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Lorentzian fuzzy spheres

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We show that fuzzy spheres are solutions of Lorentzian Ishibashi-Kawai-Kitazawa-Tsuchiya-type matrix models. The solutions serve as toy models of closed noncommutative cosmologies where big bang/crunch singularities appear only after taking the commutative limit. The commutative limit of these solutions corresponds to a sphere embedded in Minkowski space. This "sphere" has several novel features. The induced metric does not agree with the standard metric on the sphere, and, moreover, it does not have a fixed signature. The curvature computed from the induced metric is not constant, has singularities at fixed latitudes (not corresponding to the poles) and is negative. Perturbations are made about the solutions, and are shown to yield a scalar field theory on the sphere in the commutative limit. The scalar field can become tachyonic for a range of the parameters of the theory.

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I. INTRODUCTION

It is well known that fuzzy spheres and fuzzy coset spaces [1–8] are solutions to matrix models. More specifically, they are solutions to the bosonic sector of Euclidean Ishibashi, Kawai, Kitazawa and Tsuchiya (IKKT) matrix models [9]. The solutions have been applied in particle physics to make extra dimensions noncommutative [10]. Here we show that fuzzy spheres can also be solutions to IKKT-type matrix models with a Minkowski background metric tensor. This means that in addition to making extra dimensions noncommutative, fuzzy spheres, and more generally fuzzy coset spaces, can be used to make space-time noncommutative. Moreover, they can serve as toy models for noncommutative cosmological space-times.

Various aspects of Lorentzian IKKT matrix models have been discussed in the literature, including classical solutions and their implications for cosmology [11–15]. The solutions were generally written in terms of infinite-dimensional matrices, and they may or may not be associated with finite-dimensional Lie algebras. Unlike previous solutions to Lorentzian matrix models, fuzzy spheres are expressed in terms of $N \times N$ matrices, where N is finite. Upon taking $N \to \infty$, corresponding to the commutative limit, they lead to closed two-dimensional space-time cosmologies. Big bang/crunch singularities then appear in this limit, while the finite-dimensional matrix description is singularity free.

In Sec. II we write down a fuzzy sphere solution to a Lorentzian IKKT-type model. The model is written down specifically in three space-time dimensions and cubic and quadratic terms are added to the bosonic sector of the action of [9]. We show that the solution yields a closed (two-dimensional) universe in the commutative limit. While the commutative limit of the solution is topologically a two-sphere, there are a number of novel features, arising from the fact that it is embedded in a three-dimensional Minkowski space. The induced metric does not agree with the standard metric on the sphere, and, moreover, it does not have a fixed signature. The curvature computed from the induced metric is not constant, and it is negative. It is singular at two fixed latitudes (which are not located at the poles) and timelike geodesics originate and terminate at these latitudes. Thus in this toy model, the big bang/crunch singularities occur at nonzero spatial size.

We examine perturbations around the fuzzy sphere solution in Sec. III. In the commutative limit, the perturbations are described by a scalar field coupled to a gauge field. The latter can be eliminated, yielding a scalar field which can propagate in the Lorentzian region of the two-dimensional surface. Depending on the choice of parameters, the scalar field can be massive, massless or tachyonic.

Concluding remarks are given in Sec. IV.

II. FUZZY SPHERE SOLUTION TO A LORENTZIAN IKKT-TYPE MATRIX MODEL

The setting here is the bosonic sector of a Lorentzian IKKT-type matrix model in three space-time dimensions. The dynamical degrees of freedom for the matrix model are contained in three infinite-dimensional Hermitian matrices X^μ, $\mu = 0, 1, 2$, with $\mu = 0$ indicating a timelike direction. In addition to the standard Yang-Mills term, we include a cubic term and a quadratic term in the action (both of which are necessary for obtaining fuzzy sphere solutions),

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$$S(X) = \frac{1}{g^2}\mathrm{Tr}\left(-\frac{1}{4}[X_\mu, X_\nu][X^\mu, X^\nu] + \frac{2}{3}i\alpha\epsilon_{\mu\nu\lambda}X^\mu X^\nu X^\lambda + \frac{\beta}{2}X_\mu X^\mu\right), \quad (1)$$

where g, α and β are real coefficients. Our conventions are $\epsilon_{012} = 1$, and we raise and lower indices $\mu, \nu, \ldots$ with the flat metric $[\eta_{\mu\nu}] = \mathrm{diag}(-1, 1, 1)$. The resulting equations of motion are

$$[[X_\mu, X_\nu], X^\nu] + i\alpha\epsilon_{\mu\nu\lambda}[X^\nu, X^\lambda] = -\beta X_\mu. \quad (2)$$

