

Thermodynamics of Gauss-Bonnet-dilaton Lifshitz black branesM. Kord Zangeneh,^{1,*} M. H. Dehghani,^{1,2,†} and A. Sheykhi^{1,2,‡}¹*Physics Department and Biruni Observatory, Shiraz University, Shiraz 71454, Iran*²*Research Institute for Astronomy and Astrophysics of Maragha (RIAAM),
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(Received 31 July 2015; published 17 September 2015)

We explore an effective supergravity action in the presence of a massless gauge field which contains a Gauss-Bonnet term as well as a dilaton field. We construct a new class of black brane solutions of this theory with a Lifshitz asymptotic by fixing the parameters of the model such that the asymptotic Lifshitz behavior can be supported. Then we construct the well-defined finite action through the use of the counterterm method. We also obtain two independent constants along the radial coordinate by combining the equations of motion. Calculations of these two constants at infinity through the use of the large- r behavior of the metric functions show that our solution respects the no-hair theorem. Furthermore, we combine these two constants in order to get a constant C which is proportional to the energy of the black brane. We calculate this constant at the horizon in terms of the temperature and entropy and at large- r in terms of the geometrical mass. By calculating the value of the energy density through the use of the counterterm method, we obtain the relation between the energy density, the temperature, and the entropy. This relation is the generalization of the well-known Smarr formula for AdS black holes. Finally, we study the thermal stability of our black brane solution and show that it is stable under thermal perturbations.

DOI: [10.1103/PhysRevD.92.064023](https://doi.org/10.1103/PhysRevD.92.064023)

PACS numbers: 04.50.Kd, 04.70.Bw, 04.70.Dy

I. INTRODUCTION

The formulation of the quantum theory of gravity and its applications to physical systems in order to understand physics at strong gravity is one of the most interesting challenges of modern theoretical physics. The area where quantum gravity may play a significant role includes cosmology and black hole physics. Although the leading candidates are the ten-dimensional superstring theories, most analyses have been performed by using low-energy effective theories inspired by string theories. This is due to the fact that it is difficult to study geometrical settings in superstring theories. The effective theories are the supergravities which typically involve not only the metric but also a dilaton field (as well as several gauge fields). On the other hand, the effective supergravity action coming from superstrings contains higher-order curvature correction terms. The simplest correction is the Gauss-Bonnet (GB) term coupled to the dilaton field in the low-energy effective heterotic string [1]. It is then natural to ask how the black hole solutions are affected by higher-order terms in these effective theories. To our knowledge, just one exact solution of such a theory has been obtained [2]. Indeed, in Ref. [2], a dilatonic Einstein-Gauss-Bonnet (EGB) theory with a nonminimal coupling between the EGB term and dilaton field has resulted by applying the general Kaluza-Klein reduction on EGB gravity, and an exact solution has

been introduced. Also in Ref. [3], it was shown that the anti-de Sitter (AdS) metric can be an exact solution of this theory. In addition, asymptotically AdS solutions of this theory have been considered in Refs. [3,4] numerically. Here, we would like to find Lifshitz solutions of this theory. The motivation for this investigation is that Lifshitz black holes have received much attention recently. Indeed, black hole configurations in Lifshitz spacetime are dual to non-relativistic conformal field theories enjoying anisotropic conformal transformations,

$$t \rightarrow \lambda^z t, \quad \vec{x} \rightarrow \lambda \vec{x}, \quad (1)$$

where the constant $z > 1$ is called the dynamical exponent and shows the anisotropy between space and time. In other words, while AdS black holes are dual to scale invariant relativistic field theories which respect the isotropic conformal transformation [5]

$$t \rightarrow \lambda t, \quad \vec{x} \rightarrow \lambda \vec{x}, \quad (2)$$

Lifshitz black holes are dual to nonrelativistic field theories. Furthermore, there are many situations where isotropic conformal transformation (2) is not respected. For instance, in many condensed matter systems, there are phase transitions governed by fixed points which exhibit dynamical scaling (1). The gravity models dual to such systems are no longer AdS and one needs a spacetime that its boundary respects anisotropic conformal transformation (1). This spacetime, which is known as Lifshitz spacetime, was first introduced in [6] as

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$$ds^2 = -\frac{r^{2z}}{l^2} dt^2 + \frac{l^2 dr^2}{r^2} + r^2 d\vec{x}^2. \quad (3)$$

From the beginning of the introduction of the Lifshitz spacetime, it was known that this is not a vacuum solution of Einstein gravity or even the Einstein equation with the cosmological constant in the case of an arbitrary value of z . Therefore, one needs some matter sources or higher-curvature corrections in order to guarantee that the asymptotic behavior of spacetime is Lifshitz. One of the matter sources considered in many works is a massive gauge field [7–11]. In the latter case, some exact solutions have been obtained, more often for fixed z [7,8]. However, it seems impossible in some cases to find an exact solution of the field equations of motion especially for arbitrary z , either in Einstein [8,9] or in Lovelock gravities [10,11]. But, it is possible to study the thermodynamics of these black hole configurations by using a conserved quantity along the radial coordinate [9,10]. Another way to guarantee the Lifshitz asymptotic behavior is by considering higher-curvature corrections [12]. In addition, the asymptotic supporting matter source can also be chosen to be a dilaton field and a massless gauge field [13–15]. One of the advantages of the latter matter source over the massive gauge field is that in this case it is possible to find analytic Lifshitz black hole solutions in Einstein gravity for arbitrary z . Some charged exact solutions have been presented in [13,14]. In [15], the thermal behavior of uncharged dilaton Lifshitz black branes has been studied by using perturbation theory.

