$\eta - \eta'$ mixing

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The $\eta - \eta'$ mixing mass term due to the derivative coupling $SU(3) \times SU(3)$ symmetry breaking term, produces an additional momentum-dependent pole term for processes with η' but is suppressed in the η amplitude by a factor $m_\eta^2/m_{\eta'}^2$. This seems to be the origin of the two-angle description of the pseudoscalar decay constants used in the literature. In this paper, by diagonalizing both the mixing mass term and the momentum-dependent mixing term, we show that the $\eta - \eta'$ system can be described by a meson field renormalization and a new mixing angle θ which differs from the usual mixing angle θ_P by a small momentum-dependent mixing d term. This new mixing scheme with exact treatment of the momentum-dependent mixing term is actually simpler than the perturbation treatment and should be used in any determination of the $\eta - \eta'$ mixing angle and the momentum-dependent mixing term. Assuming nonet symmetry for the η_0 singlet amplitude, from the sum rules relating θ and d to the measured vector meson radiative decay amplitudes, we obtain consistent solutions: $\theta = -(13.99 \pm 3.1)^\circ$, $d = 0.12 \pm 0.03$ from $\rho \to \eta\gamma$ and $\eta' \to \rho\gamma$ decays, for ω, $\theta = -(15.47 \pm 3.1)^\circ$, $d = 0.11 \pm 0.03$, and for ϕ, $\theta = -(12.66 \pm 2.1)^\circ$, $d = 0.10 \pm 0.03$. It seems that vector meson radiative decays would favor a small $\eta - \eta'$ mixing angle and a small momentum-dependent mixing term.

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I. INTRODUCTION

The $\eta - \eta'$ mixing angle used in the past to describe the $\eta - \eta'$ system is based on the assumption that the off-diagonal octet-singlet mixing mass term does not depend significantly on the energy of the state [1]. However, as with the derivative coupling $SU(3) \times SU(3)$ breaking terms used in the derivation of the f_K/f_π ratio and the Callan-Treiman relation for the vector currents in K_{l3} decays [2], recent work [3,4] shows that a quadratic derivative off-diagonal octet-singlet mixing term could exist and requires two angles θ_8 and θ_0 to describe the pseudoscalar meson decay constants. One could also describe the $\eta - \eta'$ system by the usual mixing angle θ_P with the additional off-diagonal derivative $SU(3)$ breaking mass term treated as a perturbation [5] in which the momentum-dependent off-diagonal mass term produces an additional contribution which is suppressed by $O(m_\eta^2/m_{\eta'}^2)$ for processes involving η. Thus, the quadratic momentum-dependent off-diagonal mixing mass term, while leaving the amplitude with η almost unaffected, could enhance or suppress the η' amplitude. Since the mixing angle contains a higher-order $SU(3)$ breaking term, to be consistent, we need to include also higher-order terms in the momentum-dependent mixing terms by diagonalizing both the momentum-independent and momentum-dependent mixing terms. In the past 20 years, there have been only two papers considering diagonalizing the $\eta - \eta'$ Lagrangian with both the off-diagonal mass term and the full off-diagonal kinetic terms [6,7], which, however, produces an $\eta - \eta'$ Lagrangian with two mixing angles and two field renormalization parameters. Actually, it is not necessary to use their full off-diagonal kinetic terms, since the coefficients of the $\partial_\mu\eta_8\partial_\mu\eta_8$ and $\partial_\mu\eta_0\partial_\mu\eta_0$ terms can be absorbed into the mass terms after rescaling so that the $\eta - \eta'$ Lagrangian contains the usual canonical kinetic terms and only one off-diagonal $\partial_\mu\eta_8\partial_\mu\eta_0$ term. With this most general kinetic term, in this paper, we will show that the $\eta - \eta'$ system could be described by the new mixing angle and the renormalization of the η and η' meson fields. The new mixing angle contains the usual mixing angle and a small additional term coming from d. This new mixing scheme with exact treatment of the momentum-dependent mixing terms is actually simpler than the perturbation treatment in [5] and should be used in any determination of the $\eta - \eta'$ mixing angle and the momentum-dependent mixing term. In this paper, we shall apply this new mixing scheme to vector meson radiative decays. From the sum rules relating the pure octet and singlet vector meson radiative decay amplitudes to that for the measured decays amplitudes, and using nonet symmetry for the pure octet and singlet amplitudes, we obtain consistent solutions for the new mixing angle θ and the momentum-dependent mixing term d. For $\rho \to \eta\gamma$ and $\eta' \to \rho\gamma$ decays, $\theta = -(13.99 \pm 3.1)^\circ$, $d = 0.12 \pm 0.03$, for $\omega \to \eta\gamma$ and $\eta' \to \omega\gamma$ decays, $\theta = -(15.47 \pm 3.1)^\circ$, $d = 0.11 \pm 0.03$ for $\phi \to \eta\gamma$ and $\phi \to \eta'\gamma$. It is remarkable that these values are consistent with each other, to within experimental errors. For $\phi \to \eta'\gamma$ decays, with $SU(3)$ breaking from the s quark magnetic coupling included, we get $\theta = -(12.66 \pm 2.1)^\circ$, $d = 0.10 \pm 0.03$, consistent with