The dynamics is invariant under three-dimensional Lorentz transformations, $X^\mu \to L^\mu{}_\nu X^\nu$, where L is a 3×3 Lorentz matrix, and unitary "gauge" transformations, $X^\mu \to UX^\mu U^\dagger$, where U is an infinite-dimensional unitary matrix. The equations of motion also have discrete symmetries, namely proper reflections. An example is

$$(X^0, X^1, X^2) \to (-X^0, X^1, -X^2). \quad (3)$$

Translation invariance in the three-dimensional Minkowski space is broken when $\beta \neq 0$.

When $\beta \neq 0$, there exist finite-dimensional matrix solutions to the equations of motion (2), which are associated with the $su(2)$ algebra in an N-dimensional representation. Say the latter is spanned by $N \times N$ Hermitian matrices $J_i, i = 1, 2, 3$, satisfying $[J_i, J_j] = i\alpha\epsilon_{ijk}J_k$.[1] Let us set

$$X^0 = \frac{w_3}{\alpha}J_3 \qquad X^1 = \frac{w_1}{\alpha}J_1 \qquad X^2 = \frac{w_2}{\alpha}J_2, \quad (4)$$

where w_i are real. Upon substituting this expression into the equations of motion one gets

$$\begin{aligned}(w_1^2 + w_2^2 + \beta)w_3 + 2\alpha w_1 w_2 &= 0\\ (w_2^2 - w_3^2 + \beta)w_1 - 2\alpha w_2 w_3 &= 0\\ (w_1^2 - w_3^2 + \beta)w_2 - 2\alpha w_1 w_3 &= 0, \end{aligned} \quad (5)$$

which has nontrivial solutions. Lorentz symmetry is in general broken by the solutions, unlike the case with de Sitter and anti–de Sitter solutions [14,15]. The $su(2)$ Casimir operator for any of the solutions can be written as $\frac{1}{w_3^2}(X^0)^2 + \frac{1}{w_1^2}(X^1)^2 + \frac{1}{w_2^2}(X^2)^2$, which has the value $\frac{1}{4}(N^2 - 1)$ in the N-dimensional representation, thereby defining a fuzzy sphere [or, actually, a fuzzy ellipsoid, since rotational invariance in the (X^0, X^1, X^2) space does not in general hold].

[1] The Levi-Civita symbol here is associated with Euclidean space, unlike the ones appearing in (1) and (2), which are associated with Minkowski space.

In the special case where $w_1^2 = w_2^2 = w_3^2$, the solution is invariant under the full three-dimensional rotation group (and not the Lorentz group). Let us more generally restrict to the case of rotational invariance in the (X^1, X^2) plane, which means $w_1^2 = w_2^2$. Two simple solutions exist in this case,

$$X^0 = 2J_3 \qquad X^1 = \frac{\sqrt{-\beta}}{\alpha}J_1 \qquad X^2 = -\frac{\sqrt{-\beta}}{\alpha}J_2, \quad (6)$$

and

$$X^0 = -2J_3 \qquad X^1 = \frac{\sqrt{-\beta}}{\alpha}J_1 \qquad X^2 = \frac{\sqrt{-\beta}}{\alpha}J_2. \quad (7)$$

