Light propagation in the gravitational field of N arbitrarily moving bodies in 1PN approximation for high-precision astrometry

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The light-trajectory in the gravitational field of N extended bodies in arbitrary motion is determined in the first post-Newtonian approximation. According to the theory of reference systems, the gravitational fields of these massive bodies are expressed in terms of their intrinsic multipoles, allowing for the arbitrary shape and inner structure of these bodies. The results of this investigation aim towards a consistent general-relativistic theory of light propagation in the Solar System for high-precision astrometry at the sub-microarcsecond level of accuracy.

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I. INTRODUCTION

The primary objective of astrometry is the determination of the positions and motions of celestial objects, like stars or Solar system objects, from angular observations, that is to say to trace a light ray detected by an observer back to the celestial light source. Consequently, one fundamental assignment in relativistic astrometry concerns the precise description of the trajectory of a light signal, which is emitted by the celestial object and propagates through the gravitational field of the Solar system towards the observer. The growing accuracy of observations and new observational techniques have made it necessary to take subtle relativistic effects into account. In this respect, a breakthrough in astrometric precision has been achieved by the space-mission Hipparcos (launch: 8 August 1989) of European Space Agency (ESA), which has accomplished an astrometric precision of up to 1 milliarcsecond (mas) in measuring the positions of stars [1,2]. The next milestone in astrometry is established by the ESA astrometry mission Gaia (launch: 19 December 2013), where the positions of celestial objects can be determined within an accuracy of several microarcseconds (μas) in the ideal case (bright stars) [3].

While microarcsecond astrometry has been realized both theoretically and technologically within the Gaia mission, the dawning of sub-microarcsecond (sub-μas) or even nanoarcsecond (nas) astrometry is going to pass into the strategic focus of astronomers. For instance, NEAT [4,5] has been proposed to ESA as a candidate for one of the M-size missions within the Cosmic Vision 2015–2025, and is intended to reach a precision of about 50 nas. To achieve such accuracy, NEAT utilizes a pair of spacecraft that would fly in formation at a separation of 40 meters. This provides the long focal length necessary to generate high angular resolution to detect Earth-like planets. Further space missions like ASTROD [6,7], LATOR [8,9], ODYSSEY [10], SAGAS [11], or TIPO [12] are under discussion by ESA which require the knowledge of light propagation through the Solar System at the sub-μas or even at the nas level of accuracy. These missions are designed for a highly precise measurement of the spatial distance between two spacecrafts in order to determine the gravitational field within the Solar System. Also feasibility studies of earth-bound telescopes are presently under consideration which aim at an accuracy of about 10 nas [13]. In view of these technological advancements, a corresponding development in the theory of high-precision astrometry and especially in the theory of light propagation is indispensable.

In the limit of geometrical optics the path of a light signal (photons) is a null geodesic, governed by the geodesic equation which is valid in any coordinate system and reads in the exact form [14,15]:

$$\frac{d^2x^\alpha(\lambda)}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu(\lambda)}{d\lambda}\frac{dx^\nu(\lambda)}{d\lambda} = 0, \tag{1a}$$

$$g_{\alpha\beta}\frac{dx^\alpha(\lambda)}{d\lambda}\frac{dx^\beta(\lambda)}{d\lambda} = 0, \tag{1b}$$

where (1a) represents the geodesic equation, while the so-called isotropic condition (1b) must be imposed as additional constraint for null geodesics; $x^\alpha(\lambda)$ are the four-coordinates of the photon which depend on the affine curve parameter λ, and the Christoffel symbols are functions of the metric of curved spacetime,

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\frac{\partial g_{\beta\mu}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right), \tag{2}$$

where $g^{\alpha\beta}$ and $g_{\alpha\beta}$ are the contravariant and covariant components of the metric tensor, respectively.

Facing the fact that in the Solar System the gravitational fields are weak, $\frac{GM_A}{c^2R_A} \ll 1$, and the orbital velocities of the bodies are slow, $\frac{v_A}{c} \ll 1$, one is allowed for utilizing the post-Newtonian (PN) approximation for the metric tensor $g_{\alpha\beta}$ which is based on both of these assumptions [16]; here M_A, R_A, and v_A being mass, radius, and velocity, respectively, of some massive Solar System body (e.g., A = Sun, planets, moons, planetoids). This so-called weak-field slow-motion approximation admits an expansion of the metric of Solar System in powers of these small parameters, that means in inverse powers of the speed of light, e.g. [17]:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h^{(2)}_{\alpha\beta} + h^{(3)}_{\alpha\beta} + h^{(4)}_{\alpha\beta} + \mathcal{O}(c^{-5}), \qquad (3)$$

where $\eta_{\alpha\beta} = \mathrm{diag}(-1, +1, +1, +1)$ is the metric tensor of flat Minkowski spacetime, and $h^{(n)}_{\alpha\beta} \ll 1$ are small perturbations of it, which scale as follows, $h^{(n)}_{\alpha\beta} \sim \mathcal{O}(c^{-n})$, while their detailed structure will be considered later.

In doing so one has to bear in mind that such a post-Newtonian expansion assumes from the very beginning that all retardations are small. Therefore, the expansion in (3) is only valid inside the so-called near zone of the Solar System, $|\boldsymbol{x}| \ll \lambda_{\rm gr}$, characterized by the length of gravitational waves, $\lambda_{\rm gr}$, emitted by the Solar System [14,18–20]. To get an idea about the magnitude, one can relate this wavelength to a typical orbital period $T_{\rm orbit}$ of the Solar System bodies by $\lambda_{\rm gr} \sim cT_{\rm orbit} \sim 10^{17}$ meter, where we have considered as orbital period one revolution of Jupiter around the Sun. Hence, the boundary of the near zone, $|\boldsymbol{x}| \ll 10^{17}$ meter, is still beyond the most outer border of the Solar System and especially encompasses all Solar System objects.

Since one can define the position of any object only with respect to a concrete reference system, such description necessarily implies to introduce global coordinates which cover the entire curved spacetime and in respect to which the positions of the massive bodies, celestial objects and photons along their trajectories can be well defined. According to the recommendations of the International Astronomical Union (IAU) [21,22], the standard global reference system adopted in modern astrometry is the Barycentric Celestial Reference System (BCRS) with coordinates $(ct, \boldsymbol{x})$, where t is the coordinate time and $\boldsymbol{x}$ are Cartesian-like spatial coordinates from the origin of the global system (barycenter of the Solar System) to some field point.

In BCRS coordinates the exact light trajectory from the light source through the Solar System towards the observer, that means the exact solution of geodesic equation (1a), can be written as follows,

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + c(t - t_0)\boldsymbol{\sigma} + \Delta\boldsymbol{x}(t, t_0), \qquad (4)$$

where $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$ is the position of the light source at the moment t_0 of emission of the light signal, $\boldsymbol{\sigma} = \frac{\dot{\boldsymbol{x}}(-\infty)}{c}$ is the unit-direction of the light ray at past-null infinity, and $\Delta\boldsymbol{x}$ are gravitational corrections to the unperturbed light trajectory. These corrections are complicated expressions which depend on all parameters which characterize the metric of the Solar System. According to the post-Newtonian expansion (3), the gravitational corrections to the unperturbed light trajectory admit a corresponding expansion,

$$\Delta\boldsymbol{x} = \Delta\boldsymbol{x}_{\rm 1PN} + \Delta\boldsymbol{x}_{\rm 1.5PN} + \Delta\boldsymbol{x}_{\rm 2PN} + \mathcal{O}(c^{-5}), \qquad (5)$$

where the terms $\Delta\boldsymbol{x}_{\rm 1PN}$, $\Delta\boldsymbol{x}_{\rm 1.5PN}$, and $\Delta\boldsymbol{x}_{\rm 2PN}$ are of the order $\mathcal{O}(c^{-2})$, $\mathcal{O}(c^{-3})$, and $\mathcal{O}(c^{-4})$, respectively.

As mentioned, today's astrometric accuracy has reached a level of a few microarcseconds in angular observations, and the next scale of precision is the sub-microarcsecond level; for an historical survey see [23]. In order to analyze such highly precise astrometric data, a comprehensive and systematic relativistic procedure of data reduction is required [15,24]. Among several aspects of modern astrometry, two specific issues have carefully to be treated:

(**A**) First, the most fundamental concept in astrometric data reduction concerns the accurate definition of a set of several reference systems plus the coordinate transformations among them. In particular, for the determination of the light trajectory through the Solar System (N-body system), the following $N + 1$ coordinate systems are of primary importance: one global reference system (BCRS) with coordinates $(ct, \boldsymbol{x})$ and N local coordinate systems with coordinates $(cT_A, \boldsymbol{X}_A)$, one for each massive body $(A = 1, \ldots, N)$ and comoving with it. These $N + 1$ reference systems are fully defined by the form of their metric tensor. Furthermore, it is well known that the global metric (3) of an N-body system in the region exterior to the massive bodies admits a decomposition into two families of global multipoles, namely global mass multipoles m_L and global spin multipoles s_L [20,25–28]:

$$h^{(n)}_{\alpha\beta} = h^{(n)}_{\alpha\beta}(m_L, s_L), \quad \text{for } n = 2, 3, 4. \qquad (6)$$

These global multipoles describe the multipole structure of the entire Solar System as a whole. On the other side, from the theory of reference systems and in accordance with the IAU resolutions [21,22], it is clear that physically meaningful multipole moments of some massive body A have to be defined in the body's local reference system, namely local (also called intrinsic) mass multipoles M^A_L and local spin multipoles S^A_L. These intrinsic multipoles describe the multipole structure of each individual body separately. Consequently, the problem arises about how to express the global metric (3) in terms of local multipoles:

$$h^{(n)}_{\alpha\beta} = h^{(n)}_{\alpha\beta}(M^A_L, S^A_L), \quad \text{for } n = 2, 3, 4. \qquad (7)$$

In this respect, there are two advanced approaches in the relativistic theory of reference systems: the *Brumberg-Kopeikin* formalism (BK) [15,19,29–32] and the *Damour-Soffel-Xu* (DSX) approach [33–36]. Both these approaches coincide for all practical problems in celestial mechanics and astrometry [37] and have become a part of the IAU resolutions [21,22]. Thus it appears that the explicit form of the metric perturbations $h^{(2)}_{\alpha\beta}$, $h^{(3)}_{\alpha\beta}$, and $h^{(4)}_{00}$ in (7) are well-established expressions in celestial mechanics and modern astrometry, while the spatial components $h^{(4)}_{ij}$ in (7) deserve special attention in the case of extended bodies with full multipole structure and is presently an active field of research [38–41]; note that $h^{(4)}_{0i} = 0$.

(**B**) Second, the Solar System can be described as an isolated N-body system, where the bodies move under the influence of their mutual gravitational interaction, therewith associated are orbital motions of the bodies which are highly complicated. One has to be aware that the metric (3) and, therefore, the light trajectory (4) are functions of these complicated world lines $\boldsymbol{x}_A(t)$ of the massive bodies. In order to simplify this problem, one might want to expand the world line of some body A around some time moment t_A as follows,

$$\boldsymbol{x}_A(t) = \boldsymbol{x}_A + \frac{\boldsymbol{v}_A}{1!}(t - t_A) + \frac{\boldsymbol{a}_A}{2!}(t - t_A)^2 + \mathcal{O}(\dot{a}_A), \quad (8)$$

where $\boldsymbol{x}_A = \boldsymbol{x}_A(t_A)$, $\boldsymbol{v}_A = \boldsymbol{v}_A(t_A)$ and $\boldsymbol{a}_A = \boldsymbol{a}_A(t_A)$ are the position, velocity and acceleration of body A at time moment t_A, respectively, which are constant parameters. While terms like v_A/c are beyond 1PN approximation in the geodesic equation, one has to realize that the above series expansion is not an expansion in powers over c, thus all terms in (8) will contribute in 1PN approximation to the light ray metric, at least as long as no further assumptions like $a_A \sim G$ are asserted; cf. text below Eq. (80). In principle, the expression in (8) can be implemented into the metric tensor of the Solar System in (3). But such an approach leads rapidly to involved integrals when solving the geodesic equation in (1a), and implies an infinite series of integrals that apparently cannot be summed. Also the time-moment t_A is actually an open parameter and remains uncertain without further assumptions. Consequently, instead to apply for such an approximative expansion in (8), it is much preferable to find a solution for the light trajectory in terms of arbitrary world lines $\boldsymbol{x}_A(t)$. The actual world line of some massive body can finally be concretized by means of Solar System ephemerides; e.g. the JPL DE421 [42]. Accordingly, an important point which has to be carefully considered concerns the arbitrary motion of the massive bodies.

In this investigation we will account for both of these fundamental aspects addressed above: issue (**A**) is incorporated by the DSX approach, while issue (**B**) is accounted for by integration by parts of geodesic equation plus the evidence that the remnants of this procedure represent terms beyond 1PN approximation. In this way, a systematic approach is developed in order to determine the light trajectory in the Solar System in (4) in the first post-Newtonian (1PN) approximation,

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + c(t - t_0)\boldsymbol{\sigma} + \Delta\boldsymbol{x}_{1\mathrm{PN}}(t, t_0) + \mathcal{O}(c^{-3}), \quad (9)$$

where the global metric of the Solar System is described from the very beginning in terms of intrinsic multipoles of the extended bodies in arbitrary motion. Such a systematic formalism is an imperative prerequisite for extending the model to higher-order terms in the post-Newtonian expansion in (5).

The article is organized as follows: In Sec. II we will motivate the inevitability for an analytical solution of light trajectory in the field of N arbitrarily moving bodies with full multipole structure in post-Newtonian order for sub-microarcsecond astrometry. In Sec. III the geodesic equation in 1PN approximation and the initial-boundary conditions are introduced which determine a unique solution of the geodesic equation. The metric of the Solar System in terms of intrinsic multipoles in accordance with the IAU resolutions is given in Sec. IV. In order to simplify the integration procedure, new variables for space and time and the corresponding transformation of geodesic equation and metric tensor are presented in Sec. V. In Sec. VI the first integration of geodesic equation is performed, while in Sec. VII some specific cases (arbitrarily moving monopoles, dipoles, quadrupoles, and one body at rest with full multipole structure) are considered. It will be demonstrated that in the limit of bodies at rest the results are in agreement with known results in the literature. In Sec. VIII the second integration of the geodesic equation is represented, while some specific cases (arbitrarily moving monopoles, dipoles, quadrupoles, and one body at rest with full multipole structure) are considered in Sec. IX. In the limit of bodies at rest an agreement with known results in the literature is shown. Expressions for the observable relativistic effects of time delay and light deflection are given in Sec. X. A summary and outlook can be found in Sec. XI.

A. Notation of impact vectors

It appears to be considerate to introduce the notation in use regarding the impact vectors, while further notations are shifted to Appendix A.

While the exact light ray $\boldsymbol{x}(t)$ in (4) is a complicated function, the unperturbed light ray in flat Minkowski spacetime is just given by a straight line,

$$\boldsymbol{x}_{\mathrm{N}}(t) = \boldsymbol{x}_0 + c(t - t_0)\boldsymbol{\sigma}, \quad (10)$$

where the subscript N stands for Newtonian limit. One may introduce the following impact vector:

$$\boldsymbol{\xi} = \boldsymbol{\sigma} \times (\boldsymbol{x}_{\mathrm{N}}(t) \times \boldsymbol{\sigma}) = \boldsymbol{\sigma} \times (\boldsymbol{x}_0 \times \boldsymbol{\sigma}), \qquad d = |\boldsymbol{\xi}|. \quad (11)$$

The impact-vector in (11) points from the origin of the global system (BCRS) towards the point of closest approach of the unperturbed light ray to that origin. The impact vector in (11) is time independent, both in the case of massive bodies at rest as well as in the case of massive bodies in motion.

1. Massive bodies at rest

Massive bodies at rest means their positions to be constant with respect to the global system: $\boldsymbol{x}_A = \text{const}$. We will make use of the following notation for the vector from the massive body at rest towards the photon propagating along the exact light trajectory,

$$\boldsymbol{r}_A = \boldsymbol{x}(t) - \boldsymbol{x}_A, \tag{12}$$

with the absolute value $r_A = |\boldsymbol{r}_A|$. The vector from the massive body at rest towards the photon along the unperturbed light trajectory reads

$$\begin{aligned} \boldsymbol{r}_A^{\rm N} &= \boldsymbol{x}_{\rm N}(t) - \boldsymbol{x}_A \\ &= \boldsymbol{x}_0 + c(t - t_0)\boldsymbol{\sigma} - \boldsymbol{x}_A, \end{aligned} \tag{13}$$

with the absolute value $r_A^{\rm N} = |\boldsymbol{r}_A^{\rm N}|$, and obviously $\boldsymbol{r}_A = \boldsymbol{r}_A^{\rm N} + \mathcal{O}(c^{-2})$. We also need the vector from the massive body at rest towards the photon at the moment of signal emission,

$$\boldsymbol{r}_A^0 = \boldsymbol{x}_0 - \boldsymbol{x}_A, \tag{14}$$

with the absolute value $r_A^0 = |\boldsymbol{r}_A^0|$. Note that in the case of massive bodies at rest there will be no time argument in $\boldsymbol{r}_A$ and $\boldsymbol{r}_A^{\rm N}$, irrespective of the fact that the distance between the photon and the body actually depends on time due to the propagation of the photon. In the case of massive bodies at rest, we introduce the following impact vector:

$$\boldsymbol{d}_A = \boldsymbol{\sigma} \times (\boldsymbol{r}_A^{\rm N} \times \boldsymbol{\sigma}), \qquad d_A = |\boldsymbol{d}_A|. \tag{15}$$

The impact-vector in (15) is time independent, $\dot{\boldsymbol{d}}_A = 0$, and points from the origin of local coordinate system of massive body A towards the unperturbed light ray at the time of closest approach to that origin, defined later by Eq. (33). Notice that the term "weak gravitational field" implies $d_A \gg \frac{GM_A}{c^2}$.

2. Massive bodies in motion

In the case of massive bodies in motion, their positions become time dependent: $\boldsymbol{x}_A(t)$. Then we will make use of the following notation for the vector from the massive body towards the photon propagating along the exact light trajectory:

$$\boldsymbol{r}_A(t) = \boldsymbol{x}(t) - \boldsymbol{x}_A(t), \tag{16}$$

with the absolute value $r_A(t) = |\boldsymbol{r}_A(t)|$. The vector from the massive body in motion towards the photon along the unperturbed light trajectory reads

$$\begin{aligned} \boldsymbol{r}_A^{\rm N}(t) &= \boldsymbol{x}_{\rm N}(t) - \boldsymbol{x}_A(t) \\ &= \boldsymbol{x}_0 + c(t - t_0)\boldsymbol{\sigma} - \boldsymbol{x}_A(t), \end{aligned} \tag{17}$$

with the absolute value $r_A^{\rm N}(t) = |\boldsymbol{r}_A^{\rm N}(t)|$ and obviously $\boldsymbol{r}_A(t) = \boldsymbol{r}_A^{\rm N}(t) + \mathcal{O}(c^{-2})$. We also will need the vector from the massive body towards the photon at the time moment of emission of the light signal, given by

$$\boldsymbol{r}_A^{\rm N}(t_0) = \boldsymbol{x}_0 - \boldsymbol{x}_A(t_0), \tag{18}$$

with the absolute value $r_A^{\rm N}(t_0) = |\boldsymbol{r}_A^{\rm N}(t_0)|$. In the case of massive bodies in motion we introduce the following impact vector:

$$\boldsymbol{d}_A(t) = \boldsymbol{\sigma} \times (\boldsymbol{r}_A^{\rm N}(t) \times \boldsymbol{\sigma}), \qquad d_A(t) = |\boldsymbol{d}_A(t)|. \tag{19}$$

The impact vector in (19) is time dependent, $\dot{\boldsymbol{d}}_A \neq 0$, and points from the origin of local coordinate system of massive body A towards the unperturbed light ray at the time of closest approach to that origin. The time dependence of the impact vector in (19) is solely caused by the motion of the massive body, that means a time derivative of (19) is proportional to the orbital velocity of this body, $\dot{\boldsymbol{d}}_A(t) = \boldsymbol{\sigma} \times (\boldsymbol{\sigma} \times \boldsymbol{v}_A(t))$. The term "weak gravitational field" implies $d_A(t_A^*) \gg \frac{GM_A}{c^2}$ for the time of closest approach of the light ray to the massive body, which will be defined later; see Eq. (33).

II. MOTIVATION

Before representing our approach, it is most appropriate to review in brief the recent advancements in the theory of light propagation in the weak gravitational field of N massive bodies. It is clear that in a short review like the present, it is impossible to consider all articles written on the subject during the last decades, and many important calculations must remain unmentioned. Instead, the brief survey is enforced to be focussed on those results, which are of upmost relevance for our considerations.

As mentioned in the introductory section, the BCRS metric of Solar System admits an expansion in terms of multipoles. By inserting the decomposition of the metric in terms of global multipoles (6) into the geodesic equation (1a) one obtains a corresponding decomposition of the light-ray perturbation (5) in terms of global multipoles:

$$\Delta \boldsymbol{x} = \sum_{l=0}^{\infty} \Delta \boldsymbol{x}(m_L, s_L) + \mathcal{O}(c^{-5}). \tag{20}$$

Likewise, inserting the decomposition of the metric in terms of local multipoles (7) into the geodesic equation (1a) one obtains a corresponding decomposition of the light-ray perturbation (5) in terms of local multipoles:

$$\Delta \boldsymbol{x} = \sum_{l=0}^{\infty} \Delta \boldsymbol{x}(M_L^A, S_L^A) + \mathcal{O}(c^{-5}). \tag{21}$$

In the subsequent survey it will be carefully distinguished whether a decomposition in terms of global multipoles (20) or in terms of local multipoles (21) is meant. Let us gradually consider these individual terms, depending on the accuracy of the astrometric measurements.

A. Astrometry at the milliarcsecond level of accuracy

For astrometry on milliarcsecond (mas) level of accuracy it is sufficient to approximate all Solar System bodies as spherically symmetric objects. In the case of N monopoles at rest, the corresponding correction term in (21) reads [15]

$$\Delta \boldsymbol{x}_{\rm 1PN}^{M}(t,t_0) = -\frac{2G}{c^2}\sum_{A=1}^{N} M_A \times \left(\frac{\boldsymbol{d}_A}{r_A^{\rm N} - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}} - \frac{\boldsymbol{d}_A}{r_A^{0} - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{0}} - \boldsymbol{\sigma}\ln\frac{r_A^{\rm N} - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}}{r_A^{0} - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{0}}\right), \tag{22}$$

where the sum in (22) runs over all massive bodies of the Solar System. For a comparison of (22) with [15] it might be useful to recall: $\ln\frac{r_A^{\rm N} - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}}{r_A^{0} - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{0}} = -\ln\frac{r_A^{\rm N} + \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}}{r_A^{0} + \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{0}}$. The magnitude of light deflection for grazing rays amounts to 1.75×10^3 mas for the Sun, 16.3 mas for Jupiter, 5.8 mas for Saturn, 2.1 mas for Uranus, and 2.5 mas for Neptune [43].

