PHYSICAL REVIEW D **92,** 061302(R) (2015)

Smooth quantum dynamics of the mixmaster universe

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(Received 22 January 2015; revised manuscript received 1 April 2015; published 11 September 2015)

We present a new approach to the vacuum Bianchi IX model by combining affine coherent state quantization with Born-Oppenheimer-type adiabatic approximation in the analogy with quantum molecular physics. The analytical treatment is carried out on both quantum and semiclassical levels. Our quantization method by itself generates a specific repulsive potential that resolves the classical singularity. The quantized oscillatory degrees of freedom behave as radiation energy density. The Friedmann-like lowest-energy eigenstates of the system are found to be dynamically stable against small anisotropy perturbations, in contrast to the classical case.

DOI: 10.1103/PhysRevD.92.061302 PACS numbers: 98.80.Qc

I. INTRODUCTION

The Friedmann-Robertson-Walker model is successfully used to describe the data of observational cosmology (see e.g., [1,2]). Nevertheless, the isotropy of space is dynamically unstable towards the big bang singularity [3]. On the other hand, if the present Universe originated from an inflationary phase, then the preinflationary universe is supposed to have been both inhomogeneous and anisotropic. As evidence suggests (see [4,5]), the dynamics of such a universe backwards in time becomes ultralocal, that is, approximately identical with the homogeneous but anisotropic one at each spatial point. In any case, quantization of the isotropic models alone appears to be insufficient. Hence, the quantum version of an anisotropic model, comprising the Friedmann model as a particular case, is expected to be better suited for describing the earliest Universe.

The mixmaster universe exhibits complex behavior [6]. As it collapses, the Universe enters chaotic oscillations producing an infinite sequence of distortions from its spherical shape [7]. Those distortions essentially correspond to the level of anisotropy and may be viewed as the effect of a gravitational wave evolving in an isotropic background [8]. The dynamics of this wave is nonlinear, and its interaction with the isotropic background fuels the gravitational contraction. Not surprisingly, the quantization of the Bianchi IX model is a difficult task. Some formulations can be found in the literature, including the Wheeler-DeWitt equation [6] or, more recently, a formulation based on loop quantum cosmology [9,10]. However, the search for solutions within these formulations is quite challenging [11,12] leaving the near big bang dynamics largely unexplored. To our knowledge, the most recent published developments, e.g., [13], do not address the singularity resolution.

In this article we advocate a new approach to quantum cosmology, which employs the affine coherent state (ACS) quantization. The idea of exploiting the affine group representation in quantum gravity was already discussed in [14,15]. We enhance this idea by combining it with coherent states, which are used both in defining the quantum model and in developing a semiclassical description. The existence of the semiclassical phase space portrait in full consistency with the quantization itself is a major advantage of our method. Our approach was already applied to the Friedmann models in [16] (see also [17] for a related work). In those works, it was shown that the occurrence of a new term in the quantum Hamiltonian is the natural consequence of our quantization method. It produces a repulsive force counteracting the contraction of the Universe.

Presently, we expand our approach to the quantization of the vacuum Bianchi type IX geometry, the mixmaster universe. As before, we exploit the affine group and the associated ACS. In order to solve the dynamics in the present, more complex setting, we implement the adiabatic approximation widely utilized in quantum molecular physics [18,19]. This idea is a novelty in the study of the singularity problem, and it enables us to identify a resoluble, deeply quantum sector of this model with relevant physics. The known adiabatic approximation in quantum gravity, e.g., discussed in [20], is of a different nature and was not devised for the study of the singularity problem.

The main result is a semiclassical Friedmann-like equation obtained from the expectation values in ACS, a description peculiar to our approach. In that equation, devoted exclusively to the quantized geometry, the expansion of the Universe is governed by two terms of quantum origin. The first one is proper to the quantum mixmaster model and corresponds to the energy of the gravity wave in an eigenstate. It is proportional to the energy level number. The other one, which is more universal, corresponds to the repulsive potential preventing the singularity. The lowest-energy eigenstates of this system are interpreted as the quantum Friedmann universe supplemented with vacuum fluctuations of the anisotropy.

We find that the quantum dynamics exhibits two novel and surprising properties. First, the anisotropic degrees of freedom remain in their lowest-energy states during the quantum phase consistent with our approximation. This implies that the quantum Friedmann-like states, unlike their classical counterpart, remain stable with respect to the anisotropy perturbation. Therefore, the classical chaos is suppressed within the considered domain. Second, during the contraction the quantum energy of anisotropy grows much slower than it does on the classical level. Namely, it effectively gravitates as radiation, leading to a significant reduction in the overall gravitational pull from anisotropy.