Nontrivial solutions require the presence of both the cubic and quadratic terms in (1), $\alpha \neq 0$ and $\beta < 0$. Solutions (6) and (7) are equivalent due to the discrete symmetry (3). For the sake of definiteness we choose to work with the former, (6). The $su(2)$ Casimir operator for this solution can be written as

$$-\frac{\beta}{4\alpha^2}(X^0)^2 + (X^1)^2 + (X^2)^2, \quad (8)$$

having the value $-\frac{\beta}{4}(N^2 - 1)$ in the N-dimensional representation. The "time" matrix X^0 then has discrete eigenvalues $2\alpha m$, where $m = \frac{-N+1}{2}, \frac{-N+3}{2}, \ldots, \frac{N-1}{2}$. For any m defining a time slice we can also define a spatial size. Call A the "space" matrix, where $A^2 = X_+X_-$ and $X_\pm = X^1 \pm iX^2$. We can identify it with $-\frac{\beta}{\alpha^2}(\vec{J}^2 - J_3^2 - J_3)$ for the solution (6). A^2 then commutes with X^0 and has eigenvalues $-\beta(\frac{N^2-1}{4} - m^2 - m)$. Thus the time and the spatial size are discrete. Examples of spectra for (X^0, A^2) for some N-dimensional representations are

$$\begin{aligned} N = 2 &\quad (-\alpha, -\beta), (\alpha, 0)\\ N = 3 &\quad (-2\alpha, -2\beta), (\alpha, -\beta), (2\alpha, 0)\\ N = 4 &\quad (-3\alpha, -3\beta), (-\alpha, -4\beta), (\alpha, -3\beta), (3\alpha, 0)\\ N = 5 &\quad (-4\alpha, -4\beta), (-2\alpha, -6\beta), (\alpha, -6\beta),\\ &\qquad (2\alpha, -4\beta), (4\alpha, 0). \end{aligned} \quad (9)$$

Say $\alpha > 0$. Then, for large N, the spatial size operator A has eigenvalue $\sqrt{-\beta N}$ for the lowest time eigenvalue $\sim -\alpha N$, i.e., the initial state. It then increases to a maximum value of $\sqrt{-\beta}N/2$ as the time goes to zero, and then decreases to zero upon approaching the highest time eigenvalue $\sim\alpha N$, i.e., the final state. This solution can thus be regarded as a discrete analogue of a closed cosmological space-time. The eigenvalues of X^0 versus those of A are plotted for $N = 100$ in Fig. 1.

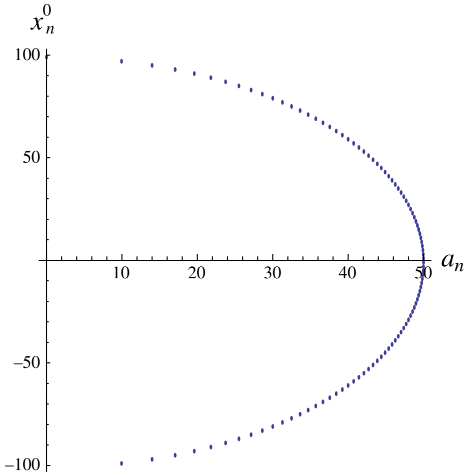

FIG. 1 (color online). Fuzzy closed universe solution. Plot of the eigenvalues x_n^0 of the time matrix X^0 versus the eigenvalues a_n of the space matrix A for $N = 100$, $\alpha = 1$ and $\beta = -1$.

Just as with the fuzzy sphere in a Euclidean background, the commutative limit of the matrix solution here is obtained by taking $N \to \infty$. Here we also need $\alpha, \beta \to 0$, with αN and $\sqrt{-\beta}N$ finite in the limit. The commutative limit of the solution is then characterized by two real parameters, which we denote by a_0 (not to be confused with an eigenvalue of A) and r^2,

$$\frac{\sqrt{-\beta}}{2\alpha} \to a_0 \qquad \frac{\sqrt{-\beta}N}{2} \to r. \tag{10}$$

One typically defines the commutative limit in an analogous fashion to the classical limit of a quantum theory, where α plays an analogous role to $\hbar$. In this limit one replaces the matrices X^μ by commuting space-time coordinates which we denote by x^μ, where x^0 and x^i, $i = 1, 2$, denote the time and space coordinates, respectively. The constraint on the $su(2)$ Casimir operator (8) means that in the commutative limit the solution satisfies

$$a_0^2(x^0)^2 + (x^1)^2 + (x^2)^2 = r^2. \tag{11}$$

While real a_0 means that the solution is topologically a two-sphere, there are a number of novel features, which we show below, due to the fact that this "sphere" is embedded in Minkowski space-time.

