Asymptotic freedom in the front-form Hamiltonian for quantum chromodynamics of gluons

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Asymptotic freedom of gluons in pure-gauge QCD is obtained in the leading terms of their renormalized Hamiltonian in the Fock space, instead of considering virtual Green's functions or scattering amplitudes. Namely, we calculate the three-gluon interaction term in the effective front-form Hamiltonian for gluons in the Minkowski space-time using the renormalization group procedure for effective particles (RGPEP), with a new generator. The resulting three-gluon vertex is a function of the scale parameter, s, that has an interpretation of the size of effective gluons. The corresponding Hamiltonian running coupling constant, g_λ, depending on the associated momentum scale $\lambda = 1/s$, is calculated in the series expansion in powers of $g_0 = g_{\lambda_0}$ up to the terms of third order, assuming some small value for g_0 at some large λ_0. The result exhibits the same finite sensitivity to small-x regularization as the one obtained in an earlier RGPEP calculation, but the new calculation is simpler than the earlier one because of a simpler generator. This result establishes a degree of universality for pure-gauge QCD in the RGPEP.

I. INTRODUCTION

This article describes a calculation of asymptotic freedom [1–4] in the leading ultraviolet terms of front form [5] (FF) Hamiltonians for gluons, using the renormalization group procedure for effective particles (RGPEP, see Sec. II) developed in recent years as an element of the program of constructing nonperturbative QCD outlined in Ref. [6]. Besides the asymptotically free ultraviolet behavior, the calculation also confirms finite dependence of the effective Hamiltonian three-gluon coupling constant on the regularization of small-x singularities in the bare theory. This dependence is of interest since the small-x behavior may in general be thought to be related to the vacuum state in the instant form [5] (IF) of dynamics and in that form the vacuum is believed to be responsible for symmetry breaking and confinement. But the third-order RGPEP calculation reported here would have to be extended to higher orders to verify if it can shed any light on the relevant mechanisms. In this work we concentrate on the gluonic part of QCD only and thus do not include quark-loop contributions. Thus, the third-order RGPEP calculation in quantum $SU(3)$ Yang-Mills gauge theory described in this article could also be described as a calculation of the third-order Hamiltonian of QCD for gluons in the quenched approximation.

In distinction from calculations of Euclidean Green's functions [1–4] and from early calculations using infinite momentum and light-front techniques [7–11], or other approaches, e.g. [12–16], some discussing three-gluon coupling [17,18], asymptotic freedom of gluons is derived here as a feature of the Minkowski space-time FF Hamiltonian that acts in the Fock space of virtual gluons obtained as a result of an explicit operator renormalization group transformation. The transformation does not involve any wave function renormalization constant and the counterterms are calculated without assuming multiplicative renormalizability. More specifically, we use the RGPEP to calculate the coefficient in front of the Hamiltonian operator interaction term that annihilates one effective gluon and creates two, or annihilates two and creates one. This coefficient is called the coupling constant. It is denoted by g_t, since it depends on the RGPEP scale parameter t that corresponds to the size s of effective gluons, $t = s^4$. In the calculated effective Hamiltonian operator, the gluon interaction vertices are softened by the form factors of width $\lambda = 1/s$ in momentum variables. Asymptotic freedom is exhibited in the behavior of the coupling constant g_t as t approaches zero, or λ tends to infinity.

Since solutions to the RGPEP equations involve interactions that are smoothed by form factors, the procedure is thought to provide a means for the understanding of the connection between quantum field theory and phenomenological models. For example, we need to find the mathematical connection between QCD and the properties of

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hadrons. This problem needs a solution irrespective of the form of dynamics one uses; e.g. see [19,20].

In the earlier RGPEP calculation [21], a generator is used that is suitable for perturbative calculations but difficult to use beyond the perturbative regime. In this article, we use a generator that is much easier to use beyond perturbative expansion. The difference between the generators is further explained in Sec. II. At the same time, our perturbative calculation demonstrates that the new RGPEP generator passes the test of producing asymptotic freedom, which any method aiming at solving QCD must pass. In particular, passing this test is a precondition for tackling nonperturbative issues, such as the ones that emerge when one allows effective gluons to have masses [6].

An additional point is thus made that two different versions of the RGPEP, defined using two different generators, yield the same behavior of the running coupling constant in the three-gluon Hamiltonian interaction term when the calculation is carried out in the third-order expansion in powers of the coupling constant. This means that a considerable change in the RGPEP generator does not influence the finite behavior of the three-gluon term. Therefore, we suggest that there exists a certain degree of universality in the behavior of FF Hamiltonians in the RGPEP: the leading terms in the Hamiltonian beta function are universal and they are universally obtained using different versions of the RGPEP.

The paper is organized as follows. The RGPEP is generally but briefly described in Sec. II. The bare Hamiltonian for gluons is introduced in Sec. III, which includes the derivation of its FF density from the Lagrangian density, solving constraints, quantization, and regularization. Section IV explains our third-order calculation of the effective Hamiltonian three-gluon interaction term as a function of the RGPEP scale. This scale is denoted by $t = \lambda^{-4}$, where $\lambda = 1/s$ is the invariant mass width of the form factors in effective Hamiltonian vertices that solve the RGPEP equation, and s is the parameter that has the interpretation of size of effective gluons. The calculation includes identification of second-order mass counterterms in Sec. IV A, a third-order counterterm for the three-gluon term in Sec. IV B, and the definition and result one obtains for the Hamiltonian running coupling constant in Sec. IV C. Comparison of the previous and current calculation is summarized in Sec. V, and Sec. VI concludes the paper. Several appendixes provide details necessary for completeness of the paper, including details of the RGPEP in Appendix A, details of the initial Hamiltonian in Appendix B, integration of the RGPEP equation order by order in Appendix C, calculation of the three-gluon vertex counterterm in Appendix D, and the method of evaluating the third-order contributions to the running coupling constant g_λ with a formula explaining the infrared stability guaranteed by the design of the RGPEP in Appendix E.

II. THE METHOD OF CALCULATION

The size parameter for gluons is introduced by solving the RGPEP differential equation,

$$\frac{d}{dt}\mathcal{H}_t = [\mathcal{G}_t, \mathcal{H}_t], \quad (1)$$

where $\mathcal{H}_t$ denotes the Hamiltonian of interest and $\mathcal{G}_t$ plays the role of a generator of the required transformation; see Appendix A. The transformation changes the bare creation and annihilation operators for pointlike gluons of canonical QCD, which are denoted by a_0 in reference to $t = 0$, to the operators for effective gluons of finite size $s = t^{1/4}$ that are denoted by a_t,

$$a_t = \mathcal{U}_t a_0 \mathcal{U}_t^\dagger, \quad (2)$$

where

$$\mathcal{U}_t = T \exp\left(-\int_0^t d\tau \mathcal{G}_\tau\right), \quad (3)$$

and T denotes ordering in τ. The initial condition for solving Eq. (1) is provided at $t = 0$ by the canonical FF Hamiltonian with modifications implied by its divergent nature. These modifications include the regularization factors in interaction vertices and the counterterms whose structure is found using solutions to Eq. (1).

Our choice of the generator has the form of a commutator, similar to but different from Wegner's [22],

$$\mathcal{G}_t = [\mathcal{H}_f, \mathcal{H}_{Pt}], \quad (4)$$

where operators $\mathcal{H}_f$ and $\mathcal{H}_{Pt}$ are defined using parts of the Hamiltonian $\mathcal{H}_t$. The operator $\mathcal{H}_f$ is the free part of $\mathcal{H}_t$ and $\mathcal{H}_{Pt}$ is defined in terms of the interaction terms as explained in Appendix A. The definition of $\mathcal{H}_{Pt}$ secures that $\mathcal{U}_t$ is invariant with respect to the seven-parameter Poincaré subgroup that forms the kinematical symmetry group of the FF of Hamiltonian dynamics. The generator $\mathcal{G}_t$ in Eq. (4) is simpler and more suitable for nonperturbative calculations than the one used in Ref. [21] (see below).

Once the generator is chosen and the counterterms that complete the definition of the initial condition for $\mathcal{H}_t$ at $t = 0$ are found, the Hamiltonian for effective gluons of size s is uniquely determined, up to the value of the coupling constant g_0 at some arbitrarily chosen value of $t = t_0$. Thus, the renormalized FF Hamiltonian for QCD could in principle be defined using the RGPEP without any reference to perturbation theory.

However, our calculation is only carried out using an expansion in powers of g_t [23] up to the third order. The reason is that too little is currently known about the counterterms required for nonperturbative calculations. Even the terms of second and third order require study.

We show in the next sections that the generator $\mathcal{G}_t$ of Eq. (4) produces the same third-order dependence of g_t on t as the one obtained in Ref. [21]. We thus obtain the Minkowski space-time Hamiltonian example of the universality of leading perturbative terms in the coupling constant in asymptotically free theories. Our perturbative calculation indicates what kind of terms are necessary to counter the ultraviolet divergences. They also show how the small-x singularities appear in addition to the ultraviolet ones, and that they cancel out, leaving behind finite effects.

The current level of knowledge about the FF Hamiltonians for QCD being quite limited, calculations of higher order than the third one discussed here are required to gain more information. However, the higher-order calculations require the third-order output reported here as an input. Given the RGPEP Eq. (1) and its systematic expansion [23], one may hope that it will eventually become possible to identify the structure of $\mathcal{H}_t$ required for obtaining nonperturbative solutions with mathematically estimable precision, perhaps using conceptual analogies between the RGPEP and procedures discussed in Refs. [24,25].

III. CANONICAL HAMILTONIAN FOR GLUONS

The initial condition for solving Eq. (1) is the canonical FF Hamiltonian for gluons in QCD plus counterterms. In this section we describe the canonical Hamiltonian for gluons. The description introduces the notation for details that appear throughout the article.

The canonical Hamiltonian is derived from the standard Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\text{tr}F^{\mu\nu}F_{\mu\nu}, \tag{5}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu]$, $A^\mu = A^{a\mu}t^a$, $[t^a, t^b] = if^{abc}t^c$ and $\text{tr}t^a t^b = \delta^{ab}/2$. The associated energy-momentum density tensor reads,

$$\mathcal{T}^{\mu\nu} = -F^{a\mu\alpha}\partial^\nu A_\alpha^a + g^{\mu\nu}F^{a\alpha\beta}F_{\alpha\beta}^a/4. \tag{6}$$

The FF Hamiltonian is obtained by integrating the component $\mathcal{T}^{+-}$ over the hyperplane defined by the condition $x^+ = x^0 + x^3 = 0$. We work in the gauge $A^+ = 0$, in which the Lagrange equations constrain A^- to

$$A^- = \frac{1}{\partial^+}2\partial^\perp A^\perp - \frac{2}{\partial^{+2}}ig[\partial^+ A^\perp, A^\perp], \tag{7}$$

so that the only degrees of freedom are the fields $A^\perp$. The first term in A^- is independent of the coupling constant g. This term is by definition included in a new constrained field, which is denoted by the same symbol A in what follows, with

$$A^- = \frac{1}{\partial^+}2\partial^\perp A^\perp, \tag{8}$$

while the second term is explicitly included in the interaction Hamiltonian that is written in terms of fields $A^\perp$, using A^- defined in Eq. (8). Employing this convention and freely integrating by parts, one obtains the FF energy of the constrained gluon field in the form

$$P^- = \frac{1}{2}\int dx^- d^2x^\perp \mathcal{H}|_{x^+=0}, \tag{9}$$

where $\mathcal{H} = \mathcal{T}^{+-}$ is a sum of four terms, denoted as in Ref. [21],

$$\mathcal{T}^{+-} = \mathcal{H}_{A^2} + \mathcal{H}_{A^3} + \mathcal{H}_{A^4} + \mathcal{H}_{[\partial AA]^2}. \tag{10}$$

The terms are [7–9]

$$\mathcal{H}_{A^2} = -\frac{1}{2}A^{\perp a}(\partial^\perp)^2 A^{\perp a}, \tag{11}$$

$$\mathcal{H}_{A^3} = gi\partial_\alpha A_\beta^a[A^\alpha, A^\beta]^a, \tag{12}$$

$$\mathcal{H}_{A^4} = -\frac{1}{4}g^2[A_\alpha, A_\beta]^a[A^\alpha, A^\beta]^a, \tag{13}$$

$$\mathcal{H}_{[\partial AA]^2} = \frac{1}{2}g^2[i\partial^+ A^\perp, A^\perp]^a\frac{1}{(i\partial^+)^2}[i\partial^+ A^\perp, A^\perp]^a. \tag{14}$$

The bare expression for the quantum gluon energy operator is obtained through replacing A^μ in $\mathcal{T}^{+-}$ by an operator four-vector $\hat{A}^\mu$, which is defined by its Fourier composition on the front corresponding to $x^+ = 0$,

$$\hat{A}^\mu = \sum_{\sigma c}\int[k][t^c\varepsilon_{k\sigma}^\mu a_{k\sigma c}e^{-ikx} + t^c\varepsilon_{k\sigma}^{\mu*}a_{k\sigma c}^\dagger e^{ikx}]_{x^+=0}. \tag{15}$$

This operator acts in the Fock space spanned by states created by products of the creation operators $a_{k\sigma c}^\dagger$ on the bare vacuum state $|0\rangle$.

