PHYSICAL REVIEW D **92,** 064004 (2015)

Critical behavior in black hole scalar field interaction

J. A. Crespo[1] and H. P. de Oliveira[1,2]

[1]*Departamento de Física Teórica—Instituto de Física A. D. Tavares, Universidade do Estado do Rio de Janeiro, R. São Francisco Xavier, 524, Rio de Janeiro, Rio de Janeiro 20550-013, Brazil*

[2]*Department of Physics and Astronomy, University of Pittsburgh, 100 Allen Hall, 3941, O'Hara Street, Pittsburgh, Pennsylvania 15260, USA*

(Received 20 July 2015; published 3 September 2015)

We study the critical behavior at the threshold of black hole formation in a model consisting of a scalar field incident to a reflector barrier enclosing a Schwarzschild black hole. Weak incident scalar field waves disturb slightly the black hole spacetime and are completely radiated by the reflector, like water waves striking against the wall of a dam. Strong incident waves produce the formation of an apparent horizon outside the barrier. In this case, a fraction of scalar field crosses the horizon together with the barrier, whereas another fraction escapes to infinity. We have integrated the field equations using a Galerkin collocation code that allowed the necessary accuracy to investigate the behavior of the black hole masses for a broad range of scalar field initial amplitude. We have shown that a scaling law describes the black hole masses for amplitudes very close to the critical value. In the limit of very strong scalar fields, the black hole masses either scale linearly with the initial amplitude or saturate depending on the existence of the initial monopole moment.

DOI: 10.1103/PhysRevD.92.064004 PACS numbers: 04.25.D-

I. INTRODUCTION

There is no doubt that critical phenomena in gravitational collapse are one of the most important discoveries of classical general relativity. In his seminal work, Choptuik [1] has studied the numerical spherical collapse of a self-gravitating massless scalar field, taking into account one-parameter initial data families. Choptuik was able to identify those data that form black holes (strong data) and those that result in complete dispersion of the scalar field (weak data). The numerical investigation towards the threshold of black hole formation revealed unexpected features that constitute the core of critical behavior in gravitational collapse. Accordingly, black holes with infinitesimal masses are formed and satisfy the following scaling relation:

$$M_{\rm BH} \simeq K(p - p_*)^\gamma. \tag{1}$$

Here p is the parameter that characterizes the strength of the initial data, and p_* is its corresponding critical value. Then, p_* separates those solutions that form black holes, $p > p_*$ (supercritical), from those that do not form black holes, $p < p_*$ (subcritical). The exponent $\gamma \simeq 0.37$ is universal in the sense of not depending on the particular initial data family. Choptuik has shown that the critical solution ($p = p_*$) is universal and has an elusive property known as discrete self-similarity (DSS). It means that the critical solution, say $\varphi_*(r, t)$, has a scale invariance symmetry expressed by $\varphi_*(r, t) = \varphi_*(e^\Delta r, e^\Delta t)$, where $\Delta \simeq 3.44$.

Critical behavior has been noticed in a large variety of collapsing fields in spherical symmetry [2], and in the collapse of axisymmetric gravitational waves [3–5]. These studies have shown two types of critical solution—namely, the type II found originally by Choptuik and the type I that can be static or periodic. The existence of critical configurations located at the threshold of black hole formation is one of the crucial ingredients for the establishment of critical behavior in gravitational collapse. All critical solutions are unstable, arising from the competition between the attractive gravitational interaction and the repulsive effective interaction present in all types of matter models.

Gomez *et al.* [6] devised a model that consists of a spherically symmetric scalar field incident on a reflecting barrier. This inner boundary corresponds to a spherical surface of radius R that encloses a Schwarzschild black hole with mass M_0 subject to the constraint $R > 2M_0$. They have set the scalar field to zero at the inner boundary so that this boundary acts as a perfect reflecting barrier. They have studied the decay of self-interacting scalar waves, confirming the previous results of Gundlach, Price, and Pullin [7] under different boundary conditions and numerical algorithm. Although their model is artificial, Gomez *et al.* presented new features regarding the role of the nonvanishing initial Newman-Penrose constant in the amplitude decay of the radiation field. They have also mentioned the presence of critical phenomena when a strong incident scalar field might form an apparent horizon outside the reflector, resulting in a new black hole with a mass greater than $R/2$ [6].

We have studied here the threshold of black hole formation in the model of Gomez *et al.* [6]. The investigation of critical phenomena when a scalar field collapse towards the reflector barrier and interacts with an existing black hole seems to be attractive. We have divided the paper as follows: Section II presents the basic equations of the model. The numerical method to integrate the field equations belongs to Sec. III. We have assembled the results in Sec. IV. The first of these is the mass scaling with a mass gap at the threshold of black hole formation that confirms the conjecture of Ref. [6]. We have found an oscillatory component superposed on the scaling law indicating that the critical solution has discrete self-similarity. The second result is the mass scaling beyond the threshold of black hole formation. Finally, in Sec. V we conclude.

II. A SIMPLE MODEL OF BLACK HOLE SCALAR FIELD INTERACTION

We have followed Ref. [6] and set the line element in Bondi coordinates,

$$ds^2 = -\frac{V}{r}\mathrm{e}^{2\beta}du^2 - 2\mathrm{e}^{2\beta}du dr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

where u is the retarded time, r is the radial coordinate, and the metric functions V and β depend on u, r. The relevant field equation is reduced to two hypersurface equations:

$$\beta_{,r} = \frac{r}{4}\varphi_{,r}^2, \quad (3)$$

$$V_{,r} = \mathrm{e}^{2\beta}, \quad (4)$$

and the scalar wave equation,

$$2(r\varphi)_{,ur} - \frac{1}{r}(rV\varphi_{,r})_{,r} = 0. \quad (5)$$

The Schwarzschild solution is given by $\varphi = 0$, $\beta = 0$, and $V = r - 2M_0$, where M_0 is the mass of the black hole. Here $r \geq R$, where R is the radius of the spherical surface identified as the reflector barrier. The scalar field vanishes at the reflector, $\phi(u, R) = 0$, together with the coordinate conditions $\beta(u, R) = 0$ and $V(u, R) = R - 2M_0$ in which $R > 2M_0$.

The field equations are evolved starting from the initial null data $\varphi_0(r) = \varphi(u_0 = 0, r)$ for $r \geq R$. We may interpret this initial distribution as an inhomogeneous cloud of a scalar field interacting with a black hole. To guarantee that the metric and the scalar field have a unique solution, we have fixed appropriate boundary conditions at the inner boundary $r = R_0$ and at the spatial infinity $r \to \infty$. We follow the approach of Gomez *et al.* [6] in establishing conditions at the boundary $r = R$,

$$\begin{aligned} \varphi(u, r_0) &= \mathcal{O}(\delta r), \\ \beta(u, r) &= \mathcal{O}(\delta r), \\ V(u, r_0) &= R - 2M_0 + \mathcal{O}(\delta r), \end{aligned} \quad (6)$$

where $\delta r = r - R$. The conditions at spatial infinity must guarantee that the spacetime is asymptotically flat:

$$\begin{aligned} \varphi(u, r) &= \mathcal{O}(r^{-1}), \\ \beta(u, r) &= H(u) + \mathcal{O}(r^{-2}), \\ V(u, r) &= r\mathrm{e}^{2H(u)} - 2\mathrm{e}^{2H(u)}M_B(u) + \mathcal{O}(r^{-1}). \end{aligned} \quad (7)$$

Here $H(u)$ and $M_B(u)$ are arbitrary functions arising from the integration of the field equations. $M_B(u)$ is the Bondi mass related to the mass function $m(u, r)$ by $M_B(u) = \lim_{r\to\infty} m(u, r)$, where

$$1 - \frac{2m(u, r)}{r} \equiv g^{\mu\nu} r_{,\mu} r_{,\nu} = \frac{V\mathrm{e}^{-2\beta}}{r}. \quad (8)$$

