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Thermodynamics of rotating thin shells in the BTZ spacetime

José P. S. Lemos, Francisco J. Lopes, Masato Minamitsuji, and Jorge V. Rocha
Centro Multidisciplinar de Astrofísica - CENTRA, Departamento de Física, Instituto Superior Técnico - IST, Universidade de Lisboa - UL, Avenida Rovisco Pais 1 1049-001, Portugal
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We investigate the thermodynamic equilibrium states of a rotating thin shell, i.e., a ring, in a $(2+1)$-dimensional spacetime with a negative cosmological constant. The inner and outer regions with respect to the shell are given by the vacuum anti-de Sitter and the rotating Bañados-Teitelbom-Zanelli spacetimes, respectively. The first law of thermodynamics on the thin shell, together with three equations of state for the pressure, the local inverse temperature and the thermodynamic angular velocity of the shell, yields the entropy of the shell, which is shown to depend only on its gravitational radii. When the shell is pushed to its own gravitational radius and its temperature is taken to be the Hawking temperature of the corresponding black hole, the entropy of the shell coincides with the Bekenstein-Hawking entropy. In addition, we consider simple *ansätze* for the equations of state, as well as a power-law equation of state where the entropy and the thermodynamic stability conditions can be examined analytically.

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I. INTRODUCTION

The origin of the Bekenstein-Hawking entropy of a black hole [1–3] is one of the greatest mysteries in modern gravitational physics. The Bekenstein-Hawking entropy is a measure of how many Planck areas there are on the event horizon, indicating that the black hole entropy has its roots in quantum gravity. However, so far there has been no fully satisfactory formulation of a quantum gravity theory and the origin of black hole entropy still remains an open question. An initial development explained the black hole and its features as the Euclideanized version of its geometry with quantum gravity making its appearance through the Euclidean functional integral of the geometry identified essentially as the partition function for hot gravity [4]. Further advancements of the path integral formalism for black holes, now put in the canonical and grand canonical ensemble formalism, for Schwarzchild and Reissner-Nordström black holes, respectively, were made by York and followers [5,6]. Schwarzchild and Reissner-Nordström black holes in spacetimes with negative cosmological constant, i.e., anti-de Sitter (AdS) spacetimes, can also have a correlated treatment [7,8].

Since black holes are vacuum solutions of the gravitational field, while our naive concepts of entropy are based on quantum properties of matter, it would be useful to study the thermodynamic properties of collapsing matter, namely whether black hole thermodynamics could emerge, or not, when we compress matter within its own gravitational radius. By taking this approach, one expects to obtain indications for how the Bekenstein-Hawking entropy springs from the final state of gravitational collapse [9]. Time-dependent collapsing solutions are difficult to follow analytically and instead one might try a sequence of static, or quasistatic, solutions up to the gravitational radius of a given configuration. As first shown in [10], by using such an approach the black hole entropy can be recovered.

Therefore it is of great interest to analyze self-gravitating matter systems, that possess both gravitational and matter degrees of freedom, and study their thermodynamics. One of the simplest such systems is an infinitesimally thin shell where the self-gravitating matter is confined, placed in an otherwise vacuum spacetime [11]. The distribution of matter on the shell fixes the extrinsic curvature, and hence the spacetime geometry outside the shell, via the junction conditions. Once the setup is given, one can check under which conditions the thin shell can be pushed quasistatically to the horizon radius.

The simplest example of a thin shell spacetime assumes staticity and spherical symmetry. Naturally, the first study on structure and thermodynamics of thin shells considered the inner spacetime to be Minkowski and the outer spacetime to be Schwarzschild [12]. By fixing the surface energy density and pressure through the junction conditions, imposing that the shell has a given local temperature and using a canonical ensemble, Martinez [12] determined the thermodynamic properties of the shell, characterized by its rest mass and radius. However, the gravitational radius, and thus the black hole limit, was never taken. This approach draws in many respects from York's work [5] where the thermodynamic properties of a pure Schwarzschild black hole were treated using a canonical ensemble, i.e., imposing a fixed temperature on some fictitious massless shell at a definite radius outside the event horizon.

The ensuing nontrivial extension of [12] was performed for an electrically charged shell [13]. In that context the inner and outer regions are given by the $(3+1)$-dimensional Minkowski and Reissner-Nortström spacetimes, respectively. The thermodynamic properties of the charged shell were characterized by the surface energy density, the pressure, the electric potential, the temperature, and the entropy. By taking the shell to its own gravitational radius, and requiring the temperature of the shell to be given by the black hole Hawking temperature, the authors of [13] found that the entropy reproduced the Bekenstein-Hawking formula.

Now, these developments appeared in static $(3+1)$-dimensional spacetimes. It is important to study other dimensions, lower and higher, as well as rotating situations. Rotating configurations with shells in $3+1$ dimensions are hopeless, as the Kerr metric defies being the metric of any reasonable rotating matter source. For small rotation, however, the matching problem has solution [14] (see also [15]). In higher dimensions, it is possible to find shell solutions in some rotating odd-dimensional spacetimes [16,17], and thus one could try to perform a thermodynamics analysis in them. On the other hand, in one lower dimension, in $2+1$ dimensions, the dynamics and thermodynamics of a thin shell is ready for such an incursion.

Indeed, in $(2+1)$-dimensional spacetimes there is the pure black hole solution, the Bañados-Teitelboim-Zanelli (BTZ) spacetime [18] that inhabits an AdS background. It is a solution that reflects in some way the rotating properties of the rotating $3+1$ Kerr solution in an AdS background. Moreover, the BTZ black hole can be formed via gravitational collapse of matter shells in a $(2+1)$-dimensional spacetime reproducing anew what happens in the $3+1$ world. For instance, thin shell, i.e., thin ring, collapse to a BTZ black hole has been studied in [19–21], and spinning string dynamics in a $(2+1)$-dimensional AdS background has also been shown to give rise to a rotating BTZ black hole [22]. Mechanical properties of a stationary shell in a BTZ background and its quasistatic collapse up to its own gravitational radius were displayed in [23]. On the other hand, the termodynamics properties of the BTZ black hole have been studied thoroughly in [24–26] (see also [18]). Thus, in this context, it is of interest to study the thermodynamics of self-gravitating thin shells. The approach of [12,13] can be applied to AdS spacetimes in $2+1$ dimensions [27,28]. In particular, the authors of [28] studied a thin shell in a static BTZ spacetime, where the inner and outer regions with respect to the shell were given by the $(2+1)$-dimensional AdS and BTZ spacetimes, respectively. The Bekenstein-Hawking entropy formula was recovered in the limiting case when the shell sat at its own gravitational radius, provided the shell's temperature coincided with the temperature of the corresponding BTZ black hole.

Having these previous studies in mind, in the present work we will investigate the thermodynamic properties of a rotating thin shell in a $(2+1)$-dimensional AdS background. In the setup we will consider, the interior and exterior of the shell are described by the static vacuum AdS spacetime in $(2+1)$ dimensions and the rotating BTZ solution, respectively. It is known that the BTZ black hole also has the Bekenstein-Hawking entropy $S = \frac{A_+}{4G}$, where $A_+ = 2\pi r_+$ is the circumference of the outer horizon (we set the Planck constant, the Boltzmann constant, and the velocity of light to unity). Our first goal is to see explicitly how adding rotation can generalize the thermodynamics on the thin shell obtained in the static case [27,28]. Our second goal is to compare the thermodynamic properties of a rotating thin shell to those obtained in [13] for the charged (nonrotating) case. It is well known that there are similarities between charged and rotating black holes, e.g., the presence of two gravitational radii, the outer and inner horizons. Our analysis will reveal substantial similarities between charged and rotating thin shells, irrespective of the dimensionality and asymptotic structure of the spacetime.

This paper is organized as follows. In Sec. II, we study the mechanical properties of a rotating thin shell in a $(2+1)$-dimensional spacetime with a BTZ exterior. In Sec. III, taking the locally measured proper mass, the circumference, and the angular momentum as the independent thermodynamic variables on the shell, we consider the first law of thermodynamics and determine the thermodynamic equations of state, as well as the entropy. Section IV is devoted to the study of the most meaningful equations of state. This is where we consider the black hole limit for rotating thin shells and show that the Bekenstein-Hawking entropy is recovered if the intrinsic temperature of the shell is equal to the Hawking temperature. In Sec. V other simple equations of state for the temperature and the electric potential inspired in the black hole case are devised. In Sec. VI, we study thermodynamic properties of the rotating thin shell for power-law equations of state, for which exact expressions for the entropy and stability conditions are provided. Finally, we conclude in Sec. VII. In the Appendix we give the BTZ black hole thermodynamic properties.

II. TIMELIKE THIN SHELLS IN THE $(2+1)$-DIMENSIONAL SPACETIME

A. The outer and inner spacetimes

We consider Einstein gravity with a cosmological constant in a $(2+1)$-dimensional spacetime. The Einstein equations are given by (the velocity of light is set to one)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where Greek indices $\mu, \nu = 0, 1, 2$ run over time and spatial coordinates. Here, $g_{\mu\nu}$ denotes the metric tensor, $G_{\mu\nu}$ represents its corresponding Einstein tensor, and $T_{\mu\nu}$ is the energy-momentum tensor for matter in the $(2+1)$-dimensional spacetime. G and Λ are the gravitational and cosmological constants, respectively, and we assume that $\Lambda < 0$, so that the spacetime is asymptotically AdS, with curvature scale

$$\ell = \sqrt{-\frac{1}{\Lambda}}. \tag{2}$$

Next we introduce a timelike shell, i.e., a ring in the $(2+1)$-dimensional spacetime, with radius R, which divides the spacetime into the outer and inner regions denoted by $M_{(o)}$ and $M_{(i)}$, respectively. We also assume that off the shell the spacetime is vacuum and hence $T_{\mu\nu}=0$ everywhere except at the location of the shell. The spacetime outside the shell ($r>R$) is described by the rotating BTZ solution, whose metric is given by [18]

$$ds^2_{(o)} = -\frac{(r^2-r_+^2)(r^2-r_-^2)}{\ell^2 r^2}dt^2_{(o)} + \frac{\ell^2 r^2}{(r^2-r_+^2)(r^2-r_-^2)}dr^2 + r^2\left(d\phi - \frac{r_+ r_-}{\ell r^2}dt_{(o)}\right)^2, \qquad r>R, \tag{3}$$

where $t_{(o)}$ is the outer time coordinate and (r,ϕ) are the radial and azimuthal coordinates. The two gravitational radii r_+ and r_- are related to the spacetime Arnowitt-Deser-Misner (ADM) mass m and the angular momentum $\mathcal{J}$, respectively, by

$$r_+^2 + r_-^2 = 8G\ell^2 m, \tag{4}$$

$$r_+ r_- = 4G\ell\mathcal{J}. \tag{5}$$

We assume

$$r_+ \geq r_-, \tag{6}$$

which then means

$$m \geq \frac{\mathcal{J}}{\ell}. \tag{7}$$

The two inequalities are saturated in the extremal case, $r_+ = r_-$, i.e., $m=\frac{\mathcal{J}}{\ell}$. We assume that the shell's character is always timelike and the shell is located outside the event horizon,

$$R > r_+. \tag{8}$$

Therefore, the outer region does not contain neither of the horizons $r=r_\pm$ nor the singularity $r=0$. The spacetime inside the shell ($r<R$) is pure AdS. It is the vacuum solution ($m=0$ and $\mathcal{J}=0$) of the outer metric, i.e.,

$$ds^2_{(i)} = -\frac{r^2}{\ell^2}dt^2_{(i)} + \frac{\ell^2}{r^2}dr^2 + r^2 d\phi^2, \qquad r<R, \tag{9}$$

where $t_{(i)}$ is the inner time coordinate, which may differ from the outer time coordinate $t_{(o)}$. Concerning the spatial coordinates (r,ϕ), we have not distinguished them from those in the outer region. But, as argued below, in order to match these two metrics smoothly across the timelike hypersurface, the shell must corotate with the outer spacetime. Consequently, an angular coordinate ψ should be introduced instead of ϕ, such that exterior spacetime in the new coordinate system $(t_{(o)}, r, \psi)$ is corotating with the shell. The entire spacetime is vacuum except at the thin shell, i.e., we are in the presence of a codimension-one distributional source at $r=R$.

The metrics (3) and (9) can be collectively expressed by

$$ds^2_{(I)} = -f_{(I)}(r)^2 dt_I^2 + g_{(I)}(r)^2 dr^2 + r^2(d\phi + h_{(I)}(r)dt_{(I)})^2, \tag{10}$$

where $I=o/i$ refers to either the outer or inner region with respect to the shell, and from (3) and (9) the functions $f_{(I)}$, $g_{(I)}$, and $h_{(I)}$ read

$$f_{(o)}(r) = \frac{\sqrt{(r^2-r_+^2)(r^2-r_-^2)}}{\ell r}, \quad g_{(o)}(r) = \frac{1}{f_{(o)}(r)},$$
$$h_o(r) = -\frac{r_+ r_-}{\ell r^2},$$
$$f_{(i)}(r) = \frac{r}{\ell}, \quad g_{(i)}(r) = \frac{1}{f_{(i)}(r)}, \quad h_{(i)}(r) = 0. \tag{11}$$

B. Junction conditions and the shell spacetime

With the outer and inner regions separated by the shell, the shell dynamics is determined by the Israel junction conditions [11]. The induced metrics on the shell, from the outer and inner regions, which we denote generically by $h^{(I)}_{ab}$, are given by

$$h^{(I)}_{ab} = e^\mu_{(I)a} e^\nu_{(I)b} g^{(I)}_{\mu\nu}, \tag{12}$$

where $e^\mu_{(I)a}$ is the projection tensor to the shell, each viewed from the (I) side. Then the first junction condition requires

$$[h_{ab}] = 0, \tag{13}$$

where $[F] = F_{(o)} - F_{(i)}$ represents the jump of a physical quantity F across the shell. Equation (13) ensures the uniqueness of the induced geometry on the shell $h^{(o)}_{ab} = h^{(i)}_{ab}$. Thus, on the shell one can define a metic h_{ab} such that

$$h_{ab} = h^{(o)}_{ab} = h^{(i)}_{ab}. \tag{14}$$

The second junction condition is given by

$$[K_{ab}] - h_{ab}[K] = -8\pi G S_{ab}, \tag{15}$$

where

$$K^{(I)}_{ab} = e^\mu_{(I)a} e^\nu_{(I)b} \nabla^{(I)}_{(\mu} n^{(I)}_{\nu)} \tag{16}$$

is the extrinsic curvature tensor defined on the shell's hypersurface, $\nabla^{(I)}_\mu$ is the covariant derivative with respect to the $(2+1)$-dimensional metric in region (I), and $n^{(I)}_\mu$ is the unit normal vector viewed from the (I) side. Moreover, $K = h^{ab}K_{ab}$ is the trace of the extrinsic curvature, and S_{ab} represents the energy-momentum tensor of matter on the thin shell, i.e., on the $(1+1)$-dimensional spacetime intrinsic to the shell. Equation (15) determines how the jump to the geometry exterior to the shell is generated by the matter on the shell.

We now apply these junction conditions to our problem. As the outer spacetime is rotating while the inner spacetime is static, in order to match these two regions, the shell at $r = R$ must corotate with the outer BTZ region [23] (see also [14]). For this purpose, we introduce a coordinate system corotating with the shell by adopting a new angular coordinate $d\psi$ such that

$$d\psi = d\phi + h_{(I)}(R)dt_{(I)}. \tag{17}$$

The line element given in Eq. (10) is then written as

$$ds^2_{(I)} = -f_{(I)}(r)^2dt^2_{(I)} + g_{(I)}(r)^2dr^2 + r^2(d\psi + \bar{h}_{(I)}(r)dt_{(I)})^2, \tag{18}$$

where we have introduced

$$\bar{h}_{(I)}(r) = h_{(I)}(r) - h_{(I)}(R). \tag{19}$$

At the position of the shell $\bar{h}_{(I)}(R) = 0$ and the effects of the spacetime rotation are hidden at the level of the induced geometry. The induced line element on the shell, which is uniquely determined by (14), is given by

$$ds^2_{\Sigma} = h_{ab}dy^a dy^b = -d\tau^2 + R^2 d\psi^2, \quad R = R(\tau), \tag{20}$$

where Latin indices $a, b = 0, 1$ run over the $(1+1)$-dimensional shell's coordinates, $y^a = (\tau, \psi)$. The proper time on the shell τ is defined by

$$\begin{aligned} d\tau &= \sqrt{f_{(o)}(R)^2dt^2_{(o)} - g_{(o)}(R)^2dR^2} \\ &= \sqrt{f_{(i)}(R)^2dt^2_{(i)} - g_{(i)}(R)^2dR^2}, \end{aligned} \tag{21}$$

which also fixes the relation between the outer and inner time coordinates, $t_{(o)}$ and $t_{(i)}$. The nonvanishing components of the projection tensor on the shell, $e^{\mu}_{(I)a}$, viewed from side (I), are given by

$$e^t_{(I)\tau} = \dot{t}_{(I)} = \frac{1}{f_{(I)}(R)}\sqrt{1 + g_{(I)}(R)^2\dot{R}^2}, \quad e^r_{(I)\tau} = \dot{R}, \quad e^{\psi}_{(I)\psi} = 1, \tag{22}$$

where an overdot represents a derivative with respect to proper time τ. The unit normal vector to the shell viewed from side (I), $n^{(I)}_{\mu}$, is

$$n^{(I)}_{\mu} = \left(-\dot{R}f_{(I)}(R)g_{(I)}(R), g_{(I)}\sqrt{1 + g_{(I)}(R)^2\dot{R}^2}, 0\right), \tag{23}$$

and obeys $n^{\mu}_{(I)}n^{(I)}_{\mu} = 1$. Since we are interested in the quasistatic process, in the rest of this paper we assume that $\dot{R} = \ddot{R} = 0$. In this case the components of the extrinsic curvature tensor are given by

$$\begin{aligned} K_{(I)}{}^{\tau}{}_{\tau} &= \frac{f'_{(I)}(R)}{f_{(I)}(R)g_{(I)}(R)}, \qquad K_{(I)}{}^{\psi}{}_{\psi} = \frac{1}{g_{(I)}(R)R}, \\ K_{(I)}{}^{\tau}{}_{\psi} &= -\frac{R^2h'_{(I)}(R)}{2f_{(I)}(R)g_{(I)}(R)}, \end{aligned} \tag{24}$$

where a prime means a derivative with respect to r. For the spacetimes we consider in this work we can use $f_{(I)}g_{(I)} = 1$ to simplify these expressions. The components of the extrinsic curvature tensor for the outer region (o) are given by

$$\begin{aligned} K_{(o)}{}^{\tau}{}_{\tau} &= \frac{R^4 - r_+^2r_-^2}{\ell R^2\sqrt{(R^2 - r_+^2)(R^2 - r_-^2)}}, \\ K_{(o)}{}^{\psi}{}_{\psi} &= \frac{\sqrt{(R^2 - r_+^2)(R^2 - r_-^2)}}{\ell R^2}, \qquad K_{(o)}{}^{\tau}{}_{\psi} = -\frac{r_+r_-}{\ell R}. \end{aligned} \tag{25}$$

Similarly, those for the inner region (i) are given by

$$K_{(i)}{}^{\tau}{}_{\tau} = \frac{1}{\ell}, \qquad K_{(i)}{}^{\psi}{}_{\psi} = \frac{1}{\ell}, \qquad K_{(i)}{}^{\tau}{}_{\psi} = 0. \tag{26}$$

At this point, we have computed all the quantities necessary to obtain the energy-momentum tensor of matter on the shell.

C. The energy-momentum tensor on the shell

We denote the nonzero components of the energy-momentum tensor of matter on the shell by

$$S^{\tau}{}_{\tau} = -\sigma, \qquad S^{\psi}{}_{\psi} = p, \qquad S^{\tau}{}_{\psi} = j, \tag{27}$$

where σ, p, and j represent the energy density, the pressure, and the angular momentum density of the shell, respectively. The second junction condition (15) plus Eqs. (25)–(26) determine the form of these components

$$\sigma = \frac{1}{8\pi G\ell}\left(1 - \frac{1}{R^2}\sqrt{(R^2 - r_+^2)(R^2 - r_-^2)}\right), \tag{28}$$

$$p = \frac{1}{8\pi G\ell}\left(\frac{R^4 - r_+^2r_-^2}{R^2\sqrt{(R^2 - r_+^2)(R^2 - r_-^2)}} - 1\right), \tag{29}$$

$$j = \frac{r_+r_-}{8\pi G\ell R}. \tag{30}$$

In the static, nonrotating, limit, $j = 0$, and so $r_- = 0$, we recover from Eqs. (28) and (29) the result of [28], $\sigma = \frac{1}{8\pi G R}(\frac{R}{\ell} - \sqrt{\frac{R^2}{\ell^2} - 8Gm})$ and $p = \frac{1}{8\pi G}\frac{R}{\ell^2}(\frac{1}{\sqrt{\frac{R^2}{\ell^2} - 8Gm}} - \frac{1}{\frac{R}{\ell}})$, where we made the replacement $r_+^2 \to 8G\ell^2 m$, arising from Eq. (4). In the presence of generic rotation, the energy-momentum tensor given by Eqs. (28)–(30) describes an imperfect fluid; see [21,23] (see also [16]). The surface energy density σ and pressure p are non-negative, and satisfy $p \geq \sigma$. Therefore, the matter shell generically obeys the weak energy condition. However, the dominant energy condition is violated, except in the extremal limit $r_+ = r_-$, in which case the inequality is saturated [23].

Defining the locally measured proper mass as $M = 2\pi R\sigma$ and the angular momentum of the shell as $J = 2\pi R j$ and using Eqs. (28)–(30) we obtain

$$M = 2\pi R\sigma = \frac{R}{4G\ell}\left(1 - \frac{1}{R^2}\sqrt{(R^2 - r_+^2)(R^2 - r_-^2)}\right), \tag{31}$$

$$J = 2\pi R j = \frac{r_+ r_-}{4G\ell}. \tag{32}$$

Thus, the angular momentum of the shell J is independent of the position of the shell R. From Eq. (4) we see that it is identical to that of the exterior BTZ spacetime,

$$\mathcal{J} = J. \tag{33}$$

This property is very similar to the case of the electrically charged shell [13], where the charge Q does not depend on the shell position R. The locally measured proper mass M is related to the ADM mass m defined in (4) by

$$m = \frac{RM}{\ell} - 2GM^2 + \frac{2G}{R^2}J^2, \tag{34}$$

where the first, second, and third terms correspond to the local rest mass, the gravitational binding energy, and the kinetic energy due to rotation, respectively.

We would like to emphasize that in our case the inner region is pure $(2+1)$-dimensional AdS spacetime and hence it has locally zero ADM mass and zero angular momentum. In the more complex case that the region inner to the shell contains instead a BTZ black hole then the total ADM mass and angular momentum of the outer spacetime defined at infinity would include in addition the ADM mass and angular momentum of the interior black hole.

III. THERMODYNAMIC ENTROPY OF THE SHELL

A. Thermodynamics on the shell

We now analyze the rotating thin shell system from a thermodynamics point of view. We assume that the shell is in thermal equilibrium, with a locally measured temperature T and entropy S. In the entropy representation, the entropy S of a system can be expressed as a function of the state independent variables. One can take as state independent thermodynamic variables for the thin shell, the proper mass M, the area of the shell A, and the angular momentum J. Then we can express the entropy as a function of these quantities, $S = S(M, A, J)$ and the first law of thermodynamics reads

$$TdS = dM + pdA - \Omega dJ, \tag{35}$$

for $T(M, A, J)$, $p(M, A, J)$, and $\Omega(M, A, J)$, representing the temperature, the pressure, and the angular velocity of the shell in terms of the state variables.

Now, since in this $(2+1)$-dimensional spacetime the area of the shell A is mathematically equivalent to the position R except for the trivial factor 2π, we make use of R as the independent variable, instead of A, as it will facilitate the calculations. Then we can express the entropy as a function of these quantities,

$$S = S(M, R, J). \tag{36}$$

Defining further the inverse local temperature β of the shell as

$$\beta = \frac{1}{T}, \tag{37}$$

the first law of thermodynamics (35) now reads

$$dS = \beta dM + 2\pi p\beta dR - \beta\Omega dJ. \tag{38}$$

The integration of this equation to yield $S = S(M, R, J)$ can then be performed once the equations of state,

$$p = p(M, R, J), \tag{39}$$

$$\beta = \beta(M, R, J), \tag{40}$$

$$\Omega = \Omega(M, R, J), \tag{41}$$

are specified. For M, p, and J we use expressions (29), (31), and (32) obtained from the junction conditions. On the other hand, β and Ω play the role of integration factors, which must be specified in order to obtain an exact expression for the entropy. However, the choice of these functions is constrained by the necessity to satisfy the integrability conditions that follow directly from the first law (38),

$$\left(\frac{\partial\beta}{\partial R}\right)_{J,M} = 2\pi\left(\frac{\partial(\beta p)}{\partial M}\right)_{J,R}, \tag{42}$$

$$2\pi\left(\frac{\partial(\beta p)}{\partial J}\right)_{R,M} = -\left(\frac{\partial(\beta\Omega)}{\partial R}\right)_{J,M}, \tag{43}$$

$$\left(\frac{\partial\beta}{\partial J}\right)_{R,M} = -\left(\frac{\partial(\beta\Omega)}{\partial M}\right)_{J,R}. \tag{44}$$

We should also require the local thermodynamic stability conditions to be satisfied,

$$\left(\frac{\partial^2 S}{\partial M^2}\right)_{R,J} \leq 0, \tag{45}$$

$$\left(\frac{\partial^2 S}{\partial R^2}\right)_{M,J} \leq 0, \tag{46}$$

$$\left(\frac{\partial^2 S}{\partial J^2}\right)_{M,R} \leq 0, \tag{47}$$

$$\left(\frac{\partial^2 S}{\partial M^2}\right)_{J,R}\left(\frac{\partial^2 S}{\partial R^2}\right)_{M,J} - \left\{\left(\frac{\partial^2 S}{\partial M\partial R}\right)_{J}\right\}^2 \geq 0, \tag{48}$$

$$\left(\frac{\partial^2 S}{\partial J^2}\right)_{M,R}\left(\frac{\partial^2 S}{\partial R^2}\right)_{M,J} - \left\{\left(\frac{\partial^2 S}{\partial J\partial R}\right)_{M}\right\}^2 \geq 0, \tag{49}$$

$$\left(\frac{\partial^2 S}{\partial M^2}\right)_{J,R}\left(\frac{\partial^2 S}{\partial J^2}\right)_{M,R} - \left\{\left(\frac{\partial^2 S}{\partial M\partial J}\right)_{R}\right\}^2 \geq 0. \tag{50}$$

For the derivation of the stability conditions with three independent variables see Appendix B of [13].

B. The three equations of state

1. Useful relations

It is useful to define the redshift function k as

$$k(r_+, r_-, R) = \frac{R}{\ell}\sqrt{\left(1-\frac{r_+^2}{R^2}\right)\left(1-\frac{r_-^2}{R^2}\right)}. \tag{51}$$

Then we can write (31) as

$$M(r_+, r_-, R) = \frac{1}{4G}\left(\frac{R}{\ell} - k(r_+, r_-, R)\right), \tag{52}$$

and repeat (32)

$$J(r_+, r_-) = \frac{r_+ r_-}{4G\ell}, \tag{53}$$

so that in thermodynamic terms we see that with the use of Eqs. (52) and (53), we can always change the independent variables from (r_+, r_-, R) to (M, R, J) and vice versa. We now prescribe the three equations of state indicated in (39).

2. The pressure equation of state

In this manner, we can express the pressure obtained in Eq. (29) as a function of (M, R, J),

$$\begin{aligned} p(M,R,J) &= \frac{1}{8\pi G\ell}\left(\frac{R^4 - r_+^2(M,R,J)r_-^2(M,R,J)}{R^3\ell k(r_+(M,R,J), r_-(M,R,J), R)} - 1\right) \\ &= \frac{MR^3 - 4G\ell J^2}{2\pi R^3(R - 4G\ell M)}, \end{aligned} \tag{54}$$

which determines the pressure equation of state of the shell. We find it more convenient to work with the equations expressed as functions of (r_+, r_-, R) but we always keep in mind that they depend implicitly on M and J through $r_\pm(M, R, J)$.

3. The temperature equation of state

Next we turn to the other equation of state, $\beta = \beta(M, R, J)$, which is constrained by the integrability condition (42). First note that

$$p = -\frac{1}{2\pi}\left(\frac{\partial M}{\partial R}\right)_{r_+, r_-}, \tag{55}$$

which expresses the conservation of the shell's stress-energy tensor. Then $\frac{1}{\beta}(\frac{\partial\beta}{\partial R})_{r_+,r_-} = \frac{1}{\beta}\{(\frac{\partial\beta}{\partial R})_{M,J} + (\frac{\partial M}{\partial R})_{r_+,r_-}(\frac{\partial\beta}{\partial M})_{R,J}\} = \frac{1}{\beta}\{2\pi\beta(\frac{\partial p}{\partial M})_{R,J} + [2\pi p + (\frac{\partial M}{\partial R})_{r_+,r_-}](\frac{\partial\beta}{\partial M})_{R,J}\} = 2\pi(\frac{\partial p}{\partial M})_{J,R} = \frac{R^4 - r_+^2 r_-^2}{R(R^2-r_+^2)(R^2-r_-^2)} = \frac{1}{k}(\frac{\partial k}{\partial R})_{r_+,r_-}$, where we used Eq. (55). Thus, in brief

$$\frac{1}{\beta}\left(\frac{\partial\beta}{\partial R}\right)_{r_+,r_-} = \frac{1}{k}\left(\frac{\partial k}{\partial R}\right)_{r_+,r_-}. \tag{56}$$

By integrating Eq. (56), we obtain the inverse local temperature equation of state of the shell

$$\beta(r_+, r_-, R) = k(r_+, r_-, R)b(r_+, r_-), \tag{57}$$

where $b(r_+, r_-)$ is an arbitrary function of $r_\pm$. The function $b(r_+, r_-)$ is interpreted as the temperature of the shell located at the radius $R = \{\frac{1}{2}(\ell^2 + r_+^2 + r_-^2 + \sqrt{\ell^4 + (r_+^2 - r_-^2)^2 + 2\ell^2(r_+^2 + r_-^2)})\}^{\frac{1}{2}}$. Since $k = \sqrt{-g_{tt}^{(o)}(R)}$, the formula (57) expresses the gravitational redshift of the temperature of the shell and is nothing but the Tolman relation for the temperature in the gravitational system. The function $b(r_+, r_-)$ depends on (M, R, J) only through $r_\pm$. The integrability condition does not yield a precise form for $b(r_+, r_-)$, which depends on the properties of matter within the shell.

