Fourth-order QCD renormalization group quantities in the V scheme and the relation of the β function to the Gell-Mann–Low function in QED

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The semianalytical $O(\alpha_s^4)$ expression for the renormalization group β function in the V scheme is obtained in the case of the $SU(N_c)$ gauge group. In the process of calculations we use the existing information about the three-loop perturbative approximation for the QCD static potential, evaluated in the $\overline{\rm MS}$ scheme. The comparison of the numerical values of the third and fourth coefficients for the QCD RG β functions in the gauge-independent V and $\overline{\rm MS}$ schemes and in the minimal momentum scheme in the Landau gauge is presented. The phenomenologically oriented comparisons for the coefficients of $O(\alpha_s^4)$ expression for the e^+e^--annihilation R-ratio in these schemes are presented. It is shown that taking into account these QCD contributions is of vital importance and leads to a drastic decrease of the scheme-dependence ambiguities of the fourth-order perturbative QCD approximations for the e^+e^--annihilation R-ratio for the number of active flavors, $n_f = 5$ in particular. We demonstrate that in the case of QED with N-types of leptons the coefficients of the β^V function are closely related to the ones of the Gell-Mann–Low Ψ function and emphasize that they start to differ from each other at the fourth order due to the appearance of the extra N^2-contribution in the V scheme. The source of this extra correction is clarified. The general all-order QED relations between the coefficients of the β^V and Ψ functions are discussed.

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I. INTRODUCTION

The renormalization group (RG) β function is one of the basic quantities of the RG method, which was developed in the classical works of Refs. [1–3]. It defines the energy behavior of the renormalized coupling constants of the renormalized quantum field models. It is known that in the case when the quantum field model under study has the single coupling constant, the perturbation theory (PT) expressions for its RG β functions depend on the choice of the scheme of subtracting ultraviolet (UV) divergences.

In QED the first and the second coefficients of the β function are scheme independent and were obtained in Ref. [3] from the analytical calculations of the two-loop approximation for the renormalized photon propagator performed in Ref. [4].

In the momentum (MOM) scheme, defined by subtractions of the UV divergences of the photon vacuum polarization function at the nonzero Euclidean point λ^2, the QED RG β function coincides with the Gell-Man–Low function $\Psi(\alpha_{\rm MOM})$, where the expression for $\alpha_{\rm MOM}(q^2)$ coincides with the QED invariant charge, uniquely defined by the combinations of the Green functions [5]. The expressions for the coefficients of the PT series for $\Psi(\alpha_{\rm MOM})$ depend on the number of leptons N.

For $N = 1$, i.e. in the case of consideration of the electron only, the three-loop term of the Ψ function was calculated analytically in Ref. [6]. This result was generalized to the case of the arbitrary number N of massless leptons in Ref. [7]. The N-dependent expressions for the four- and five-loop corrections to the Gell-Man–Low function were evaluated symbolically in Refs. [8] and [9] respectively. At $N = 1$ the result of Ref. [9] coincides with the similar expression, obtained in Ref. [10]. This feature should be considered as a strong argument in favor of the consistency of the complicated analytical five-loop calculations, performed in Ref. [9].

Another important scheme, which is used in QED, is the on-shell (OS) scheme. In this scheme the photon vacuum polarization function is defined by subtracting UV divergences at zero transferred momentum, while the renormalized on-shell masses of leptons are identified with their experimentally measured values.

In the physical OS scheme the calculations of the $\beta(\alpha_{\rm OS})$ were performed at the three-loop level in the work of [11]. The analytical expression for the corresponding four-loop correction was obtained in Ref. [12]. In the case of arbitrary N the five-loop contribution was obtained in Ref. [13]. It is in agreement with the results of the work [10], where this term was obtained at $N = 1$ with the help of the concrete

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RG relations. The agreement with the outcome of the direct five-loop calculations of Ref. [13] gives us extra confidence in the correctness and self-consistency of the results of the complicated computer calculations used in Ref. [10].

The third class of schemes, which we are interested in, is introduced when the dimensional regularization [14] is used. These schemes include the minimal subtractions (MS) scheme [15] and its modified variants, namely the $\overline{\rm MS}$ scheme [16] and the G scheme [17]. It is possible to prove that for all these modifications of the MS scheme the coefficients for the RG β functions coincide in all orders of PT.

At $N=1$ the three-loop correction to the $\beta(\alpha_{\overline{\rm MS}})$ function was evaluated in Refs. [17] and [18] independently (this result had been also presented in the review of Ref. [19]). In the case of the arbitrary N the three-loop contribution to $\beta(\alpha_{\overline{\rm MS}})$ was obtained analytically in Ref. [7]. The computation of the four-loop term was completed in Ref. [8]. The five-loop correction to the QED β function in the MS-like schemes was calculated in Ref. [9]. At $N=1$ this expression coincides with the result of nondirect analysis, performed in Ref. [10].

It is known that in QCD the MS-like schemes maintain the explicit gauge independence of various RG quantities. This property clarifies why in multiloop QCD calculations the MS-like schemes are used more often. In QCD the first coefficient of the β function was computed in Refs. [20,21] and for the number of quark flavors $n_f \leq 6$ turned out to be negative. This feature revealed the existence of the asymptotic-freedom property in the gauge theory of strong interactions. The two-loop correction to the QCD β function in the MS-like scheme was analytically evaluated in [22–24] and is also negative.[1]

At the three-loop level the QCD β function was analytically calculated in the $\overline{\rm MS}$ scheme in Ref. [26]. This result was confirmed later in Ref. [27]. The four-loop term of the QCD β function in the MS-like schemes was evaluated in Ref. [28] and confirmed in Ref. [29]. For $n_f=6$ the three-loop correction to the β function in the MS-like schemes is positive (see the numerical results presented below). Note however that the resummation of the PT series for the QCD β function in the MS-like schemes gives the argument that this feature does not affect the asymptotic freedom property [30].

In QCD one can also use another gauge-independent scheme, namely the V scheme. It was first introduced in Refs. [31,32] and is determined by perturbative high-order QCD corrections to the static potential. This scheme was used in Ref. [33] to model massive dependence of the first two coefficients of the RG β function in the V scheme and for the related analysis of the manifestation of the massive-dependent corrections in the effect of running of the QCD coupling constant from the energies above the production of charm quarks to the high energy region above the scale $s=M_Z^2$. In this case the advantage of using the V scheme and not the $\overline{\rm MS}$ scheme is contained in the possibility of modeling the smooth transition of the QCD coupling constant through the thresholds of heavy quark productions. Among other applications of the V scheme in QCD is the analysis of the perturbative QCD predictions for $\Gamma(H^0\to b\bar{b})$ [34]. It was shown in this work that within the large β_0-expansion the perturbative approximations for $\Gamma(H^0\to b\bar{b})$ in the V scheme are converging to the concrete stable value faster than in the $\overline{\rm MS}$ scheme.

However, to analyze more carefully the behavior of various perturbative QCD series for the observable physical quantities in the V scheme it is necessary to know high-order PT corrections to the QCD β function in this scheme. In the present work we will get the semianalytical result for the fourth coefficient of the QCD β function in the V scheme, i.e. for the β^V function. In Sec. II the available results of the analytical and semianalytical calculations of the PT QCD corrections to the static potential $V_{\rm QCD}(\alpha_s(\mu^2))$ in the $\overline{\rm MS}$ scheme are summarized. The concrete three-loop results, obtained by two groups of authors, are compared. Section III is devoted to the definition of the V scheme and to the presentation of the concrete results for the third and fourth coefficients of the β^V function. The problem of finding the analytical expression for the concrete known numerical contributions to the fourth-order term of the β^V function is raised. In Sec. IV the numerical values for the scheme-dependent coefficients of the QCD β function in the V scheme are compared with the similar terms, obtained in the MS-like schemes and in the gauge-dependent minimal-MOM (mMOM) scheme widely used at present, defined in Ref. [35]. We also get the $O(\alpha_s^4)$ expression for the e^+e^--annihilation R-ratio in the V scheme and compare it with gauge-independent scheme and gauge-dependent mMOM scheme results. In fact both $\overline{\rm MS}$ and mMOM schemes were applied recently for the analysis of the behavior of the R-ratio in the fourth order of PT [36]. Using the concrete physical input, we modify this analysis and emphasize that it is more consistent to perform this comparison for $n_f=4,5$ numbers of active flavors in the energy region above the production of charm-quark pairs and below $s\approx 900\ {\rm GeV}^2$, where the effects of subprocess $e^+e^-\to Z^0\to$ hadrons did not yet start to manifest itself. In Sec. V we consider the QED limit of the results obtained in Sec. III and obtain the expression for the $O(\alpha_{\rm V}^5)$ approximation of the β^V function in QED. The origin of the difference with the QED Gell-Man–Low Ψ function,

[1]Its first calculation [25] contained a bug, which resulted in the positive value of the two-loop term and in the appearance of the IR-fixed point of the two-loop PT approximation of the QCD β function. This unexpected conclusion stimulated recalculations of this scheme-independent correction [22–24]. They resulted in the disappearance of the perturbative scheme-independent IR-fixed point in QCD.

which is starting to manifest itself from the fourth term, is demonstrated and explained. The existing common features of the PT series for the Ψ and β^V functions are clarified in all orders of PT.

II. PRELIMINARIES: THE HIGH-ORDER EXPRESSION OF THE STATIC POTENTIAL IN QCD IN THE $\overline{\rm MS}$ SCHEME

Let us first summarize the available information about the perturbative QCD contributions to the static potential known at present. This physical quantity is used in various phenomenologically oriented QCD studies, e.g. in the process of theoretical determinations of the charm-quark, bottom-quark and the top-quark masses, and for the studies of the properties of different mesons, composed from the c- and b-quarks (see e.g. [37–39] and references therein).

Within PT the static potential in QCD is defined as a *renormalized expression* for the potential of interaction at a distance r between static heavy quark Q_h and antiquark $\bar{Q}_h$. It is expressed through the following Fourier representation:

$$\begin{aligned} \mathrm{V}_{\rm QCD}(\mu^2 \mathrm{r}^2, \alpha_s(\mu^2)) &= \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\,\vec{r}} \mathrm{V}(\vec{q}^{\,2}, \mu^2, \alpha_s(\mu^2)) \\ &= \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\,\vec{r}} \left(-4\pi C_F \frac{\alpha_{s,V}(\vec{q}^{\,2}/\mu_V^2)}{\vec{q}^{\,2}}\right) \end{aligned} \tag{2.1}$$

where $\alpha_{s,V}(\vec{q}^{\,2}/\mu_V^2)$ is the renormalized QCD coupling constant in the V scheme; $\alpha_s/4\pi = g^2/16\pi^2$; g is the strong coupling constant of the QCD Lagrangian; T^a is the generator of the $SU(N_c)$ group, normalized as $T^a = \lambda^a/2$; and C_F is the Casimir operator, defined as $(T^a T^a)_{ij} = C_F \delta_{ij}$. In the V scheme its coupling constant $\alpha_{s,V}(\vec{q}^{\,2}/\mu_V^2)$ is related to the numerator of the momentum representation of the static potential in the $\overline{\rm MS}$ scheme defined in Eq. (2.1) and is expressed as

$$\begin{aligned} \alpha_{s,V}(\vec{q}^{\,2}/\mu_V^2) &= \alpha_s(\mu^2) P(\alpha_s(\mu^2), L) \\ &= \alpha_s(\mu^2) \sum_{n=0}^{\infty} P_n^{\overline{\rm MS}}(L) \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^n. \end{aligned} \tag{2.2}$$

The rhs of Eq. (2.2) is expressed through higher-order PT QCD corrections to the static potential $P_n^{\overline{\rm MS}}(L)$ in the $\overline{\rm MS}$ scheme which are known at present up to $O(\alpha_s^3)$-level and will be presented below.