The commutative limit also requires replacing the commutator of functions of X^μ, evaluated for the solution (6), by $i\alpha$ times the Poisson bracket of the same functions of the coordinates x^μ. The commutators of X^μ lead to the following Poisson brackets of the coordinates:

$$\{x^0, x^1\} = -2x^2 \qquad \{x^2, x^0\} = -2x^1$$
$$\{x^1, x^2\} = -2a_0^2 x^0. \tag{12}$$

We can express x^μ in terms of angular momenta j_i, $i = 1, 2, 3$, which satisfies the $su(2)$ Poisson bracket algebra $\{j_i, j_j\} = \epsilon_{ijk} j_k$, using

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = 2 \begin{pmatrix} j_3 \\ a_0 j_1 \\ -a_0 j_2 \end{pmatrix}, \tag{13}$$

and from (11), $j_1^2 + j_2^2 + j_3^2 = (\frac{r}{2a_0})^2$. For simplicity, we set $r = 2a_0$ so that j_i spans a sphere of unit radius. We can introduce standard spherical coordinates (θ, ϕ), $0 < \theta < \pi, 0 \le \phi < 2\pi$, and write

$$j_1 = \sin\theta\cos\phi \qquad j_2 = \sin\theta\sin\phi \qquad j_3 = \cos\theta. \tag{14}$$

The $su(2)$ Poisson bracket algebra for j_i is recovered upon defining the Poisson brackets on the sphere to be

$$\{F, G\}(\theta, \phi) = \csc\theta(\partial_\theta F \partial_\phi G - \partial_\phi F \partial_\theta G), \tag{15}$$

for any two functions F and G on the sphere.

The induced metric $\mathfrak{g}_{ab} = \partial_a x^\mu \partial_b x_\mu$, $a, b, \ldots = \theta, \phi$, computed from (13) and (14) does not agree with the standard metric on the sphere, and, moreover, it does not have a fixed signature. Moreover, the curvature computed from the induced metric is not constant, and it is negative. The invariant interval constructed from the induced metric is

$$-d\tau^2 = 4(a_0^2\cos^2\theta - \sin^2\theta)d\theta^2 + 4a_0^2\sin^2\theta d\phi^2. \tag{16}$$

$\mathfrak{g}_{\theta\theta}$ vanishes at two latitudes $\theta = \theta_\pm$ on the sphere defined by $\tan\theta_\pm = \pm a_0$. Say that $\theta = \theta_+$ is contained in the northern hemisphere, $0 < \theta_+ < \frac{\pi}{2}$, while $\theta = \theta_-$ is contained in the southern hemisphere, $\frac{\pi}{2} < \theta_- < \pi$. The signature on the sphere is Euclidean for $0 < \theta < \theta_+$ and $\theta_- < \theta < \pi$, while it is Lorentzian for $\theta_+ < \theta < \theta_-$. We can regard θ as a timelike variable for the latter, with $2a_0 \sin\theta$ being the spatial radius at any time slice. $\theta = \theta_\pm$ correspond to singularities in the curvature, as opposed to coordinate singularities. The Ricci scalar computed from the induced metric is

$$R = -\frac{1}{2(a_0^2\cos\theta^2 - \sin^2\theta)^2}, \tag{17}$$

and thus it is singular at the latitudes $\theta = \theta_\pm$. Equation (17) shows that the curvature in the nonsingular regions is everywhere negative. The singularities of the Ricci tensor are analogous to big bang/crunch singularities, with the distinction that they occur at a nonzero spatial radius $2a_0 \sin\theta_\pm = \frac{2a_0^2}{\sqrt{a_0^2+1}}$. Timelike longitudinal geodesics exist in the Lorentzian region which originate and terminate at the singular latitudes $\theta = \theta_\pm$. This is because their tangent vectors $(\frac{d\theta}{d\tau}, \frac{d\phi}{d\tau}) = (\frac{1}{\sqrt{\sin^2\theta - a_0^2 \cos\theta^2}}, 0)$ are well defined in the Lorentzian region, $\theta_+ < \theta < \theta_-$, while they are imaginary in the Euclidean regions, $0 < \theta < \theta_+$ and $\theta_- < \theta < \pi$. The total elapsed proper time along these geodesics is finite and given by the elliptic integral $2\int_{\tan^{-1}a_0}^{\pi - \tan^{-1}a_0} d\theta\sqrt{\sin^2\theta - a_0^2\cos\theta^2}$.

