30] J. Wang, V. de la Incera, E. J. Ferrer, and Q. Wang, Phys. Rev. D **84**, 065014 (2011); E. J. Ferrer and J. Keith, Phys. Rev. C **86**, 035205 (2012); E. J. Ferrer, V. de la Incera, I. Portillo, and M. Quiroz, Phys. Rev. D **89**, 085034 (2014); E. J. Ferrer, V. de la Incera, J. Keith, and I. Portillo, Nucl. Phys. **A933**, 229 (2015).
[31] G. Sun, L. He, and P. Zhuang, Phys. Rev. D **75**, 096004 (2007); L. He and P. Zhuang, Phys. Rev. D **76**, 056003 (2007).