PHYSICAL REVIEW D **92,** 064033 (2015)

Solar System and stellar tests of a quantum-corrected gravity

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(Received 26 July 2015; published 22 September 2015)

The renormalization group running of the gravitational constant has a universal form and represents a possible extension of general relativity. These renormalization group effects on general relativity will cause the running of the gravitational constant, and there exists a scale of renormalization $\alpha\nu$, which depends on the mass of an astronomical system and needs to be determined by observations. We test renormalization group effects on general relativity and obtain the upper bounds of $\alpha\nu$ in the low-mass scales: the Solar System and five systems of binary pulsars. Using the supplementary advances of the perihelia provided by INPOP10a (IMCCE, France) and EPM2011 (IAA RAS, Russia) ephemerides, we obtain new upper bounds on $\alpha\nu$ in the Solar System when the Lense–Thirring effect due to the Sun's angular momentum and the uncertainty of the Sun's quadrupole moment are properly taken into account. These two factors were absent in the previous work. We find that INPOP10a yields the upper bound as $\alpha\nu = (0.3 \pm 2.8) \times 10^{-20}$ while EPM2011 gives $\alpha\nu = (-2.5 \pm 8.3) \times 10^{-21}$. Both of them are tighter than the previous result by 4 orders of magnitude. Furthermore, based on the observational data sets of five systems of binary pulsars: PSR J0737 − 3039, PSR B1534 + 12, PSR J1756 − 2251, PSR B1913 + 16, and PSR B2127 + 11C, the upper bound is found as $\alpha\nu = (-2.6 \pm 5.1) \times 10^{-17}$. From the bounds of this work at a low-mass scale and the ones at the mass scale of galaxies, we might catch an updated glimpse of the mass dependence of $\alpha\nu$, and it is found that our improvement of the upper bounds in the Solar System can significantly change the possible pattern of the relation between $\log|\alpha\nu|$ and $\log m$ from a linear one to a power law, where m is the mass of an astronomical system. This suggests that $|\alpha\nu|$ needs to be suppressed more rapidly with the decrease of the mass of low-mass systems. It also predicts that $|\alpha\nu|$ might have an upper limit in high-mass astrophysical systems, which can be tested in the future.

DOI: 10.1103/PhysRevD.92.064033 PACS numbers: 04.50.Kd, 04.25.Nx, 11.10.Hi, 96.12.De

I. INTRODUCTION

Although all of astronomical and physical experiments have proven the validation of Einstein's general relativity (GR) with increasing precision [1,2], it seems that the theory might be incomplete. In the macroscopic scale, it is difficult for GR to explain the flat rotation curves of spiral galaxies, e.g., Refs. [3–5] without introducing dark matter and the present acceleration of the Universe, e.g., Refs. [6,7], without dark energy. At the microscopic scale, a more intrinsic problem is that, within the framework of GR itself, the theory will be broken by approaching the singularities of the relevant solutions. Two distinct examples of singularities are the center of a black hole and the starting point of the big bang. A highly possible way to erase the singularities is to apply the renormalization group technique in dealing with matter on curved spacetime background, e.g., Refs. [8–17]. It may also be effectively realized by a semiclassical approach: a classical action with quantum corrections [18,19]. If GR is the classical limit of a theory of quantum gravity, there should also exist a semiclassical limit.

In the present investigation, we focus on the renormalization group effects on GR (RGGR) [20–32]. It predicts the possibility of the running of the gravitational constant G due to quantum corrections. This running effect of RGGR can be observed and tested at the low-energy scale, which was proposed phenomenologically in Ref. [29] and then justified theoretically in Ref. [33]. The variation of G was also investigated within different contexts, such as orbital motions [34–38] and cosmology [39–41]. In RGGR, there exists a scale of renormalization. The identification of this scale has some uncertainty, which is measured by a dimensionless parameter $\alpha\nu$, and it needs to be determined by astronomical observations and physical experiments.

A remarkable property of this parameter is that its value grows with the increase of the masses of the astronomical systems [22,23,33]. Several works have studied RGGR in the mass scale of galaxies. Rodrigues *et al.* [29] found $\alpha\nu \sim 10^{-7}$ by analyzing the rotation curves of some spiral galaxies without dark matter. Rodrigues [31] extended the investigation on the spiral galaxies to some elliptical galaxies, including ordinary and giant samples, and found

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RGGR could be applied with good results to these elliptical galaxies. Rodrigues *et al.* [32] considered modified gravity models and the central cusp of dark matter haloes in galaxies and found the RGGR could achieve fits with better agreement than modified Newtonian dynamics and almost as good as a Navarro–Frenk–White halo.

However, little attention has been paid for RGGR in the low-mass scales, such as the scales of the Solar System and stellar systems. One exception is that Farina *et al.* [30] calculated the dynamics of the Laplace–Runge–Lenz vector for a planet in the Solar System and found $\alpha\nu \sim 10^{-17}$ based on the uncertainty in the measurement of the precession of Mercury [42]. Therefore, in order to gain better understanding of the mass dependence of the renormalization scale $\alpha\nu$ and its upper bounds at the scale of mass much less than the one of galaxies, we focus on the Solar and stellar tests of RGGR and their upper bounds on $\alpha\nu$ in this work, shedding light on the variation of the renormalization scale from stellar mass to the galaxy's mass.