The evolution of the $\overline{\rm MS}$-scheme coupling constant $\alpha_s(\mu^2)$ (which depends on the $\overline{\rm MS}$-scheme renormalization parameter μ^2) is governed by the QCD $\overline{\rm MS}$-scheme β function:

$$\mu^2 \frac{\partial(\alpha_s/4\pi)}{\partial\mu^2} = \beta^{\overline{\rm MS}}(a_s) = -\sum_{i=0}^{\infty} \beta_i \left(\frac{\alpha_s}{4\pi}\right)^{i+2}, \tag{2.3}$$

where $a_s = \alpha_s/4\pi$ and its known four $\overline{\rm MS}$-scheme coefficients, taken from the work of Ref. [28], read

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_l \tag{2.4}$$

$$\beta_1 = \frac{34}{3} C_A^2 - 4 C_F T_F n_l - \frac{20}{3} C_A T_F n_l \tag{2.5}$$

$$\begin{aligned} \beta_2 = {} & \frac{2857}{54} C_A^3 + 2 C_F^2 T_F n_l - \frac{205}{9} C_F C_A T_F n_l \\ & - \frac{1415}{27} C_A^2 T_F n_l + \frac{44}{9} C_F T_F^2 n_l^2 + \frac{158}{27} C_A T_F^2 n_l^2 \end{aligned} \tag{2.6}$$

$$\begin{aligned} \beta_3 = {} & \left(\frac{150653}{486} - \frac{44}{9}\zeta(3)\right) C_A^4 + \left(-\frac{39143}{81} + \frac{136}{3}\zeta(3)\right) C_A^3 T_F n_l + \left(\frac{7073}{243} - \frac{656}{9}\zeta(3)\right) C_A^2 C_F T_F n_l \\ & + \left(-\frac{4204}{27} + \frac{352}{9}\zeta(3)\right) C_A C_F^2 T_F n_l + 46 C_F^3 T_F n_l + \left(\frac{7930}{81} + \frac{224}{9}\zeta(3)\right) C_A^2 T_F^2 n_l^2 \\ & + \left(\frac{1352}{27} - \frac{704}{9}\zeta(3)\right) C_F^2 T_F^2 n_l^2 + \left(\frac{17152}{243} + \frac{448}{9}\zeta(3)\right) C_A C_F T_F^2 n_l^2 + \frac{424}{243} C_A T_F^3 n_l^3 + \frac{1232}{243} C_F T_F^3 n_l^3 \\ & + \left(-\frac{80}{9} + \frac{704}{3}\zeta(3)\right) \frac{d_A^{abcd} d_A^{abcd}}{N_A} + \left(\frac{512}{9} - \frac{1664}{3}\zeta(3)\right) \frac{d_F^{abcd} d_A^{abcd}}{N_A} n_l + \left(-\frac{704}{9} + \frac{512}{3}\zeta(3)\right) \frac{d_F^{abcd} d_F^{abcd}}{N_A} n_l^2. \end{aligned} \tag{2.7}$$

The characteristic color structures of the group $SU(N_c)$ are defined as in the detailed work of Ref. [40]. In the notations of Ref. [40] we have $[T^a, T^b] = i f^{abc} T^c$, where f^{abc} are the antisymmetric (under permutations of any pair of indices) structure constants, which satisfy the well-known relation $f^{acd} f^{bcd} = C_A \delta^{ab}$; C_A and C_F are the Casimir operators; $Tr(T^a T^b) = T_F \delta^{ab}$; N_A is the number of the generators of the Lie algebra of the $SU(N_c)$; n_l is the number of quark

flavors; and $d_F^{abcd} = Tr(T^a T^{(b} T^c T^{d)})/6$ is the totally symmetric tensor. The notations $^{(\ldots)}$ are defining the procedure of symmetrization of the generators $T^b T^c T^d$; and $d_A^{abcd} = Tr(C^a C^{(b} C^c C^{d)})/6$ is the total symmetric tensor of $(C^a)_{bc} = -if^{abc}$, where C^a are the generators of the adjoint representation of the Lie algebra of the $SU(N_c)$ group. The corresponding color structures in Eqs. (2.4)–(2.7) have the following form [28]:

$$C_A = N_c, \qquad C_F = \frac{N_c^2 - 1}{2N_c}, \qquad N_A = N_c^2 - 1 \tag{2.8}$$

$$\frac{d_A^{abcd} d_A^{abcd}}{N_A} = \frac{N_c^2(N_c^2+36)}{24},$$
$$\frac{d_F^{abcd} d_A^{abcd}}{N_A} = \frac{N_c(N_c^2+6)}{48} \tag{2.9}$$

$$\frac{d_F^{abcd} d_F^{abcd}}{N_A} = \frac{N_c^4 - 6N_c^2 + 18}{96N_c^2}. \tag{2.10}$$

The terms proportional to the nth powers of $L = \ln(\mu^2/\vec{q}^2)$, $P_n^{\overline{\rm MS}}(L)$ in the polynomial $P(\alpha_s(\mu^2))$ of Eq. (2.2), are expressed as $P_0^{\overline{\rm MS}} = 1$, $P_1^{\overline{\rm MS}}(L) = a_1^{\overline{\rm MS}} + \beta_0 L$, $P_2^{\overline{\rm MS}}(L) = a_2^{\overline{\rm MS}} + (2a_1^{\overline{\rm MS}}\beta_0 + \beta_1)L + \beta_0^2 L^2$, $P_3^{\overline{\rm MS}}(L) = a_3^{\overline{\rm MS}} + (3a_2^{\overline{\rm MS}}\beta_0 + 2a_1^{\overline{\rm MS}}\beta_1 + \beta_2^{\overline{\rm MS}})L + (3a_1^{\overline{\rm MS}}\beta_0^2 + \frac{5}{2}\beta_0\beta_1)L^2 + \beta_0^3 L^3$. The powers of L in the expressions presented above arise from the solutions of the corresponding RG equations in the MS-like schemes at the three-loop level.

The coefficients $a_i^{\overline{\rm MS}}$ are calculated from the concrete Feynman diagrams. The first one, $a_1^{\overline{\rm MS}}$, was calculated a long time ago in Refs. [41,42] and has the following form:

$$a_1^{\overline{\rm MS}} = \frac{31}{9} C_A - \frac{20}{9} T_F n_l \tag{2.11}$$

where $n_l = n_f - 1$. The coefficient $a_2^{\overline{\rm MS}}$ was obtained in [31]. The bug in the pure Yang-Mills contribution to $a_2^{\overline{\rm MS}}$, evaluated in Ref. [31], was detected in Ref. [32].[2]

The final result of these analytical calculations of Refs. [31,32] is

$$a_2^{\overline{\rm MS}} = \left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3)\right) C_A^2 - \left(\frac{1798}{81} + \frac{56}{3}\zeta(3)\right) C_A T_F n_l - \left(\frac{55}{3} - 16\zeta(3)\right) C_F T_F n_l + \left(\frac{20}{9} T_F n_l\right)^2. \tag{2.12}$$

[2]This correction was confirmed later by the author in Ref. [31].

The three-loop constant perturbative contribution to the static potential in the $\overline{\rm MS}$ scheme can be presented as

$$a_3^{\overline{\rm MS}} = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)}. \tag{2.13}$$

The n_l-dependent terms were computed in Ref. [43] and have the following form:

$$a^{(3)} = -\left(\frac{20}{9}\right)^3 T_F^3 \tag{2.14}$$

$$a_3^{(2)} = \left(\frac{12541}{243} + \frac{368}{3}\zeta(3) + \frac{64\pi^4}{135}\right) C_A T_F^2 + \left(\frac{14002}{81} - \frac{416}{3}\zeta(3)\right) C_F T_F^2 \tag{2.15}$$

$$a_3^{(1)} = -709.717 C_A^2 T_F + \left(-\frac{71281}{162} + 264\zeta(3) + 80\zeta(5)\right) C_A C_F T_F + \left(\frac{286}{9} + \frac{296}{3}\zeta(3) - 160\zeta(5)\right) C_F^2 T_F - 56.83(1)\frac{d_F^{abcd} d_F^{abcd}}{N_A} \tag{2.16}$$

where the error of numerical calculation of the $C_A^2 T_F$ coefficient in Eq. (2.16) is not indicated in Ref. [43].

It is worth emphasizing that in the QED limit with $C_A = 0$, the analytical expressions of the n_l-dependent terms, which are proportional to the powers of T_F in Eqs. (2.11), (2.12) and in Eqs. (2.14)–(2.16), are in agreement with the $\overline{\rm MS}$-scheme results presented in [44] for the constant terms of the three-loop approximation of the photon vacuum polarization function in QED. They were also confirmed in Ref. [13] in the process of computation of the four-loop approximation of this quantity. The agreement with the QED results of Refs. [44] gives us extra confidence in the validity of the outcomes of calculations of Ref. [43].

The numerical expressions of the n_l-independent contributions to Eq. (2.13) were obtained in Ref. [45] and read

$$a_3^{(0)} = 502.24(1) C_A^3 - 136.39(12)\frac{d_F^{abcd} d_A^{abcd}}{N_A}. \tag{2.17}$$

These results should be compared with the results of the independent calculation of Ref. [46]

$$a_3^{(0)} = 502.22(12) C_A^3 - 136.8(14)\frac{d_F^{abcd} d_A^{abcd}}{N_A} \tag{2.18}$$

which have greater inaccuracies. Recently the more accurate result for the second term in Eq. (2.18) was obtained

with the help of the computer code used in Ref. [46]. The improved result for Eq. (2.18) is

$$a_3^{(0)} = 502.22(12)C_A^3 - 136.6(2)\frac{d_F^{abcd}d_A^{abcd}}{N_A} \qquad (2.19)$$

The numerical expression of the coefficient before the second structure in Eq. (2.19) is in agreement with the numerical expression of the same coefficient in Eq. (2.17) and demonstrates the reliability of the computer codes, created in the process of calculations, which were performed in Refs. [45] and [46].[3]

The three-loop n_l-independent correction to the static potential also contains the RG noncontrollable additional term $8\pi^2 C_A^3 L$ [47]. It is associated with the infrared (IR) divergences, which begin to manifest themselves in the static potential at the three-loop level [48,49]. In the effective theory of heavy quarkonium—nonrelativistic QCD—these IR-divergent L-terms are canceled by the concrete UV-divergent contributions (see e.g. [49]).