II. CLASSICAL HAMILTONIAN

In this paper we study the vacuum Bianchi IX model. The Friedmann equation extended to anisotropic vacuum universes, with $c = 1 = 8\pi G$, reads

$$H^2 + \frac{1}{6}{}^3R - \frac{1}{6}\Sigma^2 = 0, \tag{1}$$

where H is the Hubble rate and ${}^3R$ is the spatial curvature. The additional term Σ^2 is the total shear of the spatial section and is nonvanishing for anisotropic models. Due to its negative sign, the shear drives the gravitational collapse and it eventually dominates the dynamics.

The mixmaster describes the spacetime metric $ds^2 = -dt^2 + a^2(e^{2\beta})_{ij}\sigma^i\sigma^j$, where a is the averaged scale factor and σ^i are differential forms on a three-sphere (covering group of the rotation group) satisfying $d\sigma^i = \frac{1}{2}\epsilon_{ijk}\sigma^j \wedge \sigma^k$. The diagonal form of the metric is assumed $(e^{2\beta})_{ij} \coloneqq \mathrm{diag}(e^{2(\beta_+ + \sqrt{3}\beta_-)}, e^{2(\beta_+ - \sqrt{3}\beta_-)}, e^{-4\beta_+})$, where $\beta_\pm$ are distortion parameters [6].

In terms of these variables, the shear is the kinetic energy of anisotropic distortion, $\Sigma^2 = (p_+^2 + p_-^2)/24a^6$, where the momenta $p_\pm$ are canonical conjugates to $\beta_\pm$. The spatial curvature ${}^3R$ grows due to the overall contraction of space, but decreases due to the growth of anisotropy. This last circumstance leads to the backreaction from the spatial curvature on the shear and, as a result, oscillations in $\beta_\pm$ occur. As there is no matter content in our model, $\beta_\pm$ describe a sort of gravitational wave. The curvature can be split into isotropic and anisotropic parts: ${}^3R = 3(1 - V(\beta))/2a^2$, where $V(\beta)$ is the anisotropy curvature potential [6],

$$V(\beta) = \frac{e^{4\beta_+}}{3}((e^{-6\beta_+} - 2\cosh(2\sqrt{3}\beta_-))^2 - 4) + 1.$$

As shown in Fig. 1, this potential has three "open" C_{3v} symmetry directions. One can view them as three deep "canyons," increasingly narrow until their respective wall edges close up at the infinity, whereas their respective bottoms tend to zero. Due to its (almost) confining shape, V is expected to produce a discrete energy spectrum on the quantum level.

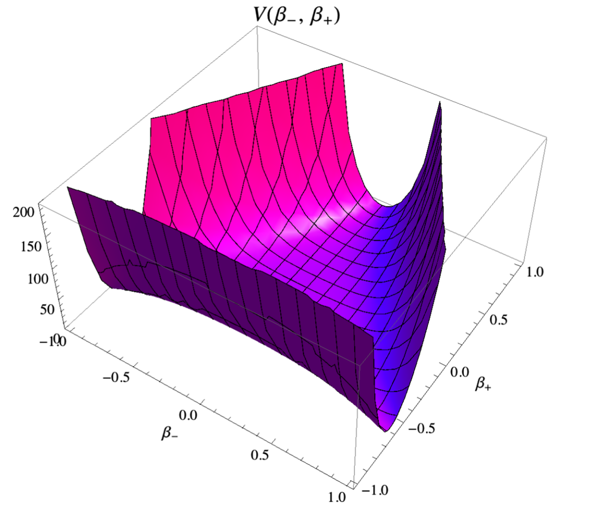

FIG. 1 (color online). Global picture of the potential $V(\beta)$ near its minimum. Boundedness from below, confining aspects, and three canyons are illustrated.

The generalized Friedmann equation (1) may be rewritten as

$$H^2 + \frac{1}{4a^2} = \frac{1}{6}\Sigma^2 + \frac{V(\beta)}{4a^2}, \tag{2}$$

where the isotropic background geometry on the lhs is pulled by the energy of anisotropic oscillations. The energy of oscillations scales roughly as a^{-6}. Although the isotropic curvature term in (2) is subdominant in the vicinity of the singularity, it is included for the completeness of the Bianchi IX geometry quantization. Its inclusion leads to the well-known classical recollapse specific to closed homogeneous cosmologies [21].