We improve and extend the previous work of Farina *et al.* [30] in the following prospectives. First, we will improve the upper bound of $\alpha\nu$ in the Solar System by making use of current and highly accurate data sets of the planetary motions. And we will also try to reduce the contamination in our investigation due to the uncertainty of the Sun's quadrupole moment, which affects the motion of Mercury significantly [43]. For these purposes, we will use the supplementary advances of the perihelia provided by the INPOP10a (IMCCE, France) [44] and EPM2011 (IAA RAS, Russia) [45] ephemerides. These two ephemerides were recently adopted in planetary science [46,47] and in detecting gravitational effects and testing modified theories of gravity [48–57]. Since INPOP10a and EPM2011 are significantly improved compared with their previous versions and the data sets they provide are much more accurate, we expect to obtain a tighter upper bound on $\alpha\nu$. To find a clearer bound, besides the uncertainty of the Sun's quadrupole moment, we will also take the Lense–Thirring effect due to the Sun's angular momentum into account. Neither of the two factors is considered in the previous work of Farina *et al.* [30].

Second, we will take binary pulsars into our investigation. These binaries usually have the masses two times larger than the mass of the Solar System. According to its mass dependence of RGGR, $\alpha\nu$ in these systems should have different values from the one in the Solar System, though the discrepancy is expected to be small but not negligible. Another reason for the inclusion of binary pulsars is their much stronger gravitational fields than the Solar System's. The relativistic periastron advances in some binary pulsars can exceed the corresponding value for Mercury by a factor of $\sim 10^5$ so that these systems are taken as an ideal and clean laboratory for testing GR, alternative relativistic theories of gravity, and modified gravity [58–79]. To test RGGR in these systems, we will adopt five well-observed binary pulsars: PSR J0737 − 3039, PSR B1534 + 12, PSR J1756 − 2251, PSR B1913 + 16, and PSR B2127 + 11C.

The rest of the paper is organized as follows. Section II is devoted to describing primary ideas of RGGR. Effects of RGGR on the dynamics in the Solar System and binary pulsars will be studied in Sec. III and will be confronted with observational data sets in Sec. IV. In Sec. V, the scale of renormalization $\alpha\nu$ is estimated in both the Solar System and the binary pulsars we taken. Our improved upper bounds on the low-mass scales significantly changes the pattern of the mass dependence of $|\alpha\nu|$, which will be represented and discussed in Sec. VI. Finally, in Sec. VII, we summarize our results and discuss their implication.

II. RENORMALIZATION GROUP EFFECTS ON GR

In this section, we will briefly review some essentials of RGGR. More details can be found in Refs. [20–32] and references therein.

From a semiclassical perspective, the Einstein–Hilbert action of gravity can be extended by applying the renormalization group effects, and the action will depend on the metric tensor of spacetime $g_{\mu\nu}$ and the renormalization group scale μ. With the convention of Ref. [80], it reads as

$$S_{\mathrm{RGGR}} = \frac{c^3}{16\pi}\int \frac{R - 2\Lambda(\mu)}{G(\mu)}\sqrt{-g}\mathrm{d}^4x, \qquad (1)$$

where c is the speed of light, $g = \det(g_{\mu\nu}) < 0$ is the determinant of the metric tensor $g_{\mu\nu}$, and R is the Ricci scalar. In the above action, G is the gravitational "constant," and $\Lambda(\mu)$ is the cosmological constant. Both of them depend on the scale of μ. Varying the action with respect to the metric tensor $g_{\mu\nu}$, we can have its field equations as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + G\Box G^{-1}g_{\mu\nu} - G(G^{-1})_{;\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \qquad (2)$$

where a semicolon denotes the covariant derivative, $\Box(\cdot) \equiv (\cdot)_{;\mu\nu}g^{\mu\nu}$, and $T_{\mu\nu}$ is the energy-momentum tensor accounting for matter. Because we only focus on the scales of the Solar System and binary pulsars, the effects of Λ will be neglected in the following parts of this work.

The dependence $G(\mu)$ is governed by the renormalization group equation

$$\mu\frac{\mathrm{d}G^{-1}}{\mathrm{d}\mu} = 2\nu G_0^{-1}, \qquad (3)$$

where $G_0 = G(\mu_0) = \hbar c M_{\mathrm{P}}^{-2}$, M_{P} is the Planck mass, and ν characterizes the strength of renormalization group effects. If ν vanishes, G remains as a constant without running, and the action in Eq. (1) recovers Einstein's GR. Equation (3) can be solved, and the solution is

$$G(\mu) = \frac{G_0}{1 + 2\nu \ln(\mu/\mu_0)}, \quad (4)$$

where μ/μ_0 needs to be specified.