Since in reality these massive bodies are moving, the question arises about how to implement the time dependence of the positions of these gravitating bodies. This particular issue has thoroughly been solved in [44] in first post-Minkowskian (1PM) approximation, and will be one aspect in the following section.

B. Astrometry at the microarcsecond level of accuracy

Meanwhile, modern space-based astrometry has accomplished the step from milliarcsecond level to the microarcsecond level of accuracy [23]. In order to determine the light trajectory at the μas level of accuracy, some further subtle relativistic effects of light propagation need to be accounted for in addition to the monopole term in (22). These include the following:

(i) the quadrupole structure of the massive bodies,
(ii) the motion of the massive bodies,
(iii) the post-post-Newtonian monopole term.

The fundamentals of the corresponding theoretical model of light propagation have been worked out in [15,17,43,45], and later be refined in [46,47]. The results of these investigations have been adopted as one of two model for the Gaia data reduction and which is called the Gaia relativistic model (GREM). Another approach has been developed in [48–52], which is the second model in use for Gaia data reduction and which is called the relativistic astrometric model (RAMOD). Both these models are designed for relativistic astrometry at the microarcsecond level of accuracy and allow for an independent check of their results. Let us consider in more detail each of these three subtle effects which are listed above.

1. Impact of the quadrupole field on light trajectory

The analytical solution for the light trajectory in a quadrupole field of a body at rest and in post-Newtonian approximation has been determined in [45], where the time dependence of the coordinates of the photon and the solution of the boundary value problem for the geodesic equations has been obtained at the first time. These results were later confirmed by different approaches in [53–55]. The formula for the quadrupole light deflection in 1PN approximation can be found in [17,43,45] and should be given here in its complete form:

$$\Delta \boldsymbol{x}_{\rm 1PN}^{Q}(t,t_0) = \frac{G}{c^2}\sum_{A=1}^{N}\frac{1}{d_A^2}[\boldsymbol{\alpha}_A(\mathcal{U}_A - \mathcal{U}_A^0) + \boldsymbol{\beta}_A(\mathcal{V}_A - \mathcal{V}_A^0) + \boldsymbol{\gamma}_A(\mathcal{F}_A - \mathcal{F}_A^0) + \boldsymbol{\delta}_A(\mathcal{E}_A - \mathcal{E}_A^0)]. \tag{23}$$

The sum in (23) runs over all massive bodies of the Solar System. In [47] it has been shown that all terms in the second line of Eq. (23) are negligible for μas astrometry, but this fact is not of much relevance in our investigation here. The time-independent vectorial coefficients in (23) are given by

$$\alpha_A^k = -M_{i_1 i_2}^A d_A^k \sigma^{i_1}\sigma^{i_2} + 2M_{i_1 k}^A d_A^{i_1} - 2M_{i_1 i_2}^A d_A^{i_2}\sigma^{i_1}\sigma^k - \frac{4}{d_A^2} M_{i_1 i_2}^A d_A^{i_1} d_A^{i_2} d_A^k, \tag{24}$$

$$\beta_A^k = +M_{i_1 i_2}^A \sigma^{i_1}\sigma^{i_2}\sigma^k - 2M_{i_1 k}^A \sigma^{i_1} + \frac{4}{d_A^2} M_{i_1 i_2}^A d_A^{i_2} d_A^k \sigma^{i_1} - \frac{2}{d_A^2} M_{i_1 i_2}^A d_A^{i_1} d_A^{i_2}\sigma^k, \tag{25}$$

$$\gamma_A^k = +M_{i_1 i_2}^A d_A^{i_1} d_A^{i_2} d_A^k - M_{i_1 i_2}^A d_A^2 d_A^k \sigma^{i_1}\sigma^{i_2} + 2M_{i_1 i_2}^A d_A^{i_2} d_A^2 \sigma^{i_1}\sigma^k, \tag{26}$$

$$\delta_A^k = -M_{i_1 i_2}^A d_A^{i_1} d_A^{i_2}\sigma^k + M_{i_1 i_2}^A d_A^2 \sigma^{i_1}\sigma^{i_2}\sigma^k + 2M_{i_1 i_2}^A d_A^{i_2} d_A^k \sigma^{i_1}, \tag{27}$$

with intrinsic mass quadrupole moments $M^A_{i_1 i_2}$. The time-dependent scalar functions in (23) are given by

$$\mathcal{U}_A = \frac{1}{r^{\rm N}_A}\frac{r^{\rm N}_A + \boldsymbol{\sigma}\cdot\boldsymbol{r}^{\rm N}_A}{r^{\rm N}_A - \boldsymbol{\sigma}\cdot\boldsymbol{r}^{\rm N}_A}, \qquad \mathcal{U}^0_A = \frac{1}{r^0_A}\frac{r^0_A + \boldsymbol{\sigma}\cdot\boldsymbol{r}^0_A}{r^0_A - \boldsymbol{\sigma}\cdot\boldsymbol{r}^0_A}, \tag{28}$$

$$\mathcal{V}_A = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}^{\rm N}_A}{r^{\rm N}_A}, \qquad \mathcal{V}^0_A = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}^0_A}{r^0_A}, \tag{29}$$

$$\mathcal{F}_A = \frac{1}{(r^{\rm N}_A)^3}, \qquad \mathcal{F}^0_A = \frac{1}{(r^0_A)^3}, \tag{30}$$

$$\mathcal{E}_A = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}^{\rm N}_A}{(r^{\rm N}_A)^3}, \qquad \mathcal{E}^0_A = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}^0_A}{(r^0_A)^3}. \tag{31}$$

The light deflection for grazing rays at giant planets due to their quadrupole-structure amounts to 240 μas for Jupiter, 95 μas for Saturn, 8 μas for Uranus, and 10 μas for Neptune [43], which clearly indicate that the effect of quadrupole light deflection must be taken into account for astrometry at the microarcsecond level of accuracy.

2. Impact of the motion of massive bodies on light trajectory

One of the most sophisticated challenges in relativistic astrometry concerns the problem of the motion of massive bodies and its impact on the light trajectories. While the solutions in (22) and (23) are valid for bodies at rest, $\boldsymbol{x}_A = 0$, in reality the global coordinates of the bodies depend on time, $\boldsymbol{x}_A(t)$, which is a highly complicated function in an N-body system due to the mutual gravitational interaction of the massive bodies. These complicated world lines of the massive bodies in the Solar System can be series expanded [17,43],

$$\boldsymbol{x}_A(t) = \boldsymbol{x}_A + \boldsymbol{v}_A(t - t_A) + \mathcal{O}(a_A), \tag{32}$$

where $\boldsymbol{x}_A$ and $\boldsymbol{v}_A$ can be thought of as the actual position and velocity of body A taken from an ephemeris for some instant of time t_A. Let us underline here that the impact of the term $\boldsymbol{v}_A$ in (32) on the light trajectory is of 1PN order, besides the fact that this term is proportional to the velocity of the body; recall that on the other side terms proportional to v_A/c are of 1.5PN order in the theory of light propagation; cf. text below Eq. (80).

An analytical integration of light trajectory in the field of an uniformly moving body (32) has been derived in closed form in 1PN approximation in [56] and later also in 1PM approximation by means of a suitable Lorentz transformation of the light trajectory [57]. As long as one considers uniformly moving bodies, the instant of time t_A in the expansion (32) remains an open parameter, but by all means heuristic arguments can be put forward for a meaningful choice for it. Perhaps the most fruitful suggestion was that given in [58], where it was supposed to accept that this parameter coincides with the time of closest approach of the light ray to the massive body, t^*_A, given by an implicit relation,

$$t^*_A = t_0 - \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{x}_0 - \boldsymbol{x}_A(t^*_A))}{c} + \mathcal{O}(c^{-2}), \tag{33}$$

$$= t_1 - \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_A(t^*_A))}{c} + \mathcal{O}(c^{-2}), \tag{34}$$

where $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$ is the global spatial coordinate of the source at the moment of emission of the light signal and $\boldsymbol{x}_1 = \boldsymbol{x}(t_1)$ is the global spatial coordinate of the space-based observer at the moment of observation of the light signal; cf. Eq. (5.13) in [17]. As a result, in the light propagation formulas (22) and (23) one would have to insert $\boldsymbol{x}_A(t^*_A)$. That educated guess was triggered by the idea that the biggest influence on the light ray the body exerts when the photon passes nearest to it. But a unique justification of this suggestion has not been evidenced at that time. Further arguments have later been put forward that partially justify the computation of the parameters of the linear model (32) to match the real position and velocity of the body at the moment of closest approach between the light ray and the real trajectory of the body [59].

A rigorous solution of the problem of light propagation in the field of arbitrarily moving pointlike monopoles and in the first post-Minkowskian approximation has been found in [44], where advanced integration methods have been applied that were originally been introduced in [53] for stationary fields and further developed in [60] for time-dependent fields. According to the solution in [44], the positions of the bodies have to be computed at the retarded instant of time, $t^{\rm ret}_A$, given by the implicit relation

$$t^{\rm ret}_A = t - \frac{|\boldsymbol{x}(t) - \boldsymbol{x}_A(t^{\rm ret}_A)|}{c}. \tag{35}$$

The expression (35) is valid for an arbitrary time, e.g. either $t = t_0$ or $t = t_1$. With the aid of this rigorous approach in [44] it has been shown that if the positions and velocities of the bodies are taken at $t^{\rm ret}_A$ then the effects of acceleration and the effects due to time dependence of velocity of the bodies are much smaller than 1 micro-arcsecond. The numerical accuracy of various approaches have been investigated in [59], where it was demonstrated for the monopole term that for an accuracy of 1 μas it is sufficient to take the 1PN solution of a motionless body in (22), if the position $\boldsymbol{x}_A$ of body A is taken at either t^*_A or $t^{\rm ret}_A$.

3. The post-post-Newtonian monopole term

Actually, corrections of post-post-Newtonian (2PN) order to the light ray in (5) will not be on the scope of the present investigation, but should briefly be mentioned here for reasons of completeness about μas astrometry.

While several post-post-Newtonian effects of light deflection due to a monopole at rest have been determined a long time ago [61–66], the determination of the explicit time dependence of the photons coordinate is mandatory in the data reduction for highly sophisticated astrometry missions like Gaia. In general, such 2PN corrections to the light ray caused by the monopole structure of one massive body are proportional to the square of its mass M_A, that means

$$\Delta \boldsymbol{x}^{M}_{2\mathrm{PN}}(t,t_0) = \frac{G^2}{c^4}\sum_{A=1}^{N} M_A^2 (\boldsymbol{C}_A - \boldsymbol{C}_A^0). \tag{36}$$

In this respect, an important progress has been made in [67], where a 2PN solution for the light trajectory in the Schwarzschild field as a function of coordinate time in a number of coordinate gauges was obtained; see also [15,17]. From the 2PN solution in [15,17,67], given in an iterative form, one can deduce the expicit form of the vectorial function $\boldsymbol{C}_A$ in Eq. (36) (see Appendix B),

$$\begin{aligned}\boldsymbol{C}_A &= 4\frac{\boldsymbol{d}_A}{r_A^{\mathrm{N}}}\frac{1}{r_A^{\mathrm{N}} - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\mathrm{N}}} - 4\frac{\boldsymbol{\sigma}}{r_A^{\mathrm{N}}}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\mathrm{N}}}{r_A^{\mathrm{N}} - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\mathrm{N}}} + \frac{1}{4}\frac{\boldsymbol{r}_A^{\mathrm{N}}}{(r_A^{\mathrm{N}})^2}\\ &\quad - 4\frac{\boldsymbol{d}_A}{r_A^{\mathrm{N}}}\frac{1}{r_A^{\mathrm{N}} - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\mathrm{N}}}\ln\left(r_A^{\mathrm{N}} - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\mathrm{N}}\right)\\ &\quad - 4\frac{\boldsymbol{\sigma}}{r_A^{\mathrm{N}}}\ln\left(r_A^{\mathrm{N}} - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\mathrm{N}}\right) - \frac{15}{4}\frac{\boldsymbol{\sigma}}{d_A}\arctan\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\mathrm{N}}}{d_A}\\ &\quad - \frac{15}{4}\boldsymbol{d}_A\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\mathrm{N}}}{d_A^3}\left[\frac{\pi}{2} + \arctan\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\mathrm{N}}}{d_A}\right],\end{aligned} \tag{37}$$

and $\boldsymbol{C}_A^0$ is deduced from (37) by the replacements $\boldsymbol{r}_A^{\mathrm{N}} \to \boldsymbol{r}_A^0$ and $r_A^{\mathrm{N}} \to r_A^0$. Generalizations of that 2PN solution for the case of the parametrized post-post-Newtonian metric have been given in [46], where the numerical magnitudes of the post-post-Newtonian terms have been estimated and a practical algorithm for highly effective computation of the post-post-Newtonian effects has been formulated.

Two alternative approaches to the calculation of propagation time and direction of the light rays have been formulated recently. Both approaches allow one to avoid explicit integration of the geodesic equations for light rays. The first approach in [68,69] is based on the use of Synge's world function. Another approach is based on the eikonal concept and has been developed in [70] in order to investigate the light propagation in the field of a spherically symmetric body.

In order to get an idea about the magnitude of 2PN effects, let us recall the well-known fact that the 2PN monopole correction for grazing light rays at the Sun is about 11 μas [15,17,61–66]. In the concrete case of ESA astrometry-mission Gaia there is a sun shield which is tilted at a 45 degree angle to the Sun, so that the telescopes observe a space region where the post-post-Newtonian effects of the Sun become negligible. However, while the Gaia mission will not observe close to the Sun, it will observe very close to the surface of giant planets. A corresponding detailed investigation in [46] has recovered the remarkable fact, that post-post-Newtonian corrections become relevant for light rays grazing the surface of the giant planets. As outlined in [46], the reason for this fact is the inevitable occurrence of coordinate-dependent enhanced terms, because real astrometric measurements incorporate the use of concrete global coordinate systems and inherit the choice of coordinate-dependent impact parameters, see also [71].

C. Astrometry at the sub-microarcsecond level of accuracy

In order to determine the light trajectory with an unprecedented accuracy at the sub-μas level of accuracy, many further subtle relativistic effects in the theory of light propagation have to be accounted for. Let us deploy just a minimal set of corresponding requirements which need to be considered:

(i) full set of mass multipoles,
(ii) spin-dipole,
(iii) some higher spin multipoles,
(iv) motion of arbitrarily moving massive bodies,
(v) post-post-Newtonian effects.

Of course, what is really necessary to implement into the final relativistic model depends on what is actually meant by the term "sub-μas level" of accuracy. For instance, for a model aiming at an accuracy of 0.1 μas level, there is no need to take into account any higher spin multipoles in 1.5PN approximation, while a model at the 0.01 μas level necessitates such terms. Let us look at the present situation in the theory of light propagation at the sub-μas level of accuracy by considering each of these five issues mentioned.

1. Impact of higher mass multipoles on light trajectory

Keeping the magnitude of quadrupole light deflection by giant planets in mind, it can easily be foreseen that a light propagation model at the sub-μas level needs to take into account the impact of higher mass multipoles beyond the well-known mass quadrupole term in (23).

A systematic approach to the integration of light geodesic equations in the stationary gravitational field of a localized source at rest, $\boldsymbol{x}_A = \mathrm{const}$, located at the origin of coordinate system and having time-independent local multipole structure, M_L^A and S_L^A, has been worked out in [53] in 1PN and 1.5PN approximation. In particular, sophisticated integration methods have been introduced in [53] allowing for analytical integrations of geodesic equations in the complex field of multipoles to arbitrary order.

Furthermore, the case of light propagation in the field of a localized source at rest which is characterized by time-dependent multipoles has been investigated in [72,73] in 1PM approximation. This solution can be interpreted in two different ways:

(i) Either the localized source is thought of to be composed of N arbitrarily moving bodies, but then the time-dependent multipoles have to be interpreted as global multipoles, $m_L(t)$ and $s_L(t)$, which characterize the entire N-body system as a whole.

(ii) Or the localized source is thought of as being just one body A at rest with intrinsic multipoles, $M_L^A(t)$ and $S_L^A(t)$, which characterize that single body.

But neither of these two interpretations allow one to consider the solution in [72,73] to be valid for the case of arbitrarily moving bodies, $\boldsymbol{x}_A(t)$, and with local multipoles $M_L^A(t)$ and $S_L^A(t)$ characterizing each individual body A of the N-body system.

The influence of time-independent intrinsic mass multipoles of higher order on a light ray by an isolated axisymmetric body at rest has also been investigated in [54], using a different approach based on the multipole expansion of time transfer function. Explicitly, a formula for the bending of light due to any order of multipole moments has been derived and numerical estimates have been presented. For instance, it has been found in [54] that the light deflection due to the mass octupole structure amounts to 0.016 μas, and due to the mass hexadecupole structure it amounts to 9.6 μas for grazing rays at Jupiter.

Recently, in [74] the light propagation in the field of an uniformly moving axisymmetric body has been determined in terms of the full multipole structure of the body. Furthermore, an analytical formula for the time delay caused by the gravitational field of a body in slow and uniform motion with arbitrary multipoles has been derived in [75].

Assessment: According to these investigations in the literature, the 1PN solution $\Delta\boldsymbol{x}_{\rm 1PN}(t, t_0)$ in (9) in the gravitational field of N arbitrarily moving bodies, $\boldsymbol{x}_A(t)$, and with time-dependent intrinsic mass multipoles, $M_L^A(t)$, has not been determined thus far, but appears to be an inevitable requirement for sub-μas astrometry.

2. Light propagation in the field of spin-dipoles

The next term beyond μas astrometry which is certainly required at sub-μas level is the impact of rotational motion of massive bodies on the light propagation; note that such a term is already of 1.5PN order. For instance, the light deflection due to rotational motion of Solar System bodies amounts to 0.7 μas for the grazing ray at the Sun, 0.2 μas for the grazing ray at Jupiter, and 0.04 μas for the grazing ray at Saturn [43,45].

The first solution of the light trajectory $\Delta\boldsymbol{x}^S_{\rm 1.5PN}(t, t_0)$ in the gravitational field of massive bodies at rest possessing a time-independent intrinsic spin dipole, $\boldsymbol{S}^A$, has been obtained in [45]. This solution provides all the details of light propagation, especially the time dependence of the coordinates of the photon and the solution of the corresponding boundary value problem.

Utilizing advanced integration methods, a solution for the light trajectory in the field of one body at rest and having time-independent local spin-dipole, $\boldsymbol{S}^A$, has also been obtained in [53] in 1.5PN approximation. Moreover, an analytical solution in 1PM approximation for the case of light propagation in the field of an arbitrarily moving pointlike spin-dipole, $\boldsymbol{s}(t)$ (expressed in terms of a global spin-tensor) has been derived in [76].

Assessment: In view of these few investigations available in the literature, the task remains to determine the light trajectory $\Delta\boldsymbol{x}^S_{\rm 1.5PN}(t, t_0)$ in the gravitational field of an arbitrarily moving body, $\boldsymbol{x}_A(t)$, carrying a time-dependent intrinsic spin-dipole, $\boldsymbol{S}^A(t)$.

3. Impact of higher spin multipoles on light trajectory

As mentioned above, a solution for the light trajectory in the stationary gravitational field of a localized source at rest, $\boldsymbol{x}_A = {\rm const}$, with time-independent local multipoles, M_L^A and S_L^A, has been determined in 1.5PN approximation in [53]. Furthermore, the light trajectory in the field of a localized source with time-dependent global multipoles, $m_L(t)$ and $s_L(t)$, has been obtained in [72,73] in 1PM approximation. As has been noticed already, the results in [72,73] can be considered a solution for the light trajectory in the field of either a system of N arbitrarily moving bodies characterized by global multipoles or in the field of one body A at rest characterized by local multipoles, but not as solution for the light trajectory in the field of arbitrarily moving bodies characterized by intrinsic multipoles.

Recent calculations [77] have revealed, that the light deflection due to spin-octupole structure of massive bodies at rest amounts to about 0.015 μas for Jupiter and about 0.006 μas for Saturn for grazing rays. Therefore, a model at the sub-μas level has to take into account at least the spin-octupole term which is of 1.5PN order in the theory of light propagation.

Assessment: According to these facts, the 1.5PN solution $\Delta\boldsymbol{x}_{\rm 1.5PN}(t, t_0)$ in (5) in the gravitational field of N arbitrarily moving bodies, $\boldsymbol{x}_A(t)$, and with time-dependent intrinsic spin multipoles, $S_L^A(t)$, has not been determined so far and remains an unavoidable task in order to achieve an astrometric accuracy at the sub-μas level.

4. Impact of the motion of the bodies on light trajectory

The Solar System bodies are moving along their individual world lines, $\boldsymbol{x}_A(t)$, which are complicated functions of time due to the mutual interaction among the bodies, implying that the metric and the light trajectory become also complicated functions of time. As explicated in Sec. II B 2, for astrometry this highly sophisticated problem can be treated by using the standard 1PN solutions of motionless bodies, $\boldsymbol{x}_A = \text{const}$, as long as the positions of the bodies are taken at either their retarded times t_A^{ret} or at their time of closest approach to the light ray t_A^*.

However, in the investigation [59] it has been shown that for an astrometric astrometry better than 0.2 μas one needs to take into account the motion of the bodies. In particular, it is not sufficient to apply for a simple series-expansion of the bodies world line, $\boldsymbol{x}_A(t) = \boldsymbol{x}_A + \boldsymbol{v}_A(t - t_A)$, as given by Eq. (32). Instead, one has to determine the light trajectory in the field of arbitrarily moving bodies $\boldsymbol{x}_A(t)$. For the case of arbitrarily moving monopoles such a solution has been provided in [44], and for the case of arbitrarily moving bodies with quadrupole structure such a solution has been found in [78]. But for arbitrarily moving bodies with higher intrinsic multipoles there are no solutions available so far.

Assessment: As a result, for sub-μas astrometry the approximative expansion in (32) is not applicable, instead of that one has to find a solution for the light trajectory in terms of arbitrary world lines $\boldsymbol{x}_A(t)$. The real world lines of the massive bodies can finally be implemented into the model by means of Solar System ephemerides [42].

5. *Post-post-Newtonian effects*

The most intricate issue in the theory of sub-microarcsecond astrometry will be the post-post-Newtonian effects $\Delta\boldsymbol{x}_{\text{2PN}}(t, t_0)$ in (5). Such 2PN corrections to the light ray will not be on the scope of this investigation, but some remarks should be in order.