Motivated by the above two paragraphs, it is interesting to study the thermodynamics of Lifshitz black brane solutions in the effective supergravity action coming from superstrings, which contains higher-order curvature correction terms and a dilaton field. In this paper we attack this problem. We shall investigate the thermodynamics of Lifshitz black branes in the presence of a dilaton, a massless gauge field, and GB term. Although exact dilatonic Lifshitz solutions have been introduced in Einstein gravity [13,14], no exact asymptotically Lifshitz solution has been obtained in dilaton EGB gravity. Therefore, we seek the thermodynamics of Lifshitz black branes in GB-dilaton gravity using a conserved quantity along the radial coordinate, as in the case of massive gauge field matter sources. We find the relation between energy density, temperature, and entropy by using the fact that the values of our constant quantity along the radial coordinate are related to the temperature and entropy at the horizon and to the energy density for large r .

The layout of this paper is as follows. In Sec. II, we obtain the one-dimensional Lagrangian and derive the field equations of motion for a general spherically symmetric spacetime. In Sec. [6], we show that this

theory can accept the Lifshitz metric as its solution. Two constants along the radial coordinate will be introduced in Sec. IV. Section V is devoted to the generalization of the counterterm method of Ref. [16] in the presence of a dilaton field. In Sec. VI, we calculate the thermodynamic and conserved quantities of asymptotic Lifshitz black branes and obtain the Smarr relation. We also show that our solution is thermally stable. We finish our paper with a summary and some concluding remarks.

II. FIELD EQUATIONS

The effective action of the heterotic string theory in the Einstein frame in the presence of a gauge field may be written as

$$I_{\text{bulk}} = \frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \left(R - 2\Lambda + \alpha e^{\eta\Phi} \mathcal{L}_2 - \frac{4}{n-1} (\partial\Phi)^2 - e^{-4\lambda\Phi/(n-1)} F \right), \quad (4)$$

where $F = F_{\mu\nu} F^{\mu\nu}$, $F_{\mu\nu} = \partial_{[\mu} U_{\nu]}$, η is a constant, λ is the coupling constant of dilaton and matter, Λ is the cosmological constant, α is the Gauss-Bonnet (GB) coefficient, $\mathcal{L}_2 = R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} - 4R_{\mu\nu} R^{\mu\nu} + R^2$ is the GB Lagrangian, and Φ is the dilaton field. While the ten-dimensional critical string theory predicts the coupling constant $\eta \neq 0$, we set it equal to zero. This is due to the fact that, as we will see later and also in the absence of the Gauss-Bonnet term considered in Ref. [14], the dilaton goes to infinity as r goes to infinity. Thus, $e^{\eta\Phi}$ becomes very large at large r which effectively rescales the coupling constant α to large values. Indeed, a large value $e^{\eta\Phi}$ leads to a large modification to general relativity which is ruled out by the weak-field approximation [17]. So, we set $\eta = 0$. In this case the GB term for $n \leq 3$ does not contribute to the field equations and is a total derivative in the action. Here we consider the GB coefficient α positive as in the heterotic string theory [18].

We write the spherically symmetric gauge fields and the metric of an $(n+1)$ -dimensional asymptotically Lifshitz static spacetime with zero curvature boundary as

$$U = q e^{K(r)} dt, \quad (5)$$

$$ds^2 = -e^{2A(r)} dt^2 + e^{2C(r)} dr^2 + l^2 e^{2B(r)} d\vec{x}^2, \quad (6)$$

where $d\vec{x}^2 = \sum_{i=2}^n (dx^i)^2$. Inserting (5) and (6) into the action (4) and integrating by part, one obtains the one-dimensional Lagrangian as $\mathcal{L}_{1D} = (n-1)l^{n-1} \tilde{\mathcal{L}}_{1D}$, where

$$\begin{aligned} \tilde{\mathcal{L}}_{1D} = & -\frac{2\Lambda}{n-1} e^{A+C+(n-1)B} + e^{A-C+(n-1)B} [2A'B' + (n-2)B'^2] - 2\tilde{\alpha} e^{A-3C+(n-1)B} \left[\frac{2}{3} B'^3 A' + \frac{n-4}{6} B'^4 \right] \\ & - \frac{4}{(n-1)^2} e^{A-C+(n-1)B} \Phi'^2 + \frac{2q^2}{n-1} e^{-A-C+(n-1)B-4\lambda\Phi/(n-1)+2K} K'^2, \end{aligned} \quad (7)$$