Observations and experiments have shown that Newton's law of gravity and GR serve quite well in small scales, such as the Solar System and binary pulsars [1,2,81]. It might be expected that the leading contribution of the RGGR departure from GR can be treated as a perturbation. In the limit of weak fields and low speeds, Eq. (2) recovers Newton's law in the form of Poisson's equation as

$$\nabla^2 \Phi_{\mathrm{N}} = 4\pi G_0 \rho, \quad (5)$$

where Φ_{N} is the Newtonian potential and ρ is the density of matter. The Newtonian potential is chosen to vanish at space infinity, according to the classical limit of GR. For maintaining the Newtonian limit, there is a choice of μ/μ_0 [29],

$$\frac{\mu}{\mu_0} = \left(\frac{\Phi_{\mathrm{N}}}{\Phi_0}\right)^{\alpha}, \quad (6)$$

where Φ_0 and α are two parameters. We can have the running of G from Eq. (4) as

$$G = \frac{G_0}{1 + 2\alpha\nu \ln(\Phi_{\mathrm{N}}/\Phi_0)}. \quad (7)$$

If we consider $\delta G = G - G_0$ is much less than G_0 in the scales of the Solar System and binary pulsars, we can obtain that, for a point mass, the variation of G has the form

$$G(r) = G_0 + \delta G(r) + \mathcal{O}(\delta G^2), \quad (8)$$

where

$$\delta G(r) = 2G_0 \alpha\nu \ln\left(\frac{r}{r_0}\right). \quad (9)$$

Here, the solution of Eq. (5) for a point mass is used, i.e., $\Phi_N \propto r^{-1}$, and it is assumed that Φ_0 depends on r_0, which makes $\mu = \mu_0$ in Eq. (6) and $G = G_0$ in Eq. (4). In obtaining Eqs. (8) and (9), we assume that the celestial body we considered can be approximated as a point mass. This assumption works well for a pulsar due to its compactness, but it might not be sufficient for the Sun. The quadrupole moment of the Sun and the uncertainty of its measured value cannot be ignored for planetary motions, and we will take them into account (see Sec. IV).

To work out the leading contribution of RGGR in the Solar System and binary pulsars, we need to know the effective potential Φ, which governs the motion of a particle by $\ddot{\boldsymbol{x}} = -\boldsymbol{\nabla}\Phi$. The "00" component of the metric tensor is connected to Φ with $g_{00} = -1 - 2c^{-2}\Phi$ so that, after applying the conformal transformation that $G_0(1 + 2c^{-2}\Phi) = G(1 + 2c^{-2}\Phi_{\mathrm{N}})$ and with the help of Eqs. (8) and (9), we can have the relation between Φ and Φ_N as

$$\Phi = \Phi_{\mathrm{N}} + \frac{c^2}{2}\frac{\delta G}{G_0} = -\frac{G_0 M}{r} + c^2 \alpha\nu \ln\left(\frac{r}{r_0}\right). \quad (10)$$

In the following parts of this work, Eq. (10) will used to formulate the dynamics of the Solar System and binary pulsars, and RGGR will be confronted with the observational data sets.

III. DYNAMICS IN THE FRAMEWORK OF RGGR

In this section, we will study the gravitational perturbed two-body problem in the framework of RGGR with its leading contribution given by Eq. (10). Although this perturbed model is definitely not sufficient for fully describing the dynamics of the Solar System which is essentially an N-body system, it can give the leading effects of the $\alpha\nu$ term in Eq. (10) on the planetary orbits around the Sun, and the coupling effects between the planet-planet interaction and the $\alpha\nu$ term are expected to be much smaller.

With the effective potential Φ, we can have the equations of relative motion of two point masses m_1 and m_2 as

$$\ddot{\boldsymbol{r}} = \boldsymbol{F}_{\mathrm{N}} + \boldsymbol{F}_{\delta G}, \quad (11)$$

where

$$\boldsymbol{F}_{\mathrm{N}} = -\frac{G_0 m}{r^3}\boldsymbol{r}, \quad (12)$$

$$\boldsymbol{F}_{\delta G} = -\frac{c^2\alpha\nu}{r}\boldsymbol{r}. \quad (13)$$

Here, $m \equiv m_1 + m_2$, and $\boldsymbol{r}$ is the vector pointing from m_2 to m_1. To work out the influence of $\boldsymbol{F}_{\delta G}$, we will apply the standard procedure of the methods of perturbation in celestial mechanics [82–84].

For a unperturbed two-body problem in the framework of Newton's law of gravity, there exist six integration constants, i.e., the Keplerian elements: a is the semimajor axis, e is the eccentricity, i is the inclination, Ω is the longitude of the ascending node, ω is the argument of periastron, and M is the mean anomaly. When the orbit of a Keplerian two-body problem is disturbed, these elements will become some functions of time. With conventional notations, they can be described by the Gauss perturbing equations as [82–84]

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{n\sqrt{1-e^2}}[Se\sin f + T(1 + e\cos f)], \quad (14)$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{na}[S\sin f + T(\cos f + \cos E)], \quad (15)$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{r\cos u}{na^2\sqrt{1-e^2}}W, \tag{16}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{r\sin u}{na^2\sqrt{1-e^2}\sin i}W, \tag{17}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{nae}\left[-S\cos f + T\left(1+\frac{r}{p}\right)\sin f\right] - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t}, \tag{18}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = n - \sqrt{1-e^2}\frac{\mathrm{d}\omega}{\mathrm{d}t} - \sqrt{1-e^2}\cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t} - \frac{2}{na^2}Sr. \tag{19}$$

Here, n is the Keplerian mean motion satisfying Kepler's third law $n^2a^3 = G_0 m$, f is the true anomaly, $u = f + \omega$, and $p = a(1-e^2)$. S, T, and W are the radial, transverse, and out-of-plane components of the perturbing force.