In the operator $\hat{A}^\mu$, the Fourier-like integral over kinematical momentum variables is carried out with the measure $[k] = \theta(k^+)dk^+ d^2k^\perp/(16\pi^3 k^+)$. Thus, the integration matches the one in the Fourier transform only in the transverse directions, the integral over k^+ being limited to positive values.

The polarization four-vectors ε have components

$$\varepsilon_{k\sigma}^\mu = (\varepsilon_{k\sigma}^+ = 0, \varepsilon_{k\sigma}^- = 2k^\perp\varepsilon_\sigma^\perp/k^+, \varepsilon_\sigma^\perp). \tag{16}$$

The symbol σ labels the gluon spin polarization and c is a color index. The creation and annihilation operators satisfy commutation relations

$$[a_{k\sigma c}, a_{k'\sigma'c'}^\dagger] = k^+\tilde{\delta}(k-k')\delta^{\sigma\sigma'}\delta^{cc'}, \tag{17}$$

where $\tilde{\delta}(p) = 16\pi^3\delta(p^+)\delta(p^1)\delta(p^2)$, and commutators among all a's, and among all $a^\dagger$'s, vanish. By definition, $a_{k\sigma c}|0\rangle = 0$ for all momenta, spins and colors.

Normal-ordering of the operator density $\mathcal{H}(\hat{A})$ defines the FF integrand in the Hamiltonian,

$$\hat{P}^- = \frac{1}{2}\int dx^- d^2x^\perp : \mathcal{H}(\hat{A}): , \qquad (18)$$

in which all annihilation operators are on the right side of all creation operators. Details of $\hat{P}^-$ are given in Appendix B.

A. Regularization

The bare Hamiltonian is regularized by introducing regulating factors, denoted by r, in the interaction terms. These factors make the interaction terms vanish [21] if the associated change of any gluon relative transverse momentum were to exceed the very large cutoff parameter Δ. Likewise, the interactions are also made to vanish if any change of any longitudinal momentum fraction x of any gluon involved in the interaction were to be smaller than the very small cutoff parameter δ. Various regularization factors can be incorporated in the interaction terms in $\hat{P}^-$ of Eq. (18) to realize these conditions. Appendix B shows how the regularization factors are introduced in $\hat{P}^-$ according to the following rules.

In every interaction Hamiltonian term every particle creation and annihilation operator is labeled by its momentum quantum numbers p^+ and $p^\perp$. Let the total momentum of all quanta annihilated in a term have components P^+ and $P^\perp$. These are the same as components of the total momentum of quanta created in the term. The relative momentum fraction x for the quantum of momentum p is defined as the ratio

$$x_{p/P} = p^+/P^+, \qquad (19)$$

and the relative transverse momentum for the quantum is defined by

$$\kappa_{p/P} = p^\perp - xP^\perp. \qquad (20)$$

Every creation and annihilation operator in every term in the entire canonical Hamiltonian of any momentum p is multiplied by the regulating factor

$$r_{\Delta\delta}(\kappa^\perp, x) = r_\Delta(\kappa^\perp)r_\delta(x)\theta(x). \qquad (21)$$

We use one transverse regulator factor

$$r_\Delta(z) = \exp(-z/\Delta^2), \qquad (22)$$

and one of the following three different small-x regulator factors,

(a) $\quad r_\delta(x) = x/(x+\delta), \qquad (23)$

(b) $\quad r_\delta(x) = \theta(x-\delta), \qquad (24)$

(c) $\quad r_\delta(x) = x^\delta\theta(x-\epsilon). \qquad (25)$

Dependence on the transverse regulator factors will be removed using the RGPEP. Effects of the small-x regularization will be described by comparing results obtained using different regulator factors in Eqs. (23) to (25).

Regularization factors in canonical QCD terms that are quartic in gluon field operators are additionally specified by treating every such term as built solely from vertices in which one quantum is changed to two or vice versa. This regularization choice also applies to the seagull terms that result from constraints on A^-, as if the constrained field component corresponded to an exchange of a quantum with a corresponding momentum. Details are available in Appendix B. In an abbreviated notation, the regularization uses symbols $\tilde{r}_{P,p} = \tilde{r}_{\Delta\delta}(P, p)$, where

$$\tilde{r}_{\Delta\delta}(P, p) = r_{\Delta\delta}(p^\perp - x_{p/P}P^\perp, x_{p/P})r_{\Delta\delta} \\ \times [P^\perp - p^\perp - (1 - x_{p/P})P^\perp, 1 - x_{p/P}]. \qquad (26)$$

B. Counterterms

The initial condition for solving the RGPEP Eq. (1) is provided by the regulated canonical Hamiltonian plus counterterms. The ultraviolet divergent parts of these counterterms, depending on the regularization parameter Δ, are found in a process of calculating Hamiltonians with finite parameter t and eliminating their dependence on Δ by adjusting the initial condition. More precisely, one adjusts the counterterms so that the coefficients of products of creation and annihilation operators in an effective theory for gluons of finite size s become independent of the regularization parameter Δ when the regularization in dynamics of gluons of size zero is being removed. The remaining unknown finite parts must be adjusted to respect symmetries of the theory and to match its predictions with experiments. In the case of pure glue theory, the only unknown parameter would be Λ_{QCD}, which could be adjusted so that, for example, the theory yields the desired value of mass for some glueball, if the mass gap is found to exist.

The question that arises is how the small-x regularization effects can be removed. The required counterterms are relevant to our understanding of the theory ground state and mechanism of confinement [6]. We shall show in the next sections that the small-x divergences cancel out in the third-order RGPEP coupling constant in the effective Hamiltonians. However, the third-order effective three-gluon interaction terms exhibit a finite small-x regularization dependence which is not yet fully understood.

One hopes that the finite small-x regularization dependence, which we illustrate using different small-x regulator factors listed in Eqs. (23) to (25), may cancel out in the mass eigenvalues for glueballs and their scattering amplitudes. Verification of such cancellation is foreseen to be difficult because it involves solving bound-state eigenvalue problems for gluons. It may turn out that even for the calculation of lightest glueballs one needs to introduce the counterterms that also secure confinement of color [6]. Before this issue is resolved, in the practice of approximate calculations of observables, one can seek finite parts of the counterterms using as a guiding rule the minimization of dependence on the renormalization scale [13], which in the RGPEP means minimization of dependence on the gluon size parameter s.

We show in the next sections that there exists a small-x regularization that yields the third-order effective coupling constant which depends on the size of effective gluons in the same way as the running coupling constant calculated using Feynman diagrams for off-shell Green's functions depends on the virtuality of external gluon lines. Moreover, we demonstrate below that two different RGPEP generators lead to the same third-order results for the effective Hamiltonian coupling constant including finite effects of the small-x regularization. These results suggest that the calculated finite sensitivity to the small-x regularization is not accidental and, being established here, should be further studied as a potentially universal feature of a whole class of FF Hamiltonians for effective gluons.

IV. CALCULATION OF THE THREE-GLUON TERM

We solve the RGPEP Eq. (1) perturbatively, expanding H_t in powers of the coupling constant g up to third order,

$$H_t = H_{11,0,t} + H_{11,g^2,t} + H_{21,g,t} + H_{12,g,t} + H_{31,g^2,t}$$
$$+ H_{13,g^2,t} + H_{22,g^2,t} + H_{21,g^3,t} + H_{12,g^3,t}. \tag{27}$$

The first subscript lists the numbers of creation and annihilation operators contained in a term, correspondingly. The second subscript indicates the order in powers of g, and the third subscript indicates dependence on the parameter t. For building intuition, we introduce symbols: μ^2 for mass terms, which have the first subscript 11; Y for three-gluon interaction terms, which have the first subscripts 12 or 21; and X for four-gluon interaction terms, which have the first subscript 22. Consequently, the powers of g are explicitly accounted for using the following notation:

$$H_{11,0} \rightarrow E, \tag{28}$$

$$H_{11,g^2} \rightarrow g^2\hat{\mu}^2, \tag{29}$$

$$H_{21,g} + H_{12,g} \rightarrow gY_{21} + gY_{12}, \tag{30}$$

$$H_{22,g^2} \rightarrow g^2X_{22}, \tag{31}$$

$$H_{31,g^2} + H_{13,g^2} \rightarrow g^2\Xi_{31} + g^2\Xi_{13}, \tag{32}$$

$$H_{21,g^3} + H_{12,g^3} \rightarrow g^3Y_{h21} + g^3Y_{h12}. \tag{33}$$

To be faithful to the difference in notation between $\mathcal{H}_t$ and H_t, associated with changing bare to effective gluon operators (see Appendix A), we could also introduce symbols like $\mathcal{E}$ instead of E, $\mathcal{Y}_t$ instead of Y_t, etc. However, it is simpler to remember the difference, and write

$$H_t = E + g^2\hat{\mu}_t^2 + gY_{21t} + gY_{12t} + g^2X_{22t} + g^2\Xi_{31t}$$
$$+ g^2\Xi_{13t} + g^3Y_{h21t} + g^3Y_{h12t}. \tag{34}$$

The initial condition Hamiltonian at $t = 0$ is written using symbols with the subscript 0 in place of t,

$$H_0 = E + g^2\hat{\mu}_0^2 + gY_{210} + gY_{120} + g^2X_{220} + g^2\Xi_{310}$$
$$+ g^2\Xi_{130} + g^3Y_{h210} + g^3Y_{h120}. \tag{35}$$

The counterterms that need to be found are included in H_0.

Besides using the parameter t and its initial value $t = 0$, we also use the parameter $\lambda = s^{-1} = t^{-1/4}$, whose initial value is ∞. The parameter λ has the interpretation of momentum-space width of the form factors that appear in solutions for H_t.

The RGPEP Eq. (1) for the Hamiltonian (34) reads

$$g^2\partial_t\hat{\mu}_t^2 + gY'_{21t} + gY'_{12t} + g^2X'_{22t} + g^2\Xi'_{31t} + g^2\Xi'_{13t}$$
$$+ g^3Y'_{h21t} + g^3Y'_{h12t}$$
$$= [[E, gY_{21Pt} + gY_{12Pt} + g^2X_{22Pt} + g^2\Xi_{31Pt} + g^2\Xi_{13Pt}$$
$$+ g^3Y_{h21Pt} + g^3Y_{h12Pt}],$$
$$\times E + g^2\hat{\mu}_t^2 + gY_{21t} + gY_{12t} + g^2X_t + g^2\Xi_{31t}$$
$$+ g^2\Xi_{13t} + g^3Y_{h21t} + g^3Y_{h12t}]. \tag{36}$$

We solve Eq. (36) order by order in series of powers of g, which eventually is translated into a series expansion in powers of g_t. Mass-squared terms are of the second order and the gluon vertex is made of terms of the first and third order. After removing powers of g from the equations, we get

$$Y'_{21t} + Y'_{12t} = [[E, Y_{21Pt} + Y_{12Pt}], E], \tag{37}$$

$$\partial_t\hat{\mu}_t^2 + X'_{22t} + \Xi'_{31t} + \Xi'_{13t}$$
$$= [[E, X_{22Pt} + \Xi_{31Pt} + \Xi_{13Pt}], E]$$
$$+ [[E, Y_{21Pt} + Y_{12Pt}], Y_{21t} + Y_{12t}], \tag{38}$$

$$Y'_{h21t} + Y'_{h12t}$$
$$= [[E, Y_{h21Pt} + Y_{h12Pt}], E]$$
$$+ [[E, X_{22Pt} + \Xi_{31Pt} + \Xi_{13Pt}], Y_{21t} + Y_{12t}]$$
$$+ [[E, Y_{21Pt} + Y_{12Pt}], \hat{\mu}_t^2 + X_{22t} + \Xi_{31t} + \Xi_{13t}].$$
$$(39)$$

The running of the Hamiltonian coupling constant g_t is encoded in the operator Eq. (39). Solving Eq. (39) requires knowledge of solutions to the operator Eqs. (37) and (38). The gluon mass-squared counterterm is obtained from the second-order equations, and the three-gluon vertex counterterm from the third-order equations.

The solution for

$$Y_t = g(Y_{12t} + Y_{21t}) + g^3(Y_{h21t} + Y_{h12t}) \qquad (40)$$

is written in terms of the bare creation and annihilation operators for canonical gluons and powers of the bare coupling constant. The last step in the RGPEP is the replacement of the bare gluon operators by the effective ones at scale t and expressing g in terms of g_t.

Details of solving Eqs. (37) to (39) are described in Appendix C. Here we list the results. The first-order terms are the same as in Ref. [21]; see Appendix C 1.