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the values for ρ and ω. After subtracting the d terms, one would get a value of $-(8-10)^\circ$ for the usual mixing angle. It seems that vector meson radiative decays would favor a small $\eta-\eta'$ mixing angle as found in a previous analysis; for example, a value between -13° and -17° or an average $\theta_P = -15.3^\circ \pm 1.3^\circ$ is obtained [8] and $\theta_P \approx -11^\circ$ is obtained in [9]. Additionally, a recent analysis [10,11] using the more precise $V \to P\gamma$ measured branching ratios [12] found $\theta_P = -13.3^\circ \pm 1.3^\circ$. Our values for d are also smaller than the chiral perturbation values and other phenomenological analyses in the two-angle mixing approach [4,13,14]. In the next section, we will obtain the diagonalized Lagrangian for the $\eta-\eta'$ system with the new $\eta-\eta'$ mixing angle θ.

II. THE DIAGONALIZED $\eta-\eta'$ LAGRANGIAN

We begin by writing down the Lagrangian for the $\eta-\eta'$ system with the usual nonderivative mixing mass term m_{08}^2, the pure octet η_8 mass m_8^2, the singlet η_0 mass m_0^2, and the derivative $\eta_0-\eta_8$ mixing term

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\eta_8\partial_\mu\eta_8 + \partial_\mu\eta_0\partial_\mu\eta_0 + m_8^2\eta_8^2 + m_0^2\eta_0^2) + d\partial_\mu\eta_8\partial_\mu\eta_0 + m_{08}^2\eta_8\eta_0, \tag{1}$$

where the strength d is given by L_5 and higher-order terms in chiral perturbation theory [3,4,15]. The $\eta_0-\eta_8$ Lagrangian in Eq. (1) contains the most general kinetic and mass term. The full off-diagonal kinetic and mass terms used in previous work [6,7] to diagonalize both the kinetic and mass terms of the $\eta_0-\eta_8$ system can be brought to the above form since the rescaling of the kinetic terms can be absorbed into the mass term so that $\mathcal{L}_0$ contains only the off-diagonal $\partial_\mu\eta_8\partial_\mu\eta_0$ and the usual canonical kinetic terms. Thus, our Lagrangian contains, as mentioned earlier, besides the usual $\eta-\eta'$ mass parameters, only two mixing parameters, the usual momentum-independent $\eta-\eta'$ mixing mass term, and the momentum-dependent $\eta-\eta'$ off-diagonal kinetic terms. This is an important difference between our approach and that of Refs. [6,7]. In a straightforward manner, we will show that the $\eta-\eta'$ system can be described by only one mixing angle and a field renormalization parameter.