Another possibility of calculating the Bondi mass is through the following integral:

$$M_B(u) = M_0 + \frac{1}{4}\int_{R_0}^{\infty} \mathrm{e}^{-2\beta} rV\varphi_{,r}^2 dr. \quad (9)$$

The Bondi mass is not a conserved quantity, but a monotonically decreasing function according to the Bondi formula [8],

$$\frac{dM_B}{du} = -\frac{1}{2}\mathrm{e}^{-2H}N^2(u), \quad (10)$$

where $N(u) = \lim_{r\to\infty}(r\varphi_{,u})$ is the news function.

Despite the apparent simple form of the field equations (3)–(5), no globally well-behaved exact solutions are known. But similarly to the case without the barrier ($M_0 = 0$), two main behaviors are recognizable [6]. If the scalar field strength only produces small distortion on the black hole spacetime, it will be completely radiated away to infinity by the reflecting boundary, resulting in a black hole of mass M_0. On the other hand, above a critical value, the scalar field collapses to form an apparent horizon outside the reflector. In this case, a fraction of the scalar field crosses the horizon while another fraction is radiated to infinity. The reflector barrier falls into the horizon, and the mass of the formed black hole must satisfy the condition

$$M_{\mathrm{BH}} > \frac{R}{2}. \quad (11)$$

Gomez *et al.* [6] have claimed that critical phenomena seem to appear for black holes whose masses are close to $R/2$, but unlike the case without the reflector [1], this system has a mass gap.

III. A DYNAMICAL SYSTEM APPROACH THROUGH THE GALERKIN COLLOCATION METHOD

We present here the confirmation of the critical behavior conjectured in Ref. [6]. To this aim, we have integrated the field equation using an improved code based on the Galerkin collocation method [9]. It is convenient to introduce a new radial coordinate η such that $r = R(1+\eta)$, where the spatial domain $r \geq R$ is equivalent to $\eta \geq 0$. We have also introduced an auxiliary scalar field Φ by

$$\Phi(u,\eta) \equiv (1+\eta)\varphi(u,\eta). \tag{12}$$

The central idea of any spectral method is to approximate the relevant fields Φ, V, and β as appropriate series with respect to sets of basis functions according to

$$\Phi_a(u,\eta) = \sum_{k=0}^{N} a_k(u)\psi_k(\eta), \tag{13}$$

$$\beta_a(u,\eta) = \sum_{k=0}^{\bar{N}} b_k(u)\chi_k(\eta), \tag{14}$$

$$V_a(u,\eta) = R - 2M_0 + \sum_{k=0}^{\bar{N}} c_k(u)\eta TL_k(\eta), \tag{15}$$

where $N, \bar{N}$ are the truncation orders that dictate the number of unknown modes $a_k(u), b_k(u), c_k(u)$. The basis functions $\psi_k(\eta), \chi_k(\eta)$ are constructed such that the boundary conditions at $\eta = 0$ and $\eta \to \infty$ [cf. Eqs. (6) and (7), respectively] are satisfied as prescribed in the Galerkin method. Thus, it is necessary to express the basis functions as proper linear combinations of the rational Chebyshev polynomials $TL_k(\eta)$ defined by

$$TL_k(\eta) = T_k\left(x = \frac{\eta - L_0}{\eta + L_0}\right), \tag{16}$$

where $T_k(x)$ represents the standard Chebyshev polynomials of k order and L_0 is the map parameter.