A. Mass-squared term and its counterterm

The second-order mass-squared term for effective gluons that solves Eq. (38) has the form

$$\hat{\mu}_t^2 = \sum_{\sigma c} \int [k] \frac{\mu_t^2}{k^+} a_{k\sigma c}^\dagger a_{k\sigma c}, \qquad (41)$$

where the only element that depends on the scale t is the parameter μ_t^2. The result for it reads

$$\mu_t^2 = \mu_\delta^2 + \frac{g^2}{(4\pi)^2} \int_0^1 dx r_{\delta\mu}(x) P(x)$$
$$\times \int_0^\infty dz \exp(-2tz^2), \qquad (42)$$

where

$$\sum_{12} |Y_{12k}|^2/\kappa^2 = N_c[1 + 1/x^2 + 1/(1-x)^2]$$
$$= P(x)/[2x(1-x)], \qquad (43)$$

and $P(x)$ is the Altarelli-Parisi gluon splitting function $P_{GG}(x)$ [26]. $N_c = 3$ denotes the number of colors. The effective mass-squared term is sensitive to the small-x regularization. The counterterm that canceled dependence on the ultraviolet cutoff $\Delta \to \infty$ contains the mass-squared factor of the form

$$\mu_0^2 = \mu_\delta^2 + \frac{g^2}{(4\pi)^2} \int_0^1 dx r_{\delta\mu}(x) P(x)$$
$$\times \int_0^\infty dz \exp[-4zx(1-x)/\Delta^2]. \qquad (44)$$

Comparison with Ref. [21] shows that the gluon mass-squared term obtained using the RGPEP generator of Eq. (A4) does not differ from the one obtained using the generator of Eq. (A5).

The ultraviolet finite part of the mass counterterm, μ_δ^2, depends on the small-x regularization parameter δ in the initial Hamiltonian. Therefore, the simplest way of choosing the ultraviolet finite part of the mass-squared counterterm is to set the mass squared for effective gluons at some value of t to a desired function of δ. Such function of δ can be fixed by demanding that the effective Hamiltonian eigenvalue for lightest states with color quantum numbers of a single gluon contains a specified mass-squared term that depends on the parameter δ in a specific way.

The right dependence for defining a complete theory of gluons is currently unknown but it is also currently not excluded that one can attempt to describe confinement of gluons by demanding that the gluon mass eigenvalue diverges in the limit $\delta \to 0$. Verification of this option requires studies far beyond the scope of this article. Namely, one needs to study terms of higher order and consider the eigenvalue problem in higher order than third, before one will know if the perturbative expansion of the RGPEP can lead to establishment of a general structure of the Hamiltonian that may solve Eq. (1) beyond perturbation theory. Here, we shall find that, once the ultraviolet divergences are removed, the small-x divergences do not appear in the third-order asymptotically free coupling constant in renormalized Hamiltonians for effective gluons.

The second-order effective gluon mass term, denoted by $\hat{m}_t^2$, is obtained from $\hat{\mu}_t^2$ in Eq. (41) by applying the transformation $\mathcal{U}_t$ and thus replacing the creation and annihilation operators for bare gluons by the ones for effective gluons of size s. Namely,

$$\hat{m}_t^2 = \mathcal{U}_t \hat{\mu}_t^2 \mathcal{U}_t^\dagger. \qquad (45)$$

This transformation amounts to the replacement in Eq. (41) of $a_{k\sigma c}^\dagger a_{k\sigma c}$ by $a_{tk\sigma c}^\dagger a_{tk\sigma c}$. The former operators correspond to thin and the latter to thick lines in Fig. 6.

B. Third-order three-gluon term and its counterterm

We focus our attention on the term $H_{21,g^3,t}$ in Eq. (27), knowing that $H_{12,g^3,t}$ is its Hermitian conjugate. The term has the structure

$$H_{21,g^3,t} = \mathcal{U}_t \gamma_{t21} \mathcal{U}_t^\dagger, \qquad (46)$$

where

$$\gamma_{t21} = f_t \sum_n \gamma_{t21(n)}. \tag{47}$$

The factor f_t in front of an operator means that the vertex functions in the operator are multiplied by the form factor defined in Eq. (A6).

The subscript n in the sum ranges over ten values, denoted by alphabet letters from a to j. Each of the summed terms results from some specific operator product in Eq. (39). In each of these terms there appears a vertex function, denoted by $\gamma_{(n)}$, in the otherwise universal pattern of the formula

$$\gamma_{t21(n)} = \sum_{123} \int [123]\tilde{\delta}(k_1 + k_2 - k_3)\frac{g^3}{16\pi^3}\frac{1}{2}\gamma_{(n)}a_1^\dagger a_2^\dagger a_3. \tag{48}$$

The vertex functions $\gamma_{(n)}$, with subscript n ranging from a to j, are given in Appendix C 3, with their operator origin in Eq. (39) being illustrated by diagrams in Fig. 7 there. The thick external lines in Fig. 7 correspond to the creation and annihilation operators that appear in the three-gluon inter-action term for effective gluons of size s. The thin internal lines correspond to the commutators that result from moving annihilation operators to the right of all creation operators for gluons of size zero, in terms of which the RGPEP Eq. (1) is solved. Lines with a transverse dash indicate instantaneous interactions in H_0 that result from the constraint of Eq. (7).

The vertex functions diverge when the ultraviolet cutoff parameter Δ is being sent to infinity. The divergences result from integration over the transverse relative momentum of virtual quanta whose creation and annihilation operators were contracted in the products that appear on the right-hand side of Eq. (39); cf. Eq. (C8). The required counter-term, calculated in Appendix D, has the form

$$\tilde{Y}_{h210} = \sum_{123} \int [123]\tilde{\delta}(k_1 + k_2 - k_3)\frac{g^3}{16\pi^3}\gamma_0 a_1^\dagger a_2^\dagger a_3, \tag{49}$$

with the vertex function γ_0 derived in Eq. (D12),

$$\gamma_0 = -Y_{123}\frac{\pi}{3}\ln\frac{\Delta}{\mu}\{N_c[11 + h(x_1)]\} + \gamma_{\text{finite}}. \tag{50}$$

For calculation of the Hamiltonian running coupling constant, the finite part of the counterterm, denoted in Eq. (50) by γ_{finite}, will not need to be specified when the subtraction of the diverging part is introduced as described in the next section.

C. Running coupling constant

Our Hamiltonian running coupling constant g_t is extracted from the three-gluon terms in H_t. These terms create two gluons and annihilate one, or vice versa. Both types yield the same result for g_t. The three-gluon term is a sum of terms denoted by (a) to (j) in the previous section (cf. Fig. 8). The vertex function of the entire sum depends on the gluon colors, polarizations and momenta. In the terms that vary with the scale parameter t, the dependence on color and polarization in the limit $\kappa_{12}^\perp \to 0$ takes the form of a combination Y_{123} shown in Appendix B in Eq. (B3). This combination is multiplied by a function $g(t, x_1)$, where $x_1 = 1 - x_2$ refers to the $+$-momentum fraction carried by one gluon of the total momentum of two gluons that are created or annihilated by the three-gluon term. The coupling constant g_t is defined as the value of $g(t, x_1)$ at some value of $x_1 = x_0$,

$$g_t = g(t, x_0). \tag{51}$$

Note that the Hamiltonian $\mathcal{H}_t$ that appears in Eq. (1) is calculated using the bare creation and annihilation operators and the effective Hamiltonian H_t is obtained from $\mathcal{H}_t$ by inserting the effective creation and annihilation operators in place of the bare ones. The vertex function is not changed. We calculate g_t using $\mathcal{H}_t$.

The counterterm contribution to the vertex function can be written as

$$\gamma_0 = -\gamma_{t_0} + \tilde{\gamma}_{\text{finite}}. \tag{52}$$

The diverging part of the counterterm is thus specified using the negative of γ_t at an arbitrary finite value of t_0. Therefore, $\tilde{\gamma}_{\text{finite}}$ may differ from γ_{finite} in Eq. (50) by terms that do not depend on t. Such terms will not contribute to the dependence of g_t on t and will not be further discussed in this article. Note that since the diverging part of the counterterm may be a function of x_1, as displayed in Eq. (50), one also has to consider the counterterm finite part that may be a function of x_1 [6].

After inclusion of the counterterm defined in Eq. (52), our result for the three-gluon interaction term in $\mathcal{H}_t$ has the form (the symbol σ stands for spin variables)

$$Y_t = gY_{1t} + g^3Y_{3t} + \cdots, \tag{53}$$

$$Y_t = \sum_{123} \int [123]\tilde{\delta}(1 + 2 - 3)f_{12t}\tilde{Y}_t(x_1, \kappa_{12}^\perp, \sigma)a_1^\dagger a_2^\dagger a_3, \tag{54}$$

$$Y_{1t} = \sum_{123} \int [123]\tilde{\delta}(1 + 2 - 3)f_{12t}\tilde{Y}_{1t}(x_1, \kappa_{12}^\perp, \sigma)a_1^\dagger a_2^\dagger a_3, \tag{55}$$

$$Y_{3t} = \sum_{123} \int [123]\tilde{\delta}(1 + 2 - 3) f_{12t} \tilde{Y}_{3t}(x_1, \kappa_{12}^{\perp}, \sigma) a_1^{\dagger} a_2^{\dagger} a_3. \tag{56}$$

The symbols $\tilde{Y}$ denote vertex functions without the form factor f_t. Assuming a counterterm that involves a subtraction at some $t = t_0$ as described above, one obtains the third-order vertex factor $\tilde{Y}$ of the structure

$$\tilde{Y}_{3t}(x_1, \kappa_{12}^{\perp}, \sigma) = \tilde{T}_{3t}(x_1, \kappa_{12}^{\perp}, \sigma) - \tilde{T}_{3t_0}(x_1, \kappa_{12}^{\perp}, \sigma) \\ + \tilde{T}_{3\text{finite}}(x_1, \kappa_{12}^{\perp}, \sigma), \tag{57}$$

where the symbol $\tilde{T}$ denotes the sum of third-order terms from (a) to (i) calculated in Appendix C 3,

$$\tilde{T} = \sum_{n=a}^{i} \gamma_{(n)}. \tag{58}$$

The diverging part of the counterterm, denoted as $\gamma_{(j)}$ in Eq. (D14) in Appendix C 3, is included here through the subtraction at $t = t_0$, and the associated change in the finite part is not needed in the discussion that follows.

Combined, all the three-gluon terms in expansion up to third order in powers of g have the form

$$\tilde{Y}_t(x_1, \kappa_{12}^{\perp}, \sigma) = g\tilde{Y}_{1t}(x_1, \kappa_{12}^{\perp}, \sigma) + g^3[\tilde{T}_{3t}(x_1, \kappa_{12}^{\perp}, \sigma) \\ - \tilde{T}_{3t_0}(x_1, \kappa_{12}^{\perp}, \sigma) + \tilde{T}_{3\text{finite}}(x_1, \kappa_{12}^{\perp}, \sigma)]. \tag{59}$$

By our definition, the Hamiltonian coupling constant g_t is found as a coefficient in front of the canonical color, spin and momentum dependent factor $Y_{123}(x_1, \kappa_{12}^{\perp}, \sigma)$ of Eq. (B3), in the limit $\kappa_{12}^{\perp} \to 0$, for some value of x_1, denoted by x_0. At some arbitrary value of $t = t_0$, g_{t_0} must be set to a specific finite value g_0 that produces agreement with experiment when one describes data using the Hamiltonian with $t = t_0$.

We obtain

$$\lim_{\kappa_{12}^{\perp} \to 0} \tilde{Y}_t(x_1, \kappa_{12}^{\perp}, \sigma) = \lim_{\kappa_{12}^{\perp} \to 0} [c_t(x_1, \kappa_{12}^{\perp}) Y_{123}(x_1, \kappa_{12}^{\perp}, \sigma) \\ + g^3 \tilde{T}_{3\text{finite}}(x_1, \kappa_{12}^{\perp}, \sigma)] \tag{60}$$

$$= \lim_{\kappa_{12}^{\perp} \to 0} g Y_{123}(x_1, \kappa_{12}^{\perp}, \sigma) \\ + \lim_{\kappa_{12}^{\perp} \to 0} g^3 [c_{3t}(x_1, \kappa_{12}^{\perp}) - c_{3t_0}(x_1, \kappa_{12}^{\perp})] \\ \times Y_{123}(x_1, \kappa_{12}^{\perp}, \sigma) \\ + \lim_{\kappa_{12}^{\perp} \to 0} g^3 \tilde{T}_{3\text{finite}}(x_1, \kappa_{12}^{\perp}, \sigma). \tag{61}$$

Removing $Y_{123}(x_1, \kappa_{12}^{\perp}, \sigma)$ from all terms besides the term $\tilde{T}_3$ that does not have to have the spin and momentum structure of $Y_{123}(x_1, \kappa_{12}^{\perp}, \sigma)$, in the limit, we get

$$\lim_{\kappa_{12}^{\perp} \to 0} c_t(x_1, \kappa_{12}^{\perp}) = g + g^3 \lim_{\kappa_{12}^{\perp} \to 0} [c_{3t}(x_1, \kappa_{12}^{\perp}) - c_{3t_0}(x_1, \kappa_{12}^{\perp})], \tag{62}$$

or

$$c_t(x_1, 0^{\perp}) = g + g^3[c_{3t}(x_1, 0^{\perp}) - c_{3t_0}(x_1, 0^{\perp})]. \tag{63}$$

Evaluation of the limit in Eq. (62) that yields Eq. (63) is described in Appendix E.