Among the aims of this work is the determination of the four-loop approximation of the RG β function in the V scheme. This can be done by application of the RG-motivated effective charges (ECH) approach, developed in all orders of PT in the works of Refs. [50,51] and independently at the next-to-leading order (NLO) in Ref. [52] (for the concrete NLO applications see e.g. the work [53]) The fourth-order approximation of the β function in the V scheme defines the evolution of $\alpha_{s,V}$ in the region of intermediate and UV values of energy scales. It does not depend on the manifestation of IR physical effects and on the RG-uncontrollable L-dependent corrections to the static potential. In view of this we will not consider them in our further analysis.

III. THE FOURTH-ORDER APPROXIMATION OF THE QCD β FUNCTION IN THE V SCHEME

A. The scale-scheme dependence ambiguities

Let us start this section by writing the RG equation for the static potential, which is defined in Eq. (2.1). In the massless limit, considered in this work, it has the following form:

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + \beta(a_s)\frac{\partial}{\partial a_s}\right)\mathrm{V}(\vec{q}^2, \mu^2, a_s(\mu^2)) = 0.$$

In QCD the scheme-dependence feature of the PT series for the RG β function is the more delicate issue than the scheme-dependence problem of the QED RG β function discussed in the Introduction. Indeed, contrary to the QED case, in this realistic theory of strong interactions it is impossible to introduce straightforwardly the gauge-invariant analog of the MOM scheme (see e.g. [54–56]) and thus to construct the invariant charge in a unique gauge-invariant manner. In QCD the number of the invariant-type charges of the MOM schemes is proportional to 4, namely to the number of vertexes of the Lagrangian (i.e. of the gluon-quark-antiquark, gluon-ghost-ghost, three-gluon and the four-gluon vertexes). Moreover, the definitions of these invariant-type charges depend on different kinematic conditions for fixing the scales of subtractions of UV divergences in the renormalized Green functions, which enter these different QCD invariant-type charges. Indeed, fixing the kinematics conditions by a different way it is possible to construct a number of MOM schemes, i.e. the symmetric MOM scheme [54], the variant of the symmetric MOM scheme with one external zero momentum [55] and the asymmetric MOM (AMOM) scheme [56]. Different gauge-dependent MOM schemes were used in the direct calculations of the massless two-loop [35,57–61] three-loop [35,59–61] and even four-loop [35,60,62] corrections to the QCD β function. These analytical calculations revealed the importance of the careful study of the dependence on the gauge parameter.[4] The classical example of the validity of this statement is the discovery that in the AMOM the nonproper choice of the gauge in the two-loop PT correction to the QCD β function can destroy the asymptotic freedom property of the perturbative QCD [57,58].

Summarizing the discussions of the gauge ambiguities in the QCD analogs of the invariant charges of various MOM schemes, we stress that in these schemes it is impossible to construct a gauge-invariant analog of the Gell-Man–Low function. In view of this it is important to study the expansions of the β function in terms of physical coupling constants, which enter the effective LO approximations of the RG-invariant physical quantities, e.g. the effective coupling constant of the V scheme defined by the QCD static potential [33].

In all these studies the ECH method, developed in Refs. [50–52], was used. To remind the basis of this approach consider first the system of Eqs. (2.1) and (2.2), which defines the expansion of the QCD coupling constant of the V scheme through the QCD coupling constant in the $\overline{\mathrm{MS}}$ scheme.

At the first step, following the NLO definition of the ECH scheme, we define the effective scale of the V scheme as

$$\mu_V^2 = \exp[a_1^{\overline{\mathrm{MS}}}/\beta_0]\mu^2_{\overline{\mathrm{MS}}} \qquad (3.1)$$

where $a_1^{\overline{\mathrm{MS}}} = \frac{31}{9}C_A - \frac{20}{9}T_F n_l$ and β_0 is the first coefficient of the QCD β function, defined in Eq. (2.3). At the next step

[3]We are grateful to Y. Sumino for informing us of this new unpublished result of his personal calculations.

[4]It is worth emphasizing that in the Landau gauge the two-loop expression of the QCD β function in the number of MOM schemes coincides with the MS-scheme results.

we fix $\vec{q}^2 = \mu_V^2$ in Eq. (2.2) and get the following relation between the effective charge of the V scheme and the QCD coupling constant $\alpha_{s,\overline{\rm MS}}$:

$$\alpha_{s,V}(\mu_V^2) = \alpha_{s,\overline{\rm MS}}(\mu_V^2) P(\alpha_{s,\overline{\rm MS}}, L=0) = \alpha_{s,\overline{\rm MS}}(\mu_V^2)\left[1 + a_2^{\overline{\rm MS}}\left(\frac{\alpha_{s,\overline{\rm MS}}(\mu_V^2)}{4\pi}\right)^2 + a_3^{\overline{\rm MS}}\left(\frac{\alpha_{s,\overline{\rm MS}}(\mu_V^2)}{4\pi}\right)^3 + O(\alpha_{s,\overline{\rm MS}}^4)\right]. \quad (3.2)$$

Now it is possible to define the ECH β function of the static potential, which is the RG β function in the V scheme

$$\mu_V^2 \frac{\partial(\alpha_{s,V}/4\pi)}{\partial \mu_V^2} = \beta^{\rm V}(a_{s,V}) = -\sum_{i=0}^{\infty} \beta_i^V \left(\frac{\alpha_{s,V}}{4\pi}\right)^{i+2} \quad (3.3)$$

where $a_{s,V} = \alpha_{s,V}/4\pi$. The standard RG equation relates the β^V function to the β function in the MS-like schemes:

$$\beta^V(a_{s,V}(a_{s,\overline{\rm MS}}(\mu_V^2))) = \beta^{\overline{\rm MS}}(a_{s,\overline{\rm MS}}(\mu_V^2)) \frac{da_{s,V}(a_{s,\overline{\rm MS}}(\mu_V^2))}{da_{s,\overline{\rm MS}}(\mu_V^2)}. \quad (3.4)$$

Consider now the relation between β functions, computed in the gauge-invariant UV subtraction schemes:

$$\tilde{\beta}(\tilde{a}_s(a_s)) = \beta(a_s)\frac{d\tilde{a}_s(a_s)}{da_s}, \quad (3.5)$$

where we use the similar normalization conditions for both $\beta(a_s)$ and $\tilde{\beta}(\tilde{a}_s)$ functions, namely

$$\mu^2 \frac{\partial(\tilde{\alpha}_s/4\pi)}{\partial \mu^2} = \tilde{\beta}_s(\tilde{a}_s) = -\sum_{i=0}^{\infty} \tilde{\beta}_i \left(\frac{\tilde{\alpha}_s}{4\pi}\right)^{i+2}, \quad (3.6)$$

with $\tilde{a}_s = \tilde{\alpha}_s/4\pi$. For these normalization conditions the coupling constant of one gauge-invariant renormalization scheme $\tilde{a}_s(\mu)$ is related to the coupling constant $\alpha_s(\mu)$ of another gauge-invariant renormalization scheme by the following expression:

$$\tilde{\alpha}_s(\mu^2) = \alpha_s(\mu^2)\left(1 + a_1\left(\frac{\alpha_s(\mu^2)}{4\pi}\right) + a_2\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2 + a_3\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^3 + O(\alpha_s^4)\right). \quad (3.7)$$

Taking into account Eq. (3.5), the definitions for $\tilde{\beta}(\tilde{a}_s)$ in Eq. (3.6) and the relation of Eq. (3.7), it is possible to get the following links between the coefficients of the β functions in two gauge-invariant schemes:

$$\tilde{\beta}_0 = \beta_0 \quad (3.8)$$

$$\tilde{\beta}_1 = \beta_1 \quad (3.9)$$

$$\tilde{\beta}_2 = \beta_2 - a_1\beta_1 + (a_2 - a_1^2)\beta_0 \quad (3.10)$$

$$\tilde{\beta}_3 = \beta_3 - 2a_1\beta_2 + a_1^2\beta_1 + (2a_3 - 6a_1a_2 + 4a_1^3)\beta_0. \quad (3.11)$$

These formulas reflect the transformation laws of the β function from one gauge-invariant renormalization scheme to another one.

B. The V scheme β function in QCD: Its $O(\alpha_{s,v}^6)$-approximation

Consider now the fourth-order approximation of the QCD β function in the V scheme. It is related to the QCD β function of the $\overline{\rm MS}$ scheme via Eq. (3.4). Its gauge-independent coefficients can be obtained from Eqs. (3.8)–(3.11), where

$$\beta_0^V = \beta_0^{\overline{\rm MS}} = \frac{11}{3}C_A - \frac{4}{3}T_F n_l, \quad (3.12)$$

$$\beta_1^V = \beta_1^{\overline{\rm MS}} = \frac{34}{3}C_A^2 - 4C_F T_F n_l - \frac{20}{3}C_A T_F n_l, \quad (3.13)$$

and $\tilde{\beta}_i = \beta_i^V$, $\beta_i = \beta_i^{\overline{\rm MS}}$ with $i = 2,3$ and $a_j = a_j^{\overline{\rm MS}}$ for $j = 1,2,3$. Using the concrete results for $\beta_i^{\overline{\rm MS}}$ (with $i = 0,1,2,3$) and $a_j = a_j^{\overline{\rm MS}}$ (with $j = 1,2,3$) from Eqs. (3.10) and (3.11) of Sec. II, we get the third and fourth coefficients β_2^V and β_3^V of the QCD β function in the V scheme:

$$\beta_2^V = \left(\frac{206}{3} + \frac{44\pi^2}{3} - \frac{11\pi^4}{12} + \frac{242}{9}\zeta(3)\right)C_A^3 - \left(\frac{445}{9} + \frac{16\pi^2}{3} - \frac{\pi^4}{3} + \frac{704}{9}\zeta(3)\right)C_A^2 T_F n_l + 2C_F^2 T_F n_l - \left(\frac{686}{9} - \frac{176}{3}\zeta(3)\right)C_A C_F T_F n_l + \left(\frac{2}{9} + \frac{224}{9}\zeta(3)\right)C_A T_F^2 n_l^2 + \left(\frac{184}{9} - \frac{64}{3}\zeta(3)\right)C_F T_F^2 n_l^2; \quad (3.14)$$

$$\beta_3^V = \left(-\frac{5914367}{4374}+\frac{22}{3}\cdot 502.24(1)-\frac{2728\pi^2}{9}+\frac{341\pi^4}{18}-\frac{15136}{27}\zeta(3)\right)C_A^4$$
$$+\left(\frac{4841537}{2187}-\frac{22}{3}\cdot 709.717-\frac{8}{3}\cdot 502.24(1)+\frac{2752\pi^2}{9}-\frac{172\pi^4}{9}+\frac{18184}{9}\zeta(3)\right)C_A^3T_Fn_l$$
$$+\left(-\frac{15290}{9}+\frac{1952}{3}\zeta(3)+\frac{1760}{3}\zeta(5)\right)C_A^2C_FT_Fn_l+\left(\frac{572}{9}+\frac{2288}{3}\zeta(3)-\frac{3520}{3}\zeta(5)\right)C_AC_F^2T_Fn_l$$
$$+46C_F^3T_Fn_l+\left(-\frac{740860}{729}+\frac{8}{3}\cdot 709.717-\frac{640\pi^2}{9}+\frac{3208\pi^4}{405}-\frac{5696}{9}\zeta(3)\right)C_A^2T_F^2n_l^2$$
$$+\left(-\frac{232}{9}-\frac{1024}{3}\zeta(3)+\frac{1280}{3}\zeta(5)\right)C_F^2T_F^2n_l^2+\left(\frac{9328}{9}-448\zeta(3)-\frac{640}{3}\zeta(5)\right)C_AC_FT_F^2n_l^2$$
$$+\left(\frac{9376}{81}-\frac{512\pi^4}{405}+\frac{128}{27}\zeta(3)\right)C_AT_F^3n_l^3+\left(-128+\frac{256}{3}\zeta(3)\right)C_FT_F^3n_l^3$$
$$+\left(-\frac{80}{9}+\frac{704}{3}\zeta(3)\right)\frac{d_A^{abcd}d_A^{abcd}}{N_A}+\left(\frac{512}{9}-\frac{1664}{3}\zeta(3)\right)\frac{d_F^{abcd}d_A^{abcd}}{N_A}n_l+\left(-\frac{704}{9}+\frac{512}{3}\zeta(3)\right)\frac{d_F^{abcd}d_F^{abcd}}{N_A}n_l^2$$
$$-\frac{22}{3}\cdot 56.83(1)C_A\frac{d_F^{abcd}d_F^{abcd}}{N_A}n_l-\frac{22}{3}\cdot 136.39(12)C_A\frac{d_F^{abcd}d_A^{abcd}}{N_A}$$
$$+\frac{8}{3}\cdot 56.83(1)\frac{d_F^{abcd}d_F^{abcd}}{N_A}T_Fn_l^2+\frac{8}{3}\cdot 136.39(12)\frac{d_F^{abcd}d_A^{abcd}}{N_A}T_Fn_l. \quad (3.15)$$

The property of the *scheme independence* of the coefficients β_i^V within the gauge-independent MS-like schemes is the consequence of application of the ECH approach to the static potential. Indeed, it is possible to show that these coefficients are related to the massless gauge-independent scheme invariants, introduced in the work of Ref. [63] (for the details of the derivation see e.g. Ref. [64]). The analytical expression for Eq. (3.14) was obtained in Ref. [32] and agrees with the similar one of Ref. [31] with the C_A^3-term corrected later on.

The result of Eq. (3.15) is new. Its semianalytical form is explained by the similar representation presented in Sec. II for the coefficients of Eqs. (2.16), (2.17) and of Eqs. (2.18), (2.19), obtained in the works [43,45] and by the authors of Ref. [46] respectively.

Consider now the real QCD case, based on the $SU(N_c=3)$ gauge group of color. In the fundamental representation its group structures are fixed as $C_A=3$, $C_F=4/3$, $T_F=1/2$, $N_A=8$, $d_A^{abcd}d_A^{abcd}=135$, $d_F^{abcd}d_A^{abcd}=15/2$ and $d_F^{abcd}d_F^{abcd}=5/12$. Converting now the $SU(N_c)$-group expressions presented above for the coefficients of the QCD β^V function into the form corresponding to the $SU(3)$ group, we get the well-known results for β_0 and β_1,

$$\beta_0 = 11-0.666666n_l, \quad (3.16)$$

$$\beta_1 = 102-12.66666n_l, \quad (3.17)$$

and the following numerical expressions for the third and fourth coefficients of the QCD β^V function:

$$\beta_2^V = 4224.181-746.0062n_l+20.87191n_l^2, \quad (3.18)$$

$$\beta_3^V = 43175.06(6.43)-12951.700(390)n_l + 706.9658(6)n_l^2-4.87214n_l^3. \quad (3.19)$$

The errors of the first three terms in Eq. (3.19) are defined as the mean square error $\sigma=\sqrt{\sum_{i=1}^k\sigma_i^2}$, where σ_i are the numerical errors that arise from the multiplication of the factor $2\beta_0$ by the computed errors of the corresponding $\overline{\text{MS}}$-scheme numbers for $a_3^{(1)}$ and $a_3^{(0)}$, given in Eqs. (2.16) and (2.17).

C. The guess about analytical representation of the numerical terms in the $SU(N_c)$ expression for β_3^V

It may be inspiring to make a guess on the possible analytical representations of the results of numerical calculations of the concrete terms in the $a_3^{(1)}$ and $a_3^{(0)}$ coefficients. There is the general rule that the rate of transcendentality structure is increasing with increasing order of PT calculations.

Following this general rule and considering the terms in the expressions for β_2^V and β_3^V, we claim that the numerically evaluated contributions in the expressions for the $a_3^{(1)}$ and $a_3^{(0)}$ coefficients, which enter the expressions for the

concrete terms in β_3^V, can be decomposed in terms of rational and transcendental numbers in the following way:

$$709.717 = R_1 + R_2\pi^2 + R_3\pi^4 + R_4\zeta(3) + R_5\pi^2\zeta(3) + R_6\zeta(5) \quad (3.20)$$

$$502.24(1) = R_7 + R_8\pi^2 + R_9\pi^4 + R_{10}\zeta(3) + R_{11}\pi^2\zeta(3) + R_{12}\zeta(5) \quad (3.21)$$

$$56.83(1) = R_{11} + R_{12}\pi^2 + R_{13}\pi^4 + R_{14}\zeta(3) \quad (3.22)$$

$$136.39(12) = R_{15} + R_{16}\pi^2 + R_{17}\pi^4 + R_{18}\zeta(3) \quad (3.23)$$

where R_i are still unknown rational numbers. Note that the rational number is any number that can be expressed as the ratio (p/q) of two integers with nonzero q. Thus, some of R_i coefficients in Eqs. (3.20)–(3.23) may be zero. There are indications that R_{12} and R_{16} may really be zero. It will be interesting to check this guess by analytical calculations of the corresponding complicated Feynman diagrams.

IV. THE APPLICATIONS OF THE V SCHEME IN PERTURBATIVE QCD AND THE RESULTS OBTAINED IN THE $\overline{\text{MS}}$ SCHEME AND THE MINIMAL MOM SCHEME

A. General discussions

In the last few years the interest in studying the perturbative expressions for the QCD β function in the gauge-independent and gauge-dependent schemes increased. This interest was pushed ahead by the considerations of the purity of the conformal windows related to the IR fixed points in the expressions for the β functions of the strong interaction theories, based on the concrete non-Abelian groups with fermions (see e.g. [65–67]).

There are also more phenomenologically motivated studies of the behavior of various PT QCD contributions to the RG-invariant quantities, evaluated in the different UV-subtraction schemes. The first study of the gauge dependence of the three-loop corrections to the e^+e^--annihilation R-ratio was made within the AMOM scheme in Ref. [68]. However, this work was based on the analysis of the gauge dependence of the AMOM version of the $O(\alpha_s^3)$ contribution to this quantity containing the bugs, evaluated in the $\overline{\text{MS}}$ scheme in Ref. [69]. It is worth recalling that this $\overline{\text{MS}}$-scheme result was corrected in Ref. [70] and confirmed in Ref. [71] and later on in Ref. [72]. In view of this it may be interesting to clarify the status of the gauge dependence of the available $O(\alpha_s^4)$ approximation for the e^+e^--annihilation R-ratio in the AMOM scheme using the $O(\alpha_s^4)$ corrections, evaluated recently in Refs. [73,74].

Quite recently a similar analysis was done at the three-loop level in different gauge-dependent MOM schemes, and at the four-loop order in the mMOM scheme, specified for the case of the Landau gauge [36]. This mMOM scheme was formulated in Ref. [35] and already used in the theoretical studies of the behavior of the gauge-dependent QCD β function for different numbers of fermion flavors n_f (see the works of Refs. [62,65,67]). In this section we will compare the expressions for the coefficients of the RG β function in the V scheme obtained in Sec. III with the similar mMOM-scheme results. In the next section we will use the results of Sec. III to study the third- and fourth-order approximations of the e^+e^--annihilation R-ratio in the V scheme and compare it with the results obtained in the $\overline{\text{MS}}$ scheme and in the mMOM scheme, which were presented in Ref. [36].

B. The definition of the minimal MOM scheme

Let us first briefly review how the mMOM scheme is defined. Using the standard notations for the renormalization constants of QCD in an arbitrary linear covariant gauge namely

$$\psi_0 = \sqrt{Z_\psi}\psi, \qquad A_0^{a\mu} = \sqrt{Z_{\rm A}}A^{a\mu}, \qquad c_0^a = \sqrt{Z_{\rm c}}c^a,$$
$$g_0 = Z_{\rm g}g, \qquad \lambda_0 = Z_{\rm A}Z_\lambda^{-1}\lambda \quad (4.1)$$

where ψ, A_μ^a, c^a are the quarks, gluons and ghosts fields respectively; g is the constant of the strong interaction; and λ is the gauge parameter, which is included in the Lagrangian QCD as $(\partial_\mu A_\mu^a)^2/2\lambda$. We first write down the nonrenormalized gluon propagator in the momentum space:

$$D_{ab}^{\mu\nu} = \frac{i\delta_{ab}}{p^2 + i\varepsilon}\left(-g^{\mu\nu} + (1-\lambda)\frac{p^\mu p^\nu}{p^2 + i\varepsilon}\right). \quad (4.2)$$

The form of the QCD Lagrangian dictates how to relate different renormalization constants. For example, the renormalization constant of the gluon-ghost-ghost vertex has the following form:

$$Z_{\rm ccg} = Z_{\rm g}Z_{\rm A}^{1/2}Z_{\rm c}. \quad (4.3)$$

The definition of the mMOM scheme is based on the consideration of this relation [35]. Taking into account Eq. (4.3) one can write down the expression for the QCD coupling constant of the mMOM scheme $\alpha_s^{\rm mMOM}$ as

$$\alpha_s^{\rm mMOM}(\mu^2) = \frac{Z_{\rm A}^{\rm mMOM}(\mu^2)(Z_{\rm c}^{\rm mMOM}(\mu^2))^2}{(Z_{\rm ccg}^{\rm mMOM}(\mu^2))^2}\alpha_s^0. \quad (4.4)$$

Following the proposals of Ref. [35] the renormalization expressions for the gluon and ghost propagators are defined by using the requirements that at $p^2 = \mu^2$ their residues are equal to unity, namely

$$D(p^2,\alpha_s^{\rm mMOM}(\mu^2))|_{p^2=\mu^2}=1,$$
$$G(p^2,\alpha_s^{\rm mMOM}(\mu^2))|_{p^2=\mu^2}=1. \quad (4.5)$$

Then the renormalized expression for the gluon propagator, defined in the Landau gauge $\lambda=0$, will take the following form:

$$D^{\mu\nu}_{ab}=i\delta_{ab}\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2+i\varepsilon}\right)\frac{D(p^2,\alpha_s^{\rm mMOM}(\mu^2))}{p^2+i\varepsilon} \quad (4.6)$$

while the expression for the ghost propagator is defined as

$$D^c_{ab}=i\delta_{ab}\frac{G(p^2,\alpha_s^{\rm mMOM}(\mu^2))}{p^2+i\varepsilon}. \quad (4.7)$$