The largest perturbation term is of course the monopole term, $\Delta\boldsymbol{x}^M_{\text{2PN}}(t, t_0)$, which in the case of pointlike bodies at rest, $\boldsymbol{x}_A = \text{const}$, has been calculated for the first time in [67]; see also [15,17]. In reality, the bodies are moving, and one has to treat the problem of moving monopoles in post-post-Newtonian approximation where, however, only very limited results are available thus far. In particular, in [79] the light deflection in 2PN approximation in the field of two moving point-like bodies has been determined, using two essential approximations: (i) both the light source and the observer are assumed to be located at infinity in an asymptotically flat space, and (ii) the relative separation distance of the bodies is assumed to be much smaller than the impact parameter of incoming light ray. These approximations are of interest in the case of studying light propagation in the field of a binary pulsar, but they are not applicable for real astrometric observations in the Solar System.

Presently it remains unknown, how large the impact of higher mass multipoles on light deflection in post-post-Newtonian order is. In order to tackle this problem, an extension of the DSX metric [33,34] towards postlinear order is mandatory; see text below Eq. (7). There are several preliminary and promising efforts to extend relativistic astrometry to post-post-Newtonian order for light rays, especially to focus on the 2PN gravitational field of arbitrarily moving bodies endowed with arbitrary intrinsic mass- and spin-multipole moments. There have been several attempts to solve this problem [38–40], but they are far from being complete. Problems, that have been ignored in these articles are related with the internal structure of extended bodies. For a single body at rest these problems are well understood for both the post-Newtonian [25,26] and the post-Minkowskian case [27,28], where many structure-dependent terms appear in intermediate calculations that cancel exactly by virtue of the local equations of motion or can be eliminated by corresponding gauge transformations. However, in post-post-Newtonian order the situation is still unclear. For a spherically symmetric body the complete derivation of the metric in the exterior of the massive body (Schwarzschild metric) was recently solved in [41], where it has been shown how such structure-dependent terms cancel so that one finally ends up with the well-known Schwarzschild solution in harmonic gauge. This work allows in principle to determine the light trajectory in the field of a spherically symmetric and extended massive body at rest in 2PN approximation.

Assessment: So far, the light trajectory in 2PN approximation, $\Delta\boldsymbol{x}_{\text{2PN}}(t, t_0)$ in (5), is only known for pointlike monopoles at rest. Moreover, the DSX metric in postlinear approximation has to be determined, in order to ascertain the impact on light deflection of terms in second post-Newtonian order beyond the monopole term, either numerically or analytically.

III. GEODESIC EQUATION IN 1PN APPROXIMATION

The description of the metric of the Solar System becomes more complex the more accurate the astrometric measurements are and one has to resort on approximation schemes to solve the geodesic equation (1a). Since the gravitational fields of Solar System are weak and the motions of the massive bodies are slow, we can utilize the so-called post-Newtonian expansion (weak-field slow-motion approximation) for the metric as given by Eq. (3). The main objective of this investigation is an analytical solution for the light trajectory in 1PN approximation, see Eq. (9). As a result, terms of the order $\mathcal{O}(c^{-2})$ in the metric tensor are required for such an approximation:

$$g_{\alpha\beta}(t,\boldsymbol{x}) = \eta_{\alpha\beta} + h^{(2)}_{\alpha\beta}(t,\boldsymbol{x}) + \mathcal{O}(c^{-3}). \tag{38}$$

Inserting (38) into (1a) by virtue of (2) yields the geodesic equation in 1PN approximation, which can be rewritten in terms of global coordinate time (cf. Refs. [15,17,59] and especially the first four terms in Eq. (A.4) in [59]),

$$\frac{\ddot{x}^i(t)}{c^2} = +\frac{1}{2}h^{(2)}_{00,i} - h^{(2)}_{00,j}\frac{\dot{x}^i(t)}{c}\frac{\dot{x}^j(t)}{c} - h^{(2)}_{ij,k}\frac{\dot{x}^j(t)}{c}\frac{\dot{x}^k(t)}{c} + \frac{1}{2}h^{(2)}_{jk,i}\frac{\dot{x}^j(t)}{c}\frac{\dot{x}^k(t)}{c} + \mathcal{O}(c^{-3}), \tag{39}$$

where $h^{(2)}_{\alpha\beta,i} = \partial h^{(2)}_{\alpha\beta}/\partial x^i$, while a dot denotes a derivative with respect to coordinate time. In order to find a unique solution of the geodesic equation in (39), so-called mixed initial-boundary conditions must be imposed, which have extensively been used in the literature, e.g. [15,17,46,53,60,67,72]:

$$\boldsymbol{x}_0 = \boldsymbol{x}(t_0), \tag{40}$$

$$\boldsymbol{\sigma} = \lim_{t\to-\infty} \frac{\dot{\boldsymbol{x}}(t)}{c}. \tag{41}$$

The first condition (40) defines the spatial coordinates of the photon at the moment t_0 of emission of light. The second condition (41) defines the unit-direction ($\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} = 1$) of the light ray at past null infinity, that means the unit-tangent vector along the light path at infinite distance in the past from the origin of the global coordinate system.

The metric perturbations in (39) are functions of the coordinates of the global reference system (BCRS). It is, however, important to realize that in the geodesic equation this coordinate dependence has always to be understood as being the coordinates of the photon $\boldsymbol{x}(t)$ at time t, which means

$$h^{(2)}_{\alpha\beta} = h^{(2)}_{\alpha\beta}(t,\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{x}(t)}. \tag{42}$$

Consequently, the spatial derivatives in (39) are taken along the light ray:

$$h^{(2)}_{\alpha\beta,i} = \frac{\partial h^{(2)}_{\alpha\beta}(t,\boldsymbol{x})}{\partial x^i}\bigg|_{\boldsymbol{x}=\boldsymbol{x}(t)}. \tag{43}$$

The geodesic equation in (39) has usually been solved by an iteration procedure. In the first iteration the right-hand side in (39) vanishes, $\ddot{x}^i = 0$, and the integration of this differential equation yields the unperturbed light ray in Eq. (10). The exact light trajectory $\boldsymbol{x}(t)$ deviates from the Newtonian approximation by terms of the order $\mathcal{O}(c^{-2})$, which means

$$\boldsymbol{x}(t) = \boldsymbol{x}_{\rm N}(t) + \mathcal{O}(c^{-2}). \tag{44}$$

Solving the geodesic equations (39) by iteration implies that $\dot{\boldsymbol{x}}(t)$ can be replaced by its Newtonian approximation, $\dot{\boldsymbol{x}}_{\rm N}(t) = c\boldsymbol{\sigma}$, which follows by time-derivative of (10), so that the geodesic equation in (39) simplifies as follows:

$$\frac{\ddot{x}^i(t)}{c^2} = +\frac{1}{2}h^{(2)}_{00,i} - h^{(2)}_{00,j}\sigma^i\sigma^j - h^{(2)}_{ij,k}\sigma^j\sigma^k + \frac{1}{2}h^{(2)}_{jk,i}\sigma^j\sigma^k + \mathcal{O}(c^{-3}). \tag{45}$$

In 1PN approximation, the metric perturbations in (45) have to be taken at the spatial coordinates of the unperturbed light ray given by (10), which means

$$h^{(2)}_{\alpha\beta} = h^{(2)}_{\alpha\beta}(t,\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{x}_{\rm N}(t)}, \tag{46}$$

and in (45) one has first to differentiate with respect to spatial coordinates and afterwards one inserts the unperturbed light ray, that means

$$h^{(2)}_{\alpha\beta,i} = \frac{\partial h^{(2)}_{\alpha\beta}(t,\boldsymbol{x})}{\partial x^i}\bigg|_{\boldsymbol{x}=\boldsymbol{x}_{\rm N}(t)}. \tag{47}$$

In our investigation we will solve the geodesic equation (45) in 1PN approximation, that means the exact light trajectory $\boldsymbol{x}(t)$ is determined up to terms of the order $\mathcal{O}(c^{-3})$:

$$\boldsymbol{x}(t) = \boldsymbol{x}_{\rm 1PN}(t) + \mathcal{O}(c^{-3}). \tag{48}$$

The first and second integral of geodesic equations (45) in 1PN approximation can formally be written as follows [15]:

$$\dot{\boldsymbol{x}}_{\rm 1PN}(t) = c\boldsymbol{\sigma} + \Delta\dot{\boldsymbol{x}}_{\rm 1PN}(t), \tag{49}$$

$$\boldsymbol{x}_{\rm 1PN}(t) = \boldsymbol{x}(t_0) + c\boldsymbol{\sigma}(t-t_0) + \Delta\boldsymbol{x}_{\rm 1PN}(t,t_0), \tag{50}$$

where $\Delta\boldsymbol{x}_{\rm 1PN}$ are small perturbations of the unperturbed light trajectory, and $\Delta\dot{\boldsymbol{x}}_{\rm 1PN}$ is the time derivative of these small perturbations.

IV. THE METRIC OF THE SOLAR SYSTEM

In order to describe and to interpret observational data in astrometry correctly, a set of several reference systems and the transformation laws among their coordinates must be introduced. In this respect, two standard reference systems are of fundamental importance, which are adopted by the IAU resolution B1.3 (2000) [21]: the Barycentric Celestial Reference System (BCRS) with coordinates $(ct,\boldsymbol{x})$ and the Geocentric Celestial Reference System (GCRS) with coordinates $(cT,\boldsymbol{X})$. Furthermore, for any massive body A of the Solar System a so-called GCRS-like reference system with coordinates $(cT_A,\boldsymbol{X}_A)$ can be introduced. In this section we will give a summary about how to combine these systems to a global metric tensor in terms of local multipoles, which is the physically adequate reference system for modeling of light trajectories through the Solar System.

A. BCRS

The harmonic coordinates of BCRS are denoted by $x^\mu = (ct, x^i)$, where $t = \mathrm{TCB}$ is the BCRS coordinate time, and cover the entire spacetime and can therefore be used to model light trajectories from distant celestial objects to the observer. The origin of the BCRS is located at the barycenter of the Solar System, and the IAU Resolution B2 (2006) [22] recommends the spatial axes of BCRS to be oriented according to the spatial axes of the International Celestial Reference System (ICRS) [80]. According to IAU resolution B1.3 (2000) [21], the Solar System is assumed to be isolated and the spacetime is asymptotically flat, that means the BCRS metric $g_{\mu\nu}(t,\boldsymbol{x})$ at spatial infinity reads

$$\lim_{|\boldsymbol{x}|\to\infty} g_{\mu\nu}(t,\boldsymbol{x}) = \eta_{\mu\nu}. \tag{51}$$

The BCRS is completely characterized by the form of its metric tensor, up to order $\mathcal{O}(c^{-3})$ given by [21]

$$g_{00}(t,\boldsymbol{x}) = -1 + \frac{2w(t,\boldsymbol{x})}{c^2} + \mathcal{O}(c^{-4}), \tag{52}$$

$$g_{0i}(t,\boldsymbol{x}) = \mathcal{O}(c^{-3}), \tag{53}$$

$$g_{ij}(t,\boldsymbol{x}) = \left(1 + \frac{2w(t,\boldsymbol{x})}{c^2}\right)\delta_{ij} + \mathcal{O}(c^{-4}). \tag{54}$$

The scalar gravitational potential in (52) and (54) is given by the integral

$$w(t,\boldsymbol{x}) = \frac{G}{c^2}\int d^3x' \frac{t^{00}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} + \mathcal{O}(c^{-2}), \tag{55}$$

which runs over the entire Solar System, and where t^{00} is the time-time component of the energy-momentum tensor $t^{\mu\nu}$ in global BCRS coordinates; recall the components of the energy-momentum tensor scale as follows: $t^{00} = \mathcal{O}(c^2), t^{0i} = \mathcal{O}(c^1), t^{ij} = \mathcal{O}(c^0)$.

The global gravitational potential in (55) admits an expansion in terms of global STF multipoles, which characterize the multipole structure of the Solar System as a whole [25–27]: [81]:

$$w(t,\boldsymbol{x}) = G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!} m_L(t)\partial_L \frac{1}{r} + \mathcal{O}(c^{-2}), \tag{56}$$

where $\partial_L = \frac{\partial}{\partial x^{a_1}}\cdots\frac{\partial}{\partial x^{a_l}}$. The global mass multipoles in (56) are Cartesian symmetric and trace-free (STF) tensors, in Newtonian approximation given by (cf. Eq. (2.34a) in [26])

$$m_L(t) = \int d^3x \hat{x}_L \frac{t^{00}(t,\boldsymbol{x})}{c^2} + \mathcal{O}(c^{-2}), \tag{57}$$

where the integral runs over the entire Solar System. The global mass monopole, i.e. $l = 0$ in Eq. (57), is just the total (Newtonian) mass, $M = \text{const}$, of the entire Solar System, while the global mass-dipole term vanishes, i.e. $m_i = 0$, because the origin of BCRS is located at the barycenter of the Solar System.

A further comment should be in order about a possible retarded time argument of the energy-momentum tensor in Eq. (55); cf. text below Eqs. (17) in [21]. One may easily recognize that such retarded time argument would be beyond 1PN approximation for the light rays. In particular, in terms of multipole expansion, one may demonstrate the following relation,

$$\begin{aligned} w(t,\boldsymbol{x}) &= G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!} m_L(t)\partial_L \frac{1}{r} + \mathcal{O}(c^{-2}) \\ &= G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L \frac{m_L(t_{\text{ret}})}{r} + \mathcal{O}(c^{-2}), \end{aligned} \tag{58}$$

where the retarded time has been defined by Eq. (35). If one expands the retarded multipoles [second line in Eq. (58)] in inverse powers of c, then one finds that all terms proportional to $1/c$ cancel against each other. This cancellation is important, because terms of odd powers $1/c$ would violate the time-reversal symmetry, cf. the corresponding statement in the text below Eq. (17) in the IAU resolutions [21]. The time-reversal symmetry is violated because of the gravitational radiation emitted by the Solar System which is, however, an effect much beyond 1PN approximation.

The expansion in (56) has two specific characteristics, which prevent a direct use for our intentions:

(1) As emphasized in [25–28], the expansion in (57) is valid outside a sphere which encloses the complete N-body system, see also [82]. However, for a description of light rays inside the Solar System (light trajectories between the massive bodies) one has to apply a metric tensor which is also valid inside this sphere, i.e. in space-regions between these massive bodies; cf. text on p. 3298 in [33].

(2) From the theory of relativistic reference systems it is clear that physically meaningful multipole moments of some body A have to be defined in the body's local reference system $(cT_A, \boldsymbol{X}_A)$.

For these reasons, in our approach we will have to express the gravitational potential in (56) by local (intrinsic) mass multipoles M_L^A, which are defined in the local coordinate system $(cT_A, \boldsymbol{X}_A)$ of the corresponding massive body. This crucial issue will be the subject in what follows.

B. GCRS

The harmonic coordinates of GCRS are denoted by $X^\mu = (cT, X^i)$, where $T = \text{TCG}$ is the GCRS coordinate time. According to IAU resolution B1.3 (2000) [21], the origin of GCRS is comoving with the Earth and located at the barycenter of the Earth, and is adequate to describe physical processes in the vicinity of the Earth. The spatial axes of GCRS are kinematically nonrotating with respect to BCRS; i.e., they are locally noninertial. The GCRS is completely characterized by the form of its metric tensor, up to order $\mathcal{O}(c^{-3})$ given by [21,33,34]

$$G_{00}(T,\boldsymbol{X}) = -1 + \frac{2W(T,\boldsymbol{X})}{c^2} + \mathcal{O}(c^{-4}), \quad (59)$$

$$G_{0i}(T,\boldsymbol{X}) = \mathcal{O}(c^{-3}), \quad (60)$$

$$G_{ij}(T,\boldsymbol{X}) = \left(1 + \frac{2W(T,\boldsymbol{X})}{c^2}\right)\delta_{ij} + \mathcal{O}(c^{-4}). \quad (61)$$

The scalar gravitational potential in (59) and (61) can uniquely be separated into two terms: a local potential, $W_{\rm loc}$, which originates from the body A itself and an external potential, $W_{\rm ext}$, which is associated with inertial effects (due to the accelerated motion of the local system) and tidal forces (caused by the other bodies of the Solar System) [21,33,34]:

$$W(T,\boldsymbol{X}) = W_{\rm loc}(T,\boldsymbol{X}) + W_{\rm ext}(T,\boldsymbol{X}). \quad (62)$$

Explicit expressions for the external potential $W_{\rm ext}$ are given in [33,34], while the potential $W_{\rm loc}$ is defined by the following integral,

$$W_{\rm loc}(T,\boldsymbol{X}) = \frac{G}{c^2}\int_{V_E} d^3X' \frac{T^{00}(T,\boldsymbol{X}')}{|\boldsymbol{X}-\boldsymbol{X}'|} + \mathcal{O}(c^{-2}), \quad (63)$$

which runs over the entire volume V_E of the Earth, and where T^{00} is the time-time component of the energy-momentum tensor $T^{\mu\nu}$ of the isolated Earth and expressed in GCRS coordinates; recall the components of energy-momentum tensor scale as follows: $T^{00} = \mathcal{O}(c^2)$, $T^{0i} = \mathcal{O}(c^1), T^{ij} = \mathcal{O}(c^0)$. The local potential (63) is generated by the Earth and can be expanded into a series of local STF multipole moments, which characterize the multipole structure of the Earth as an isolated body [21,25–28,33]:

$$W_{\rm loc}(T,\boldsymbol{X}) = G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}M_L(T)\mathcal{D}_L\frac{1}{R} + \mathcal{O}(c^{-2}), \quad (64)$$

where $\mathcal{D}_L = \frac{\partial}{\partial X^{a_1}}\cdots\frac{\partial}{\partial X^{a_l}}$.

The local mass monopole, i.e. $l=0$ in Eq. (64), is just the (Newtonian) mass of the Earth, $M = \text{const}$. Actually, the origin of the GCRS is assumed to be located at the barycenter of the Earth; hence, the dipole term in (64) vanishes: $M_i = 0$. But in real measurements of celestial mechanics the center-of-mass of massive Solar System bodies can usually not be determined exactly, so it is meaningful to keep this term and to assume $M_i \neq 0$ in general. The STF mass multipoles M_L in (64) in Newtonian approximation are given by

$$M_L(T) = \int_{V_E} d^3X\hat{X}_L\frac{T^{00}(T,\boldsymbol{X})}{c^2} + \mathcal{O}(c^{-2}). \quad (65)$$

According to the theory of reference systems, [15,21,29–36], the GCRS is the standard reference system to define local multipoles of the Earth. However, as it has been noted in [21], the detailed form of mass multipoles in (65) is not needed for practical astrometry or celestial mechanics, since these terms are related to observational quantities. That means, the gravitational potentials can be expanded in terms of vector spherical harmonics and the coefficients of such an expansion are equivalent to the local multipoles; see Appendix A in [21].

C. Metric of Solar System in terms of intrinsic multipoles in the DSX framework

Physically meaningful multipoles of the massive bodies can only be defined in their local reference systems. On these grounds, for each massive body A of the Solar System a GCRS-like reference system with coordinates $(cT_A, \boldsymbol{X}_A)$ and comoving with the body A is introduced, to permit the definition of local multipoles of this body. Hence, for an N-body system, there are in total $N+1$ reference systems, one global chart $(ct,\boldsymbol{x})$ and N local charts $(cT_A,\boldsymbol{X}_A)$, which are linked to each other via coordinate transformations, which allow the construction of one global reference system in terms of local multipoles M_L^A of the massive bodies $A = 1,\ldots,N$. That reference system is valid in the entire near zone of the Solar System, and combines the advantage of locally defined multipoles and is well defined in space-regions between the massive bodies; cf. text above Eq. (6.9a) in [33]. Such a system is also physically adequate for modeling the light trajectory from a light source through the near zone of the Solar System towards the observer. The corresponding framework has been elaborated within the DSX theory [33–36], which has originally been established for celestial mechanics and for deriving the equations of motion of a N-body system. This framework has later be reformulated in terms of PPN formalism in [83], aiming at several tests of relativity in celestial mechanics, e.g. tests of equivalence principle. One main result of the DSX formalism are these transformation rules for the coordinates $(ct,\boldsymbol{x})\longleftrightarrow(cT_A,\boldsymbol{X}_A)$ and for the metric potentials $w\longleftrightarrow W_A$. According to [33,34], the global coordinates $(ct,\boldsymbol{x})$ and the local coordinates $(cT_A,\boldsymbol{X}_A)$ of some body A are related by the following coordinate transformation; cf. Eq. (2.8a) in [33] (for the inverse transformation, we refer to [21]),

$$x^\mu = x_A^\mu(T_A) + e_a^\mu(T_A)X_A^a + \mathcal{O}(c^{-2}), \quad (66)$$

where x_A^μ is the world line of body A in BCRS coordinates (i.e. a selected point associated with body A) and e_a^μ are tetrads along the world line of this body (cf. Eqs. (2.16) in [33]),

$$e_a^0(T_A) = \frac{\dot{x}_A^a(T_A)}{c} + \mathcal{O}(c^{-3}), \tag{67}$$

$$e_a^i(T_A) = \delta_{ai} + \mathcal{O}(c^{-2}), \tag{68}$$

where in (67) a dot means derivative with respect to T_A; thus $\dot{x}_A^a(T_A)$ are the spatial components of the three-velocity of body A in the global system and given in terms of the body's local coordinate time T_A. Without going into the details, using the tensorial transformation rule for metric tensors in different coordinate systems (cf. Eq. (4.11) in [33]), it has been demonstrated in [33] that the global potential can be expressed in terms of local (intrinsic) STF multipoles M_L^A as follows (for the inverse transformation we refer to [21]):

$$w(t,\boldsymbol{x}) = \sum_{A=1}^{N} w_A(t,\boldsymbol{x}), \tag{69}$$

$$w_A(t,\boldsymbol{x}) = G \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} M_L^A(T_A) \mathcal{D}_L^A \frac{1}{R_A} + \mathcal{O}(c^{-2}), \tag{70}$$

where in (69) the sum runs over all bodies of the N-body system, $R_A = |\boldsymbol{X}_A|$ is the spatial distance from the origin of local coordinate system to some field point located outside the massive body, and $\mathcal{D}_L^A = \frac{\partial}{\partial X_A^{a_1}} \cdots \frac{\partial}{\partial X_A^{a_l}}$. The local STF mass multipoles M_L^A in (70) in Newtonian approximation are given by

$$M_L^A(T_A) = \int_{V_A} d^3 X_A \hat{X}_L^A \frac{T_A^{00}(T_A, \boldsymbol{X}_A)}{c^2} + \mathcal{O}(c^{-2}), \tag{71}$$

where the integration runs over the volume V_A of the massive body A under consideration, and where T_A^{00}, is the time-time component of the energy-momentum tensor $T_A^{\mu\nu}$ of the isolated massive body A and expressed in the coordinates of the local reference system of that envisaged body.