Setting $x_1 = x_0$ and dropping the argument $\kappa_{12}^{\perp}$ set to zero, our definition of the coupling constant g_t leads to

$$g_t \equiv c_t(x_0) \tag{64}$$

$$= g + g^3[c_{3t}(x_0) - c_{3t_0}(x_0)]. \tag{65}$$

We calculate the coupling constant g that appears in the initial Hamiltonian after inclusion of the counterterm, by demanding that at $t = t_0$ the coupling constant should have the value g_0,

$$g_{t_0} = g_0. \tag{66}$$

The value of g_0 is determined by comparison with data using the Hamiltonian corresponding to $t = t_0$. Hence, with accuracy to terms of order g_0^3 or smaller, Eq. (65) implies

$$g_t = g_0 + g_0^3[c_{3t}(x_0) - c_{3t_0}(x_0)]. \tag{67}$$

The right-hand side of this result is calculated in Appendix E. It can also be expressed as a function of momentum scale $\lambda = 1/s$, which facilitates comparison with Refs. [1,2] and [21]. For this purpose, we denote g_t by g_λ when we set $t = \lambda^{-4}$.

From all terms that contribute to the right-hand side of Eq. (67), listed as (a) to (i) in Appendix E, only the contributions of $\gamma_{(a)}$, $\gamma_{(d)}$ and $\gamma_{(g)}$ are different from zero. These terms yield

$$g_\lambda = g_0 - \frac{g_0^3}{48\pi^2} N_c[11 + h(x_0)] \ln \frac{\lambda}{\lambda_0}, \tag{68}$$

where

$$h(x_0) = \chi(x_0) + \chi(1 - x_0), \tag{69}$$

$$\chi(x_0) = 6 \int_{x_0}^{1} dx r_{\delta Y}[2/(1 - x) + 1/(x - x_0) + 1/x] \\ - 9\tilde{r}_\delta(x_0) \int_0^1 dx r_{\delta\mu}(x) \left[\frac{1}{x} + \frac{1}{1 - x}\right]. \tag{70}$$

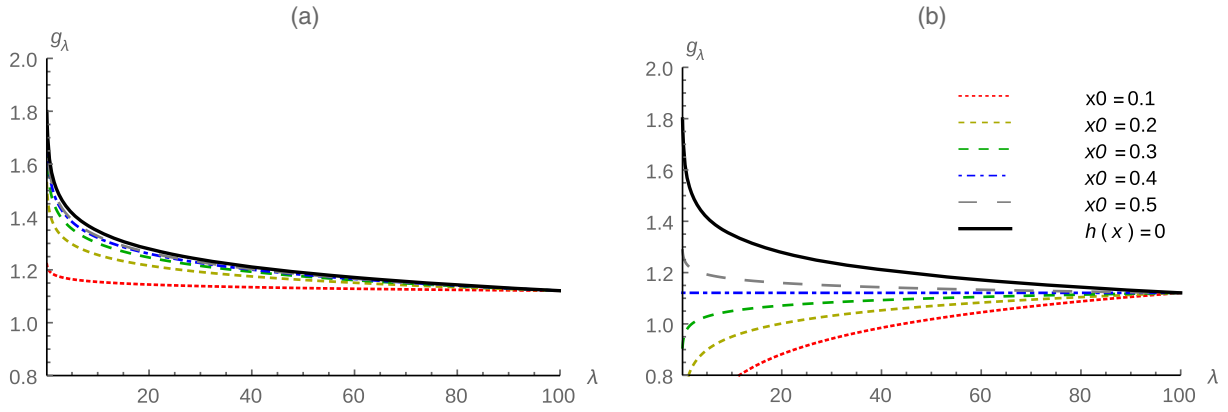This result depends in a finite way on our regularization of small-x divergences.

FIG. 1 (color online).   The FF Hamiltonian third-order RGPEP running coupling constant for effective gluons, g_λ of Eq. (68), is drawn using different dashed lines for different values of x_0, as a function of λ in GeV, starting from an arbitrarily chosen value $g_0 = 1.1$ at $\lambda_0 = 100$ GeV. Plots (a) and (b) correspond to small-x regularizations in Eqs. (23) and (24). The thick continuous lines in both plots show one and the same result for the regularization in Eq. (25), which exhibits no dependence on x_0.

For the choices of small-x regularization that are listed in Sec. III A in Eqs. (23), (24) and (25), to which we refer as versions (a), (b) and (c), the limit $\delta \to 0$ yields the Hamiltonian running coupling with the function $h(x)$ given by, correspondingly,

$$\text{(a) } h(x_0) = 12\left[3 + \frac{1 - x_0 - x_0^2}{(1 - x_0)(1 - 2x_0)}\ln x_0 \right.$$
$$\left. + \frac{(1 - x_0)^2 - x_0}{x_0(1 - 2x_0)}\ln(1 - x_0)\right], \tag{71}$$

$$\text{(b) } h(x_0) = 12\ln\min(x_0, 1 - x_0), \tag{72}$$

$$\text{(c) } h(x_0) = 0. \tag{73}$$

The running of the Hamiltonian coupling constant described by Eq. (68) for small-x regularizations (a) and (b) is illustrated by the dashed curves in plots (a) and (b) of Fig. 1, respectively. Different dashed curves correspond to different values of x_0. Only examples with x_0 between 0.1 and 0.5 are plotted, because the function $h(x_0)$ in Eq. (68) is symmetric with respect to the change $x_0 \to 1 - x_0$. For the regularization (c), we have $h(x_0) = 0$ irrespective of the value of x_0, and

$$g_\lambda = g_0 - \frac{g_0^3}{48\pi^2}N_c 11 \ln\frac{\lambda}{\lambda_0}, \tag{74}$$

which is illustrated by one and the same continuous line in both plots (a) and (b) of Fig. 1. Differentiation of Eq. (74) with respect to λ produces

$$\lambda\frac{d}{d\lambda}g_\lambda = \beta_0 g_\lambda^3, \tag{75}$$

where

$$\beta_0 = -\frac{11N_c}{48\pi^2}. \tag{76}$$

This result matches the asymptotic freedom result in Refs. [1,2], when one identifies λ with the momentum scale of external gluon lines in Feynman diagrams. Discussion of this result is provided below in Sec. V.

The increase of g_λ with λ in Fig. 1(b) for values of x_0 away from $1/2$ demands a comment on its relationship to asymptotic freedom, since asymptotic freedom is associated with a decrease rather than an increase of the coupling constant. The paradoxical appearance of $g_\lambda(x_0)$ as a function of λ reflects the fact that the effective three-gluon Hamiltonian interaction term of order g^3 depends on the small-x regularization through the vertex function $\tilde{Y}_{3t}$ defined in Eq. (56) that in the limit of $\kappa_{12}^\perp \to 0$ is a product of the canonical color, spin and momentum dependent factor $Y_{123}(x_1, \kappa_{12}^\perp, \sigma)$ of Eq. (B3) and a concave function of x_1, displayed in Eq. (63), which has a maximum at $x_1 = 1/2$ and falls off symmetrically on both sides of $1/2$. Thus, if one uses the vertex function at its maximum to define g_λ, i.e., by setting x_0 to $1/2$ in Eq. (68), one obtains g_λ that decreases when λ increases. However, if one chooses some $x_1 = x_0$ away from $1/2$, then the asymptotically free decrease of the concave term appears as an increase of g_λ. Another way of describing this effect is to say that the maximal value of the vertex function at $x_1 = 1/2$ decreases while its width increases when λ increases. Thus, Fig. 1(b) indicates that g_λ, defined using x_0 away from $1/2$, increases above g_0 when λ increases above λ_0, because it displays only terms of order g_0 and g_0^3 and the increasing width of the vertex function means that g_λ approaches from below the actually decreasing value of the vertex function at its maximum. This technical point is a fine example of the details one can trace in the Minkowski space-time interaction Hamiltonians for effective gluons using the RGPEP. In the example, one sees explicitly that in defining the

Hamiltonian running coupling constant one has to carefully account for the finite effects due to the small-x regularizations, such as in Eqs. (23) and (24), and one has to select a regularization such as in Eq. (25) in order to obtain the standard-looking result of order g_0^3. Further analysis of small-x regularization effects using the RGPEP of higher order may improve the understanding of asymptotic freedom in FF Hamiltonians of Yang-Mills theories beyond its current level.

V. UNIVERSALITY OF THE RGPEP SOLUTION

As a result of the third-order RGPEP, the three-gluon interaction term in effective FF Hamiltonians for gluons is

$$H_{tA^3} = \sum_{123} \int [123] \tilde{\delta}(p^\dagger - p) f_{12t} \tilde{Y}_t(x_1, \kappa_{12}^\perp, \sigma) a_{t1}^\dagger a_{t2}^\dagger a_{t3}$$
$$+ \text{ H.c.,} \tag{77}$$

where with accuracy to terms order g_0^3 one has

$$\tilde{Y}_t(x_1, \kappa_{12}^\perp, \sigma) = g_0 Y_{123}(x_1, \kappa_{12}^\perp, \sigma) + g_0^3 [\tilde{T}_{3t}(x_1, \kappa_{12}^\perp, \sigma)$$
$$- \tilde{T}_{3t_0}(x_1, \kappa_{12}^\perp, \sigma) + \tilde{T}_{3\text{finite}}(x_1, \kappa_{12}^\perp, \sigma)] \tag{78}$$

For infinitesimal $\kappa_{12}^\perp$,

$$\tilde{Y}_t(x_1, \kappa_{12}^\perp, \sigma) = V_t(x_1) Y_{123}(x_1, \kappa_{12}^\perp, \sigma)$$
$$+ g_0^3 \tilde{T}_{3\text{finite}}(x_1, \kappa_{12}^\perp, \sigma) + o(\kappa_{12}^\perp), \tag{79}$$

where

$$V_t(x_1) = g_t + g_t^3 [c_{3t}(x_1) - c_{3t}(x_0) - c_{3t_0}(x_1) + c_{3t_0}(x_0)], \tag{80}$$

g_t is given in Eq. (67) and the coefficients c_3 are described in Sec. IV C. Universality of this result is claimed on the basis of comparison with results obtained in Refs. [1,2] and [21].

Comparison with Refs. [1,2] shows that the FF Hamiltonian running coupling constant g_λ exhibits, in the RGPEP of third order, the same leading dependence on the momentum width of vertex form factor λ, as the running coupling constant in Refs. [1,2] exhibits as a function of the length λ of Euclidean momenta of external gluon lines in the three-point effective action. In order to compare these two results, one has to assume that the Euclidean Green's functions correspond, by some continuation procedure from imaginary to real time variable, to a Minkowski space-time quantum theory in which a renormalized Hamiltonian has a three-gluon interaction term of a specific dependence on the momentum scale parameter λ. Our calculation suggests, but does not prove, that the Euclidean scale λ corresponds to the RGPEP width λ.

Namely, the FF Hamiltonian matrix element that appears in the virtual transition amplitude between one- and two-gluon states in the Fock space of effective gluons is suggested to correspond to the continuation of the three-point Euclidean Green's function, or effective action, to the Minkowski variables. The suggestion is not verifiable by any simple continuation because we do not fully know the analytic structure of either function. However, the observed universality of asymptotic freedom in both the perturbative Euclidean Green's function calculus and Minkowskian Hamiltonian quantum mechanical operator calculus points out a direction in which one can seek a constructive demonstration that these two ways of defining a theory are equivalent.

Comparison with Ref. [21], where the Hamiltonian three-gluon vertex is calculated as a function of the momentum scale λ using the RGPEP with a different generator than the one used here, is facilitated by observing that the size of effective gluons, s, is equal to the inverse of λ. Using this relation, one sees that the present result is the same as in [21], despite the fact that the generators are different. Specifically, using $\mathcal{G}_t$ in Eq. (A5) instead of the one in Eq. (A4) does not change the third-order results for g_λ. Fig. 1 illustrates this finding by showing the present results for g_λ. The current calculation explicitly extends the universality of leading perturbative terms in the beta function to the RGPEP calculus for Hamiltonian operators in the effective particle Fock space.

Thus, the universality we claim is twofold. One universal aspect is that the third-order RGPEP Hamiltonian running coupling constants exhibit the same asymptotic freedom behavior that is known to be universal in the calculus based on the renormalized Feynman diagrams. This is of interest from the point of view that the Hamiltonian quantum mechanics in the Minkowski space-time and the Feynman diagrams for virtual transition amplitudes can be precisely related to each other in a relativistic theory including renormalization, which generally remains to be desired [24,25]. Such relation is needed for incorporating nonperturbative features of hadrons in calculations that so far remain limited to the usage of qualitative and quantitative input from the parton model.

The other universal aspect is that the third-order Hamiltonian running coupling constant depends on the size of effective gluons, or momentum width of effective Hamiltonian three-gluon vertex, in a way that does not depend on the choice of the RGPEP generator. This is of great interest in view of the fact that the presently used generator does not depend on the derivative of the Hamiltonian with respect to t, while the previously used generator does. Hence, the obtained stability of asymptotically free behavior of effective gluon interactions, with respect to change of the RGPEP generator, suggests a viable way around the difficult problem of solving for the derivative of the Hamiltonian in terms of the Hamiltonian

itself. The generator used here is thus shown to offer a way of seeking nonperturbative solutions to the RGPEP equation in a greatly simplified setup in comparison with the original one.