4. The angular velocity equation of state

Next, we consider $\Omega = \Omega(M, R, J)$. Using the integrability conditions (43) and (44), together with relation (55), we find $(\frac{\partial p}{\partial J})_{M,R} + \Omega(\frac{\partial p}{\partial M})_{J,R} = p(\frac{\partial\Omega}{\partial M})_{J,R} - \frac{1}{2\pi}(\frac{\partial\Omega}{\partial R})_{M,J} = -\frac{1}{2\pi}\{(\frac{\partial\Omega}{\partial R})_{M,J} + (\frac{\partial M}{\partial R})_{r_+,r_-}(\frac{\partial\Omega}{\partial M})_{J,R}\} = -\frac{1}{2\pi}(\frac{\partial\Omega}{\partial R})_{r_+,r_-}$. Then, taking in consideration Eq. (56), Ω obeys

$(\frac{\partial(\Omega\beta)}{\partial R})_{r_+,r_-} = -2\pi\beta(\frac{\partial p}{\partial J})_{M,R} = \frac{2r_+r_-b(r_+,r_-)}{\ell R^3}$, where we used Eq. (57) in the last step. So, in brief

$$\left(\frac{\partial(\Omega\beta)}{\partial R}\right)_{r_+,r_-} = \frac{2r_+r_-b(r_+,r_-)}{\ell R^3}. \quad (58)$$

After integrating Eq. (58) we obtain the angular velocity equation of state

$$\Omega(r_+,r_-,R) = \frac{r_+r_-}{\ell k(r_+,r_-,R)}\left(c(r_+,r_-) - \frac{1}{R^2}\right), \quad (59)$$

where $c(r_+,r_-)$ is an arbitrary function of $r_\pm$.

5. In a nutshell

In this way, we have found the three equations of state (54), (57), and (59), which are necessary to determine the entropy of the shell, as we will argue below. Up to now, the integration of the integrability conditions has introduced two integration constants, $b(r_+,r_-)$ and $c(r_+,r_-)$, which are free functions of r_+ and r_-.

C. The entropy of the shell

By changing variables from (M,R,J) to (r_+,r_-,R) and substituting Eqs. (52), (54), (57), and (59) into the first law of thermodynamics, Eq. (38), we obtain

$$dS = \frac{b}{8G\ell^2}[(1-r_-^2c(r_+,r_-))dr_+^2 + (1-r_+^2c(r_+,r_-))dr_-^2]. \quad (60)$$

This expression implies that the two integration constants must satisfy the integrability condition

$$\frac{\partial b}{\partial r_-^2}(1-r_-^2c) - br_-^2\frac{\partial c}{\partial r_-^2} = \frac{\partial b}{\partial r_+^2}(1-r_+^2c) - br_+^2\frac{\partial c}{\partial r_+^2}. \quad (61)$$

This condition can be equivalently expressed as

$$\frac{\partial b}{\partial r_+^2} - \frac{\partial b}{\partial r_-^2} = \frac{\partial(bc)}{\partial \log(r_+^2)} - \frac{\partial(bc)}{\partial \log(r_-^2)}, \quad (62)$$

which makes it manifest that any choice for which b is a function of $r_+^2 + r_-^2$ only, and bc is a function of $r_+^2r_-^2$ only, will satisfy the integrability condition. In other words, Eq. (61) is automatically obeyed whenever b and bc are only functions of the ADM mass m and the angular momentum J, respectively. This will be used below when we search for equations of state. However, in generic cases, in order to obtain a specific expression for the entropy we need to choose either $b(r_+,r_-)$ or $c(r_+,r_-)$, and then obtain the remaining function by integrating Eq. (61).

Relation (60) also indicates that the entropy S is a function of r_+ and r_- only,

$$S = S(r_+,r_-), \quad (63)$$

and hence a function of (M,R,J) only through $r_\pm(M.R,J)$,

$$S(M,R,J) = S(r_+(M,R,J), r_-(M,R,J)). \quad (64)$$

It is also worth mentioning that, from (64), shells with the same r_+ and r_-, namely with the same ADM mass m and angular momentum J but at a different position R, have the same entropy. Thus, an observer measuring m and J cannot distinguish shells with different radii by measuring the entropy.

IV. THE THIN SHELL AND THE BLACK HOLE LIMIT

A. A precisely chosen temperature equation of state and the entropy

As the equation of state for the inverse temperature $b(r_+,r_-)$, let us take it to be of the form

$$b(r_+,r_-) = b_+\gamma, \quad (65)$$

where b_+ is the inverse Hawking temperature of the BTZ black hole given by

$$b_+ = 2\pi\ell^2\frac{r_+}{r_+^2 - r_-^2} \quad (66)$$

and γ is a parameter which will depend on the properties of matter on the shell. We assume $\gamma > 0$, so that the temperature is positive. This is one of the simplest possible temperature equations of state, setting the shell's fluid temperature proportional to the black hole temperature.

We also have to specify $c(r_+,r_-)$, so that it satisfies the integrability condition (61). There is a family of solutions for c, but here we choose the following particular solution

$$c(r_+,r_-) = \frac{1}{r_+^2}, \quad (67)$$

which makes the angular velocity Ω vanish when $R \to r_+$ [see Eq. (59)]. By substituting (65) and (67) into Eq. (60), we obtain the differential for the entropy of the shell

$$dS = \frac{\gamma}{4G}dA_+, \quad (68)$$

where $A_+ = 2\pi r_+$ represents the circumference (area) of the event horizon. By integrating (68), the entropy of the shell is given by $S = S_0 + \frac{\gamma}{4G}A_+$, where S_0 is an integration constant. Requiring that when the shell is absent—or equivalently when $M = 0$ and $J = 0$ [$r_+ = r_- = 0$ from (31)]—the entropy vanishes, we fix $S_0 = 0$ and obtain

$$S = \frac{\gamma}{4G} A_+, \tag{69}$$

which shows that the entropy of the rotating shell depends on (M, R, J) only through r_+. We note that for the entropy (69) all the thermodynamic stability conditions (45)–(50) are satisfied provided $\gamma > 0$. The parameter γ should be determined by the properties of matter on the shell and cannot be determined *a priori*.