We have assumed that the residual equations—obtained when the spectral approximations (13)–(15) are inserted into the field equations (3)–(5)—vanish at the collocation or grid points. Due to the possibility of choosing distinct truncation orders, we can have two distinct sets of collocation points. Therefore, we approximate the field equations into a set of ordinary differential equations for the unknown modes $a_k(u)$, and two sets of algebraic equations for $b_k(u), c_k(u)$. The spectral representation preserves the hierarchy of the field equations. After establishing the initial values $a_k(0)$, the corresponding initial values $b_k(0), c_k(0)$ are calculated from the algebraic relations. With these modes we can determine $\frac{da_k}{du}$ at $u = 0$ and, as a consequence, the unknown modes $a_k(u)$ in the next instant. The process repeats, providing the evolution of the modes and allowing us to reconstruct the spacetime.

To evolve the self-gravitating cloud of the scalar field incident towards the reflector, we need to specify the initial data $\Phi_0(\eta) = \Phi(u = 0, \eta)$ that fix the initial values $a_k(0)$. We have chosen the following initial data families:

$$\Phi_0(\eta) = \frac{\epsilon\eta}{(\eta+1)}, \tag{17}$$

$$\Phi_0(\eta) = \frac{\epsilon\eta}{1+\eta}\exp\left(-\frac{(\eta-\eta_0)^2}{\sigma^2}\right), \tag{18}$$

where ϵ is the amplitude of the initial scalar field distribution about the black hole, and we have fixed $\sigma = 1$, $\eta_0 = 3.0$.

We have used the Bondi formula (10) as a benchmark for the convergence of the code. The numerical test consists in verifying the global energy conservation expressed by the quantity $C(u)$ defined by integrating the Bondi formula [8,9]:

$$C(u) = \left| \frac{M_B(u_0) - M_B(u)}{M_B(u_0)} - \frac{1}{2M_B(u_0)}\int_{u_0}^{u} e^{-2H(u)}N^2(u)du \right|, \tag{19}$$

where $M_B(u_0)$ is the initial Bondi mass and $M_B(u_0) - M_B(u)$ is the mass loss evaluated at the instant u. The mass loss must be equal to the integral of the news function that determines the amount of radiated mass at each instant.

The *exact* evolution produces $C(u) = 0$, while the numerical integration of the field equations implies $C(u) \neq 0$. Thus, the decrease of $C(u)$ due to the increase of the truncation orders $N, \bar{N}$ [see Eqs. (13)–(15)] constitutes a robust diagnostic for the convergence of the Galerkin collocation procedure. We have chosen the parameters $M_0 = 0.2$, $R = 0.5$ and evolved the initial data (18) with $\epsilon = 1.23$ using a fourth-order Runge-Kutta integrator with step size 5.0×10^{-4}. Although generating a subcritical solution, we have shown that $\epsilon = 1.232$ corresponds to a supercritical solution with the formation of an apparent horizon. We have proceeded by selecting truncation orders $N = 30, 40, 50, \ldots, 90$ and $\bar{N} = 1.5N$, so that after evolving the scalar field until being reflected away, we have collected the maximum deviation of $C(u)$. In Fig. 1 we have exhibited the exponential decay of these maximum values, $C_{\max}$, as expected.

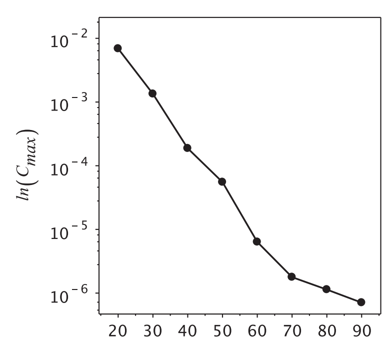

FIG. 1. Exponential decay of the maximum error in the global energy conservation with the increase of the truncation orders N.

IV. CRITICAL BEHAVIOR

We have confirmed the two main behaviors that depend on the amplitude ϵ and the location of the reflector. For very small amplitude, the field is continuously reflected until it has been radiated away completely, leaving behind a black hole of mass M_0. For ϵ greater than a certain critical value, ϵ_*, an apparent horizon forms at $\eta > 0$ ($r > R$). It signalizes the formation of a new black hole with mass $M_{\mathrm{BH}} > R/2$, where part of the scalar field crosses the apparent horizon together with the reflector. In the coordinate system we are adopting, the formation of the apparent horizon occurs when β diverges, meaning that the expansion of outgoing null rays vanishes [10].

We have evolved the supercritical data and determined the corresponding masses of the resulting black holes for a large range of $\epsilon > \epsilon_*$. To provide enough accuracy, we have set $N = 400$ and $\bar{N} = 2N$ and evolved the dynamical system for the unknown modes $a_k(u)$ using a Runge-Kutta-Fehlberg integrator with adaptive step size. The spectrum of black hole masses for the initial data (17) is shown in Fig. 2. The approach of $R/2$ is clear as ϵ approaches ϵ_*.

We have considered the formation of black holes whose mass is close to $R/2$ or $\epsilon \sim \epsilon_*$ and the following scaling law with mass gap emerges:

$$\delta M_{\mathrm{BH}} = M_{\mathrm{BH}} - \frac{R}{2} \simeq k_0 \delta\epsilon^{\gamma}, \tag{20}$$

where k_0 is a constant, $\delta\epsilon = \epsilon - \epsilon_*$, and γ is the critical exponent. In Fig. 3 we have presented the above scaling law that best fits the numerical points $(\delta M_{\mathrm{BH}k}, \delta\epsilon_k)$ obtained from the initial data (17) and (18), where $M_0 = 0.7$, $R = 2.0$ and $M_0 = 0.2, R = 0.5$, respectively. We have found that $\gamma \approx 0.11$ does not depend on the initial data, the position of the reflector, or the mass M_0 of the black hole. Another interesting feature worth of mentioning is an oscillatory component with period $\Delta \approx 6.80$ that superposes the scaling law (20).

The above features are typical of type-II critical behavior that arises in systems without a characteristic scale. A characteristic scale could be provided, for instance, by the introduction of a potential term. In the present case, we have a massless scalar field under specific conditions at the reflecting boundary surface $r = R$ such that the field equations (3)–(5) remain invariant under the change $r \to kr$ and $u \to ku$ with k a constant. Therefore, critical phenomena hold no matter if the scalar field is imploding in a reflecting boundary at $r = R$ or imploding towards the origin. With the reflecting boundary, the critical solution has mass $R/2$ and is described by the following equations:

$$\beta_{,r} = \frac{r}{4}\varphi_{,r}^2, \qquad V_{,r} = \mathrm{e}^{2\beta}, \qquad \varphi_{,r} = \frac{A_0}{rV}, \tag{21}$$

where A_0 is a constant of integration. It is possible to derive the exact solution of these equations by reducing them to $\frac{dV}{d\phi} - \frac{V^2}{A_0} + A_0 - C_1 V = 0$, where C_1 is a constant. Equivalently, this solution belongs to the general solution of Janis, Newmann, and Winicour [11] with the appropriate boundary conditions at the reflector barrier.

We have proceeded further with the numerical experiments exploring the regime of very strong self-gravitating scalar fields characterized by $\epsilon \gg \epsilon_*$. The time within which the apparent horizon forms decreases drastically with the increase of the initial amplitude. For instance, considering the initial data (17) with $\epsilon = 8.0$, the apparent