To diagonalize this Lagrangian, we shall first make the substitution

$$\eta_8 = \frac{(\eta_{01}-\eta_{81})}{\sqrt{2}}, \qquad \eta_0 = \frac{(\eta_{01}+\eta_{81})}{\sqrt{2}}. \tag{2}$$

$\mathcal{L}_0$ becomes

$$\mathcal{L}_1 = \frac{(1-d)}{2}\partial_\mu\eta_{81}\partial_\mu\eta_{81} + \frac{(1+d)}{2}\partial_\mu\eta_{01}\partial_\mu\eta_{01} + \frac{1}{2}(m_{81}^2\eta_{81}^2 + m_{01}^2\eta_{01}^2) + m_{081}^2\eta_{81}\eta_{01}. \tag{3}$$

with

$$m_{81}^2 = \frac{(m_0^2 + m_8^2 - 2m_{08}^2)}{2}, \qquad m_{01}^2 = \frac{(m_0^2 + m_8^2 + 2m_{08}^2)}{2}, \qquad m_{081}^2 = \frac{(m_0^2 - m_8^2)}{2}. \tag{4}$$

To bring the kinetic term in $\mathcal{L}_1$ to the canonical form, we now perform a renormalization of the η_{81} and η_{01} meson field operators

$$\eta_{81} = \frac{\eta_{82}}{\sqrt{1-d}}, \qquad \eta_{01} = \frac{\eta_{02}}{\sqrt{1+d}}, \tag{5}$$

and $\mathcal{L}_1$ becomes

$$\mathcal{L}_2 = \frac{1}{2}\left(\partial_\mu\eta_{82}\partial_\mu\eta_{82} + \partial_\mu\eta_{02}\partial_\mu\eta_{02} + \frac{m_{81}^2}{(1-d)}\eta_{82}^2 + \frac{m_{01}^2}{(1+d)}\eta_{02}^2\right) + \frac{m_{081}^2}{\sqrt{1-d^2}}\eta_{82}\eta_{02}, \tag{6}$$

which can now be brought back to the octet-singlet basis by the transformation

$$\eta_{82} = \frac{(\eta_{03}-\eta_{83})}{\sqrt{2}}, \qquad \eta_{02} = \frac{(\eta_{03}+\eta_{83})}{\sqrt{2}}. \tag{7}$$

We have finally,

$$\mathcal{L}_3 = \frac{1}{2}(\partial_\mu\eta_{83}\partial_\mu\eta_{83} + \partial_\mu\eta_{03}\partial_\mu\eta_{03} + m_{82}^2\eta_{83}^2 + m_{02}^2\eta_{03}^2) + m_{082}^2\eta_{83}\eta_{03}, \tag{8}$$

with

$$m_{82}^2 = \frac{(1-\sqrt{1-d^2})m_0^2 + (1+\sqrt{1-d^2})m_8^2}{2(1-d^2)} + \frac{dm_{08}^2}{(1-d^2)},$$
$$m_{02}^2 = \frac{(1+\sqrt{1-d^2})m_0^2 + (1-\sqrt{1-d^2})m_8^2}{2(1-d^2)} + \frac{dm_{08}^2}{(1-d^2)},$$
$$m_{082}^2 = \frac{m_{08}^2 - d(m_0^2+m_8^2)/2}{(1-d^2)}. \tag{9}$$

Thus, we have been able to bring the original Lagrangian of the pure octet η_8 and singlet η_0 mesons with the derivative coupling $SU(3)$ symmetry breaking momentum-dependent $\eta_8-\eta_0$ mixing term to the usual form with only the energy-independent mixing mass term with $\mathcal{L}_3$ having the same form as $\mathcal{L}_0$, except that the mass and mixing terms are modified by additional contributions from the momentum-dependent mixing term d and the renormalization of the η_8 and η_0 meson fields. In the limit of $d=0$, we recover the usual mass term in $\mathcal{L}_0$. In terms of the η_{83} and η_{03} state, the pure $SU(3)$ octet and singlet state are then given by

$$\eta_8 = \left(\frac{\sqrt{1-d}+\sqrt{1+d}}{2\sqrt{(1-d^2)}}\right)\eta_{83} + \left(\frac{\sqrt{1-d}-\sqrt{1+d}}{2\sqrt{(1-d^2)}}\right)\eta_{03},$$
$$\eta_0 = \left(\frac{\sqrt{1-d}-\sqrt{1+d}}{2\sqrt{(1-d^2)}}\right)\eta_{83} + \left(\frac{\sqrt{1-d}+\sqrt{1+d}}{2\sqrt{(1-d^2)}}\right)\eta_{03}. \tag{10}$$