III. EMERGENT FIELD DYNAMICS

Here we perturb around the matrix solution (6). Similar to [8], we find it useful to define noncommutative field strengths $F_{\mu\nu}$ on the fuzzy sphere. Here we take

$$F^{01} = \frac{1}{\alpha}[X^0, X^1] + 2iX^2$$
$$F^{02} = \frac{1}{\alpha}[X^0, X^2] - 2iX^1$$
$$F^{12} = \frac{1}{\alpha}[X^1, X^2] - \frac{i\beta}{2\alpha}X^0, \tag{18}$$

which transform covariantly under unitary gauge transformations, $F_{\mu\nu} \to UF_{\mu\nu}U^\dagger$, and vanish when evaluated on the fuzzy sphere solutions (6). The matrix action (1) can then be reexpressed in terms of the noncommutative field strengths

$$g^2 S(X) = \mathrm{Tr}\left\{-\frac{\alpha^2}{4}F_{\mu\nu}F^{\mu\nu} - \frac{4}{3}i\alpha^2(F^{01}X^2 + F^{20}X^1) + i\alpha^2\left(\frac{2}{3} - \frac{\beta}{2\alpha^2}\right)F^{12}X^0 + \left(\frac{\beta}{2} - \frac{2\alpha^2}{3}\right) \times \left((X^1)^2 + (X^2)^2\right) + \beta\left(\frac{\beta}{8\alpha^2} - \frac{5}{6}\right)(X^0)^2\right\}. \tag{19}$$

Now we perturb around the matrix solution (6) using

$$X^0 = 2\left(J_3 + \frac{\alpha^2}{\sqrt{-\beta}}A^0\right)$$
$$X^1 = \frac{\sqrt{-\beta}}{\alpha}J_1 + \alpha A^1$$
$$X^2 = -\frac{\sqrt{-\beta}}{\alpha}J_2 - \alpha A^2, \tag{20}$$

where the perturbations are functions on the fuzzy sphere, $A^\mu = A^\mu(J_1, J_2, J_3)$. If we write infinitesimal unitary gauge transformations using $U = \mathbb{1} - \frac{i\alpha}{\sqrt{-\beta}}\Lambda$, where Λ is a Hermitian matrix with infinitesimal elements, then the infinitesimal variations of A^μ read

$$\delta A^\mu = -i\left(\frac{1}{\alpha}[\Lambda, J^\mu] + \frac{\alpha}{\sqrt{-\beta}}[\Lambda, A^\mu]\right), \tag{21}$$

where we identify (J^0, J^1, J^2) with (J_3, J_1, J_2). Substituting (20) into (19) gives

$$S(X) = \frac{\alpha^2}{g^2}\mathrm{Tr}\left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{4}{3}i\alpha(F^{01}A^2 + F^{20}A^1) + \frac{2i\alpha^2}{\sqrt{-\beta}}\left(\frac{2}{3} - \frac{\beta}{2\alpha^2}\right)F^{12}A^0 + \frac{8i\alpha}{3}([J_1, A^2] - [J_2, A^1])A^0 - 2i\alpha\left(\frac{2}{3} - \frac{\beta}{2\alpha^2}\right)[A^1, A^2]J_3 + \left(\frac{\beta}{2} - \frac{2\alpha^2}{3}\right)\left((A^1)^2 + (A^2)^2\right) - 2\alpha^2\left(\frac{\beta}{4\alpha^2} - \frac{5}{3}\right)(A^0)^2\right\} + S(X|_{\mathrm{solution}}). \tag{22}$$

As stated previously, the commutative limit is obtained by taking $N \to \infty$, along with $\alpha, \beta \to 0$ and both αN and $\sqrt{-\beta}N$ are finite in the limit. Upon using (10) and (13), the commutative limit of the field strengths (18) is

$$F^{01} \to 2i\alpha(\{j_3, A^1\} - \{j_1, A^0\} - A^2)$$
$$F^{02} \to -2i\alpha(\{j_3, A^2\} - \{j_2, A^0\} + A^1)$$
$$F^{12} \to -2i\alpha a_0(\{j_1, A^2\} - \{j_2, A^1\} - A^0), \tag{23}$$