The most important additional requirements of the mMOM scheme [35,62] are the special definitions of the renormalization constant of the gluon-ghost-ghost vertex and of the renormalization constant of the gauge parameter, namely

$$Z_{\rm ccg}^{\rm mMOM}(\alpha_s^{\rm mMOM})=Z_{\rm ccg}^{\overline{\rm MS}}(\alpha_s^{\overline{\rm MS}}),$$
$$Z_{\lambda}^{\rm mMOM}(\alpha_s^{\rm mMOM})=Z_{\lambda}^{\overline{\rm MS}}(\alpha_s^{\overline{\rm MS}}). \quad (4.8)$$

Taking into account the definition of the QCD coupling constant in the $\overline{\rm MS}$ scheme through the same vertex

$$\alpha_s^{\overline{\rm MS}}(\mu^2)=\frac{Z_{\rm A}^{\overline{\rm MS}}(\mu^2)(Z_{\rm c}^{\overline{\rm MS}}(\mu^2))^2}{(Z_{\rm ccg}^{\overline{\rm MS}}(\mu^2))^2}\alpha_s^0 \quad (4.9)$$

and Eqs. (4.3) and (4.8), one can get the useful relations between the renormalization constants of the mMOM and $\overline{\rm MS}$ schemes

$$Z_{\rm g}^{\rm mMOM}\sqrt{Z_{\rm A}^{\rm mMOM}}Z_{\rm c}^{\rm mMOM}=Z_{\rm g}^{\overline{\rm MS}}\sqrt{Z_{\rm A}^{\overline{\rm MS}}}Z_{\rm c}^{\overline{\rm MS}}, \quad (4.10)$$

and the relation between the renormalized QCD coupling constants of these schemes

$$\alpha_s^{\rm mMOM}(\mu^2)=\frac{Z_{\rm A}^{\rm mMOM}}{Z_{\rm A}^{\overline{\rm MS}}}\left(\frac{Z_{\rm c}^{\rm mMOM}}{Z_{\rm c}^{\overline{\rm MS}}}\right)^2\alpha_s^{\overline{\rm MS}}(\mu^2). \quad (4.11)$$

All formulas written above are valid for any linear covariant gauge and for the Landau gauge $\lambda=0$ in particular. This choice of the gauge leads to the simplification of the final perturbative results we will be interested in. Note also that the application of the Landau gauge allows us to simplify definite lattice Yang-Mills studies (see e.g. [75]).

C. Comparison of the fourth-order approximations of the QCD β function in the V, mMOM and $\overline{\rm MS}$ schemes

The analytical expressions for the three- and four-loop coefficients of the QCD β function in the mMOM scheme in the general covariant gauge were obtained in Ref. [35]. In the process of their derivation the $\overline{\rm MS}$-scheme results of Refs. [28,29], supplemented with the explicit expressions for the relation of Eq. (4.11), and with the three-loop anomalous dimension of the gauge parameter in the $\overline{\rm MS}$ scheme, evaluated in Ref. [59], were used. The results of Ref. [35] were confirmed recently in Ref. [62] by direct symbolical three- and four-loop computations. In the Landau gauge they take the following numerical form:

$$\beta_2^{{\rm mMOM},\lambda=0}=3040.482-625.3867n_l+19.38330n_l^2 \quad (4.12)$$

$$\beta_3^{{\rm mMOM},\lambda=0}=100541.05-24423.330n_l+1625.4022n_l^2-27.49263n_l^3. \quad (4.13)$$

It is interesting to compare these results with the numerical expressions of the same coefficients of the QCD β function in the gauge-invariant V scheme [see Eqs. (3.18) and (3.19)] and in the gauge-invariant $\overline{\rm MS}$ scheme, namely with

$$\beta_2^{\overline{\rm MS}}=1428.500-279.6111n_l+6.01851n_l^2, \quad (4.14)$$

$$\beta_3^{\overline{\rm MS}}=29242.96-6946.289n_l+405.0890n_l^2+1.49931n_l^3, \quad (4.15)$$

which follow from the results of analytical calculations of Refs. [26] and [28].

For completeness, in Table I we present this comparison for all numbers of quark flavors $1\le n_f\le 6$, where $n_f=n_l+1$. These notations are identical to the ones used for fixing the numbers of heavy flavors, which are considered in the PT QCD expression for the static potential V, where n_l is the number of quarks, lighter than Q_h. They enter virtual corrections among the heavy quark and antiquark of the flavor n_f and vary in the region $3\le n_l\le 5$.

The results of this table demonstrate that the asymptotic structure of the PT series for the effective β function in the V scheme has nonregular behavior and differs from the asymptotic structure of the PT for the β function in the $\overline{\rm MS}$ scheme, which was considered in Ref. [76] using the approach developed in Ref. [77]. In view of this it is of interest whether this nonregular behavior of the PT series for the β^V function will manifest itself in the process of studies of scheme dependence of high-order coefficients for the characteristics of typical physical QCD processes, e.g.

TABLE I. The comparison of the numerical values of the third and fourth coefficients of the QCD β function in the V, $\overline{\rm MS}$ and mMOM schemes in the Landau gauge.

The numerical coefficients of the QCD β function in different schemes						
n_f	β_2^V	β_3^V	$\beta_2^{\overline{\rm MS}}$	$\beta_3^{\overline{\rm MS}}$	$\beta_2^{\rm mMOM,\lambda=0}$	$\beta_3^{\rm mMOM,\lambda=0}$
1	3499.047	30925.46 ± 6.44	1154.907	22 703.26	2434.478	77 715.63
2	2815.656	20060.55 ± 6.48	893.351	16 982.73	1867.242	57 976.06
3	2174.010	10551.11 ± 6.54	643.833	12 090.37	1338.771	41 157.38
4	1574.107	2367.90 ± 6.62	406.351	8035.18	849.068	27 094.64
5	1015.948	-4518.30 ± 6.72	180.907	4826.15	398.131	15 622.88
6	499.533	-10136.74 ± 6.84	−32.500	2472.28	−14.038	6 577.14

for the e^+e^--annihilation R-ratio in the region of direct production of the pair of heavy quarks and antiquarks with $n_f = 4, 5$ numbers of flavors. We will not consider in this work the case of $n_f = 6$, related to the direct production of the pair of $t\bar{t}$-quarks in the process $e^+e^- \to$ hadrons, which may be studied in the future if the ILC will be built. Indeed, the total cross section of this process is dominated by the subprocess $e^+e^- \to Z^0 \to$ hadrons and not by the subprocess $e^+e^- \to \gamma \to$ hadrons that interests us in this work.

D. The fourth-order approximation for the e^+e^- R-ratio in the $\overline{\rm MS}$ and V schemes

We now discuss the fourth-order PT expression for the e^+e^--annihilation R-ratio in the V scheme. The idea to study this particular expression, as well as the PT expressions for other observable physical quantities in the V scheme, was proposed some time ago in Ref. [78]. In this section we will realize this proposal, obtain the fourth-order V-scheme PT approximation for the e^+e^--annihilation ratio $R(s)$ and compare its coefficients and energy dependence with the results obtained in the $\overline{\rm MS}$ scheme and in the Landau-gauge variant of the mMOM scheme [36]. The studies to be made in this subsection supplement the ones presented above. Moreover, the results obtained in Sec. IV C will be used in the process of the numerical calculations to be presented below.

We remind the reader that the e^+e^--annihilation R-ratio is defined as

$$R(s) = \frac{\sigma(\mathrm{e^+e^-} \to \gamma \to \text{hadrons})}{\sigma_0(\mathrm{e^+e^-} \to \gamma \to \mu^+\mu^-)} = 12\pi \mathrm{Im}\Pi(s + i\varepsilon) \tag{4.16}$$

where s is the transferred energy in the Minkowskian region, $\sigma_0(\mathrm{e^+e^-} \to \gamma \to \mu^+\mu^-) = 4\pi^2\alpha/(3s)$ is the theoretical normalization factor, $\Pi(q^2)$ is the QCD expression for the photon vacuum polarization function

$$\Pi_{\mu\nu}(q^2) = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2) = i\int d^4x e^{iqx}\langle 0|Tj_\mu(x)j_\nu(0)|0\rangle \tag{4.17}$$

and $j_\mu = \sum_f Q_f \bar{\psi}_f \gamma_\mu \psi_f$ is the electromagnetic hadronic current. Since the e^+e^--annihilation R-ratio is the RG-invariant quantity, it obeys the RG equation without an anomalous dimension term, namely

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + \beta(a_s)\frac{\partial}{\partial a_s}\right)R(s) = 0. \tag{4.18}$$

In the $\overline{\rm MS}$ scheme the $O(\alpha_s^4)$ approximation for the e^+e^- R-ratio has the following form:

$$R^{\overline{\rm MS}} = 3\sum_f Q_f^2\left(1 + 4\frac{\alpha_s^{\overline{\rm MS}}}{4\pi} + r_1^{\overline{\rm MS}}\left(\frac{\alpha_s^{\overline{\rm MS}}}{4\pi}\right)^2 + r_2^{\overline{\rm MS}}\left(\frac{\alpha_s^{\overline{\rm MS}}}{4\pi}\right)^3 + r_3^{\overline{\rm MS}}\left(\frac{\alpha_s^{\overline{\rm MS}}}{4\pi}\right)^4\right) \tag{4.19}$$

where the coefficient $r_1^{\overline{\rm MS}}$ was evaluated analytically in Ref. [79] and numerically in Ref. [80] and confirmed analytically in Ref. [81]. The coefficient $r_2^{\overline{\rm MS}}$ was analytically evaluated in Ref. [70] and confirmed in Refs. [71] and [72], while the symbolical expression for the *non-singlet* and *singlet* contributions to $r_3^{\overline{\rm MS}}$ were obtained analytically only recently in Refs. [73] and [74] respectively. The coefficients $r_1^{\overline{\rm MS}}$, $r_2^{\overline{\rm MS}}$ and $r_3^{\overline{\rm MS}}$ can be expressed in the numerical form as

$$r_1^{\overline{\rm MS}} = -1.84472n_f + 31.7713, \tag{4.20}$$

$$r_2^{\overline{\rm MS}} = -0.33139n_f^2 - 76.8085n_f - 424.763 - 26.4435\delta_f, \tag{4.21}$$

$$r_3^{\overline{\rm MS}} = 5.50812n_f^3 - 204.1431n_f^2 + 4806.339n_f - 40091.67 + (49.0568n_f - 1521.214)\delta_f, \tag{4.22}$$

where the terms, proportional to $\delta_f = (\sum_f Q_f)^2/(\sum_f Q_f^2)$, are the *singlet* contributions.

In the V scheme the PT expression for the e^+e^- R-ratio is defined as

TABLE II. The comparison of the numerical values of the known coefficients for the e^+e^--annihilation R-ratio in the V and $\overline{\rm MS}$ schemes and in the Landau-gauge version of the mMOM scheme.