B. The black hole limit

Although γ should be determined by the properties of matter on the shell, there is a case in which the properties of the shell have to be adjusted to the environment. Such a situation occurs when the shell is pushed to its own gravitational radius, $R \to r_+$. In fact, as the shell approaches its gravitational radius, quantum effects would be inevitably present and their backreaction would invalidate the classical treatment we have adopted, unless we choose the black hole Hawking temperature for the temperature of the shell. Therefore, we must choose $\gamma = 1$, or equivalently $b = b_+$, see Eq. (66), with c_+ still being given by Eq. (67). In this case, (69) becomes

$$S = \frac{A_+}{4G}, \tag{70}$$

which is the same as the Bekenstein-Hawking entropy for the corresponding black hole (see the Appendix). Thus, when we push the shell to its gravitational radius the entropy coincides with the Bekenstein-Hawking entropy. In the limit $R \to r_+$, the pressure given by (54) diverges as $\frac{1}{k}$ (assuming the spacetime is not extremal, $r_+ \neq r_-$), whereas the angular velocity expressed in (59) vanishes, at least for the particular choice of the function c made in Eq. (67). Nevertheless, the local inverse temperature (57) is proportional to k, so the local temperature of the shell also diverges as $\frac{1}{k}$. These divergences cancel out precisely, so that they can reproduce the Bekenstein-Hawking entropy. In fact, the first law (38), $dS = \beta dM + 2\pi p\beta dR - \beta\Omega dJ$, reveals that in the black hole limit, the only term that survives (and remains finite) in the right-hand side is the pressure term, which neatly combines with the inverse temperature to yield an area law for the entropy [10].

Our approach adds in a nontrivial manner to the results of the static $(2+1)$-dimensional studies presented in [27,28], having also affinities to the works [12,13]. This manner of calculating the entropy of a black hole shares also certain similarities with the work [9], in the sense that both studies consider matter distributed on thin shells to determine the entropy of black holes. Here, we used a radially static thin shell that decreases its radius adiabatically toward its gravitational radius, maintaining quasistaticity of the spacetime. On the other hand, [9] considered a reversible contraction of a thin shell and found that the black hole entropy can be defined as the thermodynamic entropy stored in matter compressed into a thin layer at its own gravitational radius.

We also note that the extremal limit $r_+ = r_-$ is well defined. In this limit, we find that the temperature $\frac{1}{b(r_+, r_-)} \to 0$, but the entropy of the extremal black hole is still given by (70). It is well known that the entropy of an extremal black hole requires special care. If we had started our analysis directly using the metric for the extremal black hole, we would have found a more complicated expression for the entropy.

V. OTHER EQUATIONS OF STATE WITH $b(r_+, r_-)$ AND $c(r_+, r_-)$ OF BLACK HOLE TYPE

In the previous subsections we imposed a temperature equation of state of the Hawking type (65), as well as the specific thermodynamic angular velocity (67), and obtained that the entropy of the shell is proportional to A_+. Moreover, if we set the temperature of the shell exactly equal to the Hawking temperature [see Eq. (66), or Eq. (65) with $\gamma = 1$], then the entropy of the shell precisely reproduces the Bekenstein-Hawking entropy. The choices made for the equations of state (65) and (67), although mandatory for a shell reproducing the Bekenstein-Hawking area law when approaching its gravitational radius r_+, are just the simplest ones among a larger class of equations of state allowed by the integrability condition (61).

Here, we briefly consider some other choices for the equations of state.

Case 1. For the temperature equation of state (65), the integrability condition (61) gives a general equation of state for the thermodynamic angular velocity

$$c(r_+, r_-) = \frac{1}{r_+^2}(1 + r_+(r_+^2 - r_-^2)\tilde{c}(r_+^2 r_-^2)), \tag{71}$$

where $\tilde{c}(r_+^2 r_-^2)$ is an arbitrary function of the product $r_+^2 r_-^2$. Substituting Eqs. (65) and (71) into (60) and integrating, we obtain

$$S(r_+, r_-) = \frac{\gamma}{4G}\left(A_+ - \pi \int_0^{r_+^2 r_-^2} dx \tilde{c}(x)\right), \tag{72}$$

where we set the integration constant to zero, so that S vanishes when $r_+ \to 0$. If we set $\tilde{c}(r_+^2 r_-^2) = 0$, we recover expression (69) which reproduces the Bekenstein-Hawking entropy for $\gamma = 1$.

Case 2. On the other hand, if we start from the angular velocity equation of state (67), then the integrability condition (61) allows the quite general equation of state for the inverse temperature given by

$$b(r_+, r_-) = \frac{2\pi\ell^2 h_+(r_+^2)}{r_+^2 - r_-^2}, \tag{73}$$

where $h_+(r_+^2)$ is an arbitrary function of r_+^2. Substituting (67) and (73) into (60) and integrating, we obtain

$$S(r_+) = \frac{\pi}{4G}\int_0^{r_+^2} dx \frac{h_+(x)}{x}, \tag{74}$$

where once again we set the integration constant to zero. If we set $h_+(r_+^2) = \sqrt{r_+^2}$, we recover (69) which reproduces the Bekenstein-Hawking entropy for $\gamma = 1$.

Case 3. Similarly, if we start from the angular velocity equation of state

$$c(r_+, r_-) = \frac{1}{r_-^2}, \tag{75}$$

then the integrability condition (61) allows the more general equation of state for the inverse temperature given by

$$b(r_+, r_-) = \frac{2\pi\ell^2 h_-(r_-^2)}{r_-^2 - r_+^2}, \tag{76}$$

where $h_-(r_-^2)$ is an arbitrary function of r_-^2. Substituting (75) and (76) into (60) and integrating, we obtain

$$S(r_-) = \frac{\pi}{4G}\int_0^{r_-^2} dx \frac{h_-(x)}{x}. \tag{77}$$

Case 4. Finally, if we start from the angular velocity equation of state,

$$c(r_+, r_-) = \tilde{c}(r_+^2 r_-^2), \tag{78}$$

where $\tilde{c}(r_+^2 r_-^2)$ is an arbitrary function of the product $r_+^2 r_-^2$, then the integrability condition (61) allows

$$b(r_+, r_-) = b_0\ell^2, \tag{79}$$

where b_0 is an arbitrary constant. Substituting (78) and (79) into (60) and integrating, we obtain

$$S(r_+, r_-) = \frac{b_0}{8G}\left(r_+^2 + r_-^2 - \int_0^{r_+^2 r_-^2} dx\tilde{c}(x)\right). \tag{80}$$

The above four cases are the counterparts of those considered for the charged shell in [13]. As in [13], we will not explore them further.

VI. POWER-LAW EQUATIONS OF STATE

A. The entropy

In this section we will focus on power-law equations of state for both the inverse temperature and the thermodynamic angular velocity, for which the stability conditions can be studied explicitly. We start by specifying the temperature equation of state $b(r_+, r_-)$ in Eq. (57). One of the simple but reasonable choices for the temperature is a power-law function of $r_+^2 + r_-^2$ which is related to the ADM mass m via Eq. (4),

$$b(r_+, r_-) = 4G\ell^2 a(r_+^2 + r_-^2)^{\frac{\alpha}{2}}, \tag{81}$$

where a and α are free parameters which reflect the properties of matter on the shell. For such a temperature equation of state, the integrability condition (61) admits the following general solution for the equation of state for the thermodynamic angular velocity $c(r_+, r_-)$ in Eq. (59),

$$c(r_+, r_-) = \frac{\tilde{c}(r_+^2 r_-^2)}{(r_+^2 + r_-^2)^{\frac{\alpha}{2}}}, \tag{82}$$

where $\tilde{c}(r_+^2 r_-^2)$ is an arbitrary function of the product $r_+^2 r_-^2$. Since the product $r_+^2 r_-^2$ is related to the angular momentum J in Eq. (31), it is also reasonable that we introduce another power-law form for $\tilde{c}(r_+^2 r_-^2)$, such as

$$c(r_+, r_-) = \frac{\sigma(r_+^2 r_-^2)^{\frac{\delta}{2}}}{(r_+^2 + r_-^2)^{\frac{\alpha}{2}}}, \tag{83}$$

where σ and δ are further free parameters which also reflect the properties of matter on the shell. Substituting (81) and (83) into (60) and integrating, we obtain

$$S(r_+, r_-) = a\left(\frac{(r_+^2 + r_-^2)^{\frac{\alpha}{2}+1}}{\alpha + 2} - \frac{\sigma(r_+^2 r_-^2)^{\frac{\delta}{2}+1}}{\delta + 2}\right), \tag{84}$$

where we have set the integration constant S_0 to zero, so that the entropy S vanishes in the limit of $m \to 0$ and $J \to 0$, namely $r_+ \to 0$ and $r_- \to 0$. This requirement can be satisfied if we impose the following conditions on the exponents:

$$\alpha > -2, \qquad \delta > -2. \tag{85}$$

As in the cases discussed in the previous sections, the entropy of the rotating shell depends on (M, R, J) only through $r_\pm(M, R, J)$. We impose positivity of the temperature, which gives the following constraint on the parameter a:

$$a > 0. \tag{86}$$

B. Thermodynamic stability conditions

We now address the thermodynamic stability of the system. First, we consider the stability conditions that do not involve σ and δ. Defining

$$\xi_\pm = \frac{r_\pm}{R}, \tag{87}$$

condition (45) gives

$$\alpha \le \frac{\xi_+^2 + \xi_-^2}{\kappa^2}, \tag{88}$$

where we have also defined

$$\kappa(\xi_+, \xi_-) = \sqrt{(1 - \xi_+^2)(1 - \xi_-^2)}. \tag{89}$$

Note that $0 \le \kappa < 1$, with the first inequality being saturated only when the shell is taken to its gravitational radius, $R \to r_+$. Condition (46) gives

$$\alpha \leq -3\xi_+^2\xi_-^2 \frac{\xi_+^2 + \xi_-^2}{(1-\kappa(\xi_+,\xi_-) - \xi_+^2\xi_-^2)^2} \leq 0. \tag{90}$$

Finally, condition (48) yields

$$\alpha \leq \alpha_*(\xi_+,\xi_-) = -\frac{(1+3\xi_+^2\xi_-^2)(\xi_+^2+\xi_-^2)}{(1-\xi_+^2\xi_-^2)^2 - (1+3\xi_+^2\xi_-^2)\kappa^2(\xi_+,\xi_-)}. \tag{91}$$

We find that among the above conditions on α, condition (91) is the most stringent. Noting that $\alpha_*(\xi_+,\xi_-) \leq \alpha_*(\xi_-,\xi_-)$ [assuming $\alpha_*(\xi_-,\xi_-) > -2$], where

$$\alpha_*(\xi_-,\xi_-) = -\frac{1+3\xi_-^4}{(1-\xi_-^2)^3} < -1, \tag{92}$$

we conclude that there can be a parameter region where condition (85) is met. The corresponding value of ξ_- is given by

$$\xi_- < \sqrt{\frac{1}{2}(1+3^{\frac{1}{3}}-3^{\frac{2}{3}})} = 0.4255. \tag{93}$$

Finally, we turn to the thermodynamic stability conditions which involve σ and δ. Defining

$$\tilde{\sigma} = \sigma(1+\delta)R^{2(1+\delta)-\alpha}, \tag{94}$$

condition (47) gives

$$-(\xi_+^2+\xi_-^2)^{\frac{\alpha}{2}}(\xi_+^2+\xi_-^2+\alpha\xi_+^2\xi_-^2) + \tilde{\sigma}(\xi_+^2\xi_-^2)^{\frac{\delta}{2}}(\xi_+^2+\xi_-^2) \geq 0. \tag{95}$$

Condition (49) gives

$$\begin{aligned} &-(\xi_+^2+\xi_-^2)^{\frac{\alpha}{2}}[\xi_+^2\xi_-^2(\xi_+^2+\xi_-^2) \\ &\quad -\alpha(1-\kappa(\xi_+,\xi_-))(1-\kappa(\xi_+,\xi_-)+2\xi_+^2\xi_-^2)] \\ &-\tilde{\sigma}(\xi_+^2\xi_-^2)^{\frac{\delta}{2}}[3\xi_+^2\xi_-^2(\xi_+^2+\xi_-^2)+\alpha(1-\kappa(\xi_+,\xi_-)-\xi_+^2\xi_-^2)^2] \geq 0. \end{aligned} \tag{96}$$

Finally, condition (50) gives

$$\begin{aligned} &-(\xi_+^2+\xi_-^2)^{\frac{\alpha}{2}}[\xi_+^2+\xi_-^2-\alpha(\kappa^2(\xi_+,\xi_-)-\xi_+^2\xi_-^2)] \\ &\quad +\tilde{\sigma}(\xi_+^2\xi_-^2)^{\frac{\delta}{2}}(\xi_+^2+\xi_-^2-\alpha\kappa^2(\xi_+,\xi_-)) \geq 0. \end{aligned} \tag{97}$$

In the black hole limit, $\xi_+ \to 1$, this condition is equivalent to Eq. (95).

If one assumes that the parameter σ is dimensionless, so as to not introduce any further length scales in the problem, inspection of any of the above conditions, or of Eq. (83), shows that the exponents α and δ must be related through

$$\delta = \frac{\alpha}{2} - 1, \tag{98}$$

for dimensional consistency. This reduces the number of relevant dimensionless parameters down to four, namely $(\alpha, \tilde{\sigma}, \xi_+, \xi_-)$.

The stability conditions (95), (96), and (97) are somewhat complicated. We find numerically that all the thermodynamic stability requirements [(85), (86), (91), (95)–(97)] can be met for a certain region of the parameter space, as long as

$$\tilde{\sigma} > 0. \tag{99}$$

If that is the case, then all stability conditions tend to be satisfied for a small ξ_-, i.e., for slowly rotating shells. Thermodynamic stability can also be insured for shells close to the black hole limit, $\xi_+ \to 1$, but this happens only if ξ_- remains small, i.e., far from extremality. Moreover, we observe that these conditions are weakly dependent on α.

VII. CONCLUSIONS

In this work we have investigated the thermodynamic properties of a rotating thin shell in a $(2+1)$-dimensional asymptotically AdS spacetime, where the interior and exterior of the shell were taken to be the vacuum AdS spacetime and the BTZ black hole spacetime, respectively. These two geometries were matched across the shell using the Israel junction conditions, which in turn provide the three quantities which characterize the properties of the matter on the shell, namely, the surface energy density σ, the pressure p, and the angular momentum density j. Multiplying the surface energy density and the angular momentum density by the circumference $2\pi R$, we obtained the locally measured proper mass $M = 2\pi R\sigma$ and the angular momentum $J = 2\pi Rj$ of the shell, thus completing the study of its mechanical properties.

To address the thermodynamics of the system we adopted the locally measured mass M, the volume of the shell $2\pi R$ and the angular momentum J as the state independent variables. Using the first law of thermodynamics and the equations of state, we were able to obtain the entropy of the shell $S = S(M,R,J)$. Due to the additional variable J, the construction of the thermodynamics is more complicated than in the nonrotating case [28]. In order to obtain an expression for the entropy of the shell, one must specify equations of state for the pressure $p = p(M,R,J)$, the local inverse temperature $\beta = \beta(M,R,J)$, and the thermodynamic angular velocity $\Omega = \Omega(M,R,J)$. While the pressure equation of state is automatically determined by the junction conditions, there is a certain freedom to choose the equations of state for the local inverse temperature and thermodynamic angular velocity, which only have to satisfy an integrability condition derived from the first law of thermodynamics.

One of our main results is that the entropy of the shell must be a function of the two gravitational radii $r_\pm$ alone. In particular, shells with the same $r_{\pm}$, and thus the same ADM mass m and angular momentum J, but at different radii R share the same entropy. Thus, from the entropic properties only, it is not possible to distinguish a shell near the gravitational radius r_+ from one asymptotically far. These findings corroborate the results obtained recently for charged thin shells in $(3+1)$ dimensions [13], with the analogy becoming manifest if one replaces "electric charge" and "Coulomb potential" with "angular momentum" and "angular velocity," respectively.

The integrability conditions allow a multitude of equations of state for the local inverse temperature $\beta(M, R, J)$ and angular velocity $\Omega(M, R, J)$. Choosing a well-motivated temperature equation of state, namely setting the shell's temperature equal to the BTZ black hole Hawking temperature, and the simplest possible angular velocity equation of state that is consistent with that choice, the resulting entropy of the shell precisely agrees with the Bekenstein-Hawking entropy of the BTZ black hole.

Nevertheless, many other equations of state that do not yield a simple area law for the entropy are also consistent with the integrability condition. We have presented a large class of such examples, for which the inverse temperature and angular velocity equations of state are described by power laws. For this family of solutions the number of parameters characterizing the system is quite large. Excluding the AdS scale ℓ and the radial location of the shell R, we are still left with six parameters. Through a thermodynamic stability analysis we have obtained several constraints on the equations of state parameters.

There remains the possibility of the existence of further classes of consistent choices for the equations of state, which should require a dedicated study of their thermodynamic stability properties.

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APPENDIX: THERMODYNAMICS OF THE BTZ BLACK HOLE

Here we review the thermodynamic properties of the rotating BTZ black hole following the Hawking-Page formalism for the $(3+1)$-dimensional Schwarzschild AdS black hole [7]; see also [24–26] who applied diverse formalisms to the same BTZ black hole. The Euclidean action S_E in the $(2+1)$-dimensional Einstein gravity is given by

$$S_E = -\frac{1}{16\pi G}\int d^3x\sqrt{g_E}(R_E - 2\Lambda), \tag{A1}$$

with $\sqrt{-\Lambda} = \frac{1}{\ell}$ and where R_E is the Euclidean Ricci scalar given by $R_E = -\frac{6}{\ell^2}$. The vacuum BTZ Euclidean metric is given by metric (3) with the time Euclideanized, i.e., $t = i\tau$,

$$ds^2 = f^2(r)d\tau^2 + \frac{1}{f^2(r)}dr^2 + r^2\left(d\phi - \frac{ir_+r_-}{\ell r^2}d\tau\right)^2, \qquad 0 < r < \infty, \tag{A2}$$

where

$$f(r) = \frac{\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}}{\ell r}. \tag{A3}$$

The two radii r_+ and r_- are now the event horizon and the Cauchy horizon radii, respectively, still related to the spacetime Arnowitt-Deser-Misner (ADM) mass m and the angular momentum $\mathcal{J}$, respectively, by $r_+^2 + r_-^2 = 8G\ell^2 m$ and $r_+r_- = 4G\ell\mathcal{J}$. To have a black hole one should impose $r_+ \geq r_-$, i.e., $m \geq \frac{\mathcal{J}}{\ell}$. The inequalities are saturated in the extremal case, $r_+ = r_-$, i.e., $m = \frac{\mathcal{J}}{\ell}$.

Without introducing a regulator boundary, the on-shell Euclidean action (A1) diverges. By introducing the regulator boundary at $r = \bar{r}$, the on-shell Euclidean action reduces to

$$S_E(\bar{r}) = \frac{b_+}{4G\ell^2}(\bar{r}^2 - r_+^2), \tag{A4}$$

where b_+ is the inverse Hawking temperature

$$b_+ = \frac{2\pi\ell^2 r_+}{r_+^2 - r_-^2}, \tag{A5}$$

given by the period of the Euclidean time. The divergent part in the limit of $\bar{r} \to \infty$ arises because of the asymptotically AdS structure in the $(2+1)$-dimensional spacetime, and hence from $S_E(\bar{r})$ we should subtract the vacuum AdS counterpart

$$S_E^{(0)}(\bar{r}) = \frac{b_0(\bar{r})}{4G\ell^2}\bar{r}^2, \tag{A6}$$

for some temperature b_0. The periodicity of the vacuum AdS is chosen by requiring that the periodicity of the Euclidean time and the geometry at the section of $r = \bar{r}$ in the BTZ and AdS backgrounds should be identical, namely,

$$b_0(\bar{r})f_0(\bar{r}) = b_+f(\bar{r}), \tag{A7}$$

where $f(r)$ is given in Eq. (A3) and $f_0(r)$ is given by the vacuum case, i.e.,

$$f_0(r) = \frac{r}{\ell}. \tag{A8}$$

With (A7), defining the regularized Euclidean action by

$$\begin{aligned} S_{\rm reg}(\bar{r}) = S_E(\bar{r}) - S_E^{(0)}(\bar{r}) &= \frac{b_+}{4G\ell^2}\left(\bar{r}^2 - r_+^2 - \frac{b_0(\bar{r})}{b_+}\bar{r}^2\right) \\ &= \frac{b_+}{4G\ell^2}\left(\bar{r}^2 - r_+^2 - \frac{f(\bar{r})}{f_0(\bar{r})}\bar{r}^2\right), \end{aligned} \tag{A9}$$

and expanding it in terms of the inverse power of $\bar{r}$, we find that the divergent terms of $O(\bar{r}^2)$ are canceled out, and in the limit of $\bar{r} \to \infty$

$$S_{\rm reg} = -\frac{\pi r_+}{4G}. \tag{A10}$$

The Euclidean action $S_{\rm reg}$ is related to the free energy F by $S_{\rm reg} = b_+ F$, hence

$$F = \frac{S_{\rm reg}}{b_+} = -\frac{r_+^2 - r_-^2}{8G\ell^2}. \tag{A11}$$

Now, the first law of black hole thermodynamics is

$$dE = T_+ dS + \Omega_+ d\mathcal{J}, \tag{A12}$$

where E is the spacetime energy, T_+ the black hole temperature [given by the inverse of Eq. (A5), $T_+ = 1/b_+$], S the entropy, $\Omega_+ = \frac{r_-}{\ell r_+}$ the black hole angular velocity, and $\mathcal{J}$ the spacetime angular momentum. With the definition of the free energy as

$$F = E - T_+ S - \mathcal{J}\Omega_+, \tag{A13}$$

the first law of the black hole thermodynamics (A12) can be rewritten as

$$dF = -S dT_+ - \mathcal{J} d\Omega_+. \tag{A14}$$

Thus, the entropy and angular momentum of the black hole, conjugate to T_+ and Ω_+ respectively, are obtained as

$$\begin{aligned} S &= -\left(\frac{\partial F}{\partial T_+}\right)_{\Omega_+} = \frac{\ell^2\pi^2 T_+}{G(1-\ell^2\Omega_+^2)} = \frac{\pi r_+}{2G} = \frac{A_+}{4G}, \\ \mathcal{J} &= -\left(\frac{\partial F}{\partial \Omega_+}\right)_{T_+} = \frac{\ell^4\pi^2 T_+^2 \Omega_+}{G(1-\ell^2\Omega_+^2)^2} = \frac{r_+ r_-}{4G\ell}, \end{aligned} \tag{A15}$$

where $A_+ = 2\pi r_+$ is the area—or, in this context, the circumference—of the event horizon. The entropy S is the Bekenstein-Hawking entropy. The second relation confirms the expression for the spacetime angular momentum $\mathcal{J}$. Finally, the energy E is given by

$$E = F + T_+ S + \mathcal{J}\Omega_+ = \frac{r_+^2 + r_-^2}{8G\ell^2} = m, \tag{A16}$$

which means that the thermodynamic energy stored in the gravitational system is given by the ADM mass m.

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