From the above expressions, we see that the η_{83} and η_{03} states are a mixture of the pure η_8 and η_0 and become the pure octet and singlet state in the limit of $d = 0$. This is an example of mixing caused by renormalization of the field operators due to the momentum-dependent derivative coupling $SU(3)$ breaking terms. The Lagrangian in Eq. (8) can now be brought to the diagonal form by writing η_{83} and η_{03} in terms of the physical η and η' states and the mixing angle θ,

$$\eta_{83} = \cos(\theta)\eta + \sin(\theta)\eta',$$
$$\eta_{03} = -\sin(\theta)\eta + \cos(\theta)\eta', \tag{11}$$

with θ given by

$$\tan(2\theta) = \frac{2m_{08}^2 - d(m_0^2 + m_8^2)}{(m_0^2 - m_8^2)\sqrt{1-d^2}}, \tag{12}$$

or

$$\sin(\theta) = \left(\frac{\cos(2\theta)}{\cos(\theta)}\right)\left(\frac{m_{08}^2 - d(m_0^2 + m_8^2)/2}{(m_0^2 - m_8^2)\sqrt{1-d^2}}\right), \tag{13}$$

which takes a simple form for small θ,

$$\sin(\theta) = \left(\frac{m_{08}^2 - d(m_0^2 + m_8^2)/2}{(m_0^2 - m_8^2)\sqrt{1-d^2}}\right). \tag{14}$$

After this last step, we arrive at the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\eta\partial_\mu\eta + \partial_\mu\eta'\partial_\mu\eta' + m_\eta{}^2\eta^2 + m_{\eta'}{}^2\eta'^2), \tag{15}$$

with $m_\eta{}^2$ and $m_{\eta'}{}^2$ given by

$$m_\eta{}^2 = \frac{(m_0^2 + m_8^2 - 2dm_{08}^2)}{2(1-d^2)} - \frac{(m_0^2 - m_8^2)\cos(2\theta)}{2\sqrt{(1-d^2)}} + \frac{(d(m_0^2 + m_8^2) - 2m_{08}^2)\sin(2\theta),}{2(1-d^2)}$$
$$m_{\eta'}{}^2 = \frac{(m_0^2 + m_8^2 - 2dm_{08}^2)}{2(1-d^2)} + \frac{(m_0^2 - m_8^2)\cos(2\theta)}{2\sqrt{(1-d^2)}} - \frac{(d(m_0^2 + m_8^2) - 2m_{08}^2)\sin(2\theta)}{2(1-d^2)}. \tag{16}$$

The pure octet η_8 and singlet η_0 can now be expressed terms of η and η'. From Eqs. (10) and (11), we have

$$\eta_8 = C_{8\eta}\eta + C_{8\eta'}\eta', \qquad \eta_0 = C_{0\eta}\eta + C_{0\eta'}\eta', \tag{17}$$

with

$$C_{8\eta} = \left(-\frac{(\sqrt{1-d}-\sqrt{1+d})\sin(\theta)}{2\sqrt{(1-d^2)}} + \frac{(\sqrt{1-d}+\sqrt{1+d})\cos(\theta)}{2\sqrt{(1-d^2)}}\right),$$
$$C_{8\eta'} = \left(\frac{(\sqrt{1-d}-\sqrt{1+d})\cos(\theta)}{2\sqrt{(1-d^2)}} + \frac{(\sqrt{1-d}+\sqrt{1+d})\sin(\theta)}{2\sqrt{(1-d^2)}}\right),$$
$$C_{0\eta} = \left(-\frac{(\sqrt{1-d}+\sqrt{1+d})\sin(\theta)}{2\sqrt{(1-d^2)}} + \frac{(\sqrt{1-d}-\sqrt{1+d})\cos(\theta)}{2(1-d^2)}\right),$$
$$C_{0\eta'} = \left(\frac{(\sqrt{1-d}+\sqrt{1+d})\cos(\theta)}{2\sqrt{(1-d^2)}} + \frac{(\sqrt{1-d}-\sqrt{1+d})\sin(\theta)}{2\sqrt{(1-d^2)}}\right). \tag{18}$$

For $d = 0$, we recover the usual expression given in Eq. (11).