In the case of RGGR, the perturbing force $\boldsymbol{F}_{\delta G}$ of Eq. (13) has an S component only and no T and W parts:

$$S = -\frac{c^2\alpha\nu}{r}, \tag{20}$$

$$T = 0, \tag{21}$$

$$W = 0. \tag{22}$$

It can be immediately obtained that $\mathrm{d}i/\mathrm{d}t = 0$ and $\mathrm{d}\Omega/\mathrm{d}t = 0$. For secular evolution of the orbit, we need to average the fast changing variables over one orbital revolution in Eqs. (14)–(19). After that, we can have

$$\left\langle\frac{\mathrm{d}a}{\mathrm{d}t}\right\rangle = 0, \tag{23}$$

$$\left\langle\frac{\mathrm{d}e}{\mathrm{d}t}\right\rangle = 0, \tag{24}$$

$$\left\langle\frac{\mathrm{d}i}{\mathrm{d}t}\right\rangle = 0, \tag{25}$$

$$\left\langle\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right\rangle = 0, \tag{26}$$

$$\left\langle\frac{\mathrm{d}\omega}{\mathrm{d}t}\right\rangle = -\frac{c^2\alpha\nu\sqrt{1-e^2}}{na^2e^2}(1-\sqrt{1-e^2}), \tag{27}$$

$$\left\langle\frac{\mathrm{d}M}{\mathrm{d}t}\right\rangle = n + \frac{2c^2\alpha\nu}{na^2}. \tag{28}$$

The operator $\langle\cdot\rangle$ means calculating the average value during one Keplerian period P, i.e.,

$$\langle X\rangle \equiv \frac{1}{P}\int_0^P X\mathrm{d}t, \tag{29}$$

and there are two relations we used in the above equations:

$$\left\langle\frac{a}{r}\sin f\right\rangle = 0, \tag{30}$$

$$\left\langle\frac{a}{r}\cos f\right\rangle = -\frac{e}{1+\sqrt{1-e^2}}. \tag{31}$$

Equation (27) is the same as the result of Farina *et al.* [30], in which the dynamics of the Laplace–Runge–Lenz vector was calculated to find it out.

In the next section, we will confront the secular advance of periastron due to $\dot{\omega}_{\delta G} \equiv \langle\mathrm{d}\omega/\mathrm{d}t\rangle$ [see Eq. (27)] with the available data sets of the Solar System and binary pulsars.

IV. CONFRONTATION OF $\dot{\omega}_{\delta G}$ WITH OBSERVATIONAL DATASETS

In this section, we will confront $\dot{\omega}_{\delta G}$ with the data sets of the Solar System's ephemerides and the timing results of five binary pulsars. These two kinds of systems have their distinct characteristics so that they are expected to have different values of $\alpha\nu$. To achieve improved upper bounds, several factors need to be included in our investigation within the Solar System; meanwhile, the binary pulsars systems are much cleaner.

A. Solar System

In the case of the Solar System's planets, $\dot{\omega}_{\delta G}$ is closely connected with the supplementary advances of the perihelia $\dot{\omega}_{\mathrm{sup}}$ provided by modern ephemerides, such as INPOP10a [44,85] and EPM2011 [45,86,87].

INPOP10a and EPM2011 were obtained by fitting the "standard model" of dynamics to observational data, where standard model means Newton's law of gravity and Einstein's GR (apart from the Lense–Thirring effect; see below for details). In the INPOP10a and EPM2011 ephemerides, the standard model fitted to observations includes not only dynamics of natural bodies and artificial spacecrafts but also propagation of electromagnetic waves and how instruments onboard the spacecrafts and on Earth work. Therefore, RGGR was modeled neither in INPOP10a nor in EPM2011, and the parameter $\alpha\nu$ was not determined in these least-square fittings. In this sense, the results we obtain in the next section may not be considered as genuine "constraints" (it would be so if one solved for them in a covariance analysis by reanalyzing the data with modified software including these effects) but as preliminary indications of acceptable values to the best of the contemporary knowledge in the field of ephemerides [51]. It is necessary to stress that this method has also been adopted by other authors and has been proven valuable in several other circumstances [49,50,88]. Because, after all, such modified models are often not too analytically complex in the sense that they contain just a few parameters; if they contained several parameters, then an explicit and dedicated covariance analysis perhaps would be more necessary.

These $\dot{\omega}_{\rm sup}$ might represent possibly mismodeled or unmodeled parts of perihelion advances according to Newton's law and GR. They are almost all compatible with zero so that they can be used to draw bounds on quantities parametrizing unmodeled "forces" like the RGGR in this case. Nonetheless, the latest results by EPM2011 [86,87] returned nonzero values for Venus and Jupiter. Although the level of their statistical significance was not too high and further investigations are required, we still take them into account in this work. In the recent past, an extra nonzero effect on Saturn's perihelion was studied [89]. And, the ratios of the nonzero values of the supplementary precessions of Venus and Jupiter by EPM2011 [86,87] have been recently used to test a potential deviation from GR [52].