It follows that the Hamiltonian constraint to be quantized reads in canonical variables as

$$\mathcal{C} = \frac{3}{16}p^2 + \frac{3}{4}q^{2/3} - \mathcal{H}_q, \tag{3}$$

where $q = a^{3/2}$ and $p^2 = 16\dot{a}^2 a$ are more suitable to ACS quantization, and where

$$\mathcal{H}_q = \frac{1}{12q^2}(p_+^2 + p_-^2) + \frac{3}{4}q^{2/3}V(\beta) \tag{4}$$

is the anisotropy energy. The closed Friedmann-Robertson-Walker (FRW) geometry is obtained by putting $p_\pm = 0$ and $\beta_\pm = 0$, or simply $\mathcal{H}_q = 0$.

The Hamiltonian constraint (3) resembles a diatomic molecular Hamiltonian with the pairs (q, p) and $(\beta_\pm, p_\pm)$ playing the role of the reduced nuclear and electronic variables, respectively. In molecules, the motion of nuclei is slow enough in comparison with electrons, so the motion of electrons may be approximated as becoming immediately adjusted to varying positions of nuclei. However, the coupling between the nucleuslike and electronlike degrees of freedom in the present model differs from the usual molecular case for which the validity of the approximation rests upon the smallness of the ratio between the nuclei and electron masses. In the present case, described by Eqs. (3) and (4), the "mass" of the degrees of freedom $\beta_\pm$ evolves as q^2. Thus, it goes to zero near the singularity, $q = 0$. On the other hand, the "mass" of the degree of freedom q in Eq. (3) is constant. Thus, close to singularity, the latter may be regarded as "heavy" in comparison with the anisotropic variables that can be treated as "light."

III. QUANTUM HAMILTONIAN

The six-dimensional phase space of the mixmaster universe is quantized as follows: (A) The isotropic variables form the canonical pair (q, p) living in a half-plane. That half-plane can be viewed as the affine group. We resort to one of its two unitary irreducible representations, denoted by U, to build from a suitable fiducial vector $|\nu\rangle$ (where $\nu > 0$ is a free parameter) a family of affine coherent states (i.e., wavelets) $|q, p\rangle \coloneqq U(q, p)|\nu\rangle$. These ACS's are then used to consistently quantize the isotropic variables. While the method provides the usual $\hat{p} = -i\hbar\partial_q$, and $\hat{q}$ defined as the multiplication by q, its interest lies in the regularization of the Hamiltonian [22]. This approach together with $|\nu\rangle$ was introduced for cosmological models in [16]. Next, we use the ACS's to obtain a semiclassical description, which enables us to analyze the effective dynamics of isotropic variables. (B) For the anisotropic variables, each canonical pair $(\beta_\pm, p_\pm)$ lives in the plane. Thus, it is natural to proceed with the usual canonical quantization which yields $\hat{p}_\pm = -i\hbar\partial_{\beta_\pm}$, and $\hat{\beta}_\pm$ being the multiplication by $\beta_\pm$.

The quantized Hamiltonian corresponding to (3) and issued from quantizations (A) and (B) above reads

$$\hat{\mathcal{C}} = \frac{3}{16}\left(\hat{p}^2 + \frac{\hbar^2 \mathfrak{K}_1}{\hat{q}^2}\right) + \frac{3}{4}\mathfrak{K}_3 \hat{q}^{2/3} - \hat{\mathcal{H}}_q, \qquad (5)$$

where

$$\hat{\mathcal{H}}_q = \frac{1}{12}\mathfrak{K}_2 \frac{\hat{p}_+^2 + \hat{p}_-^2}{\hat{q}^2} + \frac{3}{4}\mathfrak{K}_3 q^{2/3} V(\hat{\beta}). \qquad (6)$$

The $\mathfrak{K}_i$'s are purely positive numerical constants dependent on the choice of the ACS. With the choice made in our previous paper [16], all these constants are simple rational functions of modified Bessel functions $K_l(\nu)$. We note in (5) the appearance of the repulsive centrifugal potential term $\hbar^2 \mathfrak{K}_1 \hat{q}^{-2}$. It is the signature of the ACS quantization, which is consistent with the half-plane geometry, and it regularizes the quantum Hamiltonian for small q [22]. As the Universe approaches the singularity, $q \to 0$, this centrifugal term sharply grows in dynamical significance.