To first order in d, we have

$$\eta_8 = (d\sin(\theta)/2 + \cos(\theta))\eta + (-d\cos(\theta)/2 + \sin(\theta))\eta',$$
$$\eta_0 = (-\sin(\theta) - d\cos(\theta)/2)\eta + (\cos(\theta) - d\sin(\theta)/2)\eta'. \tag{19}$$

Consider now the d terms in Eq. (19). The contribution to the η' amplitude from the pure η_8 term is proportional to $(-d\cos(\theta)/2 + \sin(\theta))$ which gives $-d/2$ from the first term, while another $-d/2$ from the $\sin(\theta)$ term. Similarly, the d term in the η amplitude coming from the pure singlet η_0 term $(\sin(\theta) + d\cos(\theta)/2)$ cancels out [$\sin(\theta)$ having the same d term with opposite sign]. More precisely, to first order in d, and neglecting also the $\sin(\theta/2)^2$ term in $\cos(\theta)$, we have from Eq. (14),

$$\eta_8 = (d\sin(\theta)/2 + \cos(\theta))\eta + \left(\sin(\theta_P) + \frac{dm_0^2}{(m_0^2 - m_8^2)}\right)\eta',$$
$$\eta_0 = \left(-\sin(\theta_P) + \frac{dm_8^2}{(m_0^2 - m_8^2)}\right)\eta + \left(\cos(\theta) - d\sin(\theta)/2\right)\eta', \tag{20}$$

where θ_P is the mixing angle for $d = 0$ (the usual mixing angle).

This agrees with the perturbation treatment of the derivative $SU(3) \times SU(3)$ symmetry breaking terms given in [5], except for the $d\sin(\theta)$ term which is second order in $SU(3)$ breaking.

We see that in the presence of the momentum-dependent mixing term d, the η and η' amplitudes now depend on both θ and d and are given completely by Eq. (18). Obviously, this simple expression should be used in any physical processes with η and η' rather than the perturbation treatment of the momentum-dependent mixing term used in [5]. Given A_8, A_0, the octet and singlet amplitude for η_8 and η_0, respectively, the physical amplitudes are then

$$A_\eta = C_{8\eta}A_8 + C_{0\eta}A_0,$$
$$A_{\eta'} = C_{8\eta'}A_8 + C_{0\eta'}A_0. \tag{21}$$

Following the two-angle mixing approach [4], consider now the quantity

$$P_{08} = A_8A_0(C_{8\eta}C_{0\eta} + C_{8\eta'}C_{0\eta'}). \tag{22}$$

Using Eq. (18), we find

$$P_{08} = -A_8A_0\frac{d}{(1-d^2)} = -A_8A_0\sin(\theta_0 - \theta_8), \tag{23}$$

which is precisely the expression obtained in chiral perturbation theory. To first order in d, $\sin(\theta_0 - \theta_8) = d$. This shows clearly that the parameter d is directly proportional to the coefficient L_A in the derivative expansion [4]. For $d = 0$, $P_{08} = 0$, we recover the orthogonality of the unitarity transformation between physical and unmixed states with the usual mixing angle.

Using Eq. (12) to express m_{08}^2 in terms of $\tan(2\theta)$, the expressions for η and η' masses in Eq. (16) are then

$$m_\eta{}^2 = \frac{(m_0^2 + m_8^2)}{2} - \frac{(m_0^2 - m_8^2)}{2\sqrt{(1-d^2)}\cos(2\theta)} - \frac{(d\tan(2\theta))(m_0^2 - m_8^2)}{2\sqrt{(1-d^2)}},$$
$$m_{\eta'}{}^2 = \frac{(m_0^2 + m_8^2)}{2} + \frac{(m_0^2 - m_8^2)}{2\sqrt{(1-d^2)}\cos(2\theta)} - \frac{(d\tan(2\theta))(m_0^2 - m_8^2)}{2\sqrt{(1-d^2)}}, \tag{24}$$