where A^μ are now functions on the commutative sphere. The trace on functions of the fuzzy sphere is replaced by the corresponding integration on the sphere in the commutative limit. The relevant integration measure $d\mu(\theta, \phi)$ should be such that the standard trace identities survive in the limit; i.e., for any three functions G, H and K on the sphere we want $\int d\mu(\theta,\phi)\{G, H\}K = \int d\mu(\theta,\phi)G\{H, K\}$. From (15) we need to choose the standard integration measure on the sphere $d\mu(\theta,\phi) = \sin\theta d\theta d\phi$ (rather than, say, $\sqrt{-\mathfrak{g}}d\theta d\phi$, where $\mathfrak{g}$ is the determinant of the induced metric). Then the action (22) reduces to

$$S(X) - S(X|_{\mathrm{solution}}) \to \frac{2\alpha^4}{g_c^2}\int \sin\theta d\theta d\phi\{-(\{j_3, A^1\} - \{j_1, A^0\})^2 - (\{j_3, A^2\} - \{j_2, A^0\})^2 + a_0^2(\{j_1, A^2\} - \{j_2, A^1\})^2 + 2(a_0^2 + 1)\{j_3, A^1\}A^2 + (A^0)^2 - a_0^2((A^1)^2 + (A^2)^2)\}, \tag{24}$$

where g_c is the commutative limit of the constant g. Following [8] we write the perturbations A^μ in terms of commutative gauge potentials $(\mathcal{A}_\theta, \mathcal{A}_\phi)$ and a scalar field ψ on the sphere using

$$\begin{aligned} A^0 &= \mathcal{A}_\phi + j_3\psi \\ A^1 &= -\sin\phi\mathcal{A}_\theta - \cot\theta\cos\phi\mathcal{A}_\phi + j_1\psi \\ A^2 &= \cos\phi\mathcal{A}_\theta - \cot\theta\sin\phi\mathcal{A}_\phi + j_2\psi. \end{aligned} \tag{25}$$

Then from the fundamental Poisson bracket (15), gauge variations $(\delta\mathcal{A}_\theta, \delta\mathcal{A}_\phi) = (\partial_\theta\Lambda, \partial_\phi\Lambda)$ agree with the commutative limit of (21), where Λ is now an infinitesimal function on the commutative sphere. Substituting (25) in (24) gives

$$\begin{aligned} S(X) - S(X|_{\text{solution}}) \to \frac{2\alpha^4}{g_c^2}\int \sin\theta d\theta d\phi\{&(a_0^2\cot^2\theta - 1)\mathcal{F}_{\theta\phi}^2 - \csc^2\theta(\partial_\phi\psi)^2 \\ &+ (a_0^2\sin^2\theta - \cos^2\theta)(\partial_\theta\psi)^2 - (3 - 2(a_0^2+1)\sin^2\theta)\psi^2 \\ &+ 2\csc\theta((a_0^2+1)\sin^2\theta - 2a_0^2 + 1)\mathcal{F}_{\theta\phi}\psi - 2\cos\theta(a_0^2+1)\mathcal{F}_{\theta\phi}\partial_\theta\psi\}, \end{aligned} \tag{26}$$

where $\mathcal{F}_{\theta\phi} = \partial_\theta\mathcal{A}_\phi - \partial_\phi\mathcal{A}_\theta$ is the commutative $U(1)$ field strength on the surface. We remark that the gauge-field and scalar-field kinetic energies have opposite signs, a feature that was present in similar two-dimensional systems [15]. However, gauge fields are nondynamical in two dimensions. We can solve for $\mathcal{F}_{\theta\phi}$ from the field equations, yielding

$$\mathcal{F}_{\theta\phi} = \frac{\cos\theta(a_0^2+1)\partial_\theta\psi - ((a_0^2+1)\sin^2\theta - 2a_0^2 + 1)\csc\theta\psi}{a_0^2\cot^2\theta - 1} + \text{const}, \tag{27}$$

and substitute back into the action. Upon setting the constant equal to zero, we get

$$\begin{aligned} S(X) - S(X|_{\text{solution}}) &\to \frac{2\alpha^4 a_0^2}{g_c^2}\int \sin\theta d\theta d\phi\left\{\frac{(\partial_\theta\psi)^2}{(a_0^2+1)\sin^2\theta - a_0^2} - \frac{\csc^2\theta}{a_0^2}(\partial_\phi\psi)^2 - 4m_{\text{eff}}^2\psi^2\right\} \\ &= \frac{16\alpha^4 a_0^2}{g_c^2}\int \sin\theta d\theta d\phi\left\{-\frac{1}{2}\partial^a\psi\partial_a\psi - \frac{1}{2}m_{\text{eff}}^2\psi^2\right\}, \end{aligned} \tag{28}$$

where the index $a = (\theta, \phi)$ is raised and lowered using the induced metric given in (16). The effective mass squared of the scalar field is θ dependent,

$$m_{\text{eff}}^2 = \frac{(a_0^2 - 1)((a_0^2+1)\sin^2\theta - 3a_0^2)}{4a_0^2((a_0^2+1)\sin^2\theta - a_0^2)^2}. \tag{29}$$

As stated before, the signature of the induced metric is Euclidean when $\sin^2\theta < \frac{a_0^2}{a_0^2+1}$, and Lorentzian when $\sin^2\theta > \frac{a_0^2}{a_0^2+1}$. Therefore (28) describes a Euclidean field theory for the former and a Lorentzian field theory for the latter. There are three different possibilities for the Lorentzian field theory:

(i) The action describes a tachyon when $a_0^2 > 1$. This is because the factor $(a_0^2+1)\sin^2\theta - 3a_0^2$ in (29) is negative in this case.

(ii) The scalar field is massless when $a_0^2 = 1$.

(iii) The effective mass squared for the scalar field is positive when

$$a_0^2 < 1 \quad \text{and} \quad \frac{a_0^2}{a_0^2+1} < \sin^2\theta < \frac{3a_0^2}{a_0^2+1}. \tag{30}$$

It follows that the action (28) describes a massive scalar field throughout the entire Lorentzian region when $\frac{1}{2} \le a_0^2 < 1$. On the other hand, when $a_0^2 < \frac{1}{2}$ the scalar field becomes tachyonic in the region where $\sin^2\theta > \frac{3a_0^2}{a_0^2+1}$.

IV. CONCLUDING REMARKS

We found fuzzy sphere solutions to a Lorentzian IKKT-type model which provide toy models of a noncommutative two-dimensional closed universe, where time and spatial size have discrete values. Singularities in the Ricci tensor appear in the large N (i.e., commutative) limit. They are analogous to big bang/crunch singularities, with the novel feature that they occur at nonzero spatial size. Perturbations around the fuzzy sphere solution are described by a scalar field in the commutative limit which can propagate in the Lorentzian region of the manifold. (Additional field degrees of freedom would appear if one started with a ten-dimensional matrix model.) The scalar field can be massive, massless or tachyonic, the choice depending on the parameter a_0^2 (and also on the range of θ when $a_0^2 < \frac{1}{2}$). For $\frac{1}{2} \leq a_0^2 < 1$ the scalar field is always massive, ensuring the stability of the commutative field theory in this case. Corrections to the commutative limit are obtained by expressing the matrix product in the action (22) in terms of the star product on the sphere [4–7] and keeping the next order terms in the $1/N$ expansion.

For a more realistic model of a noncommutative cosmological space-time, one can look for fuzzy coset space solutions to Lorentzian IKKT-type matrix models associated with dimension $d > 2$ [7]. One possible example worth consideration is the fuzzy analogue of the four-dimensional coset $SU(3)/U(2)$ or CP^2. For coset spaces with $d > 4$ one may be able to make both four-dimensional space-time and extra dimensions noncommutative. Just as with the example of the fuzzy sphere, the commutative limit may lead to a manifold divided up into regions with different signatures of the metric. Perturbations about such solutions are expected to be described by a coupled gauge-scalar theory in the commutative limit. A common feature of the emergent field theories in previous examples [15] is that scalar-field and gauge-field kinetic energies can appear with opposite signs, which is also seen in (26). This sign discrepancy was harmless for $d = 2$, since the gauge field could be eliminated. On the other hand, it is of concern for $d > 2$, so it would be interesting to see if this discrepancy can be cured upon taking the commutative limit of higher-dimensional fuzzy coset space solutions.

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