In order to complete the transformation, also the partial derivatives in (70) have to be transformed, which follow from the coordinate transformations (66) and read explicitly (cf. Eqs. (2.10) by virtue of Eqs. (2.16) in [33])

$$\frac{\partial}{\partial c T_A} = \frac{\partial}{\partial c t} + \frac{v_A^a(T_A)}{c} \frac{\partial}{\partial x^a} + \mathcal{O}(c^{-2}), \tag{72}$$

$$\frac{\partial}{\partial X_A^a} = \frac{\partial}{\partial x^a} + \frac{v_A^a(T_A)}{c} \frac{\partial}{\partial c t} + \mathcal{O}(c^{-2}). \tag{73}$$

Let us note already here that the second term in (72) and (73) yields terms of the order $\mathcal{O}(c^{-4})$ in the global metric; hence, these terms do not finally appear in Eq. (77). Furthermore, we note that from (66) follows the relation [19,21,33,34]

$$R_A = |\boldsymbol{x} - \boldsymbol{x}_A(t)| + \mathcal{O}(c^{-2}), \tag{74}$$

where according to (46) the field point $\boldsymbol{x}$ in (74) will later be replaced by the photon's light trajectory. The coordinate time in the global and local systems is related via [19,21,33,34]:

$$T_A = t + \mathcal{O}(c^{-2}). \tag{75}$$

Actually, a constant b_A^0 could be added on the right-hand side in (75), which would indicate different initial times of the clocks in the global and local systems (cf. Eq. (4) in [84]), but has been omitted in favor of simpler notation and could formally be added at any stage of the calculations; concerning the general problem of clock synchronization in the gravitational field of the Solar System, we refer to [85]. From (75) we conclude

$$M_L^A(T_A) = M_L^A(t) + \mathcal{O}(c^{-2}), \tag{76}$$

where the neglected terms in (76) are beyond 1PN approximation for light rays. By inserting (72)–(76) into (69)–(70), we arrive at the global gravitational potential in terms of local mass multipoles M_L^A,

$$w_A(t,\boldsymbol{x}) = G \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} M_L^A(t) \partial_L \frac{1}{r_A(t)} + \mathcal{O}(c^{-2}), \tag{77}$$

where $r_A(t) = |\boldsymbol{x} - \boldsymbol{x}_A(t)|$, and $\partial_L = \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_l}}$ are partial derivatives in the global system. In summary of this section, the metric perturbation in the near zone of the Solar System and expressed in terms of local multipoles is given by

$$h_{00}^{(2)}(t,\boldsymbol{x}) = \sum_{A=1}^{N} h_{00}^{(2)A}(t,\boldsymbol{x}), \tag{78}$$

$$h_{00}^{(2)A}(t,\boldsymbol{x}) = \frac{2G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} M_L^A(t) \partial_L \frac{1}{r_A(t)}, \tag{79}$$

$$h_{ij}^{(2)}(t,\boldsymbol{x}) = \delta_{ij} h_{00}^{(2)}(t,\boldsymbol{x}), \tag{80}$$

where the sum in (78) runs over all massive bodies of the Solar System and the metric perturbation caused by one individual body is given by (79). The metric perturbation in (78)–(80) has to be implemented into the geodesic equation in (45).

At this stage let us underline again that an implementation of the infinite series expansion (8) into (79) via $r_A(t) = |\boldsymbol{x} - \boldsymbol{x}_A(t)|$, would more explicitly elucidate the fact that an arbitrary world line of the body, $\boldsymbol{x}_A(t)$, implicitly generates terms in the metric tensor (79) which are proportional to the velocity and acceleration of the body. However, such terms would be proportional either to

$v_A(t - t_A)$ or $a_A(t - t_A)^2$, but neither to v_A/c nor a_A/c; hence, they would not be beyond 1PN approximation for the light rays. From this consideration it becomes obvious that an arbitrary world line $\boldsymbol{x}_A(t)$ implies a summation over all terms in the series expansion (8) and, therefore, a solution of the geodesic equation in terms of arbitrary world lines $\boldsymbol{x}_A(t)$ is much preferable compared to a solution in terms of approximative world lines (8).

V. TRANSFORMATION OF GEODESIC EQUATION

According to Eqs. (46)–(47), the geodesic equation in (45) has to be integrated along the unperturbed light trajectory (10). In view of this fact, it is meaningful to express the geodesic equation, i.e. the metric tensor and the derivatives, in terms of new parameters which characterize the unperturbed light trajectory from the very beginning of the integration procedure. In this respect, the investigations in [60,72,73] have recovered the remarkable efficiency of the following two independent variables τ and $\boldsymbol{\xi}$:

$$c\tau = \boldsymbol{\sigma} \cdot \boldsymbol{x}_{\rm N}(t), \qquad c\tau_0 = \boldsymbol{\sigma} \cdot \boldsymbol{x}_{\rm N}(t_0), \tag{81}$$

$$\xi^i = P^i_j x^j_{\rm N}(t), \tag{82}$$

where $P^i_j = P_{ij} = P^{ij}$ is the operator of projection onto the plane perpendicular to the vector $\boldsymbol{\sigma}$,

$$P^{ij} = \delta_{ij} - \sigma^i \sigma^j. \tag{83}$$

The three-vector $\boldsymbol{\xi} = \boldsymbol{\sigma} \times (\boldsymbol{x}_{\rm N}(t) \times \boldsymbol{\sigma}) = \boldsymbol{\sigma} \times (\boldsymbol{x}_0 \times \boldsymbol{\sigma})$ in (82) is the impact vector of the unperturbed light ray, see also Eq. (11). In particular, $\boldsymbol{\xi}$ is time independent and directed from the origin of global coordinate system to the point of closest approach of the unperturbed light trajectory; its absolute value is denoted by $d = |\boldsymbol{\xi}|$.

While some detailed explanations and geometrical elucidations can be found in [60], two comments should be in order about these new variables.

(i) First, one can easily recognize that (81) can also be written in the form $c\tau = c(t - t^*)$ and $c\tau_0 = c(t_0 - t^*)$, where

$$t^* = t_0 - \frac{\boldsymbol{\sigma} \cdot \boldsymbol{x}_0}{c}, \tag{84}$$

is the time of closest approach of unperturbed light ray to the origin of the global coordinate system; note that (84) differs from (33) which is the time of closest approach of the light ray to the origin of the local coordinate system of some massive body A. With the aid of these new variables $\boldsymbol{\xi}$ and τ, the mixed initial-boundary conditions (40) and (41) take the form

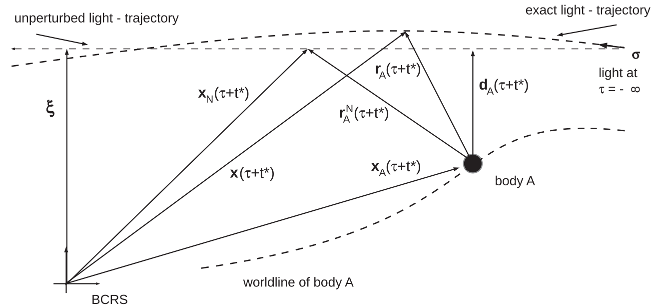

FIG. 1. A geometrical representation of the light trajectory through the Solar System in terms of the new variables $\boldsymbol{\xi}$ and τ. The impact vector $\boldsymbol{\xi}$ is defined by Eq. (82) and points from the origin of global system to the point of closest approach of the unperturbed light ray to that origin, and is time independent. The impact vector $\boldsymbol{d}_A(\tau + t^*)$ is defined by Eq. (90) and points from the origin of local system of body A towards the point of closest approach of unperturbed light ray to that origin, and is time dependent due to the motion of the body. Furthermore, $\boldsymbol{x}(\tau + t^*)$ and $\boldsymbol{x}_{\rm N}(\tau + t^*)$ are the global spatial positions of the photon of the exact light trajectory and unperturbed light trajectory, respectively. The world line of massive body A in the global system is given by $\boldsymbol{x}_A(\tau + t^*)$, and $\boldsymbol{r}_A(\tau + t^*)$ points from the origin of local system towards the exact position of the photon, while $\boldsymbol{r}^{\rm N}_A(\tau + t^*)$ points from the origin of local system towards the unperturbed light ray.

$$\boldsymbol{x}_0 = \boldsymbol{x}(\tau_0 + t^*), \tag{85}$$

$$\boldsymbol{\sigma} = \lim_{\tau \to -\infty} \frac{\dot{\boldsymbol{x}}(\tau + t^*)}{c}, \tag{86}$$

where a dot means derivative with respect to variable τ. In terms of the new variables the interpretation of these initial-boundary conditions remains the same: the first condition (85) defines the spatial coordinates of the photon at the moment of emission of light, while the second condition (86) defines the unit-direction ($\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 1$) at infinite past and infinite distance from the origin of global coordinate system, that means at the so-called past null infinity.

(ii) Second, it is important to mention that with the aid of the new variable (81) and (82), the unperturbed light ray in (10) transforms as follows [86]:

$$\boldsymbol{x}_{\rm N}(\tau + t^*) = \boldsymbol{\xi} + c\tau\boldsymbol{\sigma}. \tag{87}$$

In these new variables, the vector from the arbitrarily moving body and the light trajectory in (16) transforms as follows:

$$\boldsymbol{r}_A(\tau + t^*) = \boldsymbol{x}(\tau + t^*) - \boldsymbol{x}_A(\tau + t^*), \tag{88}$$

with the absolute value $r_A(\tau + t^*) = |\boldsymbol{r}_A(\tau + t^*)|$, while the distance between the unperturbed light ray and the arbitrarily moving body in (17) now reads

$$\boldsymbol{r}_A^{\rm N}(\tau+t^*) = \boldsymbol{x}_{\rm N}(\tau+t^*) - \boldsymbol{x}_A(\tau+t^*) = \boldsymbol{\xi} + c\tau\boldsymbol{\sigma} - \boldsymbol{x}_A(\tau+t^*), \quad (89)$$

with the absolute value $r_A^{\rm N}(\tau+t^*) = |\boldsymbol{r}_A^{\rm N}(\tau+t^*)|$, and we note $\boldsymbol{r}_A(\tau+t^*) = \boldsymbol{r}_A^{\rm N}(\tau+t^*) + \mathcal{O}(c^{-2})$. The impact parameter in (19) for arbitrarily moving bodies in these new variables reads

$$\boldsymbol{d}_A(\tau+t^*) = \boldsymbol{\sigma}\times(\boldsymbol{r}_A^{\rm N}(\tau+t^*)\times\boldsymbol{\sigma}), \quad (90)$$

with the absolute value $d_A(\tau+t^*) = |\boldsymbol{d}_A(\tau+t^*)|$. For an illustration of the expressions in Eqs. (82) and (87)–(90) see Fig. 1.

Furthermore, it has been outlined in [60,76] that by means of the new variables (81) and (82), the following relation is valid for a smooth function $F(t,\boldsymbol{x})$ (cf. Eq. (33) in [60] or Eq. (C4) in [76]):

$$\left(\frac{\partial}{\partial x^i} + \sigma^i\frac{\partial}{\partial ct}\right)F(t,\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{x}_{\rm N}(t)} = \left(P^{ij}\frac{\partial}{\partial\xi^j} + \sigma^i\frac{\partial}{\partial c\tau}\right)F(\tau+t^*,\boldsymbol{\xi}+c\tau\boldsymbol{\sigma}). \quad (91)$$

It is important to realize that on the left-hand side in (91) one has first to differentiate with respect to the fieldpoint $\boldsymbol{x}$ and global coordinate time t and afterwards one has to substitute the unperturbed light ray $\boldsymbol{x}_{\rm N}(t) = \boldsymbol{x}_0 + c(t-t_0)\boldsymbol{\sigma}$, while on the right-hand side in (91) one has first to substitute $\tau + t^*$ and $\boldsymbol{x}_{\rm N}(\tau+t^*) = \boldsymbol{\xi} + c\tau\boldsymbol{\sigma}$ and afterwards to perform the differentiation with respect to $\boldsymbol{\xi}$ and τ.

From now on, the smooth function $F(t,\boldsymbol{x})$ in relation (91) is considered to be one of the components of the metric perturbation $h_{\alpha\beta}^{(2)}(t,\boldsymbol{x})$. Then, the derivatives with respect to variable ct on the left-hand side of relation (91) yield only terms of higher-order beyond 1PN approximation,

$$\left.\frac{\partial h_{\alpha\beta}^{(2)}(t,\boldsymbol{x})}{\partial ct}\right|_{\boldsymbol{x}=\boldsymbol{x}_{\rm N}(t)} = \mathcal{O}(c^{-3}), \quad (92)$$

because they are proportional to either $\dot{M}_L^A/c$ or $\boldsymbol{v}_A/c$; for the same reason, there is no time derivative in the geodesic equation either [see (39) or (45)]. However, one has to keep the differentiation with respect to variable $c\tau$ in the right-hand side of relation (91), because that derivative does not only act on the multipoles $M_L^A(\tau+t^*)$ and spatial coordinates of the massive bodies $\boldsymbol{x}_A(\tau+t^*)$, but also on the unperturbed light ray $\boldsymbol{\xi}+c\tau\boldsymbol{\sigma}$. Therefore, in 1PN approximation the relation (91) simplifies as follows:

$$\left.\frac{\partial h_{\alpha\beta}^{(2)}(t,\boldsymbol{x})}{\partial x^i}\right|_{\boldsymbol{x}=\boldsymbol{x}_{\rm N}(t)} = \left(P^{ij}\frac{\partial}{\partial\xi^j} + \sigma^i\frac{\partial}{\partial c\tau}\right)h_{\alpha\beta}^{(2)}(\tau+t^*,\boldsymbol{\xi}+c\tau\boldsymbol{\sigma}) + \mathcal{O}(c^{-3}). \quad (93)$$

If the derivative with respect to variable $c\tau$ in (93) acts on the multipoles or spatial coordinates of the massive bodies, then terms will be generated which are beyond 1PN approximation, namely terms proportional to either $\dot{M}_L^A/c$ or $\boldsymbol{v}_A/c$, respectively, which, however, can easily be identified.

By means of relation (93), the geodesic equation in 1PN approximation in (45) transforms as follows:

$$\frac{\ddot{x}^i(\tau+t^*)}{c^2} = +\frac{1}{2}P^{ij}\frac{\partial}{\partial\xi^j}h_{00}^{(2)} - \frac{1}{2}\sigma^i\frac{\partial}{\partial c\tau}h_{00}^{(2)} + \frac{1}{2}\sigma^k\sigma^l P^{ij}\frac{\partial}{\partial\xi^j}h_{kl}^{(2)} + \frac{1}{2}\sigma^i\sigma^j\sigma^k\frac{\partial}{\partial c\tau}h_{jk}^{(2)} - \sigma^j\frac{\partial}{\partial c\tau}h_{ij}^{(2)} + \mathcal{O}(c^{-3}), \quad (94)$$

where the double-dot on the left-hand side in (94) means twice of the total derivative with respect to the new variable τ. By taking into account (80), the geodesic equation further simplifies:

$$\frac{\ddot{x}^i(\tau+t^*)}{c^2} = P^{ij}\frac{\partial}{\partial\xi^j}h_{00}^{(2)} - \sigma^i\frac{\partial}{\partial c\tau}h_{00}^{(2)} + \mathcal{O}(c^{-3}). \quad (95)$$

In the next step, the metric perturbations in (78)–(80) have to be transformed in terms of these new variables $\boldsymbol{\xi}$ and τ. Since the metric perturbations in (79) contain spatial derivatives, $\partial_L r_A^{-1}(t)$, we will have to transform these differential operators in terms of these new variables. For that one might want to use relation (91), which is valid for any smooth function, but a possible time derivative on the left-hand side of (91) generates only terms beyond 1PN approximation,

$$\left.\frac{\partial}{\partial ct}\frac{1}{r_A(t)}\right|_{\boldsymbol{x}=\boldsymbol{x}_{\rm N}(t)} = \mathcal{O}(c^{-1}). \quad (96)$$

Therefore, like in (93), we may use the simpler relation,

$$\left.\frac{\partial}{\partial x^i}\frac{1}{r_A(t)}\right|_{\boldsymbol{x}=\boldsymbol{x}_{\rm N}(t)} = \left(P^{ij}\frac{\partial}{\partial\xi^j} + \sigma^i\frac{\partial}{\partial c\tau}\right)\frac{1}{r_A^{\rm N}(\tau+t^*)} + \mathcal{O}(c^{-1}), \quad (97)$$

where we have taken into account that the derivative with respect to $c\tau$ in the right-hand side of (97) must be kept because of [cf. relation (F6)]

$$\frac{\partial}{\partial c\tau}\frac{1}{r_A^{\rm N}(\tau+t^*)} = -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)}{(r_A^{\rm N}(\tau+t^*))^3} + \mathcal{O}\left(\frac{v_A}{c}\right). \quad (98)$$

The outcome of (97) and (98) is, that the metric perturbation in (79) for one massive body A and in terms of these new variables $\boldsymbol{\xi}$ and τ is given by

$$h_{00}^{(2)}(\tau, \boldsymbol{\xi}) = \sum_{A=1}^{N} h_{00}^{(2)A}(\tau, \boldsymbol{\xi}), \tag{99}$$

$$h_{00}^{(2)A}(\tau, \boldsymbol{\xi}) = \frac{2G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} M_L^A(\tau + t^*) \partial_L \frac{1}{r_A^{\rm N}(\tau + t^*)}, \tag{100}$$

where, by means of binomial theorem, the spatial derivatives in (100) in terms of new variables can be written in the following form (cf. Eq. (24) in [53]):

$$\partial_L = \sum_{p=0}^{l} \frac{l!}{(l-p)!p!} \sigma^{i_1} \ldots \sigma^{i_p} P^{i_{p+1} j_{p+1}} \ldots P^{i_l j_l} \times \frac{\partial}{\partial \xi^{j_{p+1}}} \cdots \frac{\partial}{\partial \xi^{j_l}} \left(\frac{\partial}{\partial c\tau} \right)^p. \tag{101}$$

The insertion of metric perturbation (99)–(100) into the geodesic equation (95) finally yields the geodesic equation for light rays which propagate in the gravitational field of one arbitrarily moving body A:

$$\begin{aligned} \frac{\ddot{x}^{iA}(\tau + t^*)}{c^2} &= + \frac{2G}{c^2} P^{ij} \frac{\partial}{\partial \xi^j} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} M_L^A(\tau + t^*) \partial_L \frac{1}{r_A^{\rm N}(\tau + t^*)} \\ &\quad - \frac{2G}{c^2} \sigma^i \frac{\partial}{\partial c\tau} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} M_L^A(\tau + t^*) \partial_L \frac{1}{r_A^{\rm N}(\tau + t^*)} \\ &\quad + \mathcal{O}(c^{-3}), \end{aligned} \tag{102}$$

where the derivative operator ∂_L is given by (101). Equation (102) completes the transformation of geodesic equation in 1PN approximation and for the case of one arbitrarily moving massive body having arbitrary shape and structure. Due to the linearity of post-Newtonian equations, the case of N arbitrarily moving bodies is easily obtained by a summation over all massive bodies $A = 1, 2, \ldots, N$.

In the limit of (i) one massive body at rest, (ii) time-independent multipoles, and (iii) assuming that the center of mass is located at the origin of the global coordinate-system, the geodesic equation (102) agrees with the geodesic equation given in [53]; recall that there are no spin-multipole terms in (102) because they contribute to the order $\mathcal{O}(c^{-3})$.

VI. FIRST INTEGRATION OF THE GEODESIC EQUATION

The first integral of geodesic equation determines the coordinate-velocity of the photon and is formally written as follows [cf. Eq. (49)],

$$\dot{\boldsymbol{x}}_{\rm 1PN}(\tau + t^*) = c\boldsymbol{\sigma} + \sum_{A=1}^{N} \Delta \dot{\boldsymbol{x}}_{\rm 1PN}^A(\tau + t^*), \tag{103}$$

where the corrections to the unperturbed light ray due to one body A are given by

$$\frac{\Delta \dot{\boldsymbol{x}}_{\rm 1PN}^A(\tau + t^*)}{c} = \int_{-\infty}^{\tau} dc\tau' \frac{\Delta \ddot{\boldsymbol{x}}_{\rm 1PN}^A(\tau' + t^*)}{c^2}, \tag{104}$$

where the integrand is given by Eq. (102). Accordingly, one obtains for one body A:

$$\begin{aligned} \frac{\Delta \dot{x}_{\rm 1PN}^{iA}(\tau + t^*)}{c} &= + \frac{2G}{c^2} P^{ij} \frac{\partial}{\partial \xi^j} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \mathcal{I}_A(\tau + t^*, \boldsymbol{\xi}) \\ &\quad - \frac{2G}{c^2} \sigma^i \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \mathcal{I}_B(\tau + t^*, \boldsymbol{\xi}). \end{aligned} \tag{105}$$

The integrals in (105) are defined by (the arguments of the integrals are omitted)

$$\mathcal{I}_A = \int_{-\infty}^{\tau} dc\tau' M_L^A(\tau' + t^*) \partial'_L \frac{1}{r_A^{\rm N}(\tau' + t^*)}, \tag{106}$$

$$\mathcal{I}_B = \int_{-\infty}^{\tau} dc\tau' \frac{\partial}{\partial c\tau'} M_L^A(\tau' + t^*) \partial'_L \frac{1}{r_A^{\rm N}(\tau' + t^*)}, \tag{107}$$

where the differential operator ∂'_L in (106) and (107) is given by [cf. Eq. (101)]

$$\partial'_L = \sum_{p=0}^{l} \frac{l!}{(l-p)!p!} \sigma^{i_1} \ldots \sigma^{i_p} P^{i_{p+1} j_{p+1}} \ldots P^{i_l j_l} \times \frac{\partial}{\partial \xi^{j_{p+1}}} \cdots \frac{\partial}{\partial \xi^{j_l}} \left(\frac{\partial}{\partial c\tau'} \right)^p. \tag{108}$$

In (105) we have taken into account that $dt = d\tau'$ for the total differentials because $t^* = \text{const}$ is a constant for each individual light ray. Also the following integration rule (recall that τ and $\boldsymbol{\xi}$ are independent variables) for indefinite integrals along the unperturbed light ray has been used (cf., Eq. (4.10) in [72]):

$$\int dc\tau' \frac{\partial}{\partial \xi^i} f(\tau', \boldsymbol{\xi}) = \frac{\partial}{\partial \xi^i} \int dc\tau' f(\tau', \boldsymbol{\xi}). \tag{109}$$

The integral in (106) runs over the unknown world line $\boldsymbol{x}_A(t)$ of the massive body A and, therefore, can only be integrated by parts. Such strategy intrinsically inherits to demonstrate that the nonintegrated terms of the integration procedure involve terms which are beyond 1PN approximation, that means it elaborates on the fact that the nonintegrated terms imply an additional factor c^{-1}. In this way, the integral $\mathcal{I}_A$ is determined by Eqs. (C2)–(C4) in Appendix C, while the integral $\mathcal{I}_B$ can immediately be calculated without integration by parts:

$$\mathcal{I}_B(\tau + t^*, \boldsymbol{\xi}) = M_L^A(\tau + t^*)\partial_L \frac{1}{r_A^{\rm N}(\tau + t^*)}. \tag{110}$$

Altogether one obtains the first integral of geodesic equation in the gravitational field of one extended body A:

$$\begin{aligned}\frac{\Delta\dot{\boldsymbol{x}}^A_{\rm 1PN}(\tau + t^*)}{c} = &-\frac{2G}{c^2}\sum_{l=1}^{\infty}\sum_{p=1}^{l}\frac{(-1)^l}{(l-p)!p!}M_L^A(\tau+t^*)\sigma^{i_1}\ldots\sigma^{i_p}P^{i_{p+1}j_{p+1}}\ldots P^{i_l j_l}\frac{\partial}{\partial\xi^{j_{p+1}}}\cdots\frac{\partial}{\partial\xi^{j_l}}\left(\frac{\partial}{\partial c\tau}\right)^{p-1}\frac{\boldsymbol{d}_A(\tau+t^*)}{(r_A^{\rm N}(\tau+t^*))^3}\\ &-\frac{2G}{c^2}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}M_L^A(\tau+t^*)P^{i_1j_1}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_1}}\cdots\frac{\partial}{\partial\xi^{j_l}}\frac{\boldsymbol{d}_A(\tau+t^*)}{r_A^{\rm N}(\tau+t^*)-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)}\frac{1}{r_A^{\rm N}(\tau+t^*)}\\ &-\frac{2G}{c^2}\boldsymbol{\sigma}\sum_{l=0}^{\infty}\sum_{p=0}^{l}\frac{(-1)^l}{(l-p)!p!}M_L^A(\tau+t^*)\sigma^{i_1}\ldots\sigma^{i_p}P^{i_{p+1}j_{p+1}}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_{p+1}}}\cdots\frac{\partial}{\partial\xi^{j_l}}\left(\frac{\partial}{\partial c\tau}\right)^{p}\frac{1}{r_A^{\rm N}(\tau+t^*)},\end{aligned} \tag{111}$$

where we recall the notation $M_L^A = M^A_{i_1\ldots i_l}$. It should be underlined that after performing of the differentiations in (111) one can replace $\tau + t^*$ by the global coordinate time t. Let us also note that the following relations have been used in order to obtain (111):

$$P^{ij}\frac{\partial}{\partial\xi^j}\frac{1}{r_A^{\rm N}(\tau+t^*)} = -\frac{d_A^i(\tau+t^*)}{(r_A^{\rm N}(\tau+t^*))^3}, \tag{112}$$

and

$$P^{ij}\frac{\partial}{\partial\xi^j}\ln\left[r_A^{\rm N}(\tau+t^*) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)\right] = \frac{d_A^i(\tau+t^*)}{r_A^{\rm N}(\tau+t^*)}\frac{1}{r_A^{\rm N}(\tau+t^*)-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)}, \tag{113}$$

and the relation $P^{ij}(\xi^j - x_A^j(\tau+t^*)) = d_A^i(\tau+t^*)$.