VI. CONCLUSION

Knowledge of the third-order RGPEP result for the Hamiltonian of effective gluons is not sufficient for setting up any physical eigenvalue problem, such as the eigenvalue problem for a glueball. At least fourth-order terms are needed, which describe interactions among two effective gluons including the effect of running of the coupling constant. Such calculations are considerably more involved than the third-order calculations described here.

However, the presently used RGPEP generator, demonstrated here to imply the Hamiltonian running coupling constant of the form that is familiar from other formalisms and renormalization schemes, turns out to lead to a considerably simpler third-order calculation than the previously used generator did. The consequence of this result is that the required fourth-order calculations with the presently used generator are expected to be considerably simpler than they could have been with the previously used generator.

In particular, the same finite effects of small-x regularization are found using the present generator and the previous one. Since the FF Hamiltonian mechanisms of confinement and chiral symmetry breaking are expected to be related to the gluon dynamics at small-x, the simpler generator than the one used before is welcome as a tool for studying the small-x dynamics in full QCD.

ACKNOWLEDGMENTS

APPENDIX A: DETAILS OF THE RGPEP

The effective Hamiltonian is related to the regulated canonical one with counterterms by the condition of no dependence on the arbitrary RGPEP scale parameter t,

$$\mathcal{H}_t(a_t) = \mathcal{H}_0(a_0). \quad (A1)$$

This condition implies via Eq. (2) that

$$\mathcal{H}_t(a_0) = \mathcal{U}_t^\dagger \mathcal{H}_0(a_0) \mathcal{U}_t. \quad (A2)$$

Differentiation of Eq. (A2) with respect to t yields

$$\mathcal{H}_t'(a_0) = [\mathcal{G}_t(a_0), \mathcal{H}_t(a_0)], \quad (A3)$$

where $\mathcal{G}_t = -\mathcal{U}_t^\dagger \mathcal{U}_t'$ is called the RGPEP generator and the related solution for $\mathcal{U}_t$ is given in Eq. (3).

We consider two different generators, one from Ref. [21],

$$\mathcal{G}_t = \{(1 - f_t^{-1})\mathcal{H}_t\}_{\mathcal{H}_f}, \quad (A4)$$

and another one from Ref. [23],

$$\mathcal{G}_t = [\mathcal{H}_f, \mathcal{H}_{Pt}]. \quad (A5)$$

The curly bracket in Eq. (A4) indicates that $\mathcal{G}_t$ satisfies the equation $[\mathcal{G}_t, \mathcal{H}_f] = (1 - f_t^{-1})\mathcal{H}_t$, which is designed according to the similarity renormalization group procedure described in Ref. [27]. The form factor f_t in Eq. (A4) is chosen in the form that also appears in lowest-order solutions obtained using the generator defined in Eq. (A5). Namely, in an interaction Hamiltonian term in which R and L refer to the effective particles that enter and emerge from the interaction, respectively, the form factor is

$$f_t = e^{-t(\mathcal{M}_L^2 - \mathcal{M}_R^2)^2}, \quad (A6)$$

where $\mathcal{M}_L$ and $\mathcal{M}_R$ denote the free invariant masses of the corresponding particles.

The operator $\mathcal{H}_f$ in Eq. (A5) is called the free Hamiltonian. It is the part of $\mathcal{H}_0(a_0)$ that does not depend on the coupling constants,

$$\mathcal{H}_f = \sum_i p_i^- a_{0i}^\dagger a_{0i}, \quad (A7)$$

where i denotes the quantum numbers of gluons and p_i^- is the free FF energy for the gluon kinematical momentum components p_i^+ and $p_i^\perp$,

$$p_i^- = \frac{p_i^{\perp 2}}{p_i^+}. \quad (A8)$$

The operator $\mathcal{H}_{Pt}$ is defined in terms of $\mathcal{H}_t$, the latter considered an arbitrary series of normal-ordered powers of the creation and annihilation operators,

$$\mathcal{H}_t(a_0) = \sum_{n=2}^\infty \sum_{i_1,i_2,\dots,i_n} c_t(i_1,\dots,i_n) a_{0i_1}^\dagger \dots a_{0i_n}. \quad (A9)$$

Namely, $\mathcal{H}_{Pt}$ differs from $\mathcal{H}_t$ only by multiplication of each and every term in it by a square of a total + momentum involved in a term,

$$\mathcal{H}_{Pt}(a_0) = \sum_{n=2}^\infty \sum_{i_1,i_2,\dots,i_n} c_t(i_1,\dots,i_n) \left(\frac{1}{2}\sum_{k=1}^n p_{i_k}^+\right)^2 a_{0i_1}^\dagger \dots a_{0i_n}.$$

$$(A10)$$

This multiplication implies that $\mathcal{H}_t$ for all values of t possesses *seven* kinematical symmetries of the FF

Hamiltonian dynamics: three translations within the front, rotation around the z-axis, two transformations generated by $K^1 + J^2$ and $K^2 - J^1$, and the boost generated by K^3. The latter is the seventh symmetry generator that does not have a counterpart in the commonly used instant form of dynamics, which has only six kinematical symmetries.

Solutions to the RGPEP equation can be found using expansion in powers of the coupling constant g. Such expansion is used in Sec. IV. The last step in the RGPEP is the replacement of bare creation and annihilation operators by effective ones. One obtains

$$\hat{H}_t = \mathcal{U}_t \mathcal{H}_t(a_0) \mathcal{U}_t^\dagger. \qquad (A11)$$

This operator can be used for approximate, i.e., neglecting quarks, computations of the states of hadrons made of gluons and the transition amplitudes for scattering, decay and production processes that involve such hadrons. The arbitrary parameter t can be adjusted in order to reduce the complexity of any such calculation to minimum. In practice, it means that the size of gluons s is chosen to match the inverse of the momentum scale that characterizes a process of interest.

APPENDIX B: DETAILS OF THE INITIAL HAMILTONIAN

All interaction terms in the operator $\hat{P}^-$ of Eq. (18), are regulated as described in Sec. III A. The regularization does not change the field operator $\hat{A}^\mu$ and the associated free part $\mathcal{H}_f$ of the Hamiltonian, besides the initial condition that $k^+ > \epsilon^+ \to 0$. The latter condition eliminates the terms that contain only annihilation or only creation operators. If they were present in $\hat{P}^-$, it would produce non-normalizable states by acting on the vacuum $|0\rangle$ and all other states built using action of creation operators on $|0\rangle$ [28].

The regularized Hamiltonian terms corresponding to densities given in Eqs. (11) to (13) are listed below in the same order and represented graphically in Figs. 2–4:

$$H_{A^2} = \sum_{\sigma c} \int [k] \frac{k^{\perp 2}}{k^+} a_{k\sigma c}^\dagger a_{k\sigma c}, \qquad (B1)$$

$$H_{A^3} = \sum_{123} \int [123] \tilde{\delta}(p^\dagger - p) \tilde{r}_{\Delta\delta}(3,1)$$
$$\times [g Y_{123} a_1^\dagger a_2^\dagger a_3 + g Y_{123}^* a_3^\dagger a_2 a_1], \qquad (B2)$$
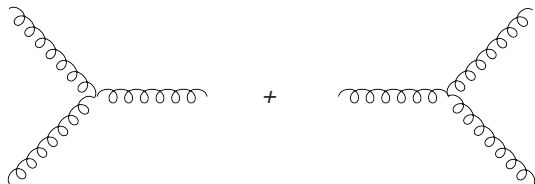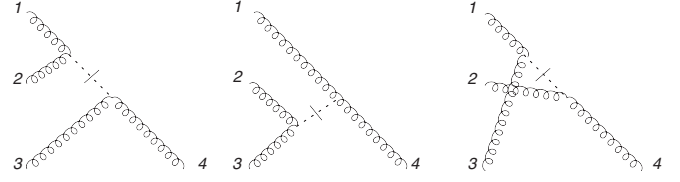

FIG. 2. Bare three-gluon vertex.

FIG. 3. Graphical representation of terms (B5) and (B8).

where

$$Y_{123} = i f^{c_1 c_2 c_3} \left[\varepsilon_1^* \varepsilon_2^* \cdot \varepsilon_3 \kappa - \varepsilon_1^* \varepsilon_3 \right.$$
$$\left. \cdot \varepsilon_2^* \kappa \frac{1}{x_{2/3}} - \varepsilon_2^* \varepsilon_3 \cdot \varepsilon_1^* \kappa \frac{1}{x_{1/3}} \right], \qquad (B3)$$

with $\varepsilon \equiv \varepsilon^\perp$ and $\kappa \equiv \kappa_{1/3}^\perp$. Symbols $p^\dagger$ and p denote the total momenta of created and annihilated particles, respectively.

$$H_{A^4} = \sum_{1234} \int [1234] \tilde{\delta}(p^\dagger - p) \frac{g^2}{4}$$
$$\times [\Xi_{A^4 1234} a_1^\dagger a_2^\dagger a_3^\dagger a_4 + X_{A^4 1234} a_1^\dagger a_2^\dagger a_3 a_4$$
$$+ \Xi_{A^4 1234}^* a_4^\dagger a_3 a_2 a_1]. \qquad (B4)$$

$$\Xi_{A^4 1234} = \frac{2}{3} \left[\tilde{r}_{1+2,1} \tilde{r}_{4,3} (\varepsilon_1^* \varepsilon_3^* \cdot \varepsilon_2^* \varepsilon_4 - \varepsilon_1^* \varepsilon_4 \cdot \varepsilon_2^* \varepsilon_3^*) f^{ac_1 c_2} f^{ac_3 c_4} \right.$$
$$+ \tilde{r}_{1+3,1} \tilde{r}_{4,2} (\varepsilon_1^* \varepsilon_2^* \cdot \varepsilon_3^* \varepsilon_4 - \varepsilon_1^* \varepsilon_4 \cdot \varepsilon_2^* \varepsilon_3^*) f^{ac_1 c_3} f^{ac_2 c_4}$$
$$\left. + \tilde{r}_{3+2,3} \tilde{r}_{4,1} (\varepsilon_1^* \varepsilon_3^* \cdot \varepsilon_2^* \varepsilon_4 - \varepsilon_3^* \varepsilon_4 \cdot \varepsilon_2^* \varepsilon_1^*) f^{ac_3 c_2} f^{ac_1 c_4} \right]. \qquad (B5)$$

$$X_{A^4 1234} = \tilde{r}_{1+2,1} \tilde{r}_{3+4,3} (\varepsilon_1^* \varepsilon_3 \cdot \varepsilon_2^* \varepsilon_4 - \varepsilon_1^* \varepsilon_4 \cdot \varepsilon_2^* \varepsilon_3) f^{ac_1 c_2} f^{ac_3 c_4}$$
$$+ [\tilde{r}_{3,1} \tilde{r}_{2,4} + \tilde{r}_{1,3} \tilde{r}_{4,2}] (\varepsilon_1^* \varepsilon_2^* \cdot \varepsilon_3 \varepsilon_4 - \varepsilon_1^* \varepsilon_4 \cdot \varepsilon_2^* \varepsilon_3)$$
$$\times f^{ac_1 c_3} f^{ac_2 c_4} + [\tilde{r}_{3,2} \tilde{r}_{1,4} + \tilde{r}_{2,3} \tilde{r}_{4,1}]$$
$$\times (\varepsilon_1^* \varepsilon_2^* \cdot \varepsilon_3 \varepsilon_4 - \varepsilon_1^* \varepsilon_3 \cdot \varepsilon_2^* \varepsilon_4) f^{ac_1 c_4} f^{ac_2 c_3}. \qquad (B6)$$
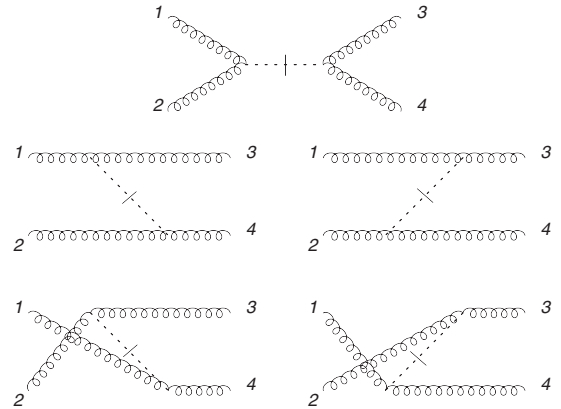

FIG. 4. Graphical representation of terms (B6) and (B9).