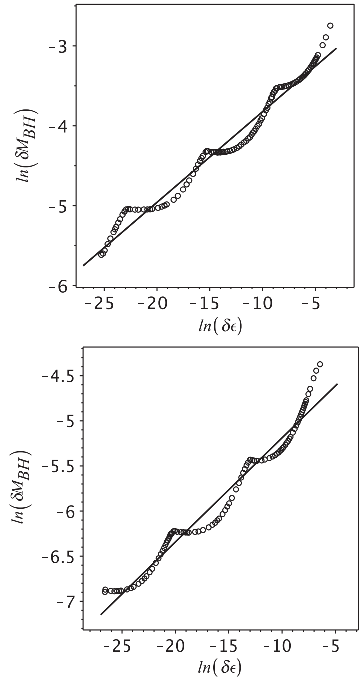

FIG. 3. Scaling law (20) for black hole formation with masses close to $R/2$. The numerical data can in this region can be fitted with $\gamma \approx 0.113$ and $\gamma \approx 0.115$ for the initial data using Eqs. (17) (upper plot) and (18) (lower plot). The critical parameters are, respectively, $\epsilon_* \approx 4.4010758346450$ and $\epsilon_* \approx 1.231345571457$. There is also an oscillatory-like component that superposes the scaling law.

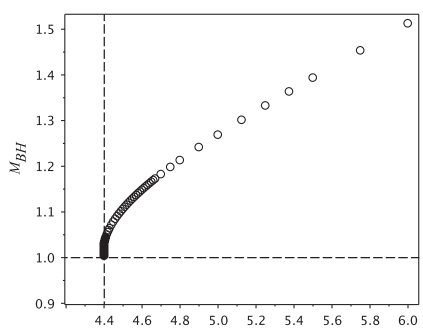

FIG. 2. Black holes mass as a function of the initial amplitude ϵ for the initial data (17), where $R/2 = 1.0$.

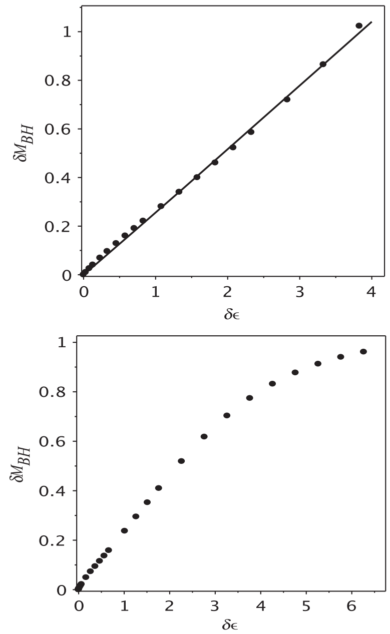

FIG. 4. Plots of the black hole masses $\delta M_{\mathrm{BH}} = M_{\mathrm{BH}} - M_{\mathrm{BH}}(\bar{\epsilon})$ with respect to $\delta\epsilon = \epsilon - \bar{\epsilon}$, where $\bar{\epsilon} = 4.670$ and 1.2355 for the initial data (17) and (18), respectively. In the first panel $\delta M_{\mathrm{BH}} \propto \delta\epsilon^{\nu}$ where $\nu \approx 1.011$, whereas in the second panel the black hole mass saturates for higher amplitudes.

horizon forms after $\Delta u \approx 0.047$, which is much smaller than $\Delta u \simeq 3$–3.5 for cases near the critical solutions. Nevertheless, the questions we have addressed are the following: Is there a relation between the large black hole masses and the initial amplitude ϵ? If yes, is this relation universal?

The numerical experiments have indicated that for the initial data (17) there exists a value of the amplitude, say $\bar{\epsilon}$, such that $\delta M_{\mathrm{BH}} = M_{\mathrm{BH}} - M_{\mathrm{BH}}(\bar{\epsilon}) \propto \delta\epsilon^{\nu}$, with $\delta\epsilon = \epsilon - \bar{\epsilon}$. We have found $\nu \approx 1.011$, implying that the black hole mass scales linearly with the amplitude ϵ in this regime. For the initial data (18), the black hole mass tends to saturate for higher amplitudes. The results are presented in Fig. 4.

Winicour *et al.* [8] have studied the formation of black holes for very high amplitudes of the initial data in the absence of the reflector barrier ($R = 0$ and $M_0 = 0$). They have shown the following features: First, the black hole mass scales linearly with the amplitude if the initial data have nonvanishing monopole moment $Q = \lim_{r\to\infty} r\varphi = \lim_{\eta\to\infty}\Phi$. For those initial data of compact support as in Eq. (18) ($Q = 0$), they have argued that the black hole masses saturate for higher amplitudes. Second, the high-amplitude dynamics does not depend on the details of the interior structure.