In the construction of $\dot{\omega}_{\rm sup}$ (see Ref. [85] for details), the effects caused by the Sun's quadrupole mass moment $J_2^{\odot}$ are considered and isolated in the final results, but the perihelion shifts caused by the Lense–Thirring effect [90] due to the Sun's angular momentum $S_{\odot}$ are absent. Therefore, by assuming these leading effects can be linearly added together, we can have the entire relation between $\dot{\omega}_{\delta G}$ and $\dot{\omega}_{\rm sup}$ as

$$\dot{\omega}_{\rm sup} = \dot{\omega}_{\delta G} + \dot{\omega}_{\rm LT} + \dot{\omega}_{\mathcal{J}_{\odot}}. \tag{32}$$

Here, the RGGR term $\dot{\omega}_{\delta G}$ can be rewritten in a more convenient form as

$$\dot{\omega}_{\delta G} = -c^2 \alpha \nu (G_0 M_{\odot} a)^{-1/2} e^{-2} \sqrt{1-e^2}(1-\sqrt{1-e^2}), \tag{33}$$

where Kepler's third law is applied and we neglect the masses of the planets: $m = M_{\odot} + M_{\rm Planet} \approx M_{\odot}$ due to $M_{\odot} \gg M_{\rm Planet}$.

The Lense–Thirring term $\dot{\omega}_{\rm LT}$ is [90]

$$\dot{\omega}_{\rm LT} = -\frac{6 G_0 S_{\odot} \cos i}{c^2 a^3 (1-e^2)^{3/2}}, \tag{34}$$

where $S_{\odot} = 1.9 \times 10^{41}\ {\rm kg\, m^2\, s^{-1}}$ [91] and i is the inclination of the planetary orbit to the equator of the Sun. The uncertainty of $S_{\odot}$ is currently about 1% [91]. This effect of the Sun on the planetary motions has been studied in several works [92–94]. Equation (34) only holds in a coordinate system of which the z axis is aligned with the Sun's angular momentum. A general formula for an arbitrary orientation can be found in Refs. [95,96]. It is useful in extrasolar planets and black holes, for which the orientation of the spin axis is generally unknown.

We add the third term in Eq. (32) to include the dimensionless uncertainty of the Sun's quadrupole moment $\mathcal{J}_{\odot}$ [97], which is currently about $\pm 10\%$ [98–102]. The Sun's quadrupole moment in INPOP10a is fitted to observations as $J_2^{\odot} = (2.40 \pm 0.25) \times 10^{-7}$ [44], and its value in EPM2011 is $J_2^{\odot} = (2.0 \pm 0.2) \times 10^{-7}$ [45]. This uncertainty of $J_2^{\odot}$ can cause an extra precession for a planet, which is [103]

$$\dot{\omega}_{\mathcal{J}_{\odot}} = \frac{3}{2} \mathcal{J}_{\odot} \frac{J_2^{\odot} R_{\odot}^2}{p^2} n \left(2 - \frac{5}{2}\sin^2 i\right), \tag{35}$$

where $R_{\odot}$ is the Sun's radius. It is clearly shown [55,104] that, although the uncertainty of $J_2^{\odot}$ can barely affect the outer planets, such as Jupiter and Saturn, it will significantly change the dynamics of the inner planets, especially Mercury. The higher-order multipoles like $J_3^{\odot}$ and $J_4^{\odot}$ have negligible impacts on the perihelion precessions, e.g., Refs. [105,106]. There are also post-Newtonian GR effects driven by $J_2^{\odot}$ [107,108]. While they may have an impact in other systems like close extrasolar planets with highly eccentric orbits, they can be left aside in the present case of the Solar System.

The effect of the cosmological constant Λ, which should be considered as somewhat "standard" in GR in view of the observed acceleration of the Universe, e.g., Refs. [6,7], has not been included in INPOP10a and EPM2011 so that it should appear in Eq. (32) as well. Its effects on the perihelion of planets were studied [50,104,109–118], and it was found that the perihelion shift caused by Λ is

$$\dot{\omega}_{\Lambda} = \frac{\sqrt{1-e^2}}{2n} \Lambda c^2. \tag{36}$$

Measurements of the cosmic microwave background radiation [119,120] imply the ratio between the energy density of Λ and the critical density of the Universe has a value of $\Omega_{\Lambda} \approx 0.7$, which means $\Lambda \sim 10^{-52}\ {\rm m^{-2}}$. It yields that, for the planets from Mercury to Saturn, $\dot{\omega}_{\Lambda}$ ranges from $\sim 10^{-12}$ to $\sim 10^{-10}$ milliarcsecond per century (mas ${\rm cy^{-1}}$), which are smaller than the $\dot{\omega}_{\rm sup}$ provided by INPOP10a [44,85] and EPM2011 [45,86,87] by about 8 orders of magnitude (see Table I for details). Therefore, given its extremely small influences, Λ can be left out from the analysis of the present work, in which we only take the $\dot{\omega}_{\rm sup}$ of the planets from Mercury to Saturn into account (see Sec. V for details).