$$H_{[\partial AA]^2} = \sum_{1234} \int [1234] \tilde{\delta}(p^\dagger - p) g^2 [(\Xi_{[\partial AA]^2 1234} a_1^\dagger a_2^\dagger a_3^\dagger a_4 + \text{H.c.}) + X_{[\partial AA]^2 1234} a_1^\dagger a_2^\dagger a_3 a_4]. \tag{B7}$$

$$\Xi_{[\partial AA]^2 1234} = -\frac{1}{6} \left[\tilde{r}_{1+2,1} \tilde{r}_{4,3} \varepsilon_1^* \varepsilon_2^* \cdot \varepsilon_3^* \varepsilon_4 \frac{(x_1 - x_2)(x_3 + x_4)}{(x_1 + x_2)^2} f^{ac_1 c_2} f^{ac_3 c_4} + \tilde{r}_{1+3,1} \tilde{r}_{4,2} \varepsilon_1^* \varepsilon_3^* \right.$$
$$\left. \cdot \varepsilon_2^* \varepsilon_4 \frac{(x_1 - x_3)(x_2 + x_4)}{(x_1 + x_3)^2} f^{ac_1 c_3} f^{ac_2 c_4} + \tilde{r}_{3+2,3} \tilde{r}_{4,1} \varepsilon_3^* \varepsilon_2^* \cdot \varepsilon_1^* \varepsilon_4 \frac{(x_3 - x_2)(x_1 + x_4)}{(x_3 + x_2)^2} f^{ac_3 c_2} f^{ac_1 c_4} \right]. \tag{B8}$$

$$X_{[\partial AA]^2 1234} = \frac{1}{4} \left[\tilde{r}_{1+2,1} \tilde{r}_{3+4,3} \varepsilon_1^* \varepsilon_2^* \cdot \varepsilon_3 \varepsilon_4 \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 + x_2)^2} f^{ac_1 c_2} f^{ac_3 c_4} - [\tilde{r}_{3,1} \tilde{r}_{2,4} + \tilde{r}_{1,3} \tilde{r}_{4,2}] \varepsilon_1^* \varepsilon_3 \right.$$
$$\left. \cdot \varepsilon_2^* \varepsilon_4 \frac{(x_1 + x_3)(x_2 + x_4)}{(x_2 - x_4)^2} f^{ac_1 c_3} f^{ac_2 c_4} - [\tilde{r}_{3,2} \tilde{r}_{1,4} + \tilde{r}_{2,3} \tilde{r}_{4,1}] \varepsilon_1^* \varepsilon_4 \cdot \varepsilon_2^* \varepsilon_3 \frac{(x_2 + x_3)(x_1 + x_4)}{(x_1 - x_4)^2} f^{ac_1 c_4} f^{ac_2 c_3} \right]. \tag{B9}$$

In all these formulas, the dot $\cdot$ is used merely to visually separate factors comprised of scalar products of transverse polarization vectors.

APPENDIX C: INTEGRATION OF THE RG EQUATIONS ORDER BY ORDER

The expansion in a series of powers of the coupling constant g is inserted in Eq. (1) and solved for the first four terms, i.e., including terms of order 1, g, g^2 and g^3. We use notation adopted in Eqs. (37) to (39).

1. First-order terms

Integration of the terms of order g yields

$$Y_{21t} + Y_{12t} = f_t [Y_{210} + Y_{120}], \tag{C1}$$

where the form factor f_t is given in Eq. (A6). The corresponding Hamiltonian term is

$$H_{(1)} = \sum_{123} \int [123] \tilde{\delta}(p^\dagger - p) f_t \tilde{r}_\delta(x_1)$$
$$\times [g Y_{123} a_{t1}^\dagger a_{t2}^\dagger a_{t3} + g Y_{123}^* a_{t3}^\dagger a_{t2} a_{t1}]. \tag{C2}$$

Note the absence of ultraviolet regularization factors and the presence of small-x regularization factor $\tilde{r}_\delta(x_1)$. The reason is that the form factor f_t removes sensitivity to transverse momenta much larger than $1/s$ but, for massless gluons, does not regulate small-x divergences.
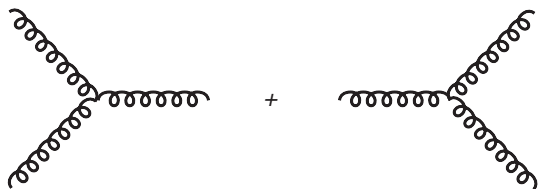

FIG. 5.   First-order term of the three-gluon vertex.

The effective creation and annihilation operators correspond to the scale parameter t (cf. Fig. 5).

2. Second-order terms

Solutions for second-order terms are

$$\hat{\mu}_t^2 = \hat{\mu}_0^2 + \int_0^t d\tau [[E, f_\tau (Y_{21P0} + Y_{12P0})], f_\tau (Y_{210} + Y_{120})]_\mu, \tag{C3}$$

$$X_{22t} = f_t \tilde{X}_0 + f_t \int_0^t d\tau f_\tau^{-1} [[E, f_\tau (Y_{21P0} + Y_{12P0})], f_\tau (Y_{210} + Y_{120})]_{X_{22}}, \tag{C4}$$

$$\Xi_{31t} = f_t \tilde{\Xi}_{310} + f_t \int_0^t d\tau f_\tau^{-1} [[E, f_\tau Y_{21P0}], f_\tau Y_{210}]_{\Xi_{31}}, \tag{C5}$$

$$\Xi_{13t} = f_t \tilde{\Xi}_{130} + f_t \int_0^t d\tau f_\tau^{-1} [[E, f_\tau Y_{21P0}], f_\tau Y_{210}]_{\Xi_{13}}. \tag{C6}$$

The gluon-mass term consists of the product of two bare vertices (see Fig. 6):

$$\hat{\mu}_t^2 = \hat{\mu}_0^2 - p^+ \frac{1 - f_t^2}{\mathcal{M}_2^2} [Y_{120} Y_{210}]_\mu. \tag{C7}$$
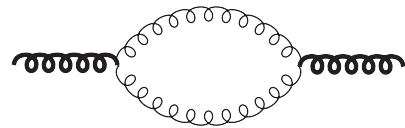

FIG. 6.   Graphical representation of the second-order RGPEP contribution to the Hamiltonian effective gluon mass term. Thin internal lines correspond to intermediate bare gluons and thick external to the effective gluons.

The subscript μ indicates that one extracts the mass-squared term from the product of operators in the bracket. This leads to Eq. (42).

3. Third-order terms

The third-order term needed for evaluation of the Hamiltonian running coupling is

$$
Y_{h21t} = f_t \tilde{Y}_{h210} + f_t \int_0^t d\tau f_\tau^{-1}[[E, X_{22P\tau}]f_\tau(Y_{210})]_{Y_{h21}} - f_t \int_0^t d\tau f_\tau^{-1}[f_\tau(Y_{120})[E, \Xi_{31P\tau}]]_{Y_{h21}}
$$

$$
+ f_t \int_0^t d\tau f_\tau^{-1}[[E, f_\tau(Y_{21P0})], \hat{\mu}_\tau^2]_{Y_{h21}} - f_t \int_0^t d\tau f_\tau^{-1}[X_{22\tau}[E, f_\tau(Y_{21P0})]]_{Y_{h21}}
$$

$$
+ f_t \int_0^t d\tau f_\tau^{-1}[[E, f_\tau(Y_{12P0})]\Xi_{31\tau}]_{Y_{h21}}.
\tag{C8}
$$

In an abbreviated notation,

$$
Y_{h21t} = f_t \sum_n \gamma_{t21(n)}
\tag{C9}
$$

where n ranges from a to i and $n = j$ for the vertex counterterm. One has

$$
\gamma_{t21(n)} = \sum_{123} \int [123]\tilde{\delta}(k_1 + k_2 - k_3) \frac{g^3}{16\pi^3} \frac{1}{2} \gamma_{(n)} a_1^\dagger a_2^\dagger a_3.
\tag{C10}
$$

We list below results for the vertex functions $\gamma_{(n)}$ for all values of n from a to i. The counterterm $\gamma_{(j)}$ is described afterwards in the next section.

a. Vertex function $\gamma_{(a)}$

$$
\gamma_{(a)} = 8\frac{N_c}{2} i f^{c_1 c_2 c_3} \int_{x_1}^1 \frac{dx\, r_{\delta t}(x)}{x(1-x)(x-x_1)}
$$

$$
\times \int d^2\kappa^\perp r_{\Delta t}(\kappa^\perp) \frac{\mathcal{B}_{t(a)}}{k_3^{+2}} \kappa_{68}^i \kappa_{16}^j \kappa^k \varepsilon_{(a)}^{ijk} + (1 \leftrightarrow 2),
\tag{C11}
$$

where

$$
r_{\delta t}(x) = r_\delta(x) r_\delta(1-x) r_\delta(x_1/x) r_\delta[(x-x_1)/x]
$$

$$
\times r_\delta[(x-x_1)/x_2] r_\delta[(1-x)/x_2],
\tag{C12}
$$

$$
r_{\Delta t}(\kappa^\perp) = \exp\left[-2(\kappa_{68}^{\perp 2} + \kappa_{16}^{\perp 2} + \kappa^{\perp 2})/\Delta^2\right],
\tag{C13}
$$

$$
\varepsilon_{(a)}^{ijk} = \varepsilon_1^{*j} \varepsilon_2^{*i} \varepsilon_3^k \left[1 - \frac{x}{x-x_1} + \frac{1}{x_1} - \frac{2x}{x_1} + \frac{x}{(1-x)x_1} + \frac{xx_2}{(1-x)x_1} + \frac{xx_2}{(x-x_1)x_1}\right] + \varepsilon_1^{*k} \varepsilon_2^{*i} \varepsilon_3^j \left[\frac{1}{x-x_1} - \frac{1}{1-x}\right]
$$

$$
+ \varepsilon_1^{*k} \varepsilon_2^{*j} \varepsilon_3^i \left[\frac{-x_2}{(1-x)(x-x_1)}\right] + \varepsilon_1^{*i} \varepsilon_2^{*k} \varepsilon_3^j \left[\frac{x_2}{(1-x)(x-x_1)}\right] + \varepsilon_1^{*i} \varepsilon_2^{*j} \varepsilon_3^k \left[\frac{-x_2}{x-x_1} + \frac{xx_2}{(1-x)(x-x_1)}\right]
$$

$$
+ \varepsilon_1^{*j} \varepsilon_2^{*k} \varepsilon_3^i \left[\frac{-x_2}{(1-x)x_1} - \frac{xx_2}{(1-x)(x-x_1)x_1}\right] + \varepsilon_1^* \varepsilon_2^* \left[\delta^{ik} \varepsilon_3^j \frac{x_2}{(1-x)^2} + \delta^{jk} \varepsilon_3^i \frac{x_2}{(1-x)x} + \delta^{ij} \varepsilon_3^k \left(\frac{xx_2}{(x-x_1)^2} - \frac{x_2}{1-x}\right)\right]
$$

$$
+ \varepsilon_1^* \varepsilon_3 \left[\delta^{jk} \varepsilon_2^{*i} \left(\frac{x}{(1-x)(x-x_1)} - \frac{1}{x}\right) - \delta^{ij} \varepsilon_2^{*k} \frac{xx_2}{(1-x)(x-x_1)^2} - \delta^{ik} \varepsilon_2^{*j} \frac{xx_2}{(1-x)^2(x-x_1)}\right]
$$

$$
+ \varepsilon_2^* \varepsilon_3 \left[\delta^{ij} \varepsilon_1^{*k} \frac{-x_2}{(x-x_1)^2} + \delta^{jk} \varepsilon_1^{*i} \frac{x_2}{x(x-x_1)} - \delta^{ik} \varepsilon_1^{*j} \left(\frac{xx_2}{(1-x)^2 x_1} + \frac{x_2}{(x-x_1)x_1}\right)\right],
\tag{C14}
$$

and

$$
\frac{\mathcal{B}_{t(a)}}{k_3^{+2}} = -\frac{x\mathcal{M}_{16}^2 - \mathcal{M}^2}{\mathcal{M}_{16}^4 + \mathcal{M}^4 - \mathcal{M}_{bd}^4}(x_2 \mathcal{M}_{68}^2 + \mathcal{M}_{bd}^2)\left(\frac{f_{16} f f_{68}/f_{12} - 1}{\mathcal{M}_{16}^4 + \mathcal{M}^4 + \mathcal{M}_{68}^4 - \mathcal{M}_{12}^4} - \frac{f_{68} f_{bd}/f_{12} - 1}{\mathcal{M}_{68}^4 + \mathcal{M}_{bd}^4 - \mathcal{M}_{12}^4}\right)
$$

$$
+ \frac{x_2 \mathcal{M}_{68}^2 + x\mathcal{M}_{16}^2}{\mathcal{M}_{68}^4 + \mathcal{M}_{16}^4 - (\mathcal{M}^2 - \mathcal{M}_{12}^2)^2}(2\mathcal{M}^2 - \mathcal{M}_{12}^2)\left(\frac{f f_{68} f_{16}/f_{12} - 1}{\mathcal{M}^4 - \mathcal{M}_{12}^4 + \mathcal{M}_{68}^4 + \mathcal{M}_{16}^4} - \frac{f_{ca} f/f_{12} - 1}{2\mathcal{M}^2(\mathcal{M}^2 - \mathcal{M}_{12}^2)}\right),
\tag{C15}
$$

with $f_{uv} = \exp[-uv^2/\lambda^4]$, $bd = \mathcal{M}_{bd}^2 = \mathcal{M}_{68}^2/x_2 + \mathcal{M}_{12}^2$, $ca = \mathcal{M}^2 - \mathcal{M}_{12}^2$.