which now depend only on m_0^2, m_8^2, and d. By taking the mass difference $m_{\eta'}{}^2 - m_8^2$ and $m_\eta{}^2 - m_8^2$, we obtain

$$m_\eta{}^2 - m_8^2 = \frac{(m_0^2 - m_8^2)}{2}\left(1 - \frac{1}{\sqrt{(1-d^2)}\cos(2\theta)} - \frac{d\tan(2\theta)}{\sqrt{(1-d^2)}}\right),$$
$$m_{\eta'}{}^2 - m_8^2 = \frac{(m_0^2 - m_8^2)}{2}\left(1 + \frac{1}{\sqrt{(1-d^2)}\cos(2\theta)} - \frac{d\tan(2\theta)}{\sqrt{(1-d^2)}}\right). \tag{25}$$

This implies

$$m_\eta{}^2 - m_8^2 = R(m_{\eta'}{}^2 - m_8^2), \tag{26}$$

with R given by

$$R = -(1 - \sqrt{(1-d^2)}\cos(2\theta) + d\sin(2\theta))(1 + \sqrt{(1-d^2)}\cos(2\theta) - d\sin(2\theta))^{-1}. \tag{27}$$

As d is a small $SU(3) \times SU(3)$ breaking parameter, putting $d = \sin(\alpha)$ and $\sqrt{1-d^2} = \cos(\alpha)$, the above expression Eq. (27) takes a simple form,

$$R = -\tan(\theta + \alpha/2)^2. \tag{28}$$

For small d, $\alpha \approx \sin(\alpha) = d$, $\theta + \alpha/2 \approx \theta_P$, and R is essentially the usual relation $R = -\tan(\theta_P)^2$, which is not affected by the presence of a momentum-dependent mixing term.

III. MIXING ANGLE FROM VECTOR MESON RADIATIVE DECAYS

With our Lagrangian in the diagonal form, we shall now try to determine θ and d using the sum rules [5] obtained by equating the vector meson radiative decay matrix element for the pure octet η_8 and singlet η_0 with the expressions for these quantities extracted from the measured matrix elements with η and η' given by Eq. (17). Defining as in [5] the electromagnetic form factor $V \to P$ by

$$\langle P(p_P)|J_\mu^{\rm em}|V(p_V)\rangle = \epsilon_{\mu p_P p_V \epsilon_V} g_{VP\gamma}, \tag{29}$$

where $g_{VP\gamma}$ is the on-shell $VP\gamma$ coupling constant with dimension the inverse of energy. We have for the radiative decay rates [16]

$$\Gamma(V \to P\gamma) = \frac{\alpha}{24} g_{VP\gamma}^2 \left(\frac{m_V^2 - m_P^2}{m_V}\right)^3,$$
$$\Gamma(P \to V\gamma) = \frac{\alpha}{8} g_{VP\gamma}^2 \left(\frac{m_P^2 - m_V^2}{m_P}\right)^3. \tag{30}$$

For convenience, we give in Table I the measured radiative branching ratios together with the extracted coupling constant $g_{VP\gamma}$ in units of GeV^{-1} and its theoretical value derived either from an $SU(3)$ effective Lagrangian with nonet symmetry for the $V \to \eta_0\gamma$ amplitude or from the quark counting rule with the coupling constant $g_{VP\gamma}$ given in terms of the quark coupling constant g_q, $(q = u, d, s)$ for the magnetic transition $(q\bar{q})(1^-) \to (q\bar{q})(0^-)\gamma$ [8,10,16]. More details on the theoretical values for $V \to \eta_8\gamma$ and $V \to \eta_0\gamma$ can be found in Ref. [5].

In terms of $g_{VP\gamma}$, the sum rules read

$$S(V \to \eta\gamma) = g_{V\eta\gamma}C_{8\eta} + g_{V\eta'\gamma}C_{8\eta'} = \left(\frac{g_{V\eta_8\gamma}}{g_{V\pi^0\gamma}}\right)_{\rm th.} g_{V\pi^0\gamma},$$
$$S(\eta' \to V\gamma) = g_{V\eta\gamma}C_{0\eta} + g_{V\eta'\gamma}C_{0\eta'} = \left(\frac{g_{V\eta_0\gamma}}{g_{V\pi^0\gamma}}\right)_{\rm th.} g_{V\pi^0\gamma}, \tag{31}$$

and similarly for other vector meson radiative decays. Thus, with the updated values of the measured values for $g_{VP\gamma}$ in Table I, we have for ρ meson radiative decay

$$S(\rho \to \eta\gamma) = 1.59C_{8\eta} + 1.35C_{8\eta'} = 1.12,$$
$$S(\eta' \to \rho\gamma) = 1.59C_{0\eta} + 1.35C_{0\eta'} = 1.63, \tag{32}$$

and for the ω meson,

$$S(\omega \to \eta\gamma) = 0.45C_{8\eta} + 0.44C_{8\eta'} = 0.29,$$
$$S(\eta' \to \omega\gamma) = 0.45C_{0\eta} + 0.44C_{0\eta'} = 0.53. \tag{33}$$

From the above sets of equations, we obtain the following solutions for θ and d:

$$\theta = -(13.99 \pm 3.1)^\circ, \quad d = 0.12 \pm 0.03, \quad \text{for } \rho,$$
$$\theta = -(15.48 \pm 3.1)^\circ, \quad d = 0.11 \pm 0.03, \quad \text{for } \omega. \tag{34}$$

Since $\eta - \eta'$ mixing is an $SU(3)$ breaking effect not present in the η_8 and η_0 decay amplitudes, ρ meson radiative decays in which only the u, d quarks are active, offer a rare opportunity to determine the mixing angle free from uncertainties from $SU(3)$ breaking due to s quark magnetic coupling, which is present in radiative ϕ meson radiative decays. With an almost ideal mixing, the ω meson radiative decays are also rather insensitive to the s quark magnetic coupling $SU(3)$ breaking, which is rather small, of the order of 1.5%. In fact, as shown in Ref. [5], instead of ω radiative decay amplitudes alone, one can use a linear combination for an ideal mixing state, the ω_0 state with the decay amplitudes with only the u, d quarks active. We have $(\varphi_V = (3.2 \pm 0.1)^\circ)$

$$S(\omega_0 \to \eta\gamma) = \cos\varphi_V S(\omega \to \eta\gamma) + \sin\varphi_V S(\phi \to \eta\gamma), \tag{35}$$

and a similar expression for $S(\eta' \to \omega_0\gamma)$. The solutions for this ideal mixing case is then

TABLE I. Theoretical values for $V \to P\gamma$ with $P = \eta_8, \eta_0$ together with the measured branching ratios and the extracted $g_{VP\gamma}$ taken from Ref. [5].

Decay	$g_{VP\gamma}$, $P = \eta_8, \eta_0$	$g_{VP\gamma}$(exp)	BR(exp) [12]
$\rho^\pm \to \pi^\pm\gamma$	$(1/3)g_u$	0.72 ± 0.04	$(4.5 \pm 0.5) \times 10^{-4}$
$\rho^0 \to \pi^0\gamma$	$(1/3)g_u$	0.83 ± 0.05	$(6.0 \pm 0.8) \times 10^{-4}$
$\rho^0 \to \eta\gamma$	$0.58g_u(f_\pi/f_{\eta_0})$	1.59 ± 0.06	$(3.00 \pm 0.20) \times 10^{-4}$
$\omega \to \pi^0\gamma$	$0.99g_u$	2.29 ± 0.03	$(8.28 \pm 0.28)\%$
$\omega \to \eta\gamma$	$0.17g_u(f_\pi/f_{\eta_0})$	0.45 ± 0.02	$(4.6 \pm 0.4) \times 10^{-4}$
$\phi \to \pi^0\gamma$	$0.06g_u$	0.13 ± 0.003	$(1.27 \pm 0.06) \times 10^{-3}$
$\phi \to \eta\gamma$	$0.47g_u(f_\pi/f_{\eta_0})$	0.71 ± 0.01	$(1.309 \pm 0.024)\%$
$\phi \to \eta'\gamma$	$-0.31g_u(f_\pi/f_{\eta_0})$	$-(0.72 \pm 0.01)$	$(6.25 \pm 0.21) \times 10^{-5}$
$\eta' \to \rho^0\gamma$	$0.82g_u(f_\pi/f_{\eta_0})$	1.35 ± 0.02	$(29.1 \pm 0.5)\%$
$\eta' \to \omega\gamma$	$0.29g_u(f_\pi/f_{\eta_0})$	0.44 ± 0.02	$(2.75 \pm 0.23)\%$
$K^{*\pm} \to K^\pm\gamma$	$0.38g_u(f_\pi/f_K)$	0.84 ± 0.04	$(9.9 \pm 0.9) \times 10^{-4}$
$K^{*0} \to K^0\gamma$	$-0.62g_u(f_\pi/f_K)$	$-(1.27 \pm 0.05)$	$(2.46 \pm 0.22) \times 10^{-3}$

$$\theta = -(15.40 \pm 2.1)^\circ, \qquad d = 0.12 \pm 0.03, \quad \text{for } \omega_0, \tag{36}$$

which is very close to the solution for ω. This indicates that $SU(3)$ breaking due to s quark magnetic coupling in ω radiative decay is, indeed, quite small and can be neglected. This ideal mixing solution is also consistent with the solution for ρ. Taking an average of the solution for ρ and ω_0, we have

$$\theta = -(14.68 \pm 3.1)^\circ, \qquad d = 0.115 \pm 0.03. \tag{37}$$

For the ϕ meson, from the sum rules

$$S(\phi \to \eta\gamma) = 0.71 C_{8\eta} - 0.72 C_{8\eta'} = 0.88,$$
$$S(\phi \to \eta'\gamma) = 0.71 C_{0\eta} - 0.72 C_{0\eta'} = -0.59, \tag{38}$$

the solution is then

$$\theta = -(12.66 \pm 2.1)^\circ, \qquad d = 0.10 \pm 0.03, \quad \text{for } \phi, \tag{39}$$

consistent with the corresponding values for ρ and ω given by Eqs. (33) and (37). This indicates that $SU(3)$ breaking for ϕ meson radiative decays is correctly given by $K^* \to K\gamma$ decays for which the new measured branching ratio gives $k = 0.83 \pm 0.04$, close to the value $k = 0.85$ given above. Thus, the values we obtained from ϕ meson radiative decays are used as a way to check the $SU(3)$ breaking effect in $\phi \to \eta\gamma, \eta'\gamma$ decays rather than a determination of the mixing angle. The value for d, ($d = \sin(\theta_0 - \theta_8)$) obtained above with our diagonalized Lagrangian is smaller than the values obtained in chiral perturbation and other phenomenological analyses [4,13,14], which give $(\theta_0 - \theta_8)$ in the range $(12 - 17)^\circ$ in the two-angle mixing approach.

Thus, by treating exactly the derivative coupling mixing term with our diagonalized Lagrangian, we have found a small mixing angle in vector meson radiative decays which are also found to have a small mixing angle (the usual mixing angle) in previous work [8–11]. By subtracting the d term in θ, we obtain a value $-(8 - 10)^\circ$ for the usual mixing angle. This value is smaller by a few degrees than the values we obtained in our previous work [5]. This could be due to the exact treatment of the momentum-dependent mixing term in our Lagrangian.

IV. CONCLUSION

In conclusion, we have diagonalized both the mass term and the momentum-dependent mixing term in the $\eta - \eta'$ Lagrangian and showed that the $\eta - \eta'$ system can be described by two parameters, the meson field renormalization and a new $\eta - \eta'$ mixing angle, which differs from the usual mixing angle by a small momentum-dependent mixing term. The expressions for the η and η' amplitude in our new mixing scheme are actually quite simple and should be used for any process with η and η'. Using the measured vector meson radiative decays, we have obtained consistent solutions for the mixing angle and the momentum-dependent mixing term. The small mixing angle we found is consistent with previous determinations. It seems that vector meson radiative decays would favor a small $\eta - \eta'$ mixing angle θ and a small momentum-dependent mixing term d.

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