B. Binary pulsars

Like the cases of the Solar System ephemerides, observations of binary pulsars were also obtained by fitting the standard model of dynamics of these systems to measured timing data [121–123]. In the present work, we did not

TABLE I. Supplementary advances in the perihelia $\dot{\omega}_{\rm sup}$ given by INPOP10a and EPM2011.

	$\dot{\omega}_{\rm sup}$ (mas cy^{-1})	
	INPOP10a [a]	EPM2011 [b]
Mercury	0.4 ± 0.6	-2.0 ± 3.0
Venus	0.2 ± 1.5	2.6 ± 1.6
EMB	-0.2 ± 0.9	$\cdots$
Earth	$\cdots$	0.19 ± 0.19
Mars	-0.04 ± 0.15	-0.020 ± 0.037
Jupiter	-41 ± 42	58.7 ± 28.3
Saturn	0.15 ± 0.65	-0.32 ± 0.47

[a]Taken from Table 5 in Ref. [44].
[b]Provided by Table 4 in Ref. [86] and Table 5 in Ref. [87].

modify the pulsar timing software to take RGGR into account. Instead, we confronted the theoretical prediction of RGGR with the measurements of the periastron advances $\dot{\omega}_{\rm PK}$ and their uncertainties of binary pulsars. $\dot{\omega}_{\rm PK}$ is a post-Keplerian parameter [124,125] and has been measured with great accuracy in some systems. In fact, this approach has also been used by several authors [52,71,74,75,78], and they did not actually reprocess the pulsar(s) timing data by *ad hoc* modifying the dynamics models to include the effects they were interested in, which were not solved for in covariance analyses.

Together with the leading term of the periastron advance caused by GR [126], the total periastron advance can be written as

$$\dot{\omega}_{\rm PK} = \dot{\omega}_{\rm GR} + \dot{\omega}_{\delta G}, \tag{37}$$

where the GR and RGGR parts are, respectively, in more convenient forms as

$$\dot{\omega}_{\rm GR} = 3\left(\frac{P_b}{2\pi}\right)^{-5/3}\left(\frac{G_0 m}{c^3}\right)^{2/3}(1-e^2)^{-1}, \tag{38}$$

$$\dot{\omega}_{\delta G} = -c^2\alpha\nu(G_0 m)^{-2/3}\left(\frac{P_b}{2\pi}\right)^{-1/3} \times e^{-2}\sqrt{1-e^2}(1-\sqrt{1-e^2}), \tag{39}$$

where P_b is the period of the binary. We can see from Eq. (37) that binary pulsars systems are much cleaner than the Solar System for gravitational tests.

V. UPPER BOUNDS ON $\alpha\nu$

In this section, we will estimate the upper bounds on $\alpha\nu$ based on the data sets provided by the Solar System ephemerides and five systems of binary pulsars. The method of minimizing χ^2 [127] will be used.

A. Upper bounds in the Solar System

The INPOP10a [44] ephemeris provides $\dot{\omega}_{\rm sup}$ for some planets in the Solar System: Mercury, Venus, Earth-Moon barycenter (EMB), Mars, Jupiter, and Saturn. Similarly, EPM2011 [45] also gives those values of the planets from Mercury to Saturn. These numbers are taken from Table 5 in Ref. [44] and Tables 4 and 5 in Refs. [86] and [87] (see Table I for details). It can be found that $\dot{\omega}_{\rm sup}$ of Mercury and Venus from EPM2011 are considerably larger than those of INPOP10a, while Venus and Jupiter have nonzero values of $\dot{\omega}_{\rm sup}$ in EPM2011.

We estimate the parameter $\alpha\nu$ and uncertainty of the Sun's quadrupole moment $\mathcal{J}_\odot$ simultaneously by minimizing

$$\chi^2_{\rm SS} = \sum_j \frac{1}{\sigma^2_{j,\rm sup}}(\dot{\omega}_{j,\rm sup} - \dot{\omega}_{j,\delta G} - \dot{\omega}_{j,\rm LT} - \dot{\omega}_{j,\mathcal{J}_\odot})^2, \tag{40}$$

where j enumerates each planet in Table I. We find that (i) INPOP10a yields the upper bounds as $\alpha\nu = (0.3 \pm 2.8) \times 10^{-20}$ and $\mathcal{J}_\odot = (5.7 \pm 1.1)\%$ and (ii) EPM2011 gives $\alpha\nu = (-2.5 \pm 8.3) \times 10^{-21}$ and $\mathcal{J}_\odot = (6.4 \pm 5.6)\%$. These results are summarized in Table III. The results obtained by INPOP10a and EPM2011 are compatible with each other. Furthermore, the values of $\mathcal{J}_\odot$ given by INPOP10a and EPM2011 are compatible with the current uncertainty of $\pm 10\%$.

Compared with the result of $\alpha\nu \sim 10^{-17}$ given by Farina *et al.* [30], the upper bounds of $|\alpha\nu| \sim 10^{-21}$ we obtain in the Solar System are improved by 4 orders of magnitude.