b. Vertex function $\gamma_{(b)}$

$$\gamma_{(b)} = 2\frac{N_c}{2}if^{c_1c_2c_3}\int_{x_1}^1\frac{dxr_{\delta t}(x)}{x(1-x)}\int d^2\kappa^\perp r_{\Delta t}(\kappa^\perp)\frac{\mathcal{B}_{t(b)}}{k_3^+}\varepsilon_{(b)}$$
$$+ (1 \leftrightarrow 2), \tag{C16}$$

where

$$\varepsilon_{(b)} \equiv \varepsilon_{(b)}^\perp \kappa^\perp = \varepsilon_1^*\varepsilon_2^* \cdot \varepsilon_3\kappa\left(1 - s_{(b)} - \frac{1}{x} - \frac{1}{1-x}\right)$$
$$+ \varepsilon_1^*\varepsilon_3 \cdot \varepsilon_2^*\kappa\left(\frac{1}{x} + \frac{s_{(b)}}{1-x}\right)$$
$$+ \varepsilon_2^*\varepsilon_3 \cdot \varepsilon_1^*\kappa\left(\frac{1}{1-x} + \frac{s_{(b)}}{x}\right), \tag{C17}$$

with $s_{(b)} = (x_1 + x)(x_2 + 1 - x)/(x - x_1)^2$ and

$$\frac{\mathcal{B}_{t(b)}}{k_3^+} = \frac{2\mathcal{M}^2 - \mathcal{M}_{12}^2}{2\mathcal{M}^2(\mathcal{M}^2 - \mathcal{M}_{12}^2)}(f_{ca}f/f_{12} - 1). \tag{C18}$$

c. Vertex function $\gamma_{(c)}$

$$\gamma_{(c)} = 2\frac{-N_c}{2}if^{c_1c_2c_3}\int_{x_1}^1\frac{dxr_{\delta t}(x)}{(x - x_1)(1-x)}$$
$$\times \int d^2\kappa^\perp r_{\Delta t}(\kappa^\perp)\frac{\mathcal{B}_{t(c)}}{k_3^+}\varepsilon_{(c)} + (1 \leftrightarrow 2), \tag{C19}$$

where

$$\varepsilon_{(c)} \equiv \varepsilon_{(c)}^\perp \kappa_{68}^\perp$$
$$= \varepsilon_1^*\varepsilon_2^* \cdot \varepsilon_3\kappa_{68}\left(\frac{-x_2}{x - x_1} + \frac{x_2 s_{(c)}}{1-x}\right)$$
$$+ \varepsilon_1^*\varepsilon_3 \cdot \varepsilon_2^*\kappa_{68}\left(-1 - s_{(c)} + \frac{x_2}{1-x} + \frac{x_2}{x - x_1}\right) + \varepsilon_2^*\varepsilon_3$$
$$\cdot \varepsilon_1^*\kappa_{68}\left(\frac{-x_2}{1-x} + \frac{x_2 s_{(c)}}{x - x_1}\right), \tag{C20}$$

with $s_{(c)} = (x_1 - x + x_1)(1 - x + 1)/x^2$ and

$$\frac{\mathcal{B}_{(c)}}{k_3^+} = \frac{x_2\mathcal{M}_{86}^2 + \mathcal{M}_{bd}^2}{\mathcal{M}_{68}^4 + \mathcal{M}_{bd}^4 - \mathcal{M}_{12}^4}(f_{68}f_{bd}/f_{12} - 1). \tag{C21}$$

d. Vertex functions $\gamma_{(d)}$ and $\gamma_{(f)}$

$$\gamma_{(d)} + \gamma_{(f)} = 4N_cY_{123}\int_0^1\frac{dxr_{\delta\mu}(x)}{x(1-x)}\int d^2\kappa^\perp r_{\Delta\mu}(\kappa^\perp)\kappa^{\perp 2}$$
$$\times\left[1 + \frac{1}{x^2} + \frac{1}{(1-x)^2}\right]\left[\frac{\mathcal{B}_{t(d)}}{x_2^2 k_3^{+2}} + \frac{\mathcal{B}_{t(f)}}{x_2 k_3^+\mathcal{M}^2}\right]$$
$$+ 4Y_{123}\frac{\mathcal{M}_{12}^2}{x_2}\tilde{\mu}_\delta^2 + (1 \leftrightarrow 2), \tag{C22}$$

where

$$\frac{B_{t(d)}}{k_3^{+2}} = -\frac{x_2\mathcal{M}^2 - \mathcal{M}_{12}^2}{\mathcal{M}^4 + \mathcal{M}_{12}^4 - \mathcal{M}_{bd}^4}\left[\mathcal{M}^2\left(x_2 + \frac{1}{x_2}\right) + \mathcal{M}_{12}^2\right]$$
$$\times\left(\frac{f^2 - 1}{2\mathcal{M}^4} - \frac{ff_{bd}/f_{12} - 1}{\mathcal{M}^4 + \mathcal{M}_{bd}^4 - \mathcal{M}_{12}^4}\right), \tag{C23}$$
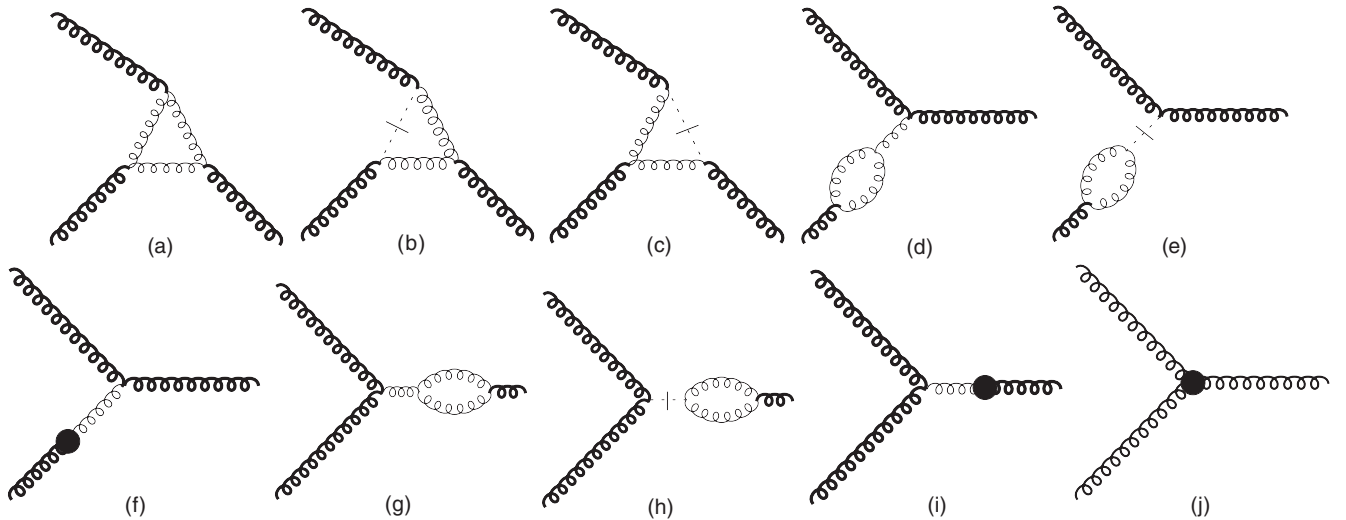
and

FIG. 7. Third-order contributions to the three-gluon vertex, including counterterm, (j). The labels (a) to (j) correspond to contributions $\gamma_{(a)}$ to $\gamma_{(j)}$ in Eq. (48).

$$\frac{\mathcal{B}_{t(f)}}{k_3^+} = \frac{\mathcal{M}_{12}^2}{\mathcal{M}^4}(f^2 - 1), \tag{C24}$$

since $\mathcal{M}_{68}^2 \equiv \mathcal{M}^2$.

e. Vertex functions $\gamma_{(g)}$ and $\gamma_{(i)}$

$$\gamma_{(g)} + \gamma_{(i)} = 2N_c Y_{123} \int_0^1 \frac{dx r_{\delta\mu}(x)}{x(1-x)} \int d^2\kappa^\perp r_{\Delta\mu}(\kappa^\perp)\kappa^{\perp 2}$$

$$\times \left[1 + \frac{1}{x^2} + \frac{1}{(1-x)^2} \right] \left[\frac{\mathcal{B}_{t(g)}}{k_3^{+2}} + \frac{\mathcal{B}_{t(i)}}{k_3^+ \mathcal{M}^2} \right]$$

$$+ 2Y_{123}\mathcal{M}_{12}^2 \tilde{\mu}_\delta^2 + (1 \leftrightarrow 2), \tag{C25}$$

where $r_{\delta\mu}(x)$ is given in Eqs. (C12), the ultraviolet regulator is (C13), $r_{\Delta\mu}(\kappa^\perp) = r_\Delta^4(\kappa^{\perp 2})$, the mass counterterm contribution stems from $2g^2\tilde{\mu}_\delta^2 = 16\pi^3\mu_\delta^2 t$, and

$$\frac{B_{t(g)}}{k_3^{+2}} = -\frac{\mathcal{M}_{12}^2 + \mathcal{M}^2}{2\mathcal{M}^2\mathcal{M}_{12}^2}(2\mathcal{M}^2 - \mathcal{M}_{12}^2)$$

$$\times \left[\frac{f^2 - 1}{2\mathcal{M}^4} - \frac{f_{ca}f/f_{12} - 1}{2\mathcal{M}^2(\mathcal{M}^2 - \mathcal{M}_{12}^2)} \right], \tag{C26}$$

and

$$\frac{\mathcal{B}_{t(i)}}{k_3^+} = \frac{-\mathcal{M}_{12}^2}{\mathcal{M}^4}(f^2 - 1). \tag{C27}$$

Graphs (e) and (h) result from products of terms Ξ_{31} or X_{22} with Y_{31}. The former are independent of the transverse momentum, and the latter is odd in transverse momentum. This leads to zero in the integration over κ.

APPENDIX D: THREE-GLUON VERTEX COUNTERTERM

The ultraviolet regularization dependence in $\gamma_{(a)}$ turns out to come only from (see below)

$$\left[\frac{\mathcal{B}_{t(a)}}{k_3^{+2}} \right]_\Delta = \frac{x_2}{\mathcal{M}^2\mathcal{M}_{68}^2}, \tag{D1}$$

which matches the result found in Ref. [21]. So,

$$\gamma_{(a)\text{div}} = 8\frac{N_c}{2} if^{c_1c_2c_3} \int_{x_1}^1 dx r_{\delta t}(x) \frac{1-x}{x_2} [I^{ijk}]_\Delta \varepsilon_{(a)}^{ijk}$$

$$+ (1 \leftrightarrow 2), \tag{D2}$$

where

$$if^{c_1c_2c_3} \frac{1-x}{x_2} [I^{ijk}]_\Delta \varepsilon_{(a)}^{ijk}$$

$$= \pi \ln \frac{\Delta}{|\kappa_{12}^\perp|} [c_{12}Y_{12} + c_{13}Y_{13} + c_{23}Y_{23}], \tag{D3}$$

and

$$c_{12} = \frac{2}{1-x} + \frac{1}{x-x_1} + \frac{1}{x} + \frac{(1-x)^2}{x_2^2} - \frac{2}{x_2}, \tag{D4}$$

$$c_{13} = \frac{2}{1-x} + \frac{1}{x-x_1} + \frac{1}{x} + \frac{(1-x)^2}{x_2^2} - 2, \tag{D5}$$

$$c_{23} = \frac{2}{1-x} + \frac{1}{x-x_1} + \frac{1}{x} - \frac{(1-x)^2}{x_2^2} - \frac{1+x^2}{x_2} - 2, \tag{D6}$$

with

$$Y_{12} = if^{c_1c_2c_3}\varepsilon_1^*\varepsilon_2^* \cdot \varepsilon_3\kappa_{12}, \tag{D7}$$

$$Y_{13} = -if^{c_1c_2c_3}\varepsilon_1^*\varepsilon_3 \cdot \varepsilon_2^*\kappa_{12}\frac{1}{x_{2/3}}, \tag{D8}$$

$$Y_{23} = -if^{c_1c_2c_3}\varepsilon_2^*\varepsilon_3 \cdot \varepsilon_1^*\kappa_{12}\frac{1}{x_{1/3}}. \tag{D9}$$

The diverging parts are $\gamma_{(b)\text{div}} = \gamma_{(c)\text{div}} = 0$ and $\gamma_{(d)\text{div}} = 2\gamma_{(g)\text{div}}$, as in Ref. [21], with

$$\gamma_{(g)\text{div}} = -N_c Y_{123} \tilde{r}_\delta(x_1) \int_0^1 dx r_{\delta\mu}(x)$$

$$\times \int_{\mu^2}^\infty \frac{\pi d\kappa^2}{\kappa^2} e^{-4\kappa^2/\Delta^2} x(1-x)$$

$$\times \left[1 + \frac{1}{x^2} + \frac{1}{(1-x)^2} \right] + (1 \leftrightarrow 2). \tag{D10}$$

These results are the same as in Ref. [21] due to the fact that the difference between old and new generators resides solely in the RGPEP factors B_t. The new generator leads to B_t's that differ from old B_t's by the additional terms in numerators, denominators and arguments of exponentials that depend on $\mathcal{M}_{12}^2$ and do not depend on $\kappa^\perp$. These additional terms do not affect the behavior of B_t s when $\kappa^\perp \to \infty$. Terms (i) and (f) are not divergent.

In summary, in the UV-limit of $\kappa^\perp \to \infty$, the integrands in all vertex functions behave in the same way as the corresponding ones obtained using the old generator [21]. Hence, the divergent part of the vertex counterterm, denoted by $\gamma_{\infty\text{div}}$, is also the same; cf. Appendix C in Ref. [21].

The divergent part of the vertex counterterm is defined by the condition

$$\gamma_{(a)\text{div}} + 3\gamma_{(g)\text{div}} + \gamma_{\infty\text{div}} + (1 \to 2) = 0. \tag{D11}$$

One thus finds the Hamiltonian vertex counterterm whose vertex function is

$$\gamma_\infty = Y_{123}\frac{-N_c\pi}{3}\ln\frac{\Delta}{\mu}[11 + h(x_1)] + \gamma_{\text{finite}}, \tag{D12}$$

where μ denotes the arbitrary separation point between the range of integration over large $\kappa^\perp$ that extends up to Δ and the finite range of integration where no dependence on Δ may arise. The finite part of the counterterm, γ_{finite}, removes the artificial dependence on μ. The function $h(x_1)$ is

$$h(x_1) = 6\int_{x_1}^1 dx r_{\delta t}(x)\left[\frac{2}{1-x} + \frac{1}{x-x_1} + \frac{1}{x}\right]$$
$$- 9\int_0^1 dx r_{\delta\mu}(x)\left[\frac{1}{x} + \frac{1}{1-x}\right] + (1 \leftrightarrow 2). \quad (D13)$$

The vertex function in the complete vertex counterterm is

$$\gamma_{(j)} = \gamma_\infty + (1 \to 2). \quad (D14)$$

APPENDIX E: THIRD-ORDER CONTRIBUTIONS TO g_t

The Hamiltonian coupling constant g_t is extracted from the term Y_t in Eq. (40) (see also Fig. 8) that is linear in $\kappa_{12}^\perp$ in the limit $\kappa_{12}^\perp \to 0$. Not every term shown in Fig. 7 contributes to the running coupling defined this way. Appendix C 3 shows that terms (e) and (h) do not contribute. Furthermore, terms (b), (c), (f) and (i) vanish faster than linearly in the limit $\kappa_{12}^\perp \to 0$. Thus, only the terms (a), (g) and (d) contribute to g_t. Contribution to g_t of each and every term is extracted in three steps: (1) calculation of the coefficient of $\kappa_{12}^\perp$ in the integrand in the limit $\kappa_{12}^\perp \to 0$; (2) integration over $\kappa^\perp$; (3) integration over x. Contributions of the counterterm are defined by the subtraction at $t = t_0$ that is described in Secs. IV B and IV C.

1. Contribution to the running coupling from $\gamma_{(a)}$

The expansion in small κ_{12}^i in term (a) concerns the factor

$$\frac{\mathcal{B}_{t(a)} - \mathcal{B}_{t_0(a)}}{k_3^{+2}} \kappa_{68}^i \kappa_{16}^j \kappa^k, \quad (E1)$$

which leads to

$$\frac{\mathcal{B}_{t(a)} - \mathcal{B}_{t_0(a)}}{k_3^{+2}}\bigg|_{\kappa_{12}^i=0}\left(\kappa^i\kappa^k\kappa_{12}^j + \frac{(1-x)}{x_2}\frac{x_1}{x}\kappa^j\kappa^k\kappa_{12}^i\right)$$
$$- \frac{x_1}{x}\frac{\mathcal{B}'_{t(a)} - \mathcal{B}'_{t_0(a)}}{k_3^{+2}}\bigg|_{\kappa_{12}^\perp=0}\kappa_{12}^\perp\kappa^i\kappa^j\kappa^k. \quad (E2)$$

This is integrated over $\kappa^\perp$, yielding

$$\int d^2\kappa^\perp \lim_{\kappa_{12}^i \to 0}\left[\frac{\mathcal{B}_{t(a)} - \mathcal{B}_{t_0(a)}}{k_3^{+2}}\kappa_{68}^i\kappa_{16}^j\kappa^k\right]$$
$$= \mathcal{A}_1(x,x_1)\pi\left(\delta^{ik}\kappa_{12}^j + \frac{(1-x)}{x_2}\frac{x_1}{x}\delta^{jk}\kappa_{12}^i\right)\frac{1}{4}\ln\frac{t}{t_0}$$
$$- \frac{x_1}{x}\mathcal{A}_2(x,x_1)\frac{1}{x-x_1}\frac{\pi}{4}$$
$$\times (\delta^{jk}\kappa_{12}^i + \delta^{ik}\kappa_{12}^k + \delta^{ij}\kappa_{12}^k)\frac{1}{4}\ln\frac{t}{t_0}, \quad (E3)$$

where $\mathcal{A}_1(x,x_1) = x^2(1-x)^2\frac{x-x_1}{xx_2}$ and $\mathcal{A}_2(x,x_1) = 2x^3(1-x)^3\frac{(x-x_1)^2}{x_2^2x^2}$. Here we have made use of the formula in Appendix E 5. The running coupling contribution is extracted from

$$\lim_{\kappa_{12}^i \to 0}\frac{g^3}{16\pi^3}\frac{1}{2}(\gamma_{t,a} - \gamma_{t_0,a})$$
$$= \frac{g^3}{16\pi^3}\frac{1}{2}8\frac{N_c}{2}if^{c_1c_2c_3}\lim_{\kappa_{12}^i \to 0}c_{t,a}(x_0,\kappa_{12}^\perp), \quad (E4)$$

where

$$\lim_{\kappa_{12}^i \to 0}c_{t,a}(x_1,\kappa_{12}^\perp) = \frac{\pi}{4}\ln\frac{t}{t_0}\int_{x_1}^1 dx r_{\delta t}(x)\frac{(1-x)}{x_2}\left[\left(1 - \frac{x_1(1-x)}{x_2x}\frac{1}{2}\right)\varepsilon_{(a)}^{iji}\kappa_{12}^j + \frac{1}{2}\frac{x_1(1-x)}{xx_2}\kappa_{12}^i\varepsilon_{(a)}^{ijj} - \frac{x_1(1-x)}{x_2x}\frac{1}{2}\varepsilon_{(a)}^{iik}\kappa_{12}^k\right]$$
$$+ (1 \leftrightarrow 2) + o(\kappa_{12}^\perp). \quad (E5)$$

The contraction of indices of the tensor structure $\varepsilon_{(a)}^{ijk}$ simplifies the limit of $c_{t,a}(x_1,\kappa_{12}^\perp)$ to

$$c_{t,a}(x_1,\kappa_{12}^\perp) \to \frac{\pi}{4}\ln\frac{t}{t_0}\left[\int_{x_1}^1 dx r_{\delta Y}f_{12}^\perp(x,x_1,\varepsilon^\perp) + (1 \leftrightarrow 2)\right]\kappa_{12}^\perp, \quad (E6)$$
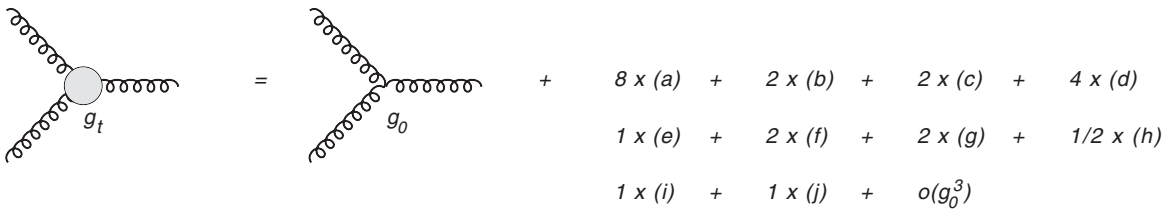

FIG. 8. Effective three-gluon vertex (expansion up to third order).

where

$$f_{12}^\perp(x, x_1, \varepsilon^\perp) = c_{12}\varepsilon_1^*\varepsilon_2^*\varepsilon_3^\perp - c_{13}\varepsilon_1^*\varepsilon_3\varepsilon_2^{*\perp}/x_2$$
$$- c_{23}\varepsilon_2^*\varepsilon_3\varepsilon_1^{*\perp}/x_1, \quad (E7)$$

with the coefficients c_{ij} given in Eqs. (D4)–(D6). Finally, the integration over x leads to the running coupling contribution of term (a),

$$g_{t,(a)} = g_{t_0,(a)} + \frac{g^3}{48\pi^2}\frac{1}{2}N_c[-11 + 3\chi_a(x_0)]\ln\frac{t}{t_0}, \quad (E8)$$

with

$$\chi_a(x_0) = \int_{x_0}^1 dx r_{\delta Y}[2/(1-x) + 1/(x - x_0) + 1/x]$$
$$+ (x_0 \to 1 - x_0). \quad (E9)$$

2. Contribution to the running coupling from $\gamma_{(d)}$

In this case $\mathcal{M}_{68}^2 = \mathcal{M}^2$, $\mathcal{M}_{bd}^2 = \mathcal{M}^2/x_2 + \mathcal{M}_{12}^2$. The limit $\kappa_{12}^\perp \to 0$ in Eq. (C23) concerns the factor,

$$\lim_{\kappa_{12}^\perp \to 0} \frac{\mathcal{B}_{t(d)} - \mathcal{B}_{t_0(d)}}{x_2^2 k_3^{+2}} = \frac{1}{1 - 1/x_2^2}\left(1 + \frac{1}{x_2^2}\right)x^2(1-x)^2$$
$$\times \left(\frac{f_0^2 - f^2}{2\kappa^4} - \frac{f_0 f_{bd,0} - f f_{bd}}{\kappa^4(1 + 1/x_2^2)}\right). \quad (E10)$$

Integration over $\kappa^\perp$ yields

$$\int d^2\kappa^\perp \kappa^{\perp 2}\lim_{\kappa_{12}^\perp \to 0}\left[\frac{\mathcal{B}_{t(d)} - \mathcal{B}_{t_0(d)}}{x_2^2 k_3^{+2}}\right] = -x^2(1-x)^2\frac{\pi}{4}\ln\frac{t}{t_0}. \quad (E11)$$

The last step is the integration over x of $\gamma_{(d)} - \gamma_{0,(d)}$ in this limit. The corresponding contribution to the running coupling is

$$g_{t,(d)} = g_{t_0,(d)} - \frac{g^3}{16\pi^2}N_c\ln\frac{t}{t_0}$$
$$\times \left\{-\frac{11}{6} + \int_0^1 dx r_{\delta\mu}(x)\left[\frac{1}{x} + \frac{1}{1-x}\right]\right\}. \quad (E12)$$

3. Contribution to the running coupling from $\gamma_{(g)}$

The calculation is analogous to the previous cases. The limit $\kappa_{12}^\perp \to 0$ concerns the difference of renormalization group factors and produces

$$\lim_{\kappa_{12}^\perp \to 0}\frac{\mathcal{B}_{t(g)} - \mathcal{B}_{t_0(g)}}{k_3^{+2}} = -t_0 f_0^2 + t f^2 - \frac{f_0^2 - f^2}{2\mathcal{M}^4}. \quad (E13)$$

Integration over $\kappa^\perp$ of the first two terms gives zero, and the only contributing part is

$$\int d^2\kappa^\perp \kappa^{\perp 2}\lim_{\kappa_{12}^\perp \to 0}\left[\frac{\mathcal{B}_{t(g)} - \mathcal{B}_{t_0(g)}}{k_3^{+2}}\right] = -x^2(1-x)^2\frac{\pi}{4}\ln\frac{t}{t_0}. \quad (E14)$$

The resulting contribution to the running coupling of term (g) is

$$g_{t,(g)} = g_{t_0,(g)} - \frac{g^3}{16\pi^2}N_c\frac{1}{2}\ln\frac{t}{t_0}$$
$$\times \left\{-\frac{11}{6} + \int_0^1 dx r_{\delta\mu}(x)\left[\frac{1}{x} + \frac{1}{1-x}\right]\right\}. \quad (E15)$$

4. Sum of contributions in Appendixes E 1, E 2 and E 3

Denoting g_{t_0} by g_0, the sum of contributions (a), (d) and (g) up to order g_0^3 gives

$$g_t = g_0 - \frac{g_0^3}{48\pi^2}N_c[11 + h(x_0)]\ln\frac{\lambda}{\lambda_0}. \quad (E16)$$

5. Useful formula

Integrals of differences of exponentials that appear in the RGPEP for massless quanta are evaluated taking advantage of the formula

$$\int d^2\kappa^\perp \frac{f - f_0}{\kappa^{\perp 2}} = \frac{\pi}{2}\ln\frac{t_0}{t}. \quad (E17)$$

This formula is a consequence of the RGPEP design that secures absence of large perturbative contributions in the matrix elements near the diagonal of the effective Hamiltonian matrix evaluated in the basis of the Fock space built using creation operators for effective particles [23]. Namely, the arguments of form factors f vanish quadratically as functions of the corresponding perturbative denominators.

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