The results of Fig. 4 confirm the above features. Notice that for the initial data (17) with $Q(0) \neq 0$, the black hole mass scales linearly with the amplitude. With the initial data of compact support as in Eq. (18) [$Q(0) = 0$], the black hole mass saturates for higher amplitudes. Therefore, in the regime of very high amplitudes it does not matter if the scalar field is imploding towards the origin or against the reflector.

V. DISCUSSION

We have studied the threshold of black hole formation in a single model of a scalar field incident to a reflector barrier enclosing a Schwarzschild black hole. A weak initial scalar field is completely radiated away, leaving behind a black hole with mass M_0. A strong initial scalar field signalized by a strength above a certain critical value collapses and forms an apparent horizon outside the reflector. In this case, we have obtained the black hole masses for a large range of the initial amplitude. A scaling law with a mass gap describes the black hole masses correctly near $R/2$. The critical exponent, $\gamma \approx 0.11$, is independent of the initial data family and the location of the reflector barrier.

The features of critical behavior are robust for the scalar field collapse under different boundary conditions. In other words, the scalar field can either implode against a reflector like water waves in a dam or towards the origin that critical behavior is present at the threshold of black hole formation.

In the region of black hole masses resulting from the collapse of very strong self-gravitating scalar fields ($\epsilon \gg \epsilon_*$), the initial data play a significant role, as pointed out by Winicour *et al.* [8]. For those data with nonvanishing initial monopole moment, the black hole mass scales linearly with the amplitude ϵ. The black hole formation turns out to be independent of the details of the interior region. For those data with compact support, the monopole moment vanishes initially, and the black hole masses tend to saturate for higher initial amplitudes. The numerical results have confirmed these two cases.

We intend to extend this investigation to the unstable kink proposed by Barreto *et al.* [12]. In this case, the incident scalar field does not vanish at the reflector that surrounds Minkowski spacetime. However, a more compelling scenario is the formation of black holes in axisymmetric spacetimes; for instance, with the implosion of gravitational waves. To date, there are few works on critical phenomena in axisymmetry and no detailed study on the dependence of the black hole masses for strong pulses of gravitational waves.

ACKNOWLEDGMENTS

We are thankful to Jeffrey Winicour for important comments and insights. We are also grateful to the Brazilian agencies CNPq and CAPES for financial support.

[1] M. W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
[2] C. Gundlach and J. M. Martín-García, Living Rev. Relativity **10**, 5 (2007).
[3] A. M. Abrahams and C. R. Evans, Phys. Rev. Lett. **70**, 2980 (1993); Phys. Rev. D **49**, 3998 (1994).
[4] E. Sorkin, Classical Quantum Gravity **28**, 025011 (2011).
[5] M. W. Choptuik, E. W. Hirschmann, S. L. Liebling, and F. Pretorius, Phys. Rev. D **68**, 044007 (2003).
[6] R. Gómez, B. G. Schmidt, and J. Winicour, Phys. Rev. D **49**, 2828 (1994).
[7] C. Gundlach, R. Price, and J. Pullin, Phys. Rev. D **49**, 883 (1994); **49**, 890 (1994).
[8] R. Gómez and J. Winicour, J. Math. Phys. (N.Y.) **33**, 1445 (1992).
[9] H. P. de Oliveira, E. L. Rodrigues, and J. E. F. Skea, Phys. Rev. D **82**, 104023 (2010).
[10] J. Winicour, Living Rev. Relativity **15**, 2 (2012).
[11] A. Janis, E. T. Newman, and J. Winicour, Phys. Rev. Lett. **20**, 878 (1968).
[12] W. Barreto, R. Goméz, L. Lehner, and J. Winicour, Phys. Rev. D **54**, 3834 (1996).