$\tilde{\alpha} = \alpha(n-2)(n-3)$, and prime denotes the derivative with respect to the r coordinate. Varying the action (7) with respect to $A(r)$, $B(r)$, $C(r)$, $\Phi(r)$, and $K(r)$, respectively, one obtains the following equations of motion:

$$\begin{aligned} E_1 = & e^{A-C+(n-1)B} \left[B'' + \frac{n}{2} B'^2 - B'C' + \frac{2}{(n-1)^2} \Phi'^2 \right] - \frac{\tilde{\alpha}n}{2} e^{A-3C+(n-1)B} \left[\frac{4}{n} B'' B'^2 + B'^4 - \frac{4}{n} B'^3 C' \right] \\ & + \frac{\Lambda}{n-1} e^{A+C+(n-1)B} + \frac{q^2}{n-1} e^{-A-C+(n-1)B-4\lambda\Phi/(n-1)+2K} K'^2 = 0, \end{aligned} \quad (8)$$

$$\begin{aligned} E_2 = & -e^{A-C+(n-1)B} \left[-(n-2)B'' - A'' - \frac{(n-1)(n-2)}{2} B'^2 - A'^2 + (n-2)B'(C' - A') + A'C' - \frac{2}{n-1} \Phi'^2 \right] \\ & - 2\tilde{\alpha} e^{A-3C+(n-1)B} \left\{ B'' B' [2A' + (n-4)B'] + B'^2 \left[A'' + \frac{(n-2)(n-3)-2}{4} B'^2 \right. \right. \\ & \left. \left. + B'((n-2)A' - (n-4)C') + A'(A' - 3C') \right] \right\} + e^{A+C+(n-1)B} \Lambda - q^2 e^{-A-C+(n-1)B-4\lambda\Phi/(n-1)+2K} K'^2 = 0, \end{aligned} \quad (9)$$

$$\begin{aligned} E_3 = & e^{A-C+(n-1)B} \left[A'B' + \frac{(n-2)}{2} B'^2 - \frac{2}{(n-1)^2} \Phi'^2 \right] - 2\tilde{\alpha} e^{A-3C+(n-1)B} \left[A'B'^3 + \frac{n-4}{4} B'^4 \right] \\ & + \frac{\Lambda}{n-1} e^{A+C+(n-1)B} + \frac{q^2}{n-1} e^{-A-C+(n-1)B-4\lambda\Phi/(n-1)+2K} K'^2 = 0, \end{aligned} \quad (10)$$

$$E_4 = \frac{4}{(n-1)^2} \left\{ -e^{A-C+(n-1)B} [\Phi'' + \Phi'(A' - C' + (n-1)B')] + q^2 \lambda e^{-A-C+(n-1)B-4\lambda\Phi/(n-1)+2K} K'^2 \right\} = 0, \quad (11)$$

$$E_5 = \frac{8q^2}{(n-1)^2} e^{-A-C+(n-1)B-4\lambda\Phi/(n-1)+2K} \left\{ K'' + K'^2 + K' \left[-\frac{4}{n-1} \Phi' - A' - C' + (n-1)B' \right] \right\} = 0. \quad (12)$$

One should also note that there is a relation between the equations of motion,

$$A'E_1 + B'E_2 + C'E_3 - E'_3 + \Phi'E_4 + K'E_5 = 0, \quad (13)$$

where prime denotes the derivative with respect to r . The above Eq. (13) reduces the number of independent field equations to four. One may note that E_5 is the Maxwell equation that can be solved for $K(r)$ as

$$(e^{K(r)})' = l^{-n} e^{4\lambda\Phi/(n-1)+A+C-(n-1)B}, \quad (14)$$

and, therefore, we leave with three independent equations of motion.

III. LIFSHITZ SOLUTION

We used the metric (6) in order to find the constants of the system along the radial coordinate r , which will be done in the next section. Here we want to find the Lifshitz solutions. In order to do this, we use the standard form of

the asymptotic Lifshitz metric through the use of the following transformations:

$$\begin{aligned} A(r) &= \frac{1}{2} \ln \left(\frac{r^{2z}}{l^{2z}} f(r) \right), \\ C(r) &= -\frac{1}{2} \ln \left(\frac{r^2}{l^2} g(r) \right), \\ B(r) &= \ln \frac{r}{l}, \\ K(r) &= \ln \frac{k(r)}{l^z}. \end{aligned} \quad (15)$$

Using the above transformation (15), the metric and gauge field can be written as

$$\begin{aligned} ds^2 &= -\frac{r^{2z}}{l^{2z}} f(r) dt^2 + \frac{l^2 dr^2}{r^2 g(r)} + r^2 d\vec{x}^2, \\ U &= \frac{q}{l^z} k(r) dt. \end{aligned}$$

In this frame, the solution of the electromagnetic equation is

$$k'(r) = e^{4\lambda\Phi/(n-1)} r^{z-n} \sqrt{\frac{f}{g}}. \quad (16)$$

Here we choose the horizon as the reference point of the potential. Thus, one obtains

$$k(r) = \int^r e^{4\lambda\Phi/(n-1)} r^{z-n} \sqrt{\frac{f}{g}} dr + D, \quad (17)$$

with

$$D = - \int^{r_0} e^{4\lambda\Phi/(n-1)} r^{z-n} \sqrt{\frac{f}{g}} dr,$$

where r_0 is the horizon radius. We first investigate the possibility of having $(n+1)$ -dimensional Lifshitz solutions. For this aim, the field equations should be satisfied for $f(r) = g(r) = 1$. In this case, by using (15) and (17), Eqs. (8)–(11) reduce to