B. Upper bound in binary pulsars

Long-term timing observations can determine the geometrical and physical parameters of binary pulsars very well. Among them, PSR J0737 − 3039 [60], PSR B1534 + 12 [128], PSR J1756 − 2251 [129], PSR B1913 + 16 [130], and PSR B2127 + 11C [131] are good samples for gravitational tests. Some of their timing parameters are listed in Table II. The uncertainties of $\dot{\omega}_{\rm PK}$ are given in the parentheses.

TABLE II. Timing parameters of five binary pulsars.

PSR	P_b (d)	m ($M_\odot$)	e	$\dot{\omega}_{\rm PK}$ (° yr^{-1})	Ref.
J0737 − 3039	0.10225156248	2.58708	0.0877775	16.89947(68)	[60]
B1534 + 12	0.420737299122	2.678428	0.2736775	1.755789(9)	[128]
J1756 − 2251	0.319633898	2.574	0.180567	2.585(2)	[129]
B1913 + 16	0.322997448911	2.828378	0.6171334	4.226598(5)	[130]
B2127 + 11C	0.33528204828	2.71279	0.681395	4.4644(1)	[131]

TABLE III. Summary of upper bounds on $\alpha\nu$.

	$\alpha\nu$	$\mathcal{J}_\odot$ (%)	Adopted data
This work	$(0.3 \pm 2.8) \times 10^{-20}$	5.7 ± 1.1	INPOP10a
	$(-2.5 \pm 8.3) \times 10^{-21}$	6.4 ± 5.6	EPM2011
	$(-2.6 \pm 5.1) \times 10^{-17}$	⋯	Binary pulsars
[30]	$\sim 10^{-17}$	⋯	Mercury
[29]	$\sim 10^{-7}$	⋯	Rotation curves

We estimate $\alpha\nu$ in these five systems by minimizing

$$\chi^2_{\mathrm{BP}} = \sum_j \frac{1}{\sigma^2_{j,\mathrm{PK}}} (\dot{\omega}_{j,\mathrm{PK}} - \dot{\omega}_{j,\mathrm{GR}} - \dot{\omega}_{j,\delta G})^2, \tag{41}$$

where j enumerates each system of binary puslars in Table II. It is found that $\alpha\nu = (-2.6 \pm 5.1) \times 10^{-17}$, which is much larger than the values we obtained in the Solar System.

A summary of our and previous results are given in Table III. It is expected that the upper bounds given by the Solar System ephemerides and binary pulsars will be tighter due to the increase of the accuracy and precision of observations in the future.

VI. IMPLICATION OF MASS DEPENDENCE OF $\alpha\nu$

With the upper bounds obtained in the Solar System, the binary pulsars, and the spiral galaxies, we might catch a glimpse of the mass dependence of $\alpha\nu$. It is worth mentioning that the mass dependence of $\alpha\nu$ we obtained in the following part of this section is very preliminary, and it is just an implication provided by the available upper bounds. The reason for this preliminary implication is that the number of the upper bounds we can obtain is quite limited. Figure 1 shows $|\alpha\nu|$ against mass. The blue squares denote the upper bounds we obtained based on the Solar System ephemerides INPOP10a and EPM2011, and the mass of Solar System is taken as $1M_\odot$. The blue circle denotes the upper bound we obtained according to five systems of binary pulsars, and the mass scale is taken as $2.6M_\odot$ by averaging the masses of these systems. The green triangle represents the bound given by Farina *et al.* [30] based on Mercury, and the red triangle represents the result obtained by Rodrigues *et al.* [29] based on the rotation curves; the mass is chosen to be the typical mass of a spiral galaxy $2 \times 10^{11} M_\odot$ (without dark matter).

We can fit these data points in two cases:

(i) *Case 1*.—It includes the upper bound of Farina *et al.* [30] based on Mercury, ours based on the binary pulsars, and the upper bound of Rodrigues *et al.* [29] based on spiral galaxies. The three points are almost in a linear pattern in Fig. 1 and can be fitted as

$$\log_{10}|\alpha\nu| = 0.88 \log_{10}\left(\frac{m}{M_\odot}\right) - 17.0. \tag{42}$$

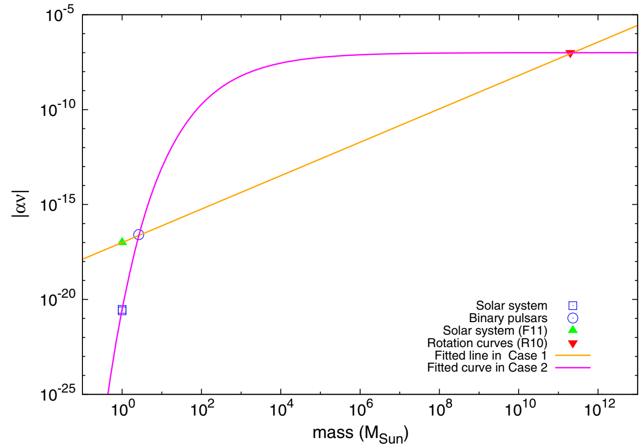

FIG. 1 (color online). Mass dependence of $\alpha\nu$. The blue squares denote the upper bounds we obtained based on the Solar System ephemerides INPOP10a and EPM2011, and the blue circle denotes the upper bound we obtained according to five systems of binary pulsars. The green triangle represents the bound given by Farina *et al.* [30] based on Mercury (F11 for short), and the red triangle represents the result obtained by Rodrigues *et al.* [29] based on the rotation curves (R10 for short). The orange line is plotted by Eq. (42) in case 1, and the magenta curve is is plotted by Eq. (43) in case 2.