The numerical coefficients of the R-ratio in different schemes									
n_f	$r_1^{\rm V}$	$r_1^{\overline{\rm MS}}$	$r_1^{\rm mMOM}$	$r_2^{\rm V}$	$r_2^{\overline{\rm MS}}$	$r_2^{\rm mMOM}$	$r_3^{\rm V}$	$r_3^{\overline{\rm MS}}$	$r_3^{\rm mMOM}$
1	−6.9622	29.9265	−21.9622	−1966.444	−528.346	−1575.567	-17815.06 ± 1.17	−36956.12	−13190.55
2	−4.3625	28.0818	−19.3625	−1830.134	−584.994	−1467.688	-14188.58 ± 1.17	−31536.11	−8632.68
3	−1.7628	26.2371	−16.7628	−1708.676	−658.171	−1374.660	-11244.00 ± 1.17	−27361.22	−4748.58
4	0.8368	24.3924	−14.1631	−1602.069	−747.876	−1296.483	-9033.31 ± 1.17	−24310.08	−1590.24
5	3.4366	22.5477	−11.5634	−1475.696	−819.494	−1198.540	-6434.41 ± 1.17	−20591.03	1575.00
6	6.0363	20.7029	−8.9637	−1369.944	−913.410	−1121.218	-4775.30 ± 1.17	−18149.16	3873.49

$$R^{\rm V} = 3\sum_f Q_f^2\left(1 + 4\frac{\alpha_{s,\rm V}}{4\pi} + r_1^{\rm V}\left(\frac{\alpha_{s,\rm V}}{4\pi}\right)^2 + r_2^{\rm V}\left(\frac{\alpha_{s,\rm V}}{4\pi}\right)^3 + r_3^{\rm V}\left(\frac{\alpha_{s,\rm V}}{4\pi}\right)^4\right). \tag{4.23}$$

Using the ECH approach of Ref. [51] and the V-scheme relations of Eqs. (3.1), (3.10) and (3.11) we obtain the following general expressions for r_i^V:

$$r_1^{\rm V} = r_1^{\overline{\rm MS}} - 4a_1^{\overline{\rm MS}}, \tag{4.24}$$

$$r_2^{\rm V} = r_2^{\overline{\rm MS}} - 4a_2^{\overline{\rm MS}} - 2a_1^{\overline{\rm MS}} r_1^{\rm V}, \tag{4.25}$$

$$r_3^{\rm V} = r_3^{\overline{\rm MS}} - 4a_3^{\overline{\rm MS}} - 3a_1^{\overline{\rm MS}} r_2^{\rm V} - (2a_2^{\overline{\rm MS}} + (a_1^{\overline{\rm MS}})^2) r_1^{\rm V} \tag{4.26}$$

and the numerical values of these coefficients, namely

$$r_1^{\rm V} = 2.59972 n_f - 9.5620, \tag{4.27}$$

$$r_2^{\rm V} = 0.50749 n_f^2 + 113.6320 n_f - 2054.140 - 26.4435\delta_f, \tag{4.28}$$

$$r_3^{\rm V} = 3.05815 n_f^3 - 144.9455 n_f^2 + 3455.279(2) n_f - 20387.90(1.17) - (39.0881 n_f + 701.466)\delta_f \tag{4.29}$$

The errors in the values of the n_f- and n_f^0-terms in Eq. (4.29) arise from the numerical errors in the values of the n_l- and n_l^0-dependent constituents $a_3^{(1)}$ and $a_3^{(0)}$ of the coefficient $a_3^{\overline{\rm MS}}$ defined in Eq. (2.16) and Ref. (2.17), which enter into the definition of r_3^V through Eq. (4.26).

E. The comparison of the fourth-order V-, $\overline{\rm MS}$- and mMOM-scheme approximations for the e^+e^- R-ratio

As the start of the study of the scheme and energy dependence of the e^+e^--annihilation R-ratio in different orders of PT in the case of applications of three different schemes we first present in Table II the comparison of the following from Eqs. (4.27)–(4.29) and Eqs. (4.20)–(4.21) numerical expressions for three PT coefficients in the V and $\overline{\rm MS}$ scheme with the numerical expressions of the same coefficients, obtained in the Landau-gauge version of the mMOM scheme in Ref. [36].

Note that the values of the coefficients $r_i^{\overline{\rm MS}}$, r_i^V and $r_i^{\rm mMOM,\lambda=0}$ with i = 2, 3 are *negative* for any number of n_f, apart from the case of the $r_3^{\rm mMOM,\lambda=0}$ value at $n_f = 5, 6$. In the $\overline{\rm MS}$ scheme this feature is related to the manifestation in the expressions for $r_2^{\overline{\rm MS}}$ and $r_3^{\overline{\rm MS}}$ of the effects proportional to π^2, which arise from analytical continuation to the Minkowskian region of energies of the PT contributions in the rhs of Eq. (4.16) (for a detailed explanation see e.g. Ref. [64]). The negative values of the V-scheme coefficients are also related to these kinematic π^2 effects, but the numerical difference with the negative values of $r_2^{\overline{\rm MS}}$- and $r_3^{\overline{\rm MS}}$-terms is related to the numerical values of the additions contributions to the r_2^V and r_3^V-terms. Note that in the case of r_2^V they are negative [see Eq. (4.25)] but in the case of r_3^V they are positive due to interplay among the third huge positive contribution to Eq. (4.26) and other negative contributions to the same equation. Note also that the values of r_2^V are very closed to $r_2^{\rm mMOM}$, but this feature does not remain at the fourth order of PT.

We now plot the energy and scheme dependence of the next-to-leading order, next-to-next-to-leading order (NNLO) and next-to-next-to-next-to-leading order ($\rm N^3LO$) approximations for the function $r(s) = R(s)/(3\sum_f Q_f^2) - 1$. It depends on $s = q^2$, where s is measured in $\rm GeV^2$. The first three plots are presented in Fig. 1 for the energy region above the threshold of charmonium production and below the threshold of the bottomonium production, i.e. in the region where $n_f = 4$ numbers of active flavors are contributing to the expression for $r(s)$. In Fig. 2 the scheme dependencies of the NLO, NNLO and $\rm N^3LO$ approximations of the same function are presented in the region with $n_f = 5$ numbers of active flavors. More definitely, we consider the energy region above the threshold of bottomonium production and up to the energies

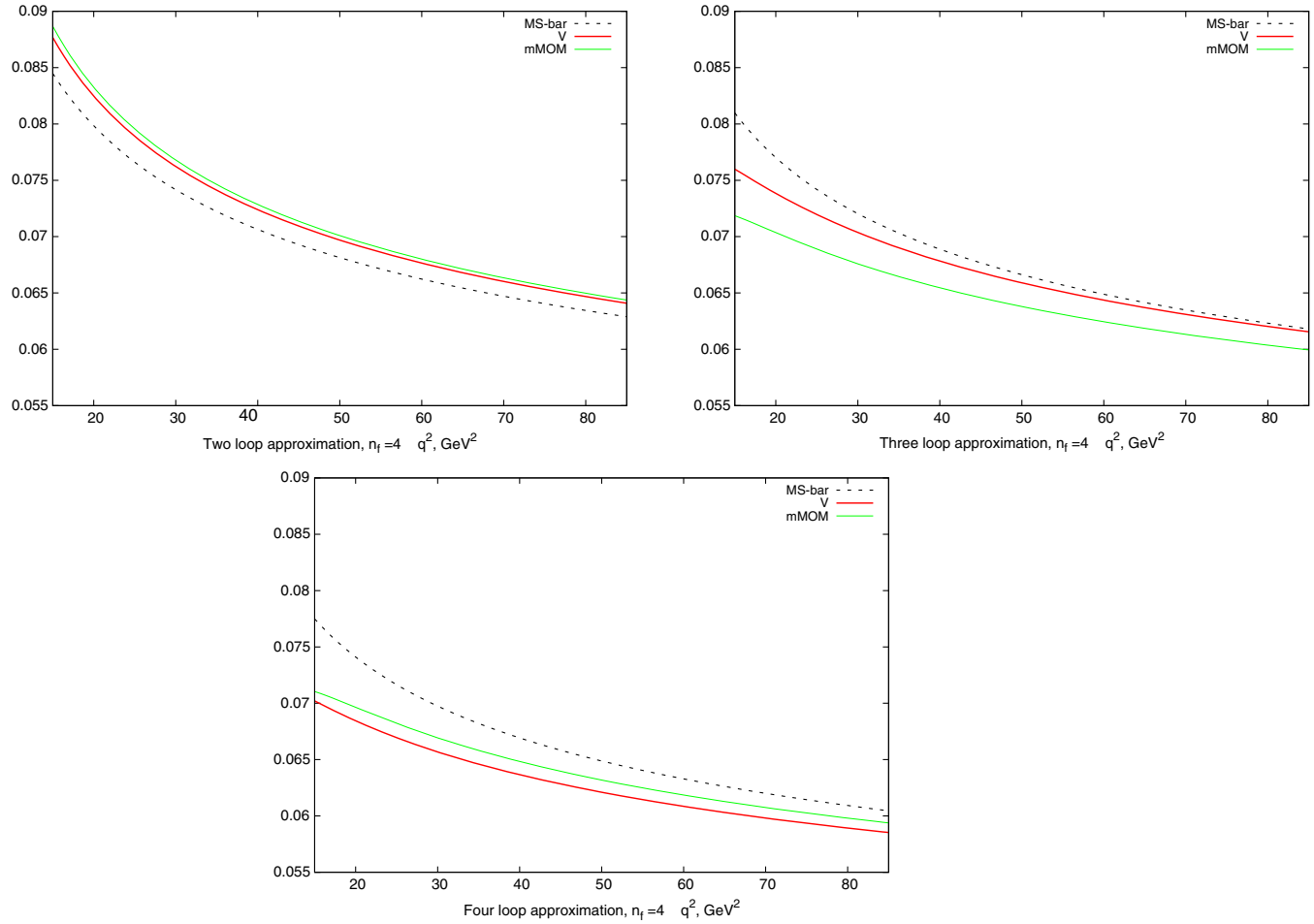

FIG. 1 (color online). Scheme dependence of the NLO (left), NNLO (right) and N^{3}LO (bottom) approximations for the e^+e^- characteristic $r(q^2) = R(q^2)/(3\sum_f Q_f^2) - 1$ in the case of $n_f = 4$ numbers of active flavors. The dashed black curve depicts the variations of the $\overline{\rm MS}$ approximants. The solid green line demonstrates the variations of the mMOM-scheme results, while the solid red line shows the V-scheme results.

$s = 900$ GeV^2, where the subprocess $e^+e^- \to Z^0 \to$ hadrons, which starts to dominate near the beginning of the left shoulder of the direct manifestation of the Z^0-boson in the e^+e^--collisions, can be safely neglected.