$$\begin{aligned} & -r^2 \Phi_{\text{lif}}'^2 - \frac{n(n-1)^2}{4} + \frac{\tilde{\alpha}n(n-1)^2}{4l^2} \\ & - \frac{n-1}{2} \left[\Lambda l^2 + \frac{q^2 e^{4\lambda\Phi_{\text{lif}}/(n-1)}}{r^{2(n-1)}} \right] = 0 \end{aligned} \quad (18)$$

$$\begin{aligned} & r^2 \Phi_{\text{lif}}'^2 + (n-1)(n-2) \left[\frac{(n-1)}{4} + \frac{z^2}{2(n-2)} + \frac{z}{2} \right] \\ & - \frac{\tilde{\alpha}(n-1)}{l^2} \left[\frac{(n-1)(n-4)}{4} + (n-2)z + z^2 \right] \\ & + \frac{n-1}{2} \left[\Lambda l^2 - \frac{q^2 e^{4\lambda\Phi_{\text{lif}}/(n-1)}}{r^{2(n-1)}} \right] = 0, \end{aligned} \quad (19)$$

$$\begin{aligned} & r^2 \Phi_{\text{lif}}'^2 - \frac{(n-1)^2((n-2) + 2z)}{4} + \frac{\tilde{\alpha}(n-1)^2}{l^2} \left[z + \frac{n-4}{4} \right] \\ & - \frac{n-1}{2} \left[\Lambda l^2 + \frac{q^2 e^{4\lambda\Phi_{\text{lif}}/(n-1)}}{r^{2(n-1)}} \right] = 0, \end{aligned} \quad (20)$$

$$r \Phi_{\text{lif}}'' + (z+n) \Phi_{\text{lif}}' - \frac{\lambda q^2 e^{4\lambda\Phi_{\text{lif}}/(n-1)}}{r^{2(n-1)+1}} = 0. \quad (21)$$

Subtracting (18) from (20) one can find

$$2r^2 \Phi_{\text{lif}}'^2 - \frac{(n-1)^2(z-1)}{2} \left(1 - \frac{2\tilde{\alpha}}{l^2} \right) = 0, \quad (22)$$

with the solution

$$\Phi_{\text{lif}}(r) = \xi \ln \left(\frac{r}{b} \right), \quad \xi = \frac{n-1}{2l} \sqrt{(z-1)(l^2 - 2\tilde{\alpha})}. \quad (23)$$

Substituting the solution (23) in the Eqs. (18)–(21), it is a matter of calculation to show that they are fully satisfied, provided

$$\begin{aligned} \lambda &= \frac{(n-1)l}{\sqrt{(z-1)(l^2 - 2\tilde{\alpha})}}, \\ q &= \frac{b^{n-1}}{\sqrt{2}l} \sqrt{(z-1)(l^2 - 2\tilde{\alpha})(z+n-1)}, \\ \Lambda &= - \frac{(z+n-1)(z+n-2)}{2l^2} \\ &+ \frac{\tilde{\alpha}[2z^2 + 2(2n-3)z + (n-2)(n-3) - 2]}{2l^4}. \end{aligned} \quad (24)$$

Please note that in the case of $\alpha = 0$ and $z = 1$, Λ reduces to $-n(n-1)/2l^2$ and in the case of $z = 1$, it reduces to $-n(n-1)(1 - \tilde{\alpha}/l^2)/2l^2$ as expected in the cases of AdS spacetimes in Einstein and Einstein-Gauss-Bonnet gravity, respectively. We can also find the asymptotic value of $k(r)$ as r goes to infinity by using (12), (23), and (24) as

$$k_{\text{lif}} = \frac{b^{2-2n} r^{z+n-1}}{z+n-1}. \quad (25)$$

IV. THE CONSTANT ALONG THE RADIAL COORDINATE

As in the case of the Einstein equation in the presence of a dilaton [15], one can find two independent constants along the radial coordinate r for this spacetime. It could be checked that there are two combinations of field equations which are exact differentials:

$$\begin{aligned} E_1 - \frac{E_2}{n-1} + E_5 &= -\frac{1}{n-1} [e^{A-C+(n-1)B} (A' - B') \\ &+ 2\tilde{\alpha} e^{A-3C+(n-1)B} (B'^3 - A'B'^2) \\ &- 2q^2 e^{-A-C+(n-1)B-4\lambda\Phi/(n-1)+2K} K']' \\ &= 0. \end{aligned} \quad (26)$$

$$\begin{aligned} E_4 + \frac{2\lambda E_5}{n-1} &= -\frac{4}{(n-1)^2} [e^{A-C+(n-1)B} \Phi' \\ &- \lambda q^2 e^{-A-C+(n-1)B-4\lambda\Phi/(n-1)+2K} K']' \\ &= 0. \end{aligned} \quad (27)$$

This fact results in two independent constants:

$$\begin{aligned} \mathcal{C}_1 &= -\frac{1}{n-1} [e^{A-C+(n-1)B} (A' - B') \\ &+ 2\tilde{\alpha} e^{A-3C+(n-1)B} (B'^3 - A'B'^2) \\ &- 2q^2 e^{-A-C+(n-1)B-4\lambda\Phi/(n-1)+2K} K'] \\ &= -\frac{1}{2(n-1)l^{n+z}} \left\{ \left[1 - \frac{2\tilde{\alpha}}{l^2} g \right] \right. \\ &\times \left. \left[r^{z+n} f' \sqrt{\frac{g}{f}} + 2r^{z+n-1} \sqrt{fg}(z-1) \right] - 4q^2 k \right\}, \end{aligned} \quad (28)$$

$$\begin{aligned} \mathcal{C}_2 = & -\frac{4}{(n-1)^2} [e^{A-C+(n-1)B} \Phi' \\ & - \lambda q^2 e^{-A-C+(n-1)B-4\lambda\Phi/(n-1)+2K} K'] \\ & - \frac{4}{(n-1)^2 l^{n+z}} [r^{n+z} \sqrt{fg} \Phi' - \lambda q^2 k]. \end{aligned} \quad (29)$$