It is clear that, according to Eq. (42), if $m \to \infty$, $|\alpha\nu| \to \infty$, which means a more massive system has a larger value of $|\alpha\nu|$.

(ii) *Case 2*.—It includes the upper bounds of both the Solar System and the binary pulsars obtained in this work and the one of Rodrigues *et al.* [29] based on spiral galaxies. Because our upper bounds in the Solar System are tighter than the one of Farina *et al.* [30] by about 4 orders of magnitude, the pattern of these points is far from a linear relation but in a curve of the power law, which can be fitted as

$$\log_{10}|\alpha\nu| = -13.5 \exp\left[-0.8 \log_{10}\left(\frac{m}{M_\odot}\right)\right] - 7.0. \tag{43}$$

Equation (43) imposes an upper bound for $|\alpha\nu|$, i.e., $|\alpha\nu| \lesssim 10^{-7}$, which is dramatically different from Eq. (42).

In fitting Eqs. (42) and (43), we only chose the simplest patterns, i.e., a linear one and a power law, because the very limited data points prevent us from exploring more complicated forms of the mass dependence of $\alpha\nu$. Nevertheless, these two cases clearly show the improvement of the upper bounds of $\alpha\nu$ in the Solar System in this work can significantly change the pattern of mass dependence between $\log_{10}(m/M_\odot)$ and $\log_{10}|\alpha\nu|$ from a linear one to a power law. This suggests that $|\alpha\nu|$ need to be suppressed more rapidly at the low-mass scale than the previous result [30] indicated. It also predicts that $|\alpha\nu|$

might have an upper limit in high-mass astrophysical systems, which can be tested in the future. It is also possible that more accurate and precise observations will change our present implication of the mass dependence of $\alpha\nu$.

VII. CONCLUSIONS AND DISCUSSION

In this work, we test RGGR in the low-mass scales and obtain the upper bounds of its renormalization scale parameter $\alpha\nu$ in the Solar System and the binary pulsars, which shed new light on the mass dependence of $\alpha\nu$.

Using the supplementary advances of the perihelia provided by INPOP10a [44] and EPM2011 [45] ephemerides and taking the Lense–Thirring effect due to the Sun's angular momentum and the uncertainty of the Sun's quadrupole moment $\mathcal{J}_\odot$ into account, we find that INPOP10a yields the upper bound as $\alpha\nu = (0.3 \pm 2.8) \times 10^{-20}$ and $\mathcal{J}_\odot = (5.7 \pm 1.1)\%$ while EPM2011 gives $\alpha\nu = (-2.5 \pm 8.3) \times 10^{-21}$ and $\mathcal{J}_\odot = (6.4 \pm 5.6)\%$. Compared with the Solar System's upper bound of $\alpha\nu \sim 10^{-17}$ by Farina *et al.* [30], our bounds of $|\alpha\nu| \sim 10^{-21}$ are improved by 4 orders of magnitude. Furthermore, based on the observation of five systems of binary pulsars, we find the upper bound is $\alpha\nu = (-2.6 \pm 5.1) \times 10^{-17}$. See Table III for a summary.

With the upper bounds we obtain in the present investigation and other results from previous works, we might catch a glimpse of the mass dependence of $\alpha\nu$, but this mass dependence we obtained is a preliminary implication provided by the available upper bounds. Figure 1 shows $|\alpha\nu|$ against the mass. Our improvement of the upper bounds of $|\alpha\nu|$ in the Solar System can significantly change the pattern of mass dependence between $\log_{10}|\alpha\nu|$ and $\log_{10}(m/M_\odot)$ from a linear one (based on the previous result of Farina *et al.* [30]) to a power law. This suggests that $\alpha\nu$ needs to be suppressed more rapidly at the low-mass scale than the previous result indicated. It also predicts that $|\alpha\nu|$ might have an upper limit in high-mass astrophysical systems, which can be tested in the future.

With tremendous advances in techniques for deep space exploration in the Solar System, ephemerides are going to be improved increasingly by high-precision data sets provided from spacecraft tracking and by sophisticated data analysis [132–134]. Future more sensitive and accurate timing observations will also improve the timing parameters of binary pulsars significantly. They will thus provide more stringent constraints on RGGR and might change our present implication of the mass dependence of $\alpha\nu$.

However, there is still a huge blank in the picture of mass dependence of $\alpha\nu$ between the scale of stellar masses and the one of galaxies (see Fig. 1). It might be necessary to make a similar analysis for these effects with other astronomical systems, such extrasolar planets [135,136], other binary systems [49], and globular clusters [137].

ACKNOWLEDGMENTS

This work is funded by the National Natural Science Foundation of China (Grant No. 11573015), the Fundamental Research Funds for the Central Universities of China (Grant No. 20620140586), and the Opening Project of Shanghai Key Laboratory of Space Navigation and Position Techniques (Grant No. 14DZ2276100).

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