VII. SOME SPECIAL CASES OF FIRST INTEGRATION

Modern computer algebra systems allow for highly efficient computation of partial differentiations which occur in the first integral (111) of geodesic equation. Here, the first few terms of (111) as instructive examples are considered and compared with known results in the literature, namely, arbitrarily moving monopoles, dipoles, quadrupoles, and the case of one massive body at rest with full mass-multipole structure. These examples can also serve as further elucidation about how the formula in (111) works.

A. Monopoles in arbitrary motion

For the case of light propagation in the gravitational field of N extended mass monopoles in arbitrary motion, we have to consider the term $l = 0$ in (111), which reads

$$\frac{\Delta\dot{\boldsymbol{x}}_M(t)}{c} = -\frac{2G}{c^2}\sum_{A=1}^{N}\frac{M_A}{r_A^{\rm N}(t)}\left(\frac{\boldsymbol{d}_A(t)}{r_A^{\rm N}(t)-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t)}+\boldsymbol{\sigma}\right), \tag{114}$$

where $\tau + t^*$ has finally been replaced by the global coordinate time t. We recall that $\boldsymbol{r}_A^{\rm N}(t) = \boldsymbol{x}_{\rm N}(t) - \boldsymbol{x}_A(t)$, with $\boldsymbol{x}_{\rm N}(t)$ being the spatial position of the unperturbed light signal and $\boldsymbol{x}_A(t)$ the spatial position of the arbitrarily moving massive monopole.

By taking the limit of monopoles at rest $\boldsymbol{x}_A = \text{const}$ in (114), one may easily recognize an agreement of (114) with Eq. (3.2.14) in [15] and with Eq. (28) in [43], where the mass monopoles are displaced by some constant vector $\boldsymbol{x}_A$ from the origin of the global coordinate system.

In [44] the light trajectory in the field of N arbitrarily moving pointlike monopoles has been determined in 1PM approximation. The 1PM approximation is a weak-field approximation, that means the pointlike monopoles could even be in ultra-relativistic motion, while (114) is for extended monopoles but in 1PN approximation, which is a weak-field slow-motion approximation. By expansion of the 1PM solution (Eqs. (32) and (34) in [44]) in powers of v_A/c, one may show an agreement with our solution in (114) up to terms of the order $\mathcal{O}(v_A/c)$.

B. Dipoles in arbitrary motion

Let us consider the dipole term, given by the term $l = 1$ in (111). Inserting the derivatives given by Eqs. (F4)–(F6) in Appendix F, we obtain

$$\frac{\Delta \dot{x}^i_D(t)}{c} = + \frac{2G}{c^2} \sum_{A=1}^{N} \frac{M^A_i(t)}{r^{\rm N}_A(t)} \frac{1}{r^{\rm N}_A(t) - \boldsymbol{\sigma} \cdot \boldsymbol{r}^{\rm N}_A(t)} + \frac{2G}{c^2} \sum_{A=1}^{N} \frac{\sigma^{i_1} M^A_{i_1}(t)}{r^{\rm N}_A(t)} \left(\frac{d^i_A(t)}{(r^{\rm N}_A(t))^2} - \frac{\sigma^i}{r^{\rm N}_A(t) - \boldsymbol{\sigma} \cdot \boldsymbol{r}^{\rm N}_A(t)} - \frac{\sigma^i (\boldsymbol{\sigma} \cdot \boldsymbol{r}^{\rm N}_A(t))}{(r^{\rm N}_A(t))^2} \right)$$
$$- \frac{2G}{c^2} \sum_{A=1}^{N} \frac{d^{i_1}_A(t) M^A_{i_1}(t)}{(r^{\rm N}_A(t))^2} \left(\frac{\sigma^i}{r^{\rm N}_A(t)} + \frac{d^i_A(t)}{r^{\rm N}_A(t)(r^{\rm N}_A(t) - \boldsymbol{\sigma} \cdot \boldsymbol{r}^{\rm N}_A(t))} + \frac{d^i_A(t)}{(r^{\rm N}_A(t) - \boldsymbol{\sigma} \cdot \boldsymbol{r}^{\rm N}_A(t))^2} \right). \tag{115}$$

If the origin of the local reference system $(cT_A, \boldsymbol{X}_A)$ is located exactly at the center of mass of the massive body A, then the dipole moment of this body vanishes, $M^A_{i_1} = 0$. However, in real high-precision astrometry, the center of mass of, for instance, a planet like Jupiter cannot be determined precisely. Therefore, for real astrometric measurements, $M^A_{i_1} \neq 0$; hence, the light deflection caused by the dipole moment of a massive body has to be taken into account, which is purely a coordinate effect; see also [19,78].

C. Quadrupoles in arbitrary motion

As further instructive example we consider the case of light propagation in the gravitational field of N arbitrarily moving quadrupoles, given by $l = 2$ in (111), which reads

$$\frac{\Delta \dot{x}^i_Q(\tau + t^*)}{c} = - \frac{2G}{c^2} \sum_{A=1}^{N} M^A_{i_1 i_2} \sigma^{i_1} P^{i_2 j_2} \frac{\partial}{\partial \xi^{j_2}} \frac{d^i_A}{(r^{\rm N}_A)^3} - \frac{G}{c^2} \sum_{A=1}^{N} M^A_{i_1 i_2} \sigma^{i_1} \sigma^{i_2} \frac{\partial}{\partial c\tau} \frac{d^i_A}{(r^{\rm N}_A)^3}$$
$$- \frac{G}{c^2} \sum_{A=1}^{N} M^A_{i_1 i_2} P^{i_1 j_1} P^{i_2 j_2} \frac{\partial}{\partial \xi^{j_1}} \frac{\partial}{\partial \xi^{j_2}} \frac{d^i_A}{r^{\rm N}_A - \boldsymbol{\sigma} \cdot \boldsymbol{r}^{\rm N}_A} \frac{1}{r^{\rm N}_A} - \frac{G}{c^2} \sum_{A=1}^{N} M^A_{i_1 i_2} \sigma^i \sigma^{i_1} \sigma^{i_2} \frac{\partial}{\partial c\tau} \frac{\partial}{\partial c\tau} \frac{1}{r^{\rm N}_A}$$
$$- \frac{2G}{c^2} \sum_{A=1}^{N} M^A_{i_1 i_2} \sigma^i \sigma^{i_1} P^{i_2 j_2} \frac{\partial}{\partial \xi^{j_2}} \frac{\partial}{\partial c\tau} \frac{1}{r^{\rm N}_A} - \frac{G}{c^2} \sum_{A=1}^{N} M^A_{i_1 i_2} \sigma^i P^{i_1 j_1} P^{i_2 j_2} \frac{\partial}{\partial \xi^{j_1}} \frac{\partial}{\partial \xi^{j_2}} \frac{1}{r^{\rm N}_A}, \tag{116}$$

where here for simpler notation the time arguments have been omitted, i.e. $r^{\rm N}_A = r^{\rm N}_A(\tau + t^*)$, $\boldsymbol{r}^{\rm N}_A = \boldsymbol{r}^{\rm N}_A(\tau + t^*)$, $\boldsymbol{d}_A = \boldsymbol{d}_A(\tau + t^*)$, and $M^A_{i_1 i_2} = M^A_{i_1 i_2}(\tau + t^*)$. The derivatives in (116) are given in Appendix F, and by inserting (F7)–(F12) into (116) one obtains the first integral of geodesic equation in the field of N arbitrarily moving quadrupoles:

$$\frac{\Delta \dot{\boldsymbol{x}}_Q(t)}{c} = \frac{G}{c^2} \sum_{A=1}^{N} \frac{1}{d^2_A(t)} \left[\boldsymbol{\alpha}_A(t) \frac{\dot{\mathcal{U}}_A(t)}{c} + \boldsymbol{\beta}_A(t) \frac{\dot{\mathcal{V}}_A(t)}{c} + \boldsymbol{\gamma}_A(t) \frac{\dot{\mathcal{F}}_A(t)}{c} + \boldsymbol{\delta}_A(t) \frac{\dot{\mathcal{E}}_A(t)}{c} \right], \tag{117}$$

where $\tau + t^*$ has finally been replaced by coordinate time t in (117), i.e. after performance of all differentiations in (116). Adopting similar notation as used in [45], the vectorial coefficients in (117) are given by

$$\alpha^k_A(t) = - M^A_{i_1 i_2}(t) d^k_A(t) \sigma^{i_1} \sigma^{i_2} + 2 M^A_{i_1 k}(t) d^{i_1}_A(t) - 2 M^A_{i_1 i_2}(t) d^{i_2}_A(t) \sigma^{i_1} \sigma^k - \frac{4}{d^2_A(t)} M^A_{i_1 i_2}(t) d^{i_1}_A(t) d^{i_2}_A(t) d^k_A(t), \tag{118}$$

$$\beta^k_A(t) = + M^A_{i_1 i_2}(t) \sigma^{i_1} \sigma^{i_2} \sigma^k - 2 M^A_{i_1 k}(t) \sigma^{i_1} + \frac{4}{d^2_A(t)} M^A_{i_1 i_2}(t) d^{i_2}_A(t) d^k_A(t) \sigma^{i_1} - \frac{2}{d^2_A(t)} M^A_{i_1 i_2}(t) d^{i_1}_A(t) d^{i_2}_A(t) \sigma^k, \tag{119}$$

$$\gamma^k_A(t) = + M^A_{i_1 i_2}(t) d^{i_1}_A(t) d^{i_2}_A(t) d^k_A(t) - M^A_{i_1 i_2}(t) d^k_A(t) d^2_A(t) \sigma^{i_1} \sigma^{i_2} + 2 M^A_{i_1 i_2}(t) d^{i_2}_A(t) d^2_A(t) \sigma^{i_1} \sigma^k, \tag{120}$$

$$\delta^k_A(t) = - M^A_{i_1 i_2}(t) d^{i_1}_A(t) d^{i_2}_A(t) \sigma^k + M^A_{i_1 i_2}(t) d^2_A(t) \sigma^{i_1} \sigma^{i_2} \sigma^k + 2 M^A_{i_1 i_2}(t) d^{i_2}_A(t) d^k_A(t) \sigma^{i_1}. \tag{121}$$

The scalar functions in (117) are given by

$$\frac{\dot{\mathcal{U}}_A(t)}{c} = \frac{d^2_A(t)}{(r^{\rm N}_A(t))^2} \frac{1}{r^{\rm N}_A(t) - \boldsymbol{\sigma} \cdot \boldsymbol{r}^{\rm N}_A(t)} \times \left(\frac{1}{r^{\rm N}_A(t)} + \frac{1}{r^{\rm N}_A(t) - \boldsymbol{\sigma} \cdot \boldsymbol{r}^{\rm N}_A(t)} \right), \tag{122}$$

$$\frac{\dot{\mathcal{V}}_A(t)}{c} = \frac{d^2_A(t)}{(r^{\rm N}_A(t))^3}, \tag{123}$$

$$\frac{\dot{\mathcal{F}}_A(t)}{c} = -3\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t)}{(r_A^{\rm N}(t))^5}, \tag{124}$$

$$\frac{\dot{\mathcal{E}}_A(t)}{c} = \frac{1}{(r_A^{\rm N}(t))^3} - 3\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t))^2}{(r_A^{\rm N}(t))^5}. \tag{125}$$

In the limit of quadrupoles at rest, $\boldsymbol{x}_A = \text{const}$, and time-independent quadrupole moments, $M^A_{i_1 i_2} = \text{const}$, the expression in (117)–(125) coincides with the corresponding results in [17,43,45] [cf. Eqs. (23)–(31)].

D. Body at rest with full mass-multipole structure

The light trajectory in the gravitational field of one massive body A at rest and located at the origin of coordinate system, $\boldsymbol{x}_A = 0$, has been determined in [53] in post-Newtonian approximation for the case of time-independent multipoles. In such situation, we have to make the following replacements: $\boldsymbol{d}_A(\tau+t^*)\to\boldsymbol{\xi}$, $d_A(\tau+t^*)\to d$, $\boldsymbol{r}_A^{\rm N}\to\boldsymbol{r}=\boldsymbol{\xi}+c\tau\boldsymbol{\sigma}$, $r_A^{\rm N}(\tau+t^*)\to r=\sqrt{d^2+c^2\tau^2}$, and $M_L^A(\tau+t^*)\to M_L^A$. Then, our solution in (111) simplifies as follows (we omit the monopole and the dipole term, because the former one has already been considered above, while the latter one is not determined in [53]):

$$\frac{\Delta\dot{x}^i_{v_A=0}(\tau+t^*)}{c} = -\frac{2G}{c^2}\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}M_L^A P^{i_1j_1}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_1}}\cdots\frac{\partial}{\partial\xi^{j_l}}\left[\frac{\xi^i}{d^2}\left(1+\frac{c\tau}{r}\right)+\frac{\sigma^i}{r}\right]$$
$$-\frac{2G}{c^2}\sum_{l=2}^{\infty}\sum_{p=1}^{l}\frac{(-1)^l}{(l-p)!p!}M_L^A\sigma^{i_1}\ldots\sigma^{i_p}P^{i_{p+1}j_{p+1}}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_{p+1}}}\cdots\frac{\partial}{\partial\xi^{j_l}}\left(\frac{\partial}{\partial c\tau}\right)^{p-1}\frac{\xi^i-c\tau\sigma^i}{r^3}, \tag{126}$$

where we have used $\frac{1}{r-c\tau}=\frac{r+c\tau}{d^2}$ and $\left(\frac{\partial}{\partial c\tau}\right)^p\frac{1}{r}=-\left(\frac{\partial}{\partial c\tau}\right)^{p-1}\frac{c\tau}{r^3}$. The expression in (126) agrees with the time derivative of Eq. (36) in [53].

Needless to say that one cannot deduce the general expression in (111) from the specific solution in (126) by some kind of an inverse replacement procedure, because such an approach would not be unique. For instance, the above replacement $d_A\to d$ is unique, but the inverse procedure is not unique, because it could either be $d\to|\boldsymbol{\xi}|$ or $d\to|\boldsymbol{d}_A|$. Similar ambiguities would appear in inverse replacements regarding variables $\boldsymbol{\xi}$ or $c\tau$. In other words: one cannot deduce the general expression in (111) from the specific solution given by Eq. (34) in [53].

VIII. SECOND INTEGRATION OF GEODESIC EQUATION

The second integral of geodesic equation governs the trajectory of the photon and is formally written as follows [cf. Eq. (50)]:

$$\boldsymbol{x}_{\rm 1PN}(\tau+t^*) = \boldsymbol{\xi}+c\tau\boldsymbol{\sigma}+\sum_{A=1}^{N}\Delta\boldsymbol{x}^A_{\rm 1PN}(\tau+t^*,\tau_0+t^*), \tag{127}$$

where the corrections to the unperturbed light ray due to one body A are given by

$$\Delta\boldsymbol{x}^A_{\rm 1PN}(\tau+t^*,\tau_0+t^*) = \int_{\tau_0}^{\tau}dc\tau'\,\frac{\Delta\dot{\boldsymbol{x}}^A_{\rm 1PN}(\tau'+t^*)}{c}, \tag{128}$$

where the integrand in given by Eq. (111). How one goes about performing the second integration is not much different in principle from the first integration represented in Sec. VI. Using relations (112) and (113) we obtain the following expression for the second integration of geodesic equation for the light trajectory in the gravitational field of one extended body A in arbitrary motion:

$$\Delta x^{iA}_{\rm 1PN}(\tau+t^*,\tau_0+t^*) = +\frac{2G}{c^2}P^{ij}\frac{\partial}{\partial\xi^j}\sum_{l=1}^{\infty}\frac{(-1)^l}{(l-1)!}\sigma^{i_1}P^{i_2j_2}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_2}}\cdots\frac{\partial}{\partial\xi^{j_l}}\mathcal{I}_C$$
$$+\frac{2G}{c^2}P^{ij}\frac{\partial}{\partial\xi^j}\sum_{l=2}^{\infty}\sum_{p=2}^{l}\frac{(-1)^l}{(l-p)!p!}\sigma^{i_1}\ldots\sigma^{i_p}P^{i_{p+1}j_{p+1}}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_{p+1}}}\cdots\frac{\partial}{\partial\xi^{j_l}}\mathcal{I}_D$$
$$-\frac{2G}{c^2}P^{ij}\frac{\partial}{\partial\xi^j}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}P^{i_1j_1}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_1}}\cdots\frac{\partial}{\partial\xi^{j_l}}\mathcal{I}_E-\frac{2G}{c^2}\sigma^i\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}P^{i_1j_1}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_1}}\cdots\frac{\partial}{\partial\xi^{j_l}}\mathcal{I}_C$$
$$-\frac{2G}{c^2}\sigma^i\sum_{l=1}^{\infty}\sum_{p=1}^{l}\frac{(-1)^l}{(l-p)!p!}\sigma^{i_1}\ldots\sigma^{i_p}P^{i_{p+1}j_{p+1}}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_{p+1}}}\cdots\frac{\partial}{\partial\xi^{j_l}}\mathcal{I}_F. \tag{129}$$

In order to obtain the form of the first two terms and of the last two terms in (129), the summation over l, p has been separated as follows:

$$\sum_{l=1}^{\infty}\sum_{p=1}^{l} F(l,p) = \sum_{l=1}^{\infty} F(l,p=1) + \sum_{l=2}^{\infty}\sum_{p=2}^{l} F(l,p), \tag{130}$$

$$\sum_{l=0}^{\infty}\sum_{p=0}^{l} F(l,p) = \sum_{l=0}^{\infty} F(l,p=0) + \sum_{l=1}^{\infty}\sum_{p=1}^{l} F(l,p). \tag{131}$$

In (129) we encounter four kinds of integrals:

$$\mathcal{I}_C = \int_{\tau_0}^{\tau} dc\tau' \frac{M_L^A(\tau'+t^*)}{r_A^{\rm N}(\tau'+t^*)}, \tag{132}$$

$$\mathcal{I}_D = \int_{\tau_0}^{\tau} dc\tau' M_L^A(\tau'+t^*)\left(\frac{\partial}{\partial c\tau'}\right)^{p-1} \frac{1}{r_A^{\rm N}(\tau'+t^*)}, \tag{133}$$

$$\mathcal{I}_E = \int_{\tau_0}^{\tau} dc\tau' M_L^A(\tau'+t^*) \times \ln\left[r_A^{\rm N}(\tau'+t^*) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau'+t^*)\right], \tag{134}$$

$$\mathcal{I}_F = \int_{\tau_0}^{\tau} dc\tau' M_L^A(\tau'+t^*)\left(\frac{\partial}{\partial c\tau'}\right)^{p} \frac{1}{r_A^{\rm N}(\tau'+t^*)}, \tag{135}$$

which are determined in Appendix D. These integrals run over the unknown world line $\boldsymbol{x}_A(t)$ of massive body A, and can also be integrated by parts, that means the procedure it essentially based upon the fact that the nonintegrated remnants are beyond 1PN approximation, because they imply an additional factor c^{-1}.