We pause to remark that q and Φ vanish for $z = 1$, and the theory reduces to Einstein-Gauss-Bonnet (EGB) gravity. Thus, with $f(r) = g(r)$ the first constant (28) reduces to

$$\mathcal{C}_1 = \frac{r^{n+1}}{2(n-1)l^{n+1}} \left(f - \frac{\tilde{\alpha}}{l^2} f^2 \right)',$$

which is known to be constant in EGB gravity and is proportional to the mass parameter of the spacetime. Also, note that the second constant is zero in EGB gravity.

Combining the constants (28) and (29), one can get a constant which is very useful in our future discussions in this work:

$$\begin{aligned} \mathcal{C} = & -2(n-1)l^{n-1} \left(\mathcal{C}_1 - \frac{n-1}{2\lambda} \mathcal{C}_2 \right) \\ = & \frac{r^{n+z}}{l^{z+1}} \left\{ \left(1 - \frac{2\tilde{\alpha}}{l^2} g \right) \left[f' \sqrt{\frac{g}{f}} + 2r^{-1} \sqrt{fg}(z-1) \right] \right. \\ & \left. - \frac{4\sqrt{fg}\Phi'}{\lambda} \right\}. \end{aligned} \quad (30)$$

In this section, we want to calculate the constant \mathcal{C} , which is conserved along the radial coordinate r . Since there is no exact GB-Lifshitz-dilaton solution, we calculate it at the horizon and at infinity. We will use this to relate the constant that appears in the expansion at $r = \infty$ to the coefficients at the horizon.

A. \mathcal{C} at the horizon

Considering nonextreme black branes, one can assume that $f(r)$ and $g(r)$ go to zero linearly at the horizon. Also,

we have chosen the reference point of $k(r)$ at the horizon. Thus, one can write

$$\begin{aligned} f(r) = & f_1 \{ (r-r_0) + f_2(r-r_0)^2 + f_3(r-r_0)^3 \\ & + f_4(r-r_0)^4 + \dots \}, \\ g(r) = & g_1(r-r_0) + g_2(r-r_0)^2 + g_3(r-r_0)^3 \\ & + g_4(r-r_0)^4 + \dots, \\ k(r) = & k_1(r-r_0) + k_2(r-r_0)^2 + k_3(r-r_0)^3 \\ & + k_4(r-r_0)^4 + \dots, \\ \Phi(r) = & \Phi_0 + \Phi_1(r-r_0) + \Phi_2(r-r_0)^2 + \Phi_3(r-r_0)^3 \\ & + \Phi_4(r-r_0)^4 + \dots. \end{aligned} \quad (31)$$

One can solve for the various coefficients by inserting these expansions into the equations of motion arising from the action (4) for the metric (6) with the conditions (24).

The constant \mathcal{C} (30) can be evaluated at $r = r_0$ by using the above expansion. One obtains

$$\mathcal{C} = \frac{r_0^{z+n} \sqrt{f_1 g_1}}{l^{z+1}}. \quad (32)$$

This must be preserved along the flow in r .

B. \mathcal{C} at infinity

We now turn to the calculation of \mathcal{C} at large r . In order to do this, we investigate the behavior of the metric functions at large r by using straightforward perturbation theory. Using the following expansions,

$$\begin{aligned} f(r) = & 1 + \varepsilon f_1(r), \\ g(r) = & 1 + \varepsilon g_1(r), \\ \Phi(r) = & \Phi_{\text{lif}} + \varepsilon \Phi_1(r), \end{aligned}$$

and finding the field equations (8)–(10) up to the first order in ε , we obtain

$$\begin{aligned} 0 = & (n-1)[(l^2 - 2\tilde{\alpha})rf_1' + (z+n-1)l^2g_1 - 2\tilde{\alpha}(3z+n-3)g_1] + 4l\sqrt{(z-1)(l^2 - 2\tilde{\alpha})}[(z+n-1)\Phi_1 - r\Phi_1'] \\ 0 = & (l^2 - 2\tilde{\alpha})[r^2f_1'' + (2z+n-1)rf_1'] + [(z+n-2)l^2 - 2\tilde{\alpha}(3z+n-4)]rg_1' + (z+n-1) \\ & \times [(2z+n-3)l^2 - 2\tilde{\alpha}(4z+n-5)]g_1 - 4l\sqrt{(z-1)(l^2 - 2\tilde{\alpha})}[(z+n-1)\Phi_1 + r\Phi_1'] \\ 0 = & (n-1)\sqrt{(z-1)(l^2 - 2\tilde{\alpha})}[2(z+n-1)g_1 + r(f_1' + g_1')] + 4l[r^2\Phi_1'' + (z+n)r\Phi_1' - 2(n-1)(z+n-1)\Phi_1]. \end{aligned} \quad (33)$$

Demanding the fact that the solutions corresponding to these field equations should go to zero as $r \rightarrow \infty$, one can find the desired solutions of Eqs. (33) as

$$\begin{aligned}
f_1(r) &= -\frac{C_1}{r^{n+z-1}} + \frac{C_2}{r^{(n+z-1+\gamma)/2}}, \\
g_1(r) &= -\frac{(l^2 - 2\tilde{\alpha})(z+n-1)(z+n-2)C_1}{\mathcal{G}r^{n+z-1}} - \frac{\mathcal{T}\mathcal{K}_+C_2}{r^{(n+z-1+\gamma)/2}}, \\
\Phi_1(r) &= -\frac{(n-1)(z-1)^{3/2}\tilde{\alpha}\sqrt{l^2-2\tilde{\alpha}}C_1}{2l\mathcal{G}r^{n+z-1}} + \frac{(n-1)\sqrt{l^2-2\tilde{\alpha}}\mathcal{T}\mathcal{K}_+C_2}{4l\sqrt{z-1}r^{(n+z-1+\gamma)/2}},
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
\gamma &= \sqrt{(z+n-1)(9z+9n-17) + \frac{16\tilde{\alpha}(z-1)^2}{l^2-2\tilde{\alpha}}}, \\
\mathcal{G} &= (l^2 - 2\tilde{\alpha})(z+n-1)(z+n-2) - 4\tilde{\alpha}(z-1)(n-1), \\
\mathcal{K}_\pm &= (l^2 - 2\tilde{\alpha})(z+n-1)(z+n-2) - \tilde{\alpha}(z-1)(n-z+1 \pm \gamma), \\
\mathcal{T}^{-1} &= (l^2 - 2\tilde{\alpha})(z+n-1)(z+n-2) - 2\tilde{\alpha}(z-1)n - \frac{8\tilde{\alpha}^2(z-1)^2}{(l^2-2\tilde{\alpha})}, \\
\mathcal{W} &= (l^2 - 2\tilde{\alpha})(n+z-2)\gamma - \tilde{\alpha}(z-1)(3\gamma+9n+7z-15).
\end{aligned}$$

Substituting (34) in (17), one could find the large- r behavior of $k(r)$ as

$$k(r) = k_{\text{lif}} - \varepsilon \frac{\mathcal{T}(\mathcal{W} + \mathcal{K}_-)C_2}{4b^{2(n-1)}(z-1)r^{(\gamma-n-z+1)/2}}. \tag{35}$$

It is easy to check that $(\gamma - n - z + 1)/2$, which is the power of r in the denominator of (35), is always positive and, therefore, $k(r) \rightarrow k_{\text{lif}}$ as $r \rightarrow \infty$.

Now we want to calculate the conserved quantity \mathcal{C} given by (30). The constant \mathcal{C} can be obtained as

$$\mathcal{C} = \frac{(z+n-1)(l^2-2\tilde{\alpha})[(z+n-2)(z+n-1)(l^2-2\tilde{\alpha}) + 2\tilde{\alpha}(z-1)^2]C_1}{\mathcal{G}l^{z+3}}. \tag{36}$$

For $\alpha = 0$, \mathcal{C} reduces to

$$\mathcal{C} = \frac{(z+n-1)C_1}{l^{z+1}}. \tag{37}$$

It is worthwhile mentioning that although there are two constants C_1 and C_2 in the solutions (34), at infinity only the constant C_1 , which is the geometrical mass of the black hole as we will show in Sec. VI, appears in the conserved quantities along the radial coordinate. Thus, one can conclude that our solution respects the no-hair theorem.

V. FINITE ACTION FOR GB-LIFSHITZ SOLUTIONS

The action (4) is neither well defined nor finite. In order to get a finite and well-defined action, one may add a few covariant boundary terms to the action. The boundary term I_{bdy} is the sum of the boundary terms which are needed to have a well-defined variational principle and the counter-terms which guarantee the finiteness of the action. I_{bdy} , for

the case of the zero curvature boundary which is our interest, can be written as

$$\begin{aligned}
I_{\text{bdy}} &= \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-h} \left\{ \Theta + 2\alpha J - \frac{(n-1)(l^2-2\tilde{\alpha})}{l^3} \right. \\
&\quad \left. + \frac{1}{2} f(e^{-4\lambda\Phi/(n-1)} U_\gamma U^\gamma) \right\} + I_{\text{deriv}},
\end{aligned} \tag{38}$$

where the boundary $\partial\mathcal{M}$ is the hypersurface at some constant r and, therefore, the Greek indices take the values 0 and $i = 2 \dots n$. In Eq. (38), h is the determinant of the induced metric $h_{\alpha\beta}$, Θ is the trace of the extrinsic curvature $\Theta_{\alpha\beta}$, and J is the trace of [19]

$$J_{\alpha\beta} = \frac{1}{3} (2\Theta\Theta_{\alpha\gamma}\Theta^\gamma_\beta + \Theta_{\gamma\delta}\Theta^{\gamma\delta}\Theta_{\alpha\beta} - 2\Theta_{\alpha\gamma}\Theta^{\gamma\delta}\Theta_{\delta\beta} - \Theta^2\Theta_{\alpha\beta}). \tag{39}$$

For our case with the flat boundary, I_{deriv} —which is a collection of terms involving derivatives of the boundary

fields—is zero. This is due to the fact that both the curvature tensor constructed from the boundary metric and covariant derivatives of U_α will not contribute to the on-shell value of the action for the pure Lifshitz solution or its first variation around the Lifshitz background. The boundary term for the matter part of the action in the absence of the dilaton has been introduced in Ref. [16]. Here, we generalize it to the case of the Lifshitz solutions in the presence of the dilaton field. For this case, we consider the matter part of the boundary term to be a function of $e^{-4\lambda\Phi/(n-1)}U_\gamma U^\gamma$ because it is constant on the boundary. One could find that $f(e^{-4\lambda\Phi/(n-1)}U_\gamma U^\gamma) = a(-e^{-4\lambda/(n-1)\Phi}U_\gamma U^\gamma)^{1/2}$ where $a = 4q/(lb^{n-1})$ [Note that q can be substituted by (24)]. The variation of the total action $I_{\text{tot}} = I_{\text{bulk}} + I_{\text{bdy}}$ about the solutions is

$$\delta I_{\text{tot}} = \int d^n x (S_{\alpha\beta} \delta h^{\alpha\beta} + S_\alpha^L \delta U^\alpha), \quad (40)$$

where

$$S_{\alpha\beta} = \frac{\sqrt{-h}}{16\pi} \left\{ \Pi_{\alpha\beta} - \frac{a}{2} e^{-2\lambda\Phi/(n-1)} (-U_\gamma U^\gamma)^{-1/2} \times (U_\alpha U_\beta - U_\gamma U^\gamma h_{\alpha\beta}) \right\}, \quad (41)$$

$$S_\beta^L = -\frac{\sqrt{-h}}{16\pi} \left\{ 4e^{-4\lambda\Phi/(n-1)} n^\alpha F_{\alpha\beta} + a e^{-2\lambda\Phi/(n-1)} (-U_\gamma U^\gamma)^{-1/2} U_\beta \right\}, \quad (42)$$

with

$$\Pi_{\alpha\beta} = \Theta_{\alpha\beta} - \Theta h_{\alpha\beta} + 2\alpha(3J_{\alpha\beta} - Jh_{\alpha\beta}) + \frac{(n-1)(l^2 - 2\tilde{\alpha})}{l^3} h_{\alpha\beta}. \quad (43)$$

In the Lifshitz background due to cancellation between different terms, $S_{\alpha\beta} = 0$ and $S_\beta^L = 0$, and therefore the total action satisfies $\delta I_{\text{tot}} = 0$ for arbitrary variations around the Lifshitz solution. Thus, we have a finite on-shell action which defines a well-defined variational principle for our background spacetime.

After constructing a well-defined finite action, one may compute the finite stress tensor. This job has been done for asymptotically AdS spacetimes which are dual to relativistic field theory [20,21]. For asymptotically Lifshitz spacetimes, the dual field theory is nonrelativistic and, therefore, its stress tensor will not be covariant. However, one can define a stress tensor complex [16], consisting of the energy density \mathcal{E} , energy flux \mathcal{E}_i , momentum density \mathcal{P}_i , and spatial stress tensor \mathcal{P}_{ij} ,

$$\mathcal{E} = 2S^t_t - S^L_t U_t, \quad \mathcal{E}^i = 2S^i_t - S^L_t U_t, \quad (44)$$

$$\mathcal{P}_i = -2S^t_i + S^L_t U_i, \quad \mathcal{P}^j_i = -2S^j_i + S^L_t U_i, \quad (45)$$

which satisfies the following conservation equations:

$$\partial_t \mathcal{E} + \partial_i \mathcal{E}^i = 0, \quad \partial_i \mathcal{P}_j + \partial_i \mathcal{P}^j_i = 0. \quad (46)$$

In Eqs. (44) and (45), the Latin indices (i, j) go from 2 to n and $S_{\alpha\beta}$ and S^L_α are given in Eqs. (41) and (42).

VI. THERMODYNAMICS OF LIFSHITZ BLACK BRANES

Now, we are ready to consider the thermodynamics of Lifshitz black brane solutions. The entropy in Gauss-Bonnet gravity can be calculated by using [22]

$$S = \frac{1}{4} \int d^{n-1} x \sqrt{\tilde{g}} (1 + 2\alpha \tilde{R}), \quad (47)$$

where \tilde{g} is the determinant of \tilde{g}_{ij} which is the induced metric of the $(n-1)$ -dimensional spacelike hypersurface of the Killing horizon. Since we are dealing with a flat horizon, $\tilde{R} = 0$ and, therefore, the entropy per unit volume is

$$S = \frac{r_0^{n-1}}{4}. \quad (48)$$

The temperature of the event horizon is given by

$$T = \frac{1}{2\pi} \left(-\frac{1}{2} \nabla_b \chi_a \nabla^b \chi^a \right)_{r=r_0}^{1/2}, \quad (49)$$

where $\chi = \partial_t$ is the Killing vector. Using (49) and the expansions of the metric functions near the event horizon given in Sec. IV, one can obtain the temperature as

$$T = \frac{r_0^{z+1}}{4\pi l^{z+1}} (f'g')_{r=r_0}^{1/2} = \frac{r_0^{z+1}}{4\pi l^{z+1}} \sqrt{f_1 g_1}. \quad (50)$$