The energy dependence of coupling constant $a_s = \alpha_s^{\overline{\rm MS}}/(4\pi)$ of the NLO, NNLO approximations of the PT expansions of the e^+e^--annihilation ratio $R(s)$ in the $\overline{\rm MS}$ scheme, which is presented in Eq. (4.19), is defined through the powers of logarithmic terms $\mathrm{L} = \ln(\mathrm{s}/\Lambda^{(\mathrm{n_f})2}_{\overline{\rm MS}})$ as

$$a_s^{\rm NLO} = \frac{1}{\beta_0 \mathrm{L}} - \frac{\beta_1 \ln(\mathrm{L})}{\beta_0^3 \mathrm{L}^2} \tag{4.30}$$

$$a_s^{\rm NNLO} = a_s^{\rm NLO} + \Delta a_s^{\rm NNLO} \tag{4.31}$$

where

$$\Delta a_s^{\rm NNLO} = \frac{1}{\beta_0^5 \mathrm{L}^3}[\beta_1^2 \ln^2(\mathrm{L}) - \beta_1^2 \ln(\mathrm{L}) + \beta_2\beta_0 - \beta_1^2]. \tag{4.32}$$

At the fourth N^{3}LO, first studied in Ref. [82], one has

$$a_s^{\rm N^3LO} = a_s^{\rm NNLO} + \Delta a_s^{\rm N^3LO} \tag{4.33}$$

where the additional correction reads

$$\Delta a_s^{\rm N^3LO} = \frac{1}{\beta_0^7 \mathrm{L}^4}\left[\beta_1^3\left(-\ln^3(\mathrm{L}) + \frac{5}{2}\ln^2(\mathrm{L}) + 2\ln(\mathrm{L}) - \frac{1}{2}\right) - 3\beta_0\beta_1\beta_2 \ln(\mathrm{L}) + \beta_0^2\frac{\beta_3}{2}\right]. \tag{4.34}$$

In the numerical form the expressions for the $\overline{\rm MS}$ β-function coefficients β_i in Eqs. (4.30)–(4.34) are defined in

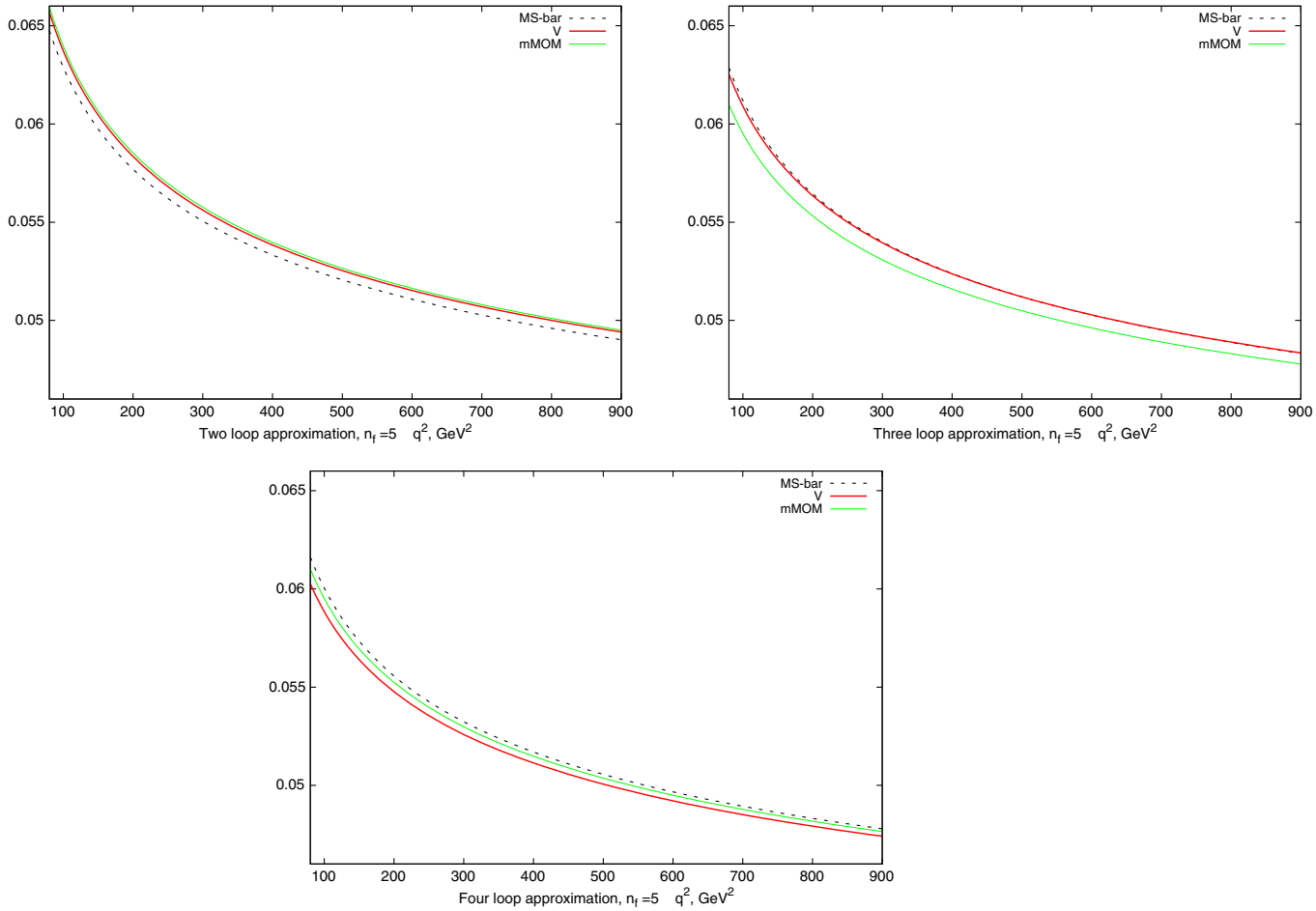

FIG. 2 (color online). Scheme dependence of the NLO (left), NNLO (right) and N^{3}LO (bottom) approximations to $r(q^2)$ are presented for $n_f = 5$ numbers of active flavors. The variation of the $\overline{\text{MS}}$-, mMOM- and V-scheme results is indicated by the three curves as in Fig. 1.

Eqs. (3.16), (3.17) and Eqs. (4.14), (4.15) respectively. For the concrete numbers of n_f flavors their values are given in Table I. Note that in the analysis of Ref. [36] the same expansion was used for the numbers of active flavors $n_f = 5$ and $n_f = 6$ and for the value of $\Lambda_{\overline{\text{MS}}} = 500$ MeV, which did not vary from order to order of the $\overline{\text{MS}}$-scheme perturbative expressions considered in Ref. [36]. In the process of obtaining our results, presented in Figs. 1 and 2, and keeping in mind physical motivations discussed above, we used $n_f = 4$ and $n_f = 5$.

Contrary to the studies of Ref. [36] the values of the parameters $\Lambda^{(n_f)}_{\overline{\text{MS}}}$, $\Lambda^{(n_f)}_{\text{mMOM}}$ and parameter $\Lambda^{(n_f)}_{V}$ (that is new to this work) were not fixed, but depend on the choice of both n_f and the order of approximations. The concrete results for the values of the parameters used are presented in Table III.

In the cases of $n_f = 4$ numbers of active flavors and $\nu = 2, 3, 4$ the values for $\Lambda^{(n_f=4)}_{\overline{\text{MS}}}$ given in Table III are fixed from the results of the fits of the Fermilab Tevatron experimental data for the xF_3 structure function of the neutrino-nucleon deep-inelastic scattering process at the N$^{(\nu-1)}$LO of the theoretical PT results, performed in Ref. [83]. In the case of $n_f = 5$ the values of $\Lambda^{(n_f=5)}_{\overline{\text{MS}}}$ at $\nu = 2, 3, 4$ were obtained in Ref. [84] from the related results for $\Lambda^{(n_f=4)}_{\overline{\text{MS}}}$ using the NLO, NNLO and N^{3}LO matching conditions, evaluated at the NNLO in

TABLE III. The dependence of the parameters, used for getting the results of Figs. 1 and 2 from the n_f, ν (order of approximation), and from the choice of the scheme.

The numerical values of the Λ_{QCD} in different schemes, MeV				
n_f	The order of approximation ν	$\Lambda^{(n_f)}_{\overline{\text{MS}}}$	$\Lambda^{(n_f)}_{V}$	$\Lambda^{(n_f)}_{\text{mMOM}}$
4	2	350	500	625
4	3	335	475	600
4	4	330	470	590
5	2	250	340	435
5	3	245	335	430
5	4	240	330	420

Refs. [85] and [86] and at the N^{3}LO in Ref. [82]. The matching point in these conditions was fixed by the on-shell b-quark mass values, extracted at different orders of PT from the analysis of the heavy quarkonium spectrum while taking into account the Padé estimated value of the coefficient a_3 from Eq. (2.13), obtained in Ref. [87]. These Padé estimates turned out to be in satisfactory agreement with the results of direct calculations of the value of a_3 obtained later (see Refs. [43,45,46]). In view of the reliability of the results of Ref. [88] we may safely use the values for $\Lambda^{(n_f=5)}_{\overline{\rm MS}}$ from Table III for transforming them to the values of the scale parameters $\Lambda^{(n_f=5)}_{\rm mMOM}$ and $\Lambda^{(n_f=5)}_{V}$ *in particular.*

In general the scale parameters $\Lambda^{(n_f)}$ of the $\overline{\rm MS}$, V and mMOM schemes considered in Table III are related by the following equations:

$$\Lambda^{(n_f)2}_{\rm V} = \Lambda^{(n_f)2}_{\overline{\rm MS}} \exp[a_1^{\overline{\rm MS}}(\mathrm{n_f})/\beta_0(\mathrm{n_f})],$$
$$\Lambda^{(\mathrm{n_f})2}_{\rm mMOM} = \Lambda^{(\mathrm{n_f})2}_{\overline{\rm MS}} \exp[(\mathrm{r}_1^{\overline{\rm MS}}(\mathrm{n_f}) - \mathrm{r}_1^{\rm mMOM}(\mathrm{n_f}))/4\beta_0(\mathrm{n_f})]. \tag{4.35}$$

They are derived by means of the ECH approach. We used these expressions to get in Table III the numerical values of $\Lambda^{(n_f)}_{V}$ and $\Lambda^{(n_f)}_{\rm mMOM}$ from the results described above for $\Lambda^{(n_f)}_{\overline{\rm MS}}$. Combining them with the numerical values for the coefficients $\beta_2^V(n_f)$, $\beta_3^V(n_f)$ and $\beta_2^{\rm mMOM,\lambda=0}(n_f)$, $\beta_3^{\rm mMOM,\lambda=0}(n_f)$ in the analogs of Eqs. (4.30), (4.32) and Eq. (4.34), and taking into account the expressions for the coefficients r_i in $r(q^2) = R(q^2)/(3\sum_f Q_f^2) - 1$ in three different schemes, we plot in Figs. 1 and 2 the energy dependence of $r(q^2)$ in three different orders of PT and three different schemes, namely $\overline{\rm MS}$, V and mMOM schemes in the cases of $n_f = 4$ and $n_f = 5$ respectively.