We consider the oscillations of $\beta_\pm$ fast in comparison with the contraction of the Universe. It legitimates the adiabatic approximation, in a way analogous to the Born-Oppenheimer approximation (BOA) [18,19] widely used in molecular physics. Due to the confining property of V, the operator $\hat{\mathcal{H}}_q$ at fixed q has a discrete spectrum. In accordance with the BOA, we assume that the anisotropy degrees of freedom $\beta_\pm$ are frozen in some eigenstate of $\hat{\mathcal{H}}_q$ with eigenenergy $e_q^{(N)}$ $(N = 0, 1, \ldots)$ evolving adiabatically. Thus, the light degrees of freedom $\beta_\pm$ can be averaged leading to the Hamiltonian:

$$\hat{\mathcal{C}}_A = \frac{3}{16}\left(\hat{p}^2 + \frac{\hbar^2 \mathfrak{K}_1}{\hat{q}^2}\right) + \frac{3}{4}\mathfrak{K}_3 \hat{q}^{2/3} - e_q^{(N)}. \qquad (7)$$

Focusing on the deep quantum domain, we look at the first energy levels near the ground state of $\hat{\mathcal{H}}_q$. Therefore, we essentially investigate the domain near the minimum of $V(\hat{\beta})$. Within the harmonic approximation $V(\hat{\beta}) \approx 8(\hat{\beta}_+^2 + \hat{\beta}_-^2)$ near its minimum $\hat{\beta}_\pm = 0$, the eigenenergies are found to be $e_q^{(N)} \simeq \hbar q^{-2/3}\sqrt{2\mathfrak{K}_2\mathfrak{K}_3}(N+1)$ with $N = n_+ + n_-$. The quantum numbers $n_\pm$ correspond to the independent harmonic oscillations in β_+ and β_- [23]. The expression for $e_q^{(N)}$ is rather a rough approximation for large values of N, since $V(\hat{\beta})$ is highly nonharmonic far away from its minimum. But for small values of N, this expression is valid at any value of q.

We notice that the discrete spectrum part in Eq. (7), $e_q^{(N)}$, becomes a small perturbation at large q, a range for which the BOA possibly loses its validity, whereas it gains all its value at small q. From the mass criterion, our approach based on the BOA is legitimate as q assumes its values near the singularity $q = 0$. Furthermore, our procedure of quantization generates a supplementary repulsive potential that prohibits the system from accessing the singularity neighborhood $q \in (0, q_m)$ with some very small bound $q_m > 0$, which depends on the initial state.

Furthermore, calculations made in molecular physics beyond the BOA (the so-called vibronic approximation) show that the mass criterion is in fact too strong: a significant breakdown of the BOA only occurs when different eigenenergy curves $q \mapsto e_q^{(N)}$ of $\hat{\mathcal{H}}_q$ are crossing. In our approach these crossings do not occur, at least for the lowest levels of $\hat{\mathcal{H}}_q$.

This reasoning based on molecular physics is robust, but qualitative in our case, due to the coupling between the q and $\beta_\pm$ degrees of freedom which is not of the molecular type. In [23] we weaken the adiabatic condition by allowing the quantized oscillations to be excited by the semiclassical dynamics of the isotropic background described below, for a fixed N. We find that the excitation is indeed very limited, which justifies our approach.

IV. SEMICLASSICAL HAMILTONIAN

Following Klauder [24], we introduce a semiclassical observable associated with the quantum Hamiltonian (7) as its expectation value $\check{\mathcal{C}}_A(q,p) \coloneqq \langle q,p|\hat{\mathcal{C}}_A|q,p\rangle$ in the ACS state $|q,p\rangle$ peaked on the classical phase space point (q,p) in the half-plane,

$$\check{\mathcal{C}}_A(q,p) = \frac{3}{16}\left(p^2 + \frac{\hbar^2\mathfrak{K}_4}{q^2}\right) + \frac{3}{4}\mathfrak{K}_5 q^{2/3} - \frac{\hbar}{q^{2/3}}\mathfrak{K}_6(N+1),$$

where the $\mathfrak{K}_i$'s are positive numerical constants [23] which are also simple rational functions of modified Bessel functions $K_l(\nu)$. With our choice of $|\nu\rangle$, at large ν, $\mathfrak{K}_i(\nu) \sim 1$, $i \neq 4$ and $\mathfrak{K}_4(\nu) \sim \nu/2$. For the consistency of our semiclassical description, we have rescaled the fiducial vector so that $\langle q,p|\hat{q}|q,p\rangle = q$ and $\langle q,p|\hat{p}|q,p\rangle = p$.

The Hamiltonian constraint imposed at the semiclassical level, $\check{\mathcal{C}}_A(q,p) = 0$, leads to the semiclassical Friedmann-like equation,

$$H^2 + \frac{4\pi^2 G^2\hbar^2}{c^4}\frac{\mathfrak{K}_4}{a^6} + \frac{\mathfrak{K}_5}{4}\left(\frac{c}{a}\right)^2 = \frac{8\pi G\hbar}{3c}(N+1)\frac{\mathfrak{K}_6}{a^4}, \tag{8}$$

where we have restored physical constants and the standard cosmological variables. In order to determine the effect of the matter field, which is absent in our model, one can simply plug effective matter terms into the rhs of the semiclassical equation (8). For more details, see our companion paper [23] (Sec. IV B) including the justification and the discussion of the correspondence between Eqs. (2) and (8). For instance, one could add an effective radiation term $\rho \propto a^{-4}$. This term has the same dependence on a as the quantized anisotropy, so it does not introduce any qualitative change to the dynamics.