Then, inserting the solutions of these four integrals, given by Eqs. (D2), (D4), (D8) and (D9), into Eq. (129) and performing the differentiations with respect to $P^{ij}\frac{\partial}{\partial \xi^j}$, the second integration of the geodesic equation for the light trajectory in the field of one body A is given by

$$\Delta\boldsymbol{x}_{\rm 1PN}^A(\tau+t^*,\tau_0+t^*) = \Delta\boldsymbol{x}_{\rm 1PN}^A(\tau+t^*) - \Delta\boldsymbol{x}_{\rm 1PN}^A(\tau_0+t^*), \tag{136}$$

where the contribution of one body A is given by

$$\begin{aligned}
\Delta\boldsymbol{x}_{\rm 1PN}^A(\tau+t^*) = &-\frac{2G}{c^2}\sum_{l=1}^{\infty}\frac{(-1)^l}{(l-1)!} M_L^A(\tau+t^*)\sigma^{i_1}P^{i_2j_2}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_2}}\cdots\frac{\partial}{\partial\xi^{j_l}}\frac{\boldsymbol{d}_A(\tau+t^*)}{r_A^{\rm N}(\tau+t^*)}\frac{1}{r_A^{\rm N}(\tau+t^*) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)}\\
&-\frac{2G}{c^2}\sum_{l=2}^{\infty}\sum_{p=2}^{l}\frac{(-1)^l}{(l-p)!p!} M_L^A(\tau+t^*)\sigma^{i_1}\ldots\sigma^{i_p}P^{i_{p+1}j_{p+1}}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_{p+1}}}\cdots\frac{\partial}{\partial\xi^{j_l}}\left(\frac{\partial}{\partial c\tau}\right)^{p-2}\frac{\boldsymbol{d}_A(\tau+t^*)}{(r_A^{\rm N}(\tau+t^*))^3}\\
&-\frac{2G}{c^2}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!} M_L^A(\tau+t^*)P^{i_1j_1}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_1}}\cdots\frac{\partial}{\partial\xi^{j_l}}\frac{\boldsymbol{d}_A(\tau+t^*)}{r_A^{\rm N}(\tau+t^*) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)}\\
&+\frac{2G}{c^2}\boldsymbol{\sigma}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!} M_L^A(\tau+t^*)P^{i_1j_1}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_1}}\cdots\frac{\partial}{\partial\xi^{j_l}}\ln\left[r_A^{\rm N}(\tau+t^*) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)\right]\\
&-\frac{2G}{c^2}\boldsymbol{\sigma}\sum_{l=1}^{\infty}\sum_{p=1}^{l}\frac{(-1)^l}{(l-p)!p!} M_L^A(\tau+t^*)\sigma^{i_1}\ldots\sigma^{i_p}P^{i_{p+1}j_{p+1}}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_{p+1}}}\cdots\frac{\partial}{\partial\xi^{j_l}}\left(\frac{\partial}{\partial c\tau}\right)^{p-1}\frac{1}{r_A^{\rm N}(\tau+t^*)},
\end{aligned} \tag{137}$$

where we recall the notation $M_L^A = M_{i_1\ldots i_l}^A$. In order to obtain (137), the relations (112) and (113) and

$$P^{ij}\frac{\partial}{\partial\xi^j}\left(r_A^{\rm N}(\tau+t^*) + \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)\ln\left[r_A^{\rm N}(\tau+t^*) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)\right]\right) = \frac{d_A^i(\tau+t^*)}{r_A^{\rm N}(\tau+t^*) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)}, \tag{138}$$

have also been used. The expression in (137) represents the solution for the second integration of geodesic equation in 1PN approximation, in the field of one arbitrarily moving body A and to any order of intrinsic mass multipoles. Like in the first integral in (111), after the differentiations in (137) the replacement of $\tau + t^*$ by the global coordinate time t can be performed. One may easily check that the time differentiation of (137) yields immediately the first integral in (111) up to terms of higher-order beyond 1PN approximation. So the solution in (137) is consistent with the solution in (111).

IX. SOME SPECIAL CASES OF SECOND INTEGRATION

Like in the case of first integration, let us consider the very first few terms of (137) as instructive examples, and compare them with research findings in the literature, namely: arbitrarily moving monopoles, dipoles, quadrupoles, and the case of one massive body at rest with full mass-multipole structure.

A. Monopoles in arbitrary motion

For the monopole term ($l = 0$), we obtain from (136) and (137):

$$\Delta \boldsymbol{x}_M(t,t_0) = -\frac{2G}{c^2}\sum_{A=1}^{N} M_A\left(\frac{\boldsymbol{d}_A(t)}{r_A^{\rm N}(t) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t)} - \frac{\boldsymbol{d}_A(t_0)}{r_A^{\rm N}(t_0) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t_0)}\right) + \frac{2G}{c^2}\boldsymbol{\sigma}\sum_{A=1}^{N} M_A \ln\frac{r_A^{\rm N}(t) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t)}{r_A^{\rm N}(t_0) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t_0)}, \qquad (139)$$

where in the final expression we have replaced $\tau + t^* = t$ and $\tau_0 + t^* = t_0$; recall $\boldsymbol{r}_A^{\rm N}(t) = \boldsymbol{x}_{\rm N}(t) - \boldsymbol{x}_A(t)$ and $\boldsymbol{r}_A^{\rm N}(t_0) = \boldsymbol{x}_0 - \boldsymbol{x}_A(t_0)$. The time derivative of (139) yields immediately (114) up to terms of order $\mathcal{O}(v_A/c)$.

In the limit of massive bodies at rest, the expression (139) coincides with Eq. (3.2.13) in [15] and with Eq. (22) in [43], where the mass monopoles are not located at the origin of the coordinate-system but displaced by some constant vector $\boldsymbol{x}_A$; cf. Eq. (22).

In [44] the light trajectory in the field of N arbitrarily moving pointlike monopoles has been determined in first post-Minkowskian approximation (1PM), that means where the pointlike monopoles could even move with ultra-relativistic speed, while (139) is for extended monopoles in 1PN approximation. By expansion of the 1PM solution (Eqs. (33) and (35) in [44]) in powers of v_A/c, one may show an agreement with our solution in (139) up to terms of the order $\mathcal{O}(v_A/c)$.

B. Dipoles in arbitrary motion

From (137) we obtain for the dipole term ($l = 1$)

$$\Delta x^i_D(t,t_0) = \Delta x^i_D(t) - \Delta x^i_D(t_0),$$

$$\Delta x^i_D(t) = +\frac{2G}{c^2}\sum_{A=1}^{N} M_i^A(t)\frac{1}{r_A^{\rm N}(t) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t)} + \frac{2G}{c^2}\sum_{A=1}^{N}\frac{M_{i_1}^A(t)}{r_A^{\rm N}(t)}\sigma^{i_1}\left(\frac{d_A^i(t)}{r_A^{\rm N}(t) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t)} - \frac{\sigma^i(\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t))}{r_A^{\rm N}(t) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t)}\right)$$
$$- \frac{2G}{c^2}\sum_{A=1}^{N}\frac{M_{i_1}^A(t)}{r_A^{\rm N}(t)}d_A^{i_1}(t)\left(\frac{\sigma^i}{r_A^{\rm N}(t) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t)} + \frac{d_A^i(t)}{(r_A^{\rm N}(t) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t))^2}\right), \qquad (140)$$

where we have used the derivatives given in Appendix G. The time derivative of (140) yields immediately (115) up to terms of order $\mathcal{O}(v_A/c)$. As mentioned above, if the origin of the local reference system $(cT_A, \boldsymbol{X}_A)$ is located exactly at the center of mass of the massive body A, then the dipole moment $M_{i_1}^A$ of this body vanishes and there would be no dipole term. But in reality one cannot determine precisely the center of mass of a massive body (e.g. giant planets) so that $M_{i_1}^A \neq 0$ and one has carefully to take into account the change in the light trajectory caused by the dipole term, which is purely a coordinate effect; cf. Ref. [19,78].

C. Quadrupoles in arbitrary motion

Now we consider the light trajectory in the gravitational field of N arbitrarily moving quadrupoles, given by the term $l = 2$ in (137), which reads

$$\Delta x^i_Q(\tau + t^*, \tau_0 + t^*) = \Delta x^i_Q(\tau + t^*) - \Delta x^i_Q(\tau_0 + t^*), \qquad (141)$$

with

$$\Delta x_Q^i(\tau+t^*) = -\frac{2G}{c^2}\sum_{A=1}^{N} M^A_{i_1 i_2}\sigma^{i_1} P^{i_2 j_2}\frac{\partial}{\partial \xi^{j_2}}\frac{d_A^i}{r_A^{\rm N}}\frac{1}{r_A^{\rm N}-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}}$$
$$-\frac{G}{c^2}\sum_{A=1}^{N} M^A_{i_1 i_2}\sigma^{i_1}\sigma^{i_2}\frac{d_A^i}{(r_A^{\rm N})^2}\frac{1}{r_A^{\rm N}} - \frac{G}{c^2}\sum_{A=1}^{N} M^A_{i_1 i_2}P^{i_1 j_1}P^{i_2 j_2}\frac{\partial}{\partial \xi^{j_1}}\frac{\partial}{\partial \xi^{j_2}}\frac{d_A^i}{r_A^{\rm N}-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}}$$
$$+\frac{G}{c^2}\sigma^i\sum_{A=1}^{N} M^A_{i_1 i_2}P^{i_1 j_1}P^{i_2 j_2}\frac{\partial}{\partial \xi^{j_1}}\frac{\partial}{\partial \xi^{j_2}}\ln\left(r_A^{\rm N}-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}\right)$$
$$-\frac{2G}{c^2}\sigma^i\sum_{A=1}^{N} M^A_{i_1 i_2}\sigma^{i_1}P^{i_2 j_2}\frac{\partial}{\partial \xi^{j_2}}\frac{1}{r_A^{\rm N}} - \frac{G}{c^2}\sigma^i\sum_{A=1}^{N} M^A_{i_1 i_2}\sigma^{i_1}\sigma^{i_2}\frac{\partial}{\partial c\tau}\frac{1}{r_A^{\rm N}}, \quad (142)$$

where here for simpler notation the time arguments have been omitted, i.e. $r_A^{\rm N} = r_A^{\rm N}(\tau+t^*)$, $\boldsymbol{r}_A^{\rm N} = \boldsymbol{r}_A^{\rm N}(\tau+t^*)$, $\boldsymbol{d}_A = \boldsymbol{d}_A(\tau+t^*)$, and $M^A_{i_1 i_2} = M^A_{i_1 i_2}(\tau+t^*)$. The derivatives in the first, fifth, and sixth term in (142) were already given in Appendix F, while the derivatives of the third and fourth term in (142) were already given in Appendix G. After performing these derivatives the replacements have to be performed: $\tau+t^* = t$ and $\tau_0+t^* = t_0$. By inserting relations (F4), (F6), (G3), (G4) into (142), one obtains the light trajectory in the field of N arbitrarily moving quadrupoles:

$$\Delta\boldsymbol{x}_Q(t,t_0) = \Delta\boldsymbol{x}_Q(t) - \Delta\boldsymbol{x}_Q(t_0), \quad (143)$$

with

$$\Delta\boldsymbol{x}_Q(t) = \frac{G}{c^2}\sum_{A=1}^{N}\frac{1}{d_A^2(t)}[\boldsymbol{\alpha}_A(t)\mathcal{U}_A(t) + \boldsymbol{\beta}_A(t)\mathcal{V}_A(t) + \boldsymbol{\gamma}_A(t)\mathcal{F}_A(t) + \boldsymbol{\delta}_A(t)\mathcal{E}_A(t)]. \quad (144)$$

The vectorial coefficients in (144) were given by Eqs. (118)–(121) and the scalar functions in (144) are given by

$$\mathcal{U}_A(t) = \frac{1}{r_A^{\rm N}(t)}\frac{r_A^{\rm N}(t)+\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t)}{r_A^{\rm N}(t)-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t)}, \quad (145)$$

$$\mathcal{V}_A(t) = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t)}{r_A^{\rm N}(t)} + 1, \quad (146)$$

$$\mathcal{F}_A(t) = \frac{1}{(r_A^{\rm N}(t))^3}, \quad (147)$$

$$\mathcal{E}_A(t) = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(t)}{(r_A^{\rm N}(t))^3}. \quad (148)$$

The time derivative of (144) yields (117), up to terms of higher order, i.e., either $\mathcal{O}(v_A/c)$ or $\mathcal{O}(\dot{M}^A_{i_1 i_2}/c)$.

In the limit of bodies at rest, $\boldsymbol{x}_A = \text{const}$, and time-independent quadrupole moments, $M^A_{i_1 i_2} = \text{const}$, the expression in (144)–(148) coincides with the corresponding results in [17,43,45] [cf. Eqs. (23)–(31)].

One should keep in view that a series expansion of the vectorial coefficients (118)–(121) does not necessarily create terms beyond 1PN approximation. For instance, a series expansion of the vectorial coefficients around some time moment t_0 implies a corresponding series expansion of the impact vector and quadrupole moment,

$$\boldsymbol{d}_A(t) = \boldsymbol{d}_A(t_0) + \boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\boldsymbol{v}_A(t_0))(t-t_0) + \mathcal{O}(a_A), \quad (149)$$

$$M^A_{i_1 i_2}(t) = M^A_{i_1 i_2}(t_0) + \dot{M}^A_{i_1 i_2}(t_0)(t-t_0) + \mathcal{O}(\ddot{M}^A_{i_1 i_2}), \quad (150)$$

which are proportional either to $v_A(t-t_0)$ or $\dot{M}^A_{i_1 i_2}(t-t_0)$, but neither to v_A/c nor $\dot{M}^A_{i_1 i_2}/c$. Consequently, the individual terms in a series expansion of vectorial coefficients are not necessarily beyond 1PN approximation.

Results for the light trajectory in the field of quadrupoles in uniform motion, $\boldsymbol{v}_A = \text{const}$, were represented in [52]. In the limit of uniform motion the expression in (143)—(148) should coincide with the results in [52]. For such a comparison the series-expansion in (32) would have to be inserted into the solution (143)—(148), which leads rapidly to cumbersome expressions. Consequently, such a comparison constitutes a rather ambitious assignment of a task and spoils the intention of the investigation.

D. Body at rest with full mass-multipole structure

As it has been mentioned above, the light trajectory in the gravitational field of one massive body at rest and located at the origin of coordinate system, $\boldsymbol{x}_A = 0$, has been determined in [53] in post-Newtonian approximation and for the case of time-independent multipoles. In such situation, we have to make the following replacements: $\boldsymbol{d}_A(\tau + t^*) \to \boldsymbol{\xi}$, $d_A(\tau + t^*) \to d$, $\boldsymbol{r}_A^{\rm N} \to \boldsymbol{r} = \boldsymbol{\xi} + c\tau\boldsymbol{\sigma}$, $r_A^{\rm N}(\tau + t^*) \to r = \sqrt{d^2 + c^2\tau^2}$, and $M_L^A(\tau + t^*) \to M_L^A$. Then our solution in (137) simplifies as follows (without the monopole and the dipole term):

$$\begin{aligned}\Delta x^i_{v_A=0}(\tau + t^*) = &-\frac{2G}{c^2}\sum_{l=2}^{\infty}\frac{(-1)^l}{(l-1)!}M_L^A\sigma^{i_1}P^{i_{p+1}j_{p+1}}\dots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_{p+1}}}\dots\frac{\partial}{\partial\xi^{j_l}}\left[\frac{\xi^i}{d^2}\frac{c\tau}{r}+\frac{\sigma^i}{r}\right]\\ &-\frac{2G}{c^2}\sum_{l=2}^{\infty}\sum_{p=2}^{l}\frac{(-1)^l}{(l-p)!p!}M_L^A\sigma^{i_1}\dots\sigma^{i_p}P^{i_{p+1}j_{p+1}}\dots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_{p+1}}}\dots\frac{\partial}{\partial\xi^{j_l}}\left(\frac{\partial}{\partial c\tau}\right)^{p-2}\left[\frac{\xi^i - c\tau\sigma^i}{r^3}\right]\\ &-\frac{2G}{c^2}\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}M_L^A P^{i_1j_1}\dots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_1}}\dots\frac{\partial}{\partial\xi^{j_l}}\left[\frac{\xi^i}{d^2}(r + c\tau) + \sigma^i\ln(r + c\tau)\right], \end{aligned} \tag{151}$$

which is in agreement with Eq. (36) in [53], and the time derivative of (151) yields (126). Let us note, that expression (151) has to be understood in combination with (136), that means in the first line the term $\frac{\xi^i}{d^2}\frac{r+c\tau}{r}$ has been replaced by $\frac{\xi^i}{d^2}\frac{c\tau}{r}$, and also $\ln\frac{r-c\tau}{r_0-c\tau_0} = -\ln\frac{r+c\tau}{r_0+c\tau_0}$ has been used.

It is of course impossible to deduce the general solution in (137) from the specific solution in (151), by reason that an inverse replacement procedure would not be unique, because it could either be $d \to |\boldsymbol{\xi}|$ or $d \to |\boldsymbol{d}_A|$; similar problems concern the variables $\boldsymbol{\xi}$ or $c\tau$. Stated somewhat differently: one cannot deduce the general expression in (137) from the specific solution given by Eq. (36) in [53]; cf. text below Eq. (126).

X. OBSERVABLE RELATIVISTIC EFFECTS

Let us consider two observable effects which are of decisive importance in relativistic astrometry: the time delay and the deflection of photons propagating through the Solar System. The observer and the celestial light source are assumed to be at rest with respect to the global system.

A. Time delay

The classical relativistic effect of time delay when a light signal propagates through the static gravitational field of a spherically symmetric massive body (monopole) has been predicted by *Shapiro* in 1963 [87] and were detected soon afterwards [88,89]. The results of these experiments have been confirmed with increasing accuracy, and the todays most accurate measurement of Shapiro delay was achieved in 2003 [90] using Cassini spacecraft. The solution in (127) allows to determine the time delay of light signals propagating through the gravitational field of a system of N arbitrarily moving massive bodies.

Let $\boldsymbol{x}_1 = \boldsymbol{x}(t_1)$ be the global spatial coordinate of the space-based observer at the moment of observation t_1 and $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$ be the global spatial coordinate of the source at the moment of emission t_0 of the light signal which is observed at $\boldsymbol{x}(t_1)$. In terms of the new variables $\boldsymbol{\xi}$ and τ, both of these spatial coordinates are given by $\boldsymbol{x}_1 = \boldsymbol{x}(\tau_1 + t^*)$ and $\boldsymbol{x}_0 = \boldsymbol{x}(\tau_0 + t^*)$. Furthermore, we introduce the following vectors:

$$\boldsymbol{R} = \boldsymbol{x}(\tau_1 + t^*) - \boldsymbol{x}(\tau_0 + t^*), \tag{152}$$

$$\boldsymbol{k} = \frac{\boldsymbol{R}}{R}, \tag{153}$$

where $R = |\boldsymbol{R}|$ with $\boldsymbol{R}$ being the vector from the source (at the moment of emission) to the observer (at the moment of observation) and $\boldsymbol{k}$ is the corresponding unit direction. Then, using the same procedure as described in [60], one obtains from Eq. (127) the following expression for the relativistic time delay (cf. [43])

$$\begin{aligned}&c(\tau_1 - \tau_0)_{\rm 1PN}\\ &\quad = R - \sum_{A=1}^{N}\boldsymbol{k}\cdot(\Delta\boldsymbol{x}^A_{\rm 1PN}(\tau_1 + t^*,\tau_0 + t^*)),\end{aligned} \tag{154}$$

where the perturbation terms $\Delta\boldsymbol{x}_{\rm 1PN}$ are given by (137); note that the below standing relation (157) has also been used. In the case of N arbitrarily moving monopoles Eq. (154) agrees with formula (51) in [44] up to order $\mathcal{O}(v_A/c)$, and in the case of N quadrupoles at rest, Eq. (154) agrees with formula (23) in [47].

The result in (154) is valid for N slowly moving bodies with full mass-multipole structure. But even for future highly precise astrometry missions aiming to determine relativity within the Solar System (e.g. ASTROD [6,7], LATOR [8,9], ODYSSEY [10], SAGAS [11], TIPO [12]) only the impact of the very first few multipoles could be detected. However, the exact determination of these relevant parts of the perturbation terms in (154) implies some remarkable effort (see for instance [47] for the efficient computation of the quadrupole term) and is beyond the scope of our present investigation.

B. Light deflection

The light deflection at the observer's position, $\boldsymbol{x}_1 = \boldsymbol{x}(t_1)$, which is assumed to be at rest with respect to the global coordinate system, is defined by the unit tangent vector of the light ray at the observer's position:

$$\boldsymbol{n}_{1\mathrm{PN}}(\tau_1 + t^*) = \frac{\dot{\boldsymbol{x}}_{1\mathrm{PN}}(\tau_1 + t^*)}{|\dot{\boldsymbol{x}}_{1\mathrm{PN}}(\tau_1 + t^*)|}. \tag{155}$$

Using (103), one obtains

$$\boldsymbol{n}_{1\mathrm{PN}}(\tau_1 + t^*) = \boldsymbol{\sigma} + \sum_{A=1}^{N} \boldsymbol{\sigma} \times \left(\frac{\Delta \dot{\boldsymbol{x}}^A_{1\mathrm{PN}}(\tau_1 + t^*)}{c} \times \boldsymbol{\sigma} \right), \tag{156}$$

where the perturbation terms $\Delta\dot{\boldsymbol{x}}_{1\mathrm{PN}}$ are given by (111). This expression is valid for light sources at far distances. In the case of N arbitrarily moving monopoles, our result in (156) agrees with Eq. (69) in [44], and in the case of N quadrupoles at rest, our result in (156) agrees with Eq. (7) in [47]. But one has to bear in mind that for astrometry within the near zone of the Solar System, where the light sources are at finite distance, one needs to determine the light deflection as function of $\boldsymbol{k}$ instead of $\boldsymbol{\sigma}$, both of which are related by (cf. [43])

$$\boldsymbol{\sigma} = \boldsymbol{k} - \frac{1}{R} \sum_{A=1}^{N} [\boldsymbol{k} \times (\Delta \boldsymbol{x}^A_{1\mathrm{PN}}(\tau_1 + t^*, \tau_0 + t^*) \times \boldsymbol{k})], \tag{157}$$

which follows from (127) and the definition in (152) and (153). Inserting (157) into (156) yields the expression for the light deflection (cf. [43])

$$\boldsymbol{n}_{1\mathrm{PN}}(\tau_1 + t^*) = \boldsymbol{k} + \sum_{A=1}^{N} \boldsymbol{k} \times \left(\frac{\Delta \dot{\boldsymbol{x}}^A_{1\mathrm{PN}}(\tau_1 + t^*)}{c} \times \boldsymbol{k} \right) - \frac{1}{R} \sum_{A=1}^{N} [\boldsymbol{k} \times (\Delta \boldsymbol{x}^A_{1\mathrm{PN}}(\tau_1 + t^*, \tau_0 + t^*) \times \boldsymbol{k})]. \tag{158}$$

In the case of quadrupoles at rest, our result in (158) agrees with Eq. (14) in [47]. Let us notice here that in order to determine the unit tangent vector of the light ray at the observer's position, one needs to ascertain both the term $\Delta\dot{\boldsymbol{x}}_{1\mathrm{PN}}$ as well as $\Delta\boldsymbol{x}_{1\mathrm{PN}}$, which are given by (111) and (137), respectively.

The formulae in (156) and (158) determine the light deflection in the field of N arbitrarily moving massive bodies with full mass-multipole structure. Like in the case of time delay, only the very first few multipoles in (156) or (158) have to be taken into account for sub-microarcsecond astrometry. But such an exact determination of the relevant multipoles implies some considerable amount of effort, see for instance [47] for the quadrupole part, and will therefore not be on the scope of the present investigation.

XI. SUMMARY AND OUTLOOK

While the precision of astrometric measurements has made an advance from the milliarcsecond to microarcsecond in the angle determination of celestial objects, prospective developments in the nearest future aim at the sub-microarcsecond or even nanoarcsecond level of accuracy. It is clear that such extremely high accuracy implies the precise determination of the light trajectory $\boldsymbol{x}(t)$ from the celestial object through the Solar System towards the observer. As a result, two aspects are of specific importance:

(**A**) In the region exterior of the massive bodies, the global metric of the Solar System (BCRS coordinates: $ct, \boldsymbol{x}$) can be expressed in terms of two families of global multipoles [25–28]: global mass multipoles m_L and global spin multipoles s_L, which define the multipole structure of the Solar System as a whole. On the other side, from the theory of relativistic reference systems follows that the multipole structure of the gravitational field of some massive body A can only be defined in a physically meaningful way within the local reference system (GCRS-like coordinates: $cT_A, \boldsymbol{X}_A$) comoving with that body. In accordance with these requirements, highly precise astrometric measurements appeal for the use of a global metric expressed in terms of intrinsic mass multipoles M_L^A and intrinsic spin multipoles S_L^A of each individual body. Such a metric is provided by the *Brumberg-Kopeikin* (BK) formalism [15,19,29–32] as well as by the *Damour-Soffel-Xu* (DSX) approach [33–36], originally been introduced for celestial mechanics, and which have become a part of the IAU resolutions B1.3 (2000) [21].

(**B**) Another aspect in the theory of light propagation concerns the fact that the massive bodies of the Solar System are moving along their world line $\boldsymbol{x}_A(t)$, which is a highly complicated function because of the mutual interaction of the massive bodies. Formally, the world line of some massive body A can be series-expanded around some time moment t_A,

$$\boldsymbol{x}_A(t) = \boldsymbol{x}_A + \frac{\boldsymbol{v}_A}{1!}(t - t_A) + \frac{\boldsymbol{a}_A}{2!}(t - t_A)^2 + \mathcal{O}(\dot{a}_A), \tag{159}$$

where $\boldsymbol{x}_A$, $\boldsymbol{v}_A$ and $\boldsymbol{a}_A$ are the position, velocity and acceleration of body A at time moment t_A, respectively.

The expansion (159) has some drawbacks:

(i) It implies to introduce an instant of time t_A, which remains an open parameter, as long as no additional arguments are put forward to identify that parameter with the time of closest approach (33) or with the retarded time (35). But so far, a unique justification of that suggestion exists only for pointlike bodies in arbitrary motion, but not for extended bodies in arbitrary motion and expressed in terms of intrinsic multipoles.

(ii) If the expansion (159) is implemented into the metric, it leads to rather cumbersome expressions when integrating the geodesic equation.

(iii) One has also to realize that (159) is not an expansion in inverse powers of the speed of light; hence, these terms are not necessarily beyond 1PN approximation of geodesic equation.

These facts make it much preferable to determine the light trajectory as function of arbitrary world lines $\boldsymbol{x}_A(t)$, that means to determine the light trajectory in the field of arbitrarily moving massive bodies. The actual world line of the massive bodies can finally be concretized and implemented by some Solar System ephemerides; e.g. the JPL DE421 [42].

As outlined in some detail by a brief survey of recent advancements in the theory of light propagation, so far there was no solution derived for the light trajectory in the gravitational field of arbitrarily shaped bodies in arbitrary motion and described in terms of their local multipoles. According to the IAU recommendations [21], in this investigation the DSX metric has been employed in order to determine the light trajectory in 1PN approximation in the gravitational field of N arbitrarily moving massive bodies with full mass-multipole structure:

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + c(t - t_0)\boldsymbol{\sigma} + \Delta\boldsymbol{x}_{1\mathrm{PN}}(t, t_0) + \mathcal{O}(c^{-3}). \tag{160}$$

The main results of this investigation are given by Eq. (111) and Eq. (137). These solutions have taken into account both of these issues (**A**) and (**B**) outlined above: expression (111) represents the first integration of geodesic equation, while expression (137) represents the second integration of geodesic equation, that means the light trajectory in the gravitational field of N arbitrarily moving and extended massive bodies and expressed in terms of their intrinsic multipoles. Furthermore, it has been shown that the results presented agree in special cases with well-established results in the literature, namely monopoles, quadrupoles, and arbitrarily shaped bodies at rest as well as monopoles in arbitrary motion.

It is clear that a comprehensive model of light propagation at the sub-μas or even at the nas level of accuracy requires at least the solution of light trajectory in 1.5PN approximation as well:

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + c(t - t_0)\boldsymbol{\sigma} + \Delta\boldsymbol{x}_{1\mathrm{PN}}(t, t_0) + \Delta\boldsymbol{x}_{1.5\mathrm{PN}}(t, t_0) + \mathcal{O}(c^{-4}). \tag{161}$$

For instance, the light deflection of a grazing ray at Jupiter amounts to about $n^{Q}_{1\mathrm{PN}} \sim 240\ \mu$as [43,45]. Such terms are already implemented in the 1PN solution. On the other side, a typical term of 1.5PN approximation would be $n^{Q}_{1.5\mathrm{PN}} \sim n^{Q}_{1\mathrm{PN}} v_A/c$, which in the case of Jupiter ($v_A/c \sim 4.5 \times 10^{-5}$) yields a light deflection of about $n^{Q}_{1.5\mathrm{PN}} \sim 0.01\ \mu$as. Another typical term of 1.5PN approximation is the light deflection due to the spin of the massive bodies, which have been determined to be about $n^{S}_{1.5\mathrm{PN}} \sim 0.7\ \mu$as, $0.2\ \mu$as, and $0.04\ \mu$as for grazing light rays at Sun, Jupiter, and Saturn, respectively [43,45]. Moreover, recent investigations [77] have recovered, that the light deflection due to the spin-octupole structure of massive bodies amounts to about $0.015\ \mu$as for Jupiter and about $0.006\ \mu$as for Saturn for grazing rays. Therefore, a model at the sub-μas level has also to account for higher spin-multipole terms which are of 1.5PN order.

Clearly, the post-Newtonian approach allows for astrometry within the boundary of the near zone of the Solar System, $|\boldsymbol{x}| \ll \lambda_{\mathrm{gr}} \sim 3$ parsec, while light rays which originate from sources lying far outside of the Solar System are subject of the far-zone astrometry. The perturbations of the light trajectory in the far zone of the Solar System are extremely weak (less than $1\ \mu$as in the light deflection), but might be of relevance for sub-microarcsecond astrometry. These effects can be investigated by means of a matching procedure of two asymptotic solutions (near-zone and far-zone solution) proposed in [17] and further elaborated in [91], and will be on the scope of a further investigation [92].

A further problem concerns the retardation effect due to the finite speed at which gravitational action travels. It has, however, been elucidated by Eq. (58) that the effect of retardation cannot be taken into account within 1PN approximation for the light rays. For this fact, the solution for the light trajectory in 1PN approximation, Eqs. (111) and (137), are functions of the instantaneous distance between the photon and massive body, as given by Eq. (17) or Eq. (89).

Furthermore, the light trajectory in 2PN approximation reads formally

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + c(t - t_0)\boldsymbol{\sigma} + \Delta\boldsymbol{x}_{1\mathrm{PN}}(t, t_0) + \Delta\boldsymbol{x}_{1.5\mathrm{PN}}(t, t_0) + \Delta\boldsymbol{x}_{2\mathrm{PN}}(t, t_0) + \mathcal{O}(c^{-5}). \tag{162}$$

The most dominant post-post-Newtonian correction is the monopole term, $\Delta\boldsymbol{x}^{M}_{2\mathrm{PN}}$, which is well known for bodies at rest. Following a suggestion in [57], for the case of uniformly moving bodies this term can be obtained by an appropriate Lorentz transformation, while for the case of arbitrarily moving bodies the solution might be acquired with the aid of sophisticated integration methods mentioned in this article. It might even be that some very few terms in 2PN approximation beyond the monopole term are required for nanoarcsecond accuracy. Such terms will rapidly decrease with increasing impact parameter d_A of the light ray and might only be of relevance for grazing rays, i.e. where d_A equals the radius R_A of the body. But for all that, the final level of ambition must include a rigorous estimation of such terms, implicating a clear understanding about whether or not some 2PN terms beyond the monopole term become relevant for astrometry at the nanoarcsecond level.

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APPENDIX A: NOTATIONS

Throughout the article the following notations are in use:

(i) G is the Newtonian constant of gravitation.

(ii) c is the vacuum speed of light in flat Minkowski space.

(iii) Lower case Latin indices $a, b, \ldots, i, j, \ldots$ take values 1,2,3.

(iv) Lower case Greek indices $\alpha, \beta, \ldots, \mu, \nu, \ldots$ take values 0,1,2,3.

(v) $\delta_{ij} = \delta^{ij} = \mathrm{diag}(+1, +1, +1)$ is Kronecker delta.

(vi) The three-dimensional coordinate quantities ("three-vectors") referred to the spatial axes of the corresponding reference system are set in boldface: $\boldsymbol{a}$.

(vii) The contravariant components of "three-vectors" are $a^i = (a^1, a^2, a^3)$.

(viii) The contravariant components of "four-vectors" are $a^\mu = (a^0, a^1, a^2, a^3)$.

(ix) Repeated indices imply the Einstein's summation irrespective of their positions (e.g. $a^i b^i = a^1 b^1 + a^2 b^2 + a^3 b^3$ and $a^\alpha b^\alpha = a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$).

(x) The absolute value (Euclidean norm) of a "three-vector" $\boldsymbol{a}$ is denoted as $|\boldsymbol{a}|$ or, simply, a and can be computed as $a = |\boldsymbol{a}| = (a^1 a^1 + a^2 a^2 + a^3 a^3)^{1/2}$.

(xi) The scalar product of any two "three-vectors" $\boldsymbol{a}$ and $\boldsymbol{b}$ with respect to the Euclidean metric δ_{ij} is denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$ and can be computed as $\boldsymbol{a} \cdot \boldsymbol{b} = \delta_{ij} a^i b^j = a^i b^i$.

(xii) The vector product of any two "three-vectors" $\boldsymbol{a}$ and $\boldsymbol{b}$ is designated by $\boldsymbol{a} \times \boldsymbol{b}$ and can be computed as $(\boldsymbol{a} \times \boldsymbol{b})^i = \varepsilon_{ijk} a^j b^k$, where $\varepsilon_{ijk} = (i - j)(j - k)(k - i)/2$ is the fully antisymmetric Levi-Civita symbol.

(xiii) The global coordinate system is denoted by lowercase letters: $(ct, \boldsymbol{x})$.

(xiv) The local coordinate system of a massive body A is denoted by uppercase letters: $(cT_A, \boldsymbol{X}_A)$.

(xv) The photon trajectory is denoted by $\boldsymbol{x}(t)$. In order to distinguish the photon's spatial coordinate $\boldsymbol{x}(t)$ from the spatial coordinate $\boldsymbol{x}$ of the global system, the time dependence of a photon's spatial coordinate will everywhere be shown explicitly throughout the article.

(xvi) The world line of massive body A is denoted by $\boldsymbol{x}_A(t)$ or $\boldsymbol{x}_A(T_A)$.

(xvii) Partial derivatives in the global coordinate system: $\partial_\mu = \frac{\partial}{\partial x^\mu}$ or $\partial_i = \frac{\partial}{\partial x^i}$.

(xviii) Partial derivatives in the local coordinate system of body A: $\mathcal{D}^A_\alpha = \frac{\partial}{\partial X^\alpha_A}$ or $\mathcal{D}^A_a = \frac{\partial}{\partial X^a_A}$.

(xix) $n! = n(n-1)(n-2)\ldots 2 \cdot 1$ is the faculty for the positive integer; $0! = 1$.

(xx) $L = i_1 i_2 \ldots i_l$ is a Cartesian multi-index of a given tensor T, that means $T_L \equiv T_{i_1 i_2 \ldots i_l}$, and each index $i_1, i_2, \ldots, i_l$ runs from 1 to 3 (i.e. over the Cartesian coordinate label).

(xxi) Two identical multi-indices imply summation, e.g.: $\partial_L T_L \equiv \sum_{i_1 \ldots i_l} \partial_{i_1 \ldots i_l} T_{i_1 \ldots i_l}$.

(xxii) The symmetric part of a Cartesian tensor T_L is (cf. Eq. (2.1) in [25])

$$T_{(L)} = T_{(i_1 \ldots i_l)} = \frac{1}{l!} \sum_\sigma A_{i_{\sigma(1)} \ldots i_{\sigma(l)}}, \qquad \text{(A1)}$$

where σ is running over all permutations of $(1, 2, \ldots, l)$.

(xxiii) The symmetric tracefree (STF) part of a Cartesian tensor T_L (notation: $\hat{T}_L \equiv \underset{L}{\mathrm{STF}} T_L$) is (cf. Eq. (2.2) in [25])

$$\hat{T}_L = \sum_{k=0}^{[l/2]} a_{lk} \delta_{(i_1 i_2} \ldots \delta_{i_{2k-1} i_{2k}} S_{i_{2k+1} \ldots i_l) a_1 a_1 \ldots a_k a_k}, \qquad \text{(A2)}$$

where $[l/2]$ means the largest integer less than or equal to $l/2$, and $S_L \equiv T_{(L)}$ abbreviates the symmetric part of tensor T_L. For instance, $T_L^{\alpha\beta}$ means STF with respect to indices L but not with respect to indices α, β. The coefficient in (A2) is given by

$$a_{lk} = (-1)^k \frac{l!}{(l-2k)!} \frac{(2l-2k-1)!!}{(2l-1)!!(2k)!!}. \qquad \text{(A3)}$$

As instructive examples of (A2), let us consider the cases $l = 2$ and $l = 3$:

$$\hat{T}_{ij} = T_{(ij)} - \frac{1}{3} \delta_{ij} T_{ss}, \qquad \text{(A4)}$$

$$\hat{T}_{ijk} = T_{(ijk)} - \frac{1}{5} (\delta_{ij} T_{(kss)} + \delta_{jk} T_{(iss)} + \delta_{ki} T_{(jss)}). \qquad \text{(A5)}$$

Throughout the article, the "hat" will be omitted for the multipoles, $M^A_L \equiv \hat{M}^A_L$, $m_L \equiv \hat{m}_L$, $S^A_L \equiv \hat{S}^A_L$, $s_L \equiv \hat{s}_L$, but kept for spatial coordinates, $\hat{x}_L$.

APPENDIX B: THE POST-POST-NEWTONIAN TERM IN EQ. (36)

The light trajectory in 2PN approximation in the field of one monopole at rest reads

$$\boldsymbol{x}^M_{2\mathrm{PN}}(t,t_0) = \boldsymbol{x}_0 + c(t-t_0)\boldsymbol{\sigma} + \Delta\boldsymbol{x}^M_{1\mathrm{PN}}(t,t_0) + \Delta\boldsymbol{x}^M_{2\mathrm{PN}}(t,t_0), \quad \text{(B1)}$$

where it has been taken into account that there is no correction term in 1.5PN order. In order to derive the expression in (36)–(37), we use the iterative solution in [15,17,46,67], according to which we have

$$\Delta\boldsymbol{x}^M_{1\mathrm{PN}}(t,t_0) + \Delta\boldsymbol{x}^M_{2\mathrm{PN}}(t,t_0) = +\frac{GM_A}{c^2}[\boldsymbol{B}_1(\boldsymbol{r}^{1\mathrm{PN}}_A) - \boldsymbol{B}_1(\boldsymbol{r}^0_A)] + \frac{G^2M_A^2}{c^4}[\boldsymbol{B}_2(\boldsymbol{r}^{\mathrm{N}}_A) - \boldsymbol{B}_2(\boldsymbol{r}^0_A)]. \quad \text{(B2)}$$

The vectorial functions in (B2) are defined by (cf. Eqs. (50) and (51) in [46]):

$$\boldsymbol{B}_1(\boldsymbol{r}^{1\mathrm{PN}}_A) = -2\frac{\boldsymbol{\sigma}\times(\boldsymbol{r}^{1\mathrm{PN}}_A\times\boldsymbol{\sigma})}{r^{1\mathrm{PN}}_A - \boldsymbol{\sigma}\cdot\boldsymbol{r}^{1\mathrm{PN}}_A} + 2\boldsymbol{\sigma}\ln(r^{1\mathrm{PN}}_A - \boldsymbol{\sigma}\cdot\boldsymbol{r}^{1\mathrm{PN}}_A), \quad \text{(B3)}$$

$$\boldsymbol{B}_2(\boldsymbol{r}^{\mathrm{N}}_A) = +4\frac{\boldsymbol{\sigma}}{r^{\mathrm{N}}_A - \boldsymbol{\sigma}\cdot\boldsymbol{r}^{\mathrm{N}}_A} + 4\frac{\boldsymbol{d}_A}{(r^{\mathrm{N}}_A - \boldsymbol{\sigma}\cdot\boldsymbol{r}^{\mathrm{N}}_A)^2} + \frac{1}{4}\frac{\boldsymbol{r}^{\mathrm{N}}_A}{(r^{\mathrm{N}}_A)^2} - \frac{15}{4}\frac{\boldsymbol{\sigma}}{d_A}\arctan\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}^{\mathrm{N}}_A}{d_A}\right) - \frac{15}{4}\boldsymbol{d}_A\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}^{\mathrm{N}}_A}{d_A^3}\left[\frac{\pi}{2} + \arctan\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}^{\mathrm{N}}_A}{d_A}\right)\right], \quad \text{(B4)}$$

where the expressions $\boldsymbol{r}^{\mathrm{N}}_A$ and $\boldsymbol{r}^{1\mathrm{PN}}_A$ are given by

$$\boldsymbol{r}^{\mathrm{N}}_A = \boldsymbol{x}_0 + c(t-t_0)\boldsymbol{\sigma} - \boldsymbol{x}_A, \quad \text{(B5)}$$

$$\boldsymbol{r}^{1\mathrm{PN}}_A = \boldsymbol{x}_0 + c(t-t_0)\boldsymbol{\sigma} - \boldsymbol{x}_A - 2\frac{GM_A}{c^2}\frac{\boldsymbol{d}_A}{r^{\mathrm{N}}_A - \boldsymbol{\sigma}\cdot\boldsymbol{r}^{\mathrm{N}}_A} + 2\frac{GM_A}{c^2}\boldsymbol{\sigma}\ln(r^{\mathrm{N}}_A - \boldsymbol{\sigma}\cdot\boldsymbol{r}^{\mathrm{N}}_A), \quad \text{(B6)}$$

while $\boldsymbol{r}^0_A$ is defined by Eq. (14). Accordingly, the expression (B3) is the source of 1PN and 2PN terms. By inserting (B6) into (B3), and by inserting (B5) into (B4), one may identify the 2PN terms uniquely and obtain the 2PN expression $\Delta\boldsymbol{x}^M_{2\mathrm{PN}}$ in (36)–(37).

APPENDIX C: INTEGRAL $\mathcal{I}_A$

The integral $\mathcal{I}_A$ in (106) reads

$$\mathcal{I}_A(\tau+t^*,\boldsymbol{\xi}) = \int_{-\infty}^{\tau} dc\tau' M^A_L(\tau'+t^*)\partial'_L\frac{1}{r^{\mathrm{N}}_A(\tau'+t^*)}. \quad \text{(C1)}$$

In order to determine the integral $\mathcal{I}_A$, it is useful to incorporate the operator $P^{ij}\frac{\partial}{\partial\xi^j}$ which stands in front of this integral according to Eq. (105). Furthermore, using the expression in (108) for the differential operator ∂'_L, the integral $\mathcal{I}_A$ can be separated into two kinds of integrals: integral $\mathcal{I}_1$ which contains differentiations with respect to the time variable (i.e. $p\geq 1$) and integral $\mathcal{I}_2$ which does not contain such differentiations (i.e. $p=0$), which means

$$P^{ij}\frac{\partial}{\partial\xi^j}\mathcal{I}_A(\tau+t^*,\boldsymbol{\xi}) = \sum_{p=1}^{l}\frac{l!}{(l-p)!p!}\sigma^{i_1}\ldots\sigma^{i_p}P^{i_{p+1}j_{p+1}}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_{p+1}}}\cdots\frac{\partial}{\partial\xi^{j_l}}P^{ij}\frac{\partial}{\partial\xi^j}\mathcal{I}_1(\tau+t^*,\boldsymbol{\xi}) + P^{i_1j_1}\ldots P^{i_lj_l}\frac{\partial}{\partial\xi^{j_1}}\cdots\frac{\partial}{\partial\xi^{j_l}}P^{ij}\frac{\partial}{\partial\xi^j}\mathcal{I}_2(\tau+t^*,\boldsymbol{\xi}). \quad \text{(C2)}$$

The integral $\mathcal{I}_1$, with the differential operation $P^{ij}\frac{\partial}{\partial\xi^j}$ in front, is given by

$$P^{ij}\frac{\partial}{\partial\xi^j}\mathcal{I}_1(\tau+t^*,\boldsymbol{\xi}) = P^{ij}\frac{\partial}{\partial\xi^j}\int_{-\infty}^{\tau} dc\tau' M^A_L(\tau'+t^*)\left(\frac{\partial}{\partial c\tau'}\right)^p\frac{1}{r^{\mathrm{N}}_A(\tau'+t^*)} = P^{ij}\frac{\partial}{\partial\xi^j}M^A_L(\tau+t^*)\left(\frac{\partial}{\partial c\tau}\right)^{p-1}\frac{1}{r^{\mathrm{N}}_A(\tau+t^*)} + \mathcal{O}\left(\frac{\dot{M}^A_L}{c}\right). \quad \text{(C3)}$$

The integral in (C3) has been integrated by part, using the integration rule for integrals along the unperturbed light ray as given by Eq. (4.9) in [72]. It is important to note that the neglected terms are of the order $\dot{M}^A_L/c$ and, therefore, they are beyond 1PN approximation because they imply an additional factor c^{-1} which is not canceled by the factor c in the differential $dc\tau'$ in the nominator. In view of (C4) the proof of this fact is rather simple.

The integral $\mathcal{I}_2$, with the differential operation $P^{ij}\frac{\partial}{\partial\xi^j}$ in front, is given by

$$P^{ij}\frac{\partial}{\partial\xi^j}\mathcal{I}_2(\tau+t^*,\boldsymbol{\xi})$$
$$=P^{ij}\frac{\partial}{\partial\xi^j}\int_{-\infty}^{\tau}dc\tau'\frac{M_L^A(\tau'+t^*)}{r_A^{\rm N}(\tau'+t^*)}$$
$$=-M_L^A(\tau+t^*)P^{ij}\frac{\partial\ln[r_A^{\rm N}(\tau+t^*)-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)]}{\partial\xi^j}$$
$$+\mathcal{O}\left(\frac{\dot{M}_L^A}{c}\right)+\mathcal{O}\left(\frac{v_A}{c}\right), \tag{C4}$$

where for the lower integration limit, we have used

$$\lim_{\tau\to-\infty}P^{ij}\frac{\partial}{\partial\xi^j}\ln[r_A^{\rm N}(\tau+t^*)-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)]=0. \tag{C5}$$

Let us note that the physical dimension of a length in the argument of the logarithm in (C4) is not a problem at all and has to be treated according to Eq. (113). The integral in (C4) has been integrated by parts, using

$$\frac{1}{r_A^{\rm N}(\tau'+t^*)}=-\frac{\partial\ln[r_A^{\rm N}(\tau'+t^*)-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau'+t^*)]}{\partial c\tau'}$$
$$+\mathcal{O}\left(\frac{v_A}{c}\right), \tag{C6}$$

where the terms proportional to v_A/c in (C6) will be given later; see Eq. (E1). The fact that the neglected terms (C4) are beyond 1PN approximation is evidenced in Appendix E.

APPENDIX D: INTEGRALS $\mathcal{I}_C$, $\mathcal{I}_D$, $\mathcal{I}_E$, $\mathcal{I}_F$

The four integrals in Eqs. (132)–(135) will be determined; in what follows, the time arguments $\tau+t^*$ and τ_0+t^* of these integrals are omitted for simpler notation. In the calculation of the integrals, all terms are neglected which are proportional to either v_A/c or $\dot{M}_A^L/c$ because they are of higher order beyond 1PN approximation. The proof for these assertions will not be given explicitly because they are very similar to the example elaborated in Appendix E.

1. Integral $\mathcal{I}_C$

The integral $\mathcal{I}_C$ reads

$$\mathcal{I}_C=\int_{\tau_0}^{\tau}dc\tau'\frac{M_L^A(\tau'+t^*)}{r_A^{\rm N}(\tau'+t^*)}. \tag{D1}$$

This integral occurs in the first and fourth term of Eq. (129).

2. Integral $\mathcal{I}_C$ for the case $l=0$

Let us first consider the integral (D1) for the case $l=0$, which occurs in the fourth term in (129). One obtains, by means of relation (C6), the following solution:

$$\mathcal{I}_C^{l=0}=\int_{\tau_0}^{\tau}dc\tau'\frac{M_A}{r_A^{\rm N}(\tau'+t^*)}$$
$$=-M_A\ln\frac{r_A^{\rm N}(\tau+t^*)-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)}{r_A^{\rm N}(\tau_0+t^*)-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau_0+t^*)}$$
$$+\mathcal{O}\left(\frac{v_A}{c}\right). \tag{D2}$$

3. Integral $\mathcal{I}_C$ for the case $l\geq 1$

Now we consider the integral (D1) for the case $l\geq 1$, which occurs in the first and fourth term in (129). In this case, we always have the differential operation $P^{ij}\frac{\partial}{\partial\xi^j}$ in front,

$$P^{ij}\frac{\partial}{\partial\xi^j}\mathcal{I}_C=P^{ij}\frac{\partial}{\partial\xi^j}\int_{\tau_0}^{\tau}dc\tau'\frac{M_L^A(\tau'+t^*)}{r_A^{\rm N}(\tau'+t^*)}. \tag{D3}$$

For evaluating this integral we can use the result in (C4), and obtain

$$P^{ij}\frac{\partial}{\partial\xi^j}\mathcal{I}_C=-M_L^A(\tau+t^*)P^{ij}\frac{\partial\ln[r_A^{\rm N}(\tau+t^*)-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)]}{\partial\xi^j}$$
$$+M_L^A(\tau_0+t^*)P^{ij}\frac{\partial\ln[r_A^{\rm N}(\tau_0+t^*)-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau_0+t^*)]}{\partial\xi^j}+\mathcal{O}\left(\frac{\dot{M}_L^A}{c}\right)+\mathcal{O}\left(\frac{v_A}{c}\right). \tag{D4}$$

4. Integral $\mathcal{I}_D$

According to expression (129), the differential operation $P^{ij}\frac{\partial}{\partial\xi^j}$ is always in front of the integral $\mathcal{I}_D$ ($p\geq 2$), so we may consider

$$P^{ij}\frac{\partial}{\partial\xi^j}\mathcal{I}_D=P^{ij}\frac{\partial}{\partial\xi^j}\int_{\tau_0}^{\tau}dc\tau' M_L^A(\tau'+t^*)\left(\frac{\partial}{\partial c\tau'}\right)^{p-1}\frac{1}{r_A^{\rm N}(\tau'+t^*)}$$
$$=+P^{ij}\frac{\partial}{\partial\xi^j}M_L^A(\tau+t^*)\left(\frac{\partial}{\partial c\tau}\right)^{p-2}\frac{1}{r_A^{\rm N}(\tau+t^*)}-P^{ij}\frac{\partial}{\partial\xi^j}M_L^A(\tau_0+t^*)\left(\frac{\partial}{\partial c\tau_0}\right)^{p-2}\frac{1}{r_A^{\rm N}(\tau_0+t^*)}+\mathcal{O}\left(\frac{\dot{M}_L^A}{c}\right), \tag{D5}$$

which has been solved using integration by parts. The proof that the correction terms are in fact of the order $\mathcal{O}(\dot{M}_L^A/c)$ is straightforward.

5. Integral $\mathcal{I}_E$

Now we consider the integral (134). According to (129), the differential operation $P^{ij}\frac{\partial}{\partial \xi^j}$ is always in front of the integral $\mathcal{I}_E$ ($p \geq 1$), so we consider

$$P^{ij}\frac{\partial}{\partial \xi^j}\mathcal{I}_E = P^{ij}\frac{\partial}{\partial \xi^j}\int_{\tau_0}^{\tau} dc\tau' M_L^A(\tau'+t^*) \times \ln\left[r_A^{\rm N}(\tau'+t^*) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau'+t^*)\right]. \tag{D6}$$

In order to solve that integral, we may use the following relation:

$$\ln\left(r_A^{\rm N}(\tau'+t^*) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau'+t^*)\right) = \frac{\partial[r_A^{\rm N}(\tau'+t^*) + \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau'+t^*)\ln\left(r_A^{\rm N}(\tau'+t^*) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau'+t^*)\right)]}{\partial c\tau'} + \mathcal{O}\left(\frac{v_A}{c}\right). \tag{D7}$$

Like in relation (C6), the form of the expressions proportional to v_A/c in (D7) can easily be determined. By inserting relation (D7) into the integral (D6), one obtains, by integration by part,

$$\begin{aligned} P^{ij}\frac{\partial}{\partial \xi^j}\mathcal{I}_E = {}& +P^{ij}\frac{\partial}{\partial \xi^j} M_L^A(\tau+t^*)[r_A^{\rm N}(\tau+t^*) + \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)\ln\left(r_A^{\rm N}(\tau+t^*) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau+t^*)\right)] \\ & - P^{ij}\frac{\partial}{\partial \xi^j} M_L^A(\tau_0+t^*)[r_A^{\rm N}(\tau_0+t^*) + \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau_0+t^*)\ln\left(r_A^{\rm N}(\tau_0+t^*) - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}(\tau_0+t^*)\right)] \\ & + \mathcal{O}\left(\frac{\dot{M}_L^A}{c}\right) + \mathcal{O}\left(\frac{v_A}{c}\right). \end{aligned} \tag{D8}$$

The proof that the neglected terms are in fact of the order v_A/c is very similar to the example elaborated in Appendix E.

6. Integral $\mathcal{I}_F$

Now we consider the integral (135). According to (129), at least one differential operation of the form $P^{ij}\frac{\partial}{\partial \xi^j}$ is always in front of the integral $\mathcal{I}_F$ ($p \geq 1$), so we consider

$$\begin{aligned} P^{ij}\frac{\partial}{\partial \xi^j}\mathcal{I}_F &= \int_{\tau_0}^{\tau} dc\tau' M_L^A(\tau'+t^*)\left(\frac{\partial}{\partial c\tau'}\right)^p \frac{1}{r_A^{\rm N}(\tau'+t^*)} \\ &= +P^{ij}\frac{\partial}{\partial \xi^j} M_L^A(\tau+t^*)\left(\frac{\partial}{\partial c\tau}\right)^{p-1}\frac{1}{r_A^{\rm N}(\tau+t^*)} - P^{ij}\frac{\partial}{\partial \xi^j} M_L^A(\tau_0+t^*)\left(\frac{\partial}{\partial c\tau_0}\right)^{p-1}\frac{1}{r_A^{\rm N}(\tau_0+t^*)} \\ &\quad + \mathcal{O}\left(\frac{\dot{M}_L^A}{c}\right), \end{aligned} \tag{D9}$$

which has been solved using integration by parts.

APPENDIX E: ESTIMATION OF NEGLECTED TERMS: AN EXAMPLE

As a typical example, let us consider the neglected terms in the solution (C4), where the relation (C6) has been used, which in its exact form reads (the variables $\tau' + t^*$ will be suppressed for simpler notation):

$$\frac{1}{r_A^{\rm N}} = -\frac{\partial \ln[r_A^{\rm N} - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}]}{\partial c\tau'} + \frac{1}{r_A^{\rm N}}\frac{\boldsymbol{v}_A}{c}\cdot\frac{r_A^{\rm N}\boldsymbol{\sigma} - \boldsymbol{r}_A^{\rm N}}{r_A^{\rm N} - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}}. \tag{E1}$$

Inserting this relation into (C4), one obtains an additional integral proportional to v_A/c, namely,

$$P^{ij}\frac{\partial}{\partial \xi^j}\int_{-\infty}^{\tau} dc\tau' \frac{M_L^A}{r_A^{\rm N}}\frac{\boldsymbol{v}_A}{c}\cdot\left(\boldsymbol{\sigma} - \frac{\boldsymbol{d}_A}{r_A^{\rm N} - \boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}}\right), \tag{E2}$$

where $\boldsymbol{r}_A^{\rm N} = \boldsymbol{d}_A + \boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N})$ has been used. The first term of this integral is identical to the integral (C4), except the additional factor $\boldsymbol{\sigma}\cdot\boldsymbol{v}_A/c$. So it remains to consider the second term in (E2); the sign in front is not relevant here,

$$\mathcal{I}_G = P^{ij}\frac{\partial}{\partial \xi^j}\int_{-\infty}^{\tau} dc\tau' \frac{M_L^A}{r_A^{\rm N}}\frac{\boldsymbol{v}_A}{c}\cdot\frac{\boldsymbol{d}_A}{r_A^{\rm N}-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}}. \tag{E3}$$

Using relation (G2), one can rewrite this integral in the following form:

$$\mathcal{I}_G = P^{ij}\frac{\partial}{\partial \xi^j}P^{ab}\frac{\partial}{\partial \xi^b}\int_{-\infty}^{\tau} dc\tau' M_L^A\frac{v_A^a}{c}\ln\,[r_A^{\rm N}-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}]. \tag{E4}$$

Using relation (D7), this integral can be integrated by parts:

$$\mathcal{I}_G = M_L^A\frac{v_A^a}{c}P^{ij}\frac{\partial}{\partial \xi^j}P^{ab}\frac{\partial}{\partial \xi^b} \times [r_A^{\rm N}+\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}\ln\,(r_A^{\rm N}-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N})]|_{-\infty}^{\tau}+\mathcal{O}(c^{-2}). \tag{E5}$$

Performing the differentiations, one finally arrives at

$$\mathcal{I}_G = M_L^A\frac{v_A^a}{c}\frac{1}{r_A^{\rm N}-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}}\left(P^{ai}-\frac{d_A^a d_A^i}{r_A^{\rm N}(r_A^{\rm N}-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N})}\right), \tag{E6}$$

up to terms of the order $\mathcal{O}(c^{-2})$, and the absolute value can be estimated by

$$|\mathcal{I}_G| \le 2M_L^A\frac{v_A}{c}\frac{1}{r_A^{\rm N}-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}}. \tag{E7}$$

As stated in relation (C4), the expression in (E6) is of the order v_A/c, hence, beyond 1PN approximation. The fact that in extreme astrometric configurations, $\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}\to r_A^{\rm N}$, the expression in (E6) becomes formally large is not of much relevance since there are many other terms of the order v_A/c which presumably cancel this term. The proof of such an assertion is, of course, beyond 1PN approximation and involves an exact consideration of all terms to that order.

APPENDIX F: PARTIAL DERIVATIVES FOR THE FIRST INTEGRATION

Throughout this section we will use the following abbreviated notation, $\boldsymbol{r}_A^{\rm N}=\boldsymbol{r}_A^{\rm N}(\tau+t^*)$, $\boldsymbol{d}_A=\boldsymbol{d}_A(\tau+t^*)$, and corresponding notation for their absolute values.

1. Example

Let us consider an example of how the differentiation is meant within the formalism:

$$P^{i_1j_1}\frac{\partial}{\partial\xi^{j_1}}\frac{1}{r_A^{\rm N}} = P^{i_1j_1}\frac{\partial}{\partial\xi^{j_1}}\frac{1}{\sqrt{\boldsymbol{\xi}^2+c^2\tau^2+\boldsymbol{x}_A^2-2\boldsymbol{\xi}\cdot\boldsymbol{x}_A-2c\tau\boldsymbol{\sigma}\cdot\boldsymbol{x}_A}}, \tag{F1}$$

where (89) has been used; recall $\boldsymbol{\xi}\cdot\boldsymbol{\sigma}=0$ and $\boldsymbol{x}_A=\boldsymbol{x}_A(\tau+t^*)$. Inserting the projector (83), one finds

$$P^{i_1j_1}\frac{\partial}{\partial\xi^{j_1}}\frac{1}{r_A^{\rm N}} = -\frac{\xi^{i_1}-x_A^{i_1}+\sigma^{i_1}(\boldsymbol{\sigma}\cdot\boldsymbol{x}_A)}{|\boldsymbol{\xi}+c\tau\boldsymbol{\sigma}-\boldsymbol{x}_A|^3} = -\frac{\xi^{i_1}+c\tau\sigma^{i_1}-x_A^{i_1}-\sigma^{i_1}(c\tau-\boldsymbol{\sigma}\cdot\boldsymbol{x}_A)}{|\boldsymbol{\xi}+c\tau\boldsymbol{\sigma}-\boldsymbol{x}_A|^3}. \tag{F2}$$

In view of $\boldsymbol{\sigma}\cdot\boldsymbol{\xi}=0$, the following term in the nominator can be rewritten as follows: $c\tau-\boldsymbol{\sigma}\cdot\boldsymbol{x}_A = \boldsymbol{\sigma}\cdot(\boldsymbol{\xi}+c\tau\boldsymbol{\sigma}-\boldsymbol{x}_A)=\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}$. Then, by using the definition of impact vector (90), we finally arrive at

$$P^{i_1j_1}\frac{\partial}{\partial\xi^{j_1}}\frac{1}{r_A^{\rm N}} = -\frac{d_A^{i_1}}{(r_A^{\rm N})^3}. \tag{F3}$$

All subsequent derivatives have been determined in a similar way.

2. Partial derivatives for the dipole term

In order to obtain the dipole term in (115), we need the following derivatives:

$$P^{i_1j_1}\frac{\partial}{\partial\xi^{j_1}}\frac{d_A^i}{r_A^{\rm N}-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}}\frac{1}{r_A^{\rm N}} = \frac{1}{r_A^{\rm N}(r_A^{\rm N}-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N})} \times\left(P^{i_1i}-\frac{d_A^{i_1}d_A^i}{(r_A^{\rm N})^2}-\frac{d_A^{i_1}d_A^i}{r_A^{\rm N}(r_A^{\rm N}-\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N})}\right), \tag{F4}$$

and

$$P^{i_1j_1}\frac{\partial}{\partial\xi^{j_1}}\frac{1}{r_A^{\rm N}} = -\frac{d_A^{i_1}}{(r_A^{\rm N})^3}, \tag{F5}$$

$$\frac{\partial}{\partial c\tau}\frac{1}{r_A^{\rm N}} = -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}_A^{\rm N}}{(r_A^{\rm N})^3}+\mathcal{O}\left(\frac{v_A}{c}\right). \tag{F6}$$

3. Partial derivatives for the quadrupole term

In (116) the derivatives $\frac{\partial M_L^A}{\partial c\tau}=\mathcal{O}(\frac{\dot{M}_L^A}{c})$ and $\frac{\partial d_A^i}{\partial c\tau}=\mathcal{O}(\frac{v_A}{c})$ are beyond 1PN approximation. Hence, we are left with the following expressions:

$$P^{i_2j_2}\frac{\partial}{\partial\xi^{j_2}}\frac{d_A^i}{(r_A^{\rm N})^3} = -3\frac{d_A^i d_A^{i_2}}{(r_A^{\rm N})^5}+\frac{P^{ii_2}}{(r_A^{\rm N})^3}, \tag{F7}$$

and

$$P^{i_1 j_1} P^{i_2 j_2} \frac{\partial}{\partial \xi^{j_1}} \frac{\partial}{\partial \xi^{j_2}} \frac{d_A^i}{r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}} \frac{1}{r_A^{\rm N}}$$

$$= -P^{i_1 i_2} \frac{d_A^i}{r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}} \frac{1}{(r_A^{\rm N})^2} \left(\frac{1}{r_A^{\rm N}} + \frac{1}{r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}} \right) - \frac{P^{i i_1}}{(r_A^{\rm N})^2} \frac{d_A^{i_2}}{r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}} \left(\frac{1}{r_A^{\rm N}} + \frac{1}{r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}} \right)$$
$$- \frac{P^{i i_2}}{(r_A^{\rm N})^2} \frac{d_A^{i_1}}{r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}} \left(\frac{1}{r_A^{\rm N}} + \frac{1}{r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}} \right) + \frac{d_A^i}{r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}} \frac{d_A^{i_1} d_A^{i_2}}{(r_A^{\rm N})^3} \left(\frac{3}{(r_A^{\rm N})^2} + \frac{3}{r_A^{\rm N} (r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N})} + \frac{2}{(r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N})^2} \right), \quad \text{(F8)}$$

$$P^{i_1 j_1} P^{i_2 j_2} \frac{\partial}{\partial \xi^{j_1}} \frac{\partial}{\partial \xi^{j_2}} \frac{1}{r_A^{\rm N}} = 3 \frac{d_A^{i_1} d_A^{i_2}}{(r_A^{\rm N})^5} - \frac{P^{i_1 i_2}}{(r_A^{\rm N})^3}, \quad \text{(F9)}$$

$$P^{i_2 j_2} \frac{\partial}{\partial \xi^{j_2}} \frac{\partial}{\partial c\tau} \frac{1}{r_A^{\rm N}} = 3 \frac{d_A^{i_2}}{(r_A^{\rm N})^5} (\boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}) + \mathcal{O}\left(\frac{v_A}{c} \right), \quad \text{(F10)}$$

$$\frac{\partial}{\partial c\tau} \frac{\partial}{\partial c\tau} \frac{1}{r_A^{\rm N}} = -\frac{1}{(r_A^{\rm N})^3} + 3 \frac{(\boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N})^2}{(r_A^{\rm N})^5} + \mathcal{O}\left(\frac{v_A}{c} \right), \quad \text{(F11)}$$

$$\frac{\partial}{\partial c\tau} \frac{1}{(r_A^{\rm N})^3} = -3 \frac{(\boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N})}{(r_A^{\rm N})^5} + \mathcal{O}\left(\frac{v_A}{c} \right), \quad \text{(F12)}$$

where $P^{i_2 j_2}(\xi^{j_2} - x_A^{j_2}) = d_A^{i_2}$ has frequently been used; note that $\delta_{j_1 j_2} P^{i_1 j_1} P^{i_2 j_2} = P^{i_1 i_2}$.

APPENDIX G: PARTIAL DERIVATIVES FOR THE SECOND INTEGRATION

Throughout this section the abbreviated notation is used: $\boldsymbol{r}_A^{\rm N} = \boldsymbol{r}_A^{\rm N}(\tau + t^*)$ and $r_A^{\rm N} = |\boldsymbol{r}_A^{\rm N}(\tau + t^*)|$.

1. Partial derivatives for the dipole term

In order to obtain the dipole term in (140), we need the following derivatives:

$$P^{i_1 j_1} \frac{\partial}{\partial \xi^{j_1}} \frac{d_A^i}{r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}} = \frac{1}{r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}} \times \left(P^{i i_1} - \frac{d_A^i d_A^{i_1}}{r_A^{\rm N} (r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N})} \right), \quad \text{(G1)}$$

$$P^{i_1 j_1} \frac{\partial}{\partial \xi^{j_1}} \ln (r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}) = \frac{d_A^{i_1}}{r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}} \frac{1}{r_A^{\rm N}}, \quad \text{(G2)}$$

where the last relation has already been given by (113).

2. Partial derivatives for the quadrupole term

In order to obtain the quadrupole term in (141), we need the following derivatives:

$$P^{i_1 j_1} P^{i_2 j_2} \frac{\partial}{\partial \xi^{j_1}} \frac{\partial}{\partial \xi^{j_2}} \frac{d_A^i}{r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}} = -\frac{P^{i i_1} d_A^{i_2}}{r_A^{\rm N} (r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N})^2} - \frac{P^{i i_2} d_A^{i_1}}{r_A^{\rm N} (r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N})^2} - \frac{P^{i_1 i_2} d_A^{i}}{r_A^{\rm N} (r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N})^2}$$
$$+ \frac{d_A^i d_A^{i_1} d_A^{i_2}}{(r_A^{\rm N})^3 (r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N})^2} + 2 \frac{d_A^i d_A^{i_1} d_A^{i_2}}{(r_A^{\rm N})^2 (r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N})^3}, \quad \text{(G3)}$$

$$P^{i_1 j_1} P^{i_2 j_2} \frac{\partial}{\partial \xi^{j_1}} \frac{\partial}{\partial \xi^{j_2}} \ln (r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N}) = \frac{P^{i_1 i_2}}{r_A^{\rm N} (r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N})} - \frac{d_A^{i_1} d_A^{i_2}}{(r_A^{\rm N})^3 (r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N})} - \frac{d_A^{i_1} d_A^{i_2}}{(r_A^{\rm N})^2 (r_A^{\rm N} - \boldsymbol{\sigma} \cdot \boldsymbol{r}_A^{\rm N})^2}. \quad \text{(G4)}$$

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