F. Discussions of the results

Considering now the plots of Figs. 1 and 2 we may conclude that in all cases the PT approximants for the function $r(s)$ related to the e^+e^--annihilation R-ratio are converging in all schemes. In the $\overline{\rm MS}$ scheme the rate of convergence of the related PT approximants is better than in the V scheme and mMOM scheme. At the NLO the results of the V scheme are closer to the mMOM ones than to the results obtained in the $\overline{\rm MS}$ scheme, while at the NNLO the situation is reversed—the V-scheme approximations are closer to the $\overline{\rm MS}$ ones, while the application of the mMOM scheme puts a lower bound on the theoretical expression for $r(s)$. However, at the N^{3}LO the lower theoretical bound on the energy dependence of $r(s)$ is changed again and the lower bound is now obtained within the V scheme. The comparison of three approximants for $r(s)$ in the case of consideration of the V-scheme results supports the conclusion, made in Sec. IV C, that the PT approximants in the V scheme have less regular behavior than the $\overline{\rm MS}$ ones. The results of Table II demonstrate the positive feature of taking into account $O(\alpha_s^4)$-corrections to e^+e^--annihilation R-ratio in all three schemes. Indeed, the scheme dependence of the expression for the e^+e^- ratio is drastically decreased at this level. This is the positive message, which supports the work presented above on the inclusion of the $O(\alpha_s^4)$ correction in the theoretical approximations in the $\overline{\rm MS}$, mMOM and V schemes.

V. THE FOUR-LOOP QED RESULT FOR THE RG β FUNCTION IN THE V SCHEME

Consider now the case of QED with N types of identically charged leptons. We will use the results of Sec. III B for the fourth-order PT approximation of the RG V-scheme β-function of the $SU(N_c)$ colour gauge group theory. Fixing the $SU(N_c)$-group weights in Eqs. (3.12), (3.13), (3.14) and (3.15) as $C_A = 0$, $C_F = 1$, $T_F = 1$, $d_A^{abcd} = 0$, $d_F^{abcd} = 1$, $N_A = 1$ and $n_f = N$, we obtain the following four-loop semianalytical expression for the RG β function in QED in the V scheme:

$$\beta^{\rm V}_{\rm QED}(a_V) = \frac{4}{3}Na_{\rm V}^2 + 4Na_{\rm V}^3 + \left(-2N + \left(\frac{64}{3}\zeta(3) - \frac{184}{9}\right)N^2\right)a_{\rm V}^4 + \left(-46N + \left(104 + \frac{512}{3}\zeta(3) - \frac{1280}{3}\zeta(5) - \frac{8}{3}\cdot 56.83(1)\right)N^2 + \left(128 - \frac{256}{3}\zeta(3)\right)N^3\right)a_{\rm V}^5 + O(a_{\rm V}^6) \tag{5.1}$$

where $a_V = \alpha_V/4\pi$ and N is the number of leptons. Comparing this result with the four-loop approximation of the QED β function in the MOM scheme, i.e. of the Gell-Man–Low Ψ function, namely with

$$\Psi(a_{\rm MOM}) = \frac{4}{3}Na_{\rm MOM}^2 + 4Na_{\rm MOM}^3 + \left(-2N + \left(\frac{64}{3}\zeta(3) - \frac{184}{9}\right)N^2\right)a_{\rm MOM}^4 + \left(-46N + \left(104 + \frac{512}{3}\zeta(3) - \frac{1280}{3}\zeta(5)\right)N^2 + \left(128 - \frac{256}{3}\zeta(3)\right)N^3\right)a_{\rm MOM}^5 + O(a_{\rm MOM}^6) \quad (5.2)$$

where $a_{\rm MOM} = \alpha_{\rm MOM}/4\pi$. We conclude that in spite of identical agreement at the third order of PT[5], the general expressions for the RG QED β-function in these two different schemes are not the same. They start to differ from the fourth order of PT due to contributing to the $0(a_{\rm V}^5)$ coefficient of the $\beta^{\rm V}$-function of the additional light-by-light-type scattering diagrams, which appear in the QED analog of the coefficient $a_3^{(1)}$ in the $\bar{\rm MS}$ scheme, given in Eq. (2.16). They enter in the definition of the N^2-term of the $\beta_3^{\rm V}$ coefficient of the V-scheme QED β-function through Eq. (3.11).

It is possible to clarify what kind of N-dependent high-order coefficients of the following expression of the QED β function in the V scheme

$$\beta^{\rm V}(a_{\rm V}) = \sum_{i=0}^{\infty}\beta_i^{\rm V}\left(\frac{\alpha_{\rm V}}{4\pi}\right)^{i+2} = \beta_0^{\rm V[1]}N\left(\frac{\alpha_{\rm V}}{4\pi}\right)^2 + \sum_{i=1}^{\infty}\sum_{l=1}^{i}\beta_i^{\rm V[l]}N^l\left(\frac{\alpha_{\rm V}}{4\pi}\right)^{i+2} \quad (5.3)$$

will also receive additional contributions and what kind of N-dependent coefficients of the QED β^V function will coincide with the similar expressions for the Ψ function, which we will define as

$$\Psi(a_{\rm MOM}) = \Psi_0^{[1]}N\left(\frac{\alpha_{\rm MOM}}{4\pi}\right)^2 + \sum_{i=1}^{\infty}\sum_{l=1}^{i}\Psi_i^{[l]}N^l\left(\frac{\alpha_{\rm MOM}}{4\pi}\right)^{i+2}. \quad (5.4)$$

Using the analogs of Eqs. (3.10) and (3.11), which can be derived using the considerations of Ref. [64], we arrive at the following relations:

$$\beta_i^{\rm V[l]} = \Psi_i^{[l]} + \Delta\beta_i^{\rm V[l]} \quad (5.5)$$

where extra terms $\Delta\beta_i^{V[l]}$ in the N-dependent contributions to the coefficients of the QED β^V function appear in the following region of indexes: $[i,l] = [i \geq 3, 2 \leq l \leq i-1]$.

In the cases of $[i,l] = [i \geq 3, l = 1 \text{ or } i]$ the proportional to $N^{[l]}$ coefficients of the β^V- and Ψ functions, defined in Eqs. (5.3) and (5.4), are the same. In the case of $i = 3$, which corresponds to the totally known for the moment fourth order results, these identical coefficients are proportional to N and N^3. At the third order the proportional to N-term was analytically evaluated in Ref. [90]. At the fourth order of PT the proportional to N and N^3 terms were evaluated in Ref. [8]. For $i = 3$ the terms under discussion can be obtained from the results of Ref. [9] and read

$$\beta_4^{\rm V[1]} = \Psi_4^{[1]} = \frac{4157}{6} + 128\zeta(3) \quad (5.6)$$

$$\beta_4^{\rm V[4]} = \Psi_4^{[4]} = -\frac{8756}{9} + \frac{3584}{9}\zeta(3) + \frac{5120}{9}\zeta(5). \quad (5.7)$$

Note that this result from Ref. [9] is in agreement with the multiloop expression for this particular contribution to the Gell-Man–Low function, evaluated in Ref. [91] up to 20 loops analytically and numerically up to 100 loops. The scheme independence of the linear-in-N-contribution to Eqs. (5.3) and (5.4) is the consequence of the *conformal symmetry* property, which is valid in QED in the perturbative quenched approximation (for the recent detailed study see Ref. [92]).

In the numerical form the scheme-dependent coefficients of the $\beta_{\rm QED}^V$ function read

$$\beta_2^{\rm V} = -2N + 5.19943N^2 \quad (5.8)$$

$$\beta_3^{\rm V} = -46N + 284.818(26)N^2 - 25.42447N^3. \quad (5.9)$$

The analogous expressions for the three- and four-loop coefficients of the QED β function in the $\overline{\rm MS}$ scheme follow from the analytical results of Ref. [8] and have the following form:

$$\beta_2^{\overline{\rm MS}} = -2N + 4.88888N^2 \quad (5.10)$$

$$\beta_3^{\overline{\rm MS}} = -46N + 82.9753N^2 + 5.06995N^3. \quad (5.11)$$

The numerical expressions for the analogous coefficients of the Ψ function (or the QED β function in the MOM scheme), which we obtain from the same work of Ref. [8], are

$$\Psi_2 = -2N + 5.19943N^2 \quad (5.12)$$

$$\Psi_3 = -46N + 133.2714N^2 - 25.42447N^3. \quad (5.13)$$

Note once more that the first three coefficients of the β^V function and of the Ψ function are the same and start to differ from the fourth order of PT in the following way:

[5]This observation was made and used in the unpublished work of A. L. Kataev and A. V. Garkusha; see Ref. [89] as well.

$$\beta_3^{\rm V} = \Psi_3 - 151.54(2)N^2. \quad (5.14)$$

This additional contribution arises from the light-by-light-type scattering contribution, which is typical of the V scheme.

For completeness we present the QED expressions for the $O(\alpha^5)$ approximations for the Ψ and $\beta^V_{\rm QED}$ functions in the case of N=1:

$$\Psi(a_{\rm MOM}) = 1.3333a^2_{\rm MOM} + 4a^3_{\rm MOM} + 3.1994a^4_{\rm MOM} - 153.8469a^5_{\rm MOM} + O(a^6_{\rm MOM}) \quad (5.15)$$

$$\beta^V_{\rm QED}(a_{\rm v}) = 1.3333a^2_{\rm V} + 4a^3_{\rm V} + 3.1994a^4_{\rm V} - 305.3936(266)a^5_{\rm V} + O(a^6_{\rm V}). \quad (5.16)$$

One can observe that even for $\mathrm{N} = 1$ the numerical effect of the light-light-scattering contribution, which is typical for the V scheme [see Eq. (5.14)], is rather sizable and almost equals the whole value of the other term in the expression of Eq. (5.14).

VI. CONCLUSIONS

In this work we consider the definition of the gauge-independent RG QCD β function in the V scheme. Using higher-order corrections to the static potential of the quark-antiquark interaction and β function in the $\overline{\rm MS}$ scheme, we compute the fourth term of the PT expression for the β function in the V scheme in the general case of the $SU(N_c)$ group in the semianalytical term. Our guess of possible expressions of the corresponding numerical contributions through concrete transcendental numbers is made. The comparison of the numerical expressions of the scheme-dependent coefficients of the β^V function of QCD with the similar coefficients of the QCD β function in the $\overline{\rm MS}$ and mMOM scheme in the Landau gauge are presented. The indication that the structure of the PT series for the effective β function in the V scheme has nonregular asymptotic behavior and differs from the asymptotic PT for the β function in the $\overline{\rm MS}$ scheme are presented. The results obtained in the V scheme are used to study the scheme dependence of the $O(\alpha_s^4)$ approximation for the e^+e^--annihilation R-ratio in the energy region above the thresholds of production of the charmonium states. The conclusion is made that the comparison between the fourth-order expressions for the e^+e^--annihilation R-ratio, obtained in the $\overline{\rm MS}$ schemes, in the Landau-gauge variant of the mMOM scheme and in the gauge-independent V scheme leads to a drastic decrease of the scheme dependence of the fourth-order perturbative QCD predictions for the case of $n_f = 5$ numbers of active flavors in particular. Considering the QED limit of the $SU(N_c)$-group β^V function we observe that its perturbative expression is starting to differ from the perturbative expression for the Gell-Mann–Low Ψ function from the level of the $O(\alpha^6_{\rm V})$-corrections. The relations between coefficients of the QED β^V function and the Ψ function are presented in all orders of PT in the case of the N-types of identical leptons. The conclusion that starting from the fourth-order perturbative approximation two N-dependent terms in the coefficients of the perturbative expansions of the β^V and Ψ functions will always coincide is made. Theoretical reasons of this foundations are presented.

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