The conserved quantities of our solution can be calculated through the use of the counterterm method of the previous section. The energy density of the black brane can be calculated by using Eq. (44) as

$$\begin{aligned} \mathcal{E} = & \frac{(n-1)\sqrt{f}(1-\sqrt{g})r^{z+n-1}}{8\pi l^{z+1}} \\ & + \frac{(n-1)\tilde{\alpha}\sqrt{f}(g^{3/2}-1)r^{z+n-1}}{12\pi l^{z+3}} \\ & + \frac{q^2 k (e^{-2\lambda\Phi/(n-1)} (\frac{l}{b})^{n-1} - 1)}{4\pi l^{z+1}}. \end{aligned} \quad (51)$$

Inserting the large r expansions given in the previous section for the metric function in the above equation, the energy density may be calculated as

$$\mathcal{E} = \frac{(n-1)(l^2 - 2\tilde{\alpha})[(z+n-2)(z+n-1)(l^2 - 2\tilde{\alpha}) + 2\tilde{\alpha}(z-1)^2]C_1}{16\pi G l^{z+3}}. \quad (52)$$

For $\alpha = 0$, \mathcal{E} reduces to

$$\mathcal{E} = \frac{(n-1)C_1}{16\pi l^{z+1}}, \quad (53)$$

which is the energy of the spacetime obtained in Ref. [14] as $\mathcal{E} = (n-1)m/16\pi l^{z+1}$, where m was the geometrical mass. This shows that in our solution, the constant C_1 is the geometrical mass too. It is remarkable to note that by using (45) one can calculate the angular momentum which is zero for our solution as one expected.

Now, using Eqs. (48)–(52), the constant \mathcal{C} can be written in terms of the thermodynamics quantities T , \mathcal{S} , and \mathcal{E} as

$$\mathcal{C} = 16\pi T\mathcal{S} = \frac{16\pi(n+z-1)\mathcal{E}}{n-1}.$$

Thus, one obtains

$$\mathcal{E} = \left(\frac{n-1}{n+z-1} \right) T\mathcal{S}. \quad (54)$$

Finally, we would like to perform thermal stability analysis in the case of asymptotically Lifshitz solution in dilaton Gauss-Bonnet gravity. Since our solution is uncharged, the positivity of the heat capacity $C = T/(dT/d\mathcal{S})$ is sufficient to ensure the local stability. In order to calculate the heat capacity, we first use the first law of thermodynamics $d\mathcal{E} = Td\mathcal{S}$ with the relation (54) for the energy density to obtain

$$\frac{dT}{d\mathcal{S}} = \frac{z}{n-1} \frac{T}{\mathcal{S}}. \quad (55)$$

Thus, the heat capacity can be obtained as

$$C = \frac{(n-1)}{z} \mathcal{S} = \frac{(n-1)r_0^{n-1}}{4z},$$

which is positive and therefore our black brane solution is thermally stable. Also, it is worth noting that the curve of $\log T$ versus $\log \mathcal{S}$ is a line with slope $z/(n-1)$,

$$\log T = \frac{z}{n-1} \log \mathcal{S} + \Gamma,$$

where Γ is an integration constant.

VII. CONCLUSION

In this paper, we considered asymptotic Lifshitz black branes of the effective supergravity action coming from superstrings, which contains a GB term and a dilaton field,

in the presence of a massless gauge field. Although it is known that the GB term is coupled to the dilaton, we considered them decoupled for simplicity. By variation of the action, we found four independent equations of motion. Then, we fixed the parameters of our model such that the asymptotic Lifshitz behavior is supported. Next, we obtained two independent constants along the radial coordinate by combining the equations of motion. We combined these two constants in order to get a constant \mathcal{C} which was proportional to the energy density. In addition, we calculated the value of this constant quantity at the horizon in terms of the thermodynamic quantities, temperature, and entropy. Also, using the large- r behaviors of the metric functions, we found the value of this constant at large r . Although there are two independent constants in our theory, we found out that only one will appear at infinity. This shows that our solution respects the no-hair theorem. In order to compute the finite stress energy tensor, we constructed the well-defined finite action. This action is the generalization of the action presented in the case where no dilaton field exists [16]. By calculating the value of the conserved quantity (energy density) in terms of the constant \mathcal{C} , we obtained the relation between energy density, temperature, and entropy. This relation is the generalization of the well-known Smarr formula for AdS black holes. Finally, we performed the thermal stability analysis on our solution and showed that our black brane solution is stable under thermal perturbations.

In this paper we studied the thermodynamics of uncharged dilaton Lifshitz black branes in the context of GB gravity where the GB term is decoupled from the dilaton. This work can be extended in various ways. First, one can consider the case where the GB term is coupled to the dilaton field. Second, one may seek the thermodynamics of the linearly and nonlinearly charged Lifshitz black branes of this theory. Another interesting case is the consideration for black holes that their horizons' geometries are not flat. A study of these solutions can also be extended to the case of the effective supergravity action coming from superstrings, which contains higher-curvature terms. We hope to address some of the above-mentioned suggestions in future works.

ACKNOWLEDGMENTS

We thank the Shiraz University Research Council. This work has been supported financially by the Research Institute for Astronomy and Astrophysics of Maragha, Iran.

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