The above semiclassical constraint admits smooth trajectories for all values of N only if $\nu \in (0, 7.19)$. For $\nu > 7.19$, Eq. (8) has no solution for the smallest values of N. The solution of (8) for a is a periodic function, $a \in [a_-, a_+]$ with $a_- > 0$ and $a_+ < \infty$, and resolves the cosmological singularity of the mixmaster universe. In Fig. 2 we plot a few trajectories in the half-plane (a, H). The classical closed FRW model is recovered at $\hbar = 0$ and large values of ν.

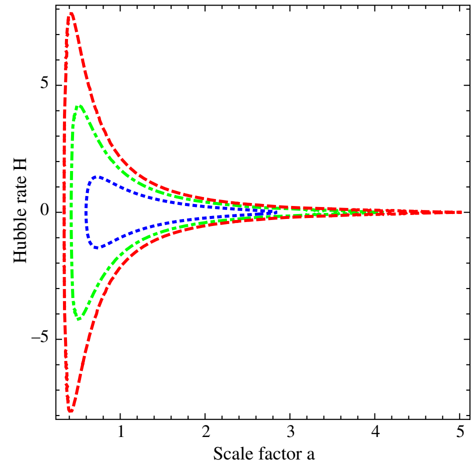

FIG. 2 (color online). Three periodic semiclassical trajectories in the half-plane (a, H) from Eq. (8). They are smooth plane curves. We use standard units $8\pi G = c = \hbar = 1$ and choose $\nu = 1$. Blue dotted curve for $N = 0$, green dot-dashed for $N = 1$, and red dashed for $N = 2$.

V. DISCUSSION

Our semiclassical analysis of the mixmaster universe leads to the modified Friedmann equation (8). The left-hand side describes the isotropic part of geometry. The Hubble parameter squared is accompanied by the repulsive potential of purely quantum origin, which grows as a^{-6} during the contraction. At small volumes, it efficiently counteracts the attraction of common forms of matter, forcing the collapsing universe to rebound. The third term is the usual isotropic spatial curvature.

The right-hand side of Eq. (8) describes the quantized energy of the anisotropy oscillations. The energy is discrete and increases linearly with integer N, as expected in our harmonic approximation. Within the adiabatic approximation, that quantum number is conserved during the evolution. The energy of the quantum oscillator evolves due to the a-dependent coefficients in front of its kinetic and potential terms given in Eq. (6). The ratio between the coefficients determines the oscillator's frequency, which is proportional to a^2. The energy of the oscillations at the quantum level is multiplied by the frequency and, consequently, scales as a^{-4}. (This becomes a poor approximation for high values of N, due to the breakdown of the harmonic approximation). It is quite a contrast to the classical wave, whose total energy is approximately unaffected by its time-dependent frequency and, therefore, scales as a^{-6}. Thus, the growth of the attractive force induced by the anisotropy is significantly reduced in the semiclassical dynamics. This essential dissimilarity between the classical and semiclassical dynamics is due to the fact that on the quantum level the contraction of space is driven by a quantum average.

Let us note that the energy of the wave does not vanish even in the ground state $N = 0$ due to the zero-point quantum fluctuations corresponding to the classical state $\beta_\pm = p_\pm = 0$. In [23] we go beyond the adiabatic approximation to check if there is a significant excitation of the wave's energy level during the semiclassical evolution of the background geometry. The method is essentially the same as the one used to discuss the generation of primordial power spectra in inflationary cosmology. We show that the wave, in fact, remains in its lowest-energy states during the quantum phase. It confirms that the quantum FRW universe, unlike its classical version, is dynamically stable with respect to the small isotropy perturbation. Therefore, it seems that the quantum closed Friedmann model may be successfully used to describe the earliest Universe as well, provided that the corresponding Hamiltonian is supplemented with the effect of the zero-point energy generated by the quantized anisotropy degrees of freedom of the mixmaster universe.

ACKNOWLEDGMENTS

We thank Nathalie Deruelle and Martin Bucher for their remarks. P. M. was supported by Ministerstwo Nauki i Szkolnictwa Wyższego under the programme "Mobilność Plus."

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