Color-kinematics duality for pure Yang-Mills and gravity at one and two loops

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(Received 3 April 2013; published 31 August 2015)

We provide evidence in favor of the conjectured duality between color and kinematics for the case of nonsupersymmetric pure Yang-Mills amplitudes by constructing a form of the one-loop four-point amplitude of this theory that makes the duality manifest. Our construction is valid in any dimension. We also describe a duality-satisfying representation for the two-loop four-point amplitude with identical four-dimensional external helicities. We use these results to obtain corresponding gravity integrands for a theory containing a graviton, dilaton, and antisymmetric tensor, simply by replacing color factors with specified diagram numerators. Using this, we give explicit forms of ultraviolet divergences at one loop in four, six, and eight dimensions, and at two loops in four dimensions.

DOI: 10.1103/PhysRevD.92.045041 PACS numbers: 11.15.-q, 12.38.Bx, 04.65.+e

I. INTRODUCTION

Recent years have seen remarkable progress in computing and understanding scattering processes in gauge and gravity theories, both for phenomenological and theoretical applications. (For various reviews see Refs. [1,2].) In particular, various new structures have been uncovered in the amplitudes of these theories (see, for example, Ref. [3]). One such structure is the duality between color and kinematics [4,5]. This Bern-Carrasco-Johansson (BCJ) duality is conjectured to hold at all loop orders in Yang-Mills theory and its supersymmetric counterparts. Besides imposing strong constraints on gauge-theory amplitudes, whenever a form of a gauge-theory loop integrand is obtained where the duality is manifest, we obtain corresponding gravity integrands simply by replacing color factors by specified gauge-theory kinematic numerator factors.

The duality between color and kinematics has been confirmed in numerous tree-level studies [6–11], including the construction of explicit representations for an arbitrary number of external legs [12]. At loop level, the duality remains a conjecture, but there is already significant nontrivial evidence in its favor for supersymmetric theories [5,13–17] and for special helicity configurations in nonsupersymmetric pure Yang-Mills theory [5,18]. Here we provide further evidence in favor of the duality at loop level, explicitly showing that it holds for pure Yang-Mills one-loop four-point amplitudes for all polarization states in D dimensions. We also present a duality-satisfying representation of the two-loop four-point identical-helicity amplitude of pure Yang-Mills. This amplitude in a non-duality-satisfying representation was first given in Ref. [19], while Ref. [5] noted the existence of a duality-satisfying form. Here we explicitly give the full duality-satisfying form, including contributions from diagrams absent from Ref. [19] that vanish under integration but are necessary to make the duality manifest.

In order to construct the one-loop four-point pure Yang-Mills amplitude, we use a D-dimensional variant [20] of the unitarity method [21]. Our construction begins by finding an ansatz for the amplitude constrained to satisfy the duality. Since the amplitude is fully determined from its D-dimensional unitarity cuts, we obtain a form of the amplitude with the duality manifest by enforcing that the ansatz has the correct unitarity cuts. The existence of such a form where both the duality and the cuts are simultaneously satisfied is rather nontrivial. We do not use helicity states tied to specific dimensions but instead use formal polarization vectors because we wish to have an expression for the amplitude valid in any dimension and for all states. The price for this generality is that the expressions are lengthier. Since the constructed integrand has manifest BCJ duality, the double-copy construction immediately gives the corresponding gravity amplitude in a theory with a graviton, dilaton, and antisymmetric tensor.

We use these results to study the ultraviolet divergences of the corresponding gravity amplitudes. Recent years have seen a renaissance in the study of ultraviolet divergences in gravity theories, in a large measure due to the greatly improved ability to carry out explicit multiloop computations in gravity theories [5,13–15,22–24]. The unitarity method also has revealed hints that multiloop supergravity theories may be better behaved in the ultraviolet than suggested by power-counting arguments based on standard symmetries [25]. Even pure Einstein gravity at one loop exhibits surprising cancellations as the number of external legs increases [26]. The question of whether it is possible to construct a finite supergravity is still an open one, though there has been enormous progress on this question in recent years, including new computations and a much better understanding of the consequences of supersymmetry and duality symmetry (see e.g. Refs. [27,28]). In half-maximal supergravity [29], two- and three-loop examples are known where the divergence vanishes, yet the understanding of the possible symmetry behind this vanishing is incomplete [23,24,28,30]. The duality between color and kinematics and its associated double-copy formula offer a new angle on the ultraviolet divergences in supergravity theories [5,13,23,24,31]. Here we explore the ultraviolet properties of nonsupersymmetric gravity from the double-copy perspective.

We use the gravity integrands constructed via the double-copy property to determine the exact form of the ultraviolet divergences. We do so at one loop in dimensions $D = 4, 6, 8$. The ultraviolet properties of one-loop four-point gravity amplitudes have already been studied in some detail over the years, including cases with scalars or antisymmetric tensors coupling to gravity [26,32–36], so no surprises should be expected, at least at four points. Nevertheless, it is useful to look in some detail at the ultraviolet properties to understand them from the double-copy perspective. Here we examine the four-point amplitudes in a theory of gravity coupled to a dilaton and an antisymmetric tensor, corresponding to the double copy of pure Yang-Mills theory. While related calculations have been carried out, we are unaware of any calculations of the ultraviolet properties in the theory corresponding to the double-copy theory.

We find that in $D = 4$, there are no one-loop divergences in amplitudes involving external gravitons, though there are divergences in the remaining amplitudes involving only external dilatons or antisymmetric tensors, as expected from simple counterterm arguments [32]. By two loops, even the four-graviton amplitudes contain divergences, as we demonstrate. In the two-loop case, the divergence is proportional to a unique R^3 operator which gives a divergence in the identical-helicity four-point amplitude. This means that the identical-helicity four-point amplitude is sufficient for determining the coefficient of the R^3 divergence. In $D = 6$ and $D = 8$, we find one-loop divergences in the four-external-graviton amplitudes. These results are not surprising and are in line with the earlier studies. Our conclusion is that, by itself, the double-copy structure is insufficient to render a gravity theory finite in $D = 4$ and requires additional ultraviolet cancellations, such as those from supersymmetry.

This paper is organized as follows. In Sec. II, we briefly review the duality between color and kinematics and the double-copy construction of gravity. In Sec. III, we present the construction of the duality-satisfying pure Yang-Mills numerators at one and two loops. Then in Sec. IV, we study the ultraviolet properties of gravity coupled to a dilaton and an antisymmetric tensor at one loop in four, six, and eight dimensions. In the same section, we also present the ultraviolet properties at two loops in four dimensions. Finally, in Sec. V we give our conclusions. Appendixes evaluating two-loop integrals needed in Sec. IV B are included. Appendix A focuses on extracting the divergences in dimensional regularization. This procedure mixes infrared and ultraviolet divergences; so, in Appendix B we give the infrared divergences that must be subtracted to obtain the ultraviolet ones. Appendix C evaluates the integrals using an alternative method for obtaining the ultraviolet divergences more directly, by introducing a mass to separate out the infrared divergences from the ultraviolet ones.

II. REVIEW OF BCJ DUALITY

An L-loop m-point gauge-theory amplitude in D dimensions, with all particles in the adjoint representation, may be written as

$$\mathcal{A}_m^{L\text{-loop}} = i^L g^{m-2+2L} \sum_{\mathcal{S}_m} \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{c_j n_j}{\prod_{\alpha_j} p_{\alpha_j}^2}, \tag{2.1}$$

where g is the gauge coupling constant. The first sum runs over the $m!$ permutations of the external legs, denoted by $\mathcal{S}_m$. The S_j symmetry factor removes any overcounting from these permutations and also from any internal automorphism symmetries of graph j. The j sum is over the set of distinct, nonisomorphic, m-point L-loop graphs with only *cubic* (i.e., trivalent) vertices. These graphs are sufficient because any diagram with quartic or higher vertices can be converted to a diagram with only cubic vertices by multiplying and dividing by the appropriate propagators. The propagators appearing in the graph are $1/\prod_{\alpha_j} p_{\alpha_j}^2$. The nontrivial kinematic information is contained in the numerators n_j and depends on momenta, polarizations, and spinors. In supersymmetric cases it will depend also on Grassmann parameters, if a superspace form is used. The loop integral is over L-independent D-dimensional loop momenta, p_l. Finally, c_j denotes the color factor obtained by dressing every vertex in graph j with the group-theory structure constant, $\tilde{f}^{abc} = i\sqrt{2} f^{abc} = \mathrm{Tr}([T^a, T^b] T^c)$, where the Hermitian generators of the gauge group are normalized via $\mathrm{Tr}(T^a T^b) = \delta^{ab}$.

The numerators appearing in Eq. (2.1) are by no means unique because of freedom in moving terms between different diagrams. Utilizing this freedom, the BCJ conjecture is that to all loop orders, representations of the amplitude exist where kinematic numerators obey the same algebraic relations that the color factors obey [4,5]. In ordinary gauge theories, this is simply the Jacobi identity,

$$c_i = c_j - c_k \Rightarrow n_i = n_j - n_k, \tag{2.2}$$

where i, j, and k label three diagrams whose color factors obey the Jacobi identity. The basic Jacobi identity is displayed in Fig. 1. The identity generalizes to any loop order with any number of external legs by embedding it in larger diagrams, where the other parts of the diagrams are identical for the three diagrams. Furthermore, if the color factor of a diagram is antisymmetric under a swap of legs, we require that the numerator obey the same antisymmetry,

$$c_i \to -c_i \Rightarrow n_i \to -n_i. \tag{2.3}$$

The duality was noticed long ago for tree-level four-point Feynman diagrams [37]; beyond this, it is rather nontrivial and no longer holds for ordinary Feynman diagrams. We note that the numerator relations are nontrivial functional relations because they depend on momenta, polarizations, and spinors, as discussed in some detail in Refs. [2,11,13].

While a complete understanding of the duality and its consequences is still lacking, a variety of studies have elucidated it, especially at tree level. In particular, this duality leads to nontrivial relations between gauge-theory color-ordered partial tree amplitudes [4,38]. The duality (2.3) has also been studied in string theory [6,9,39]. In the self-dual case, light-cone gauge Feynman rules have been shown to exhibit the duality [10]. Explicit forms of n-point tree amplitudes satisfying the duality have been found [12]. Although we do not yet have a complete Lagrangian understanding, some progress in this direction can be found in Refs. [8,10]. The duality (2.2) does not need to be expressed in terms of group structure constants but can alternatively be expressed in terms of a trace-based representation [40]. Progress has also been made in understanding the underlying infinite-dimensional Lie algebra [10,41] responsible for the duality. The duality between color and kinematics also appears to hold in three-dimensional theories based on three algebras [42], as well

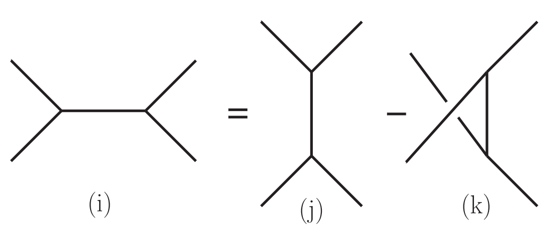

FIG. 1. The basic Jacobi relation for either color or numerator factors. These three diagrams can be embedded in a larger diagram, including loops.

as in some cases with higher-dimension operators [43]. Some initial studies of duality and its implications for gravity in the high-energy limit have also been carried out [44].

At loop level, the duality remains a conjecture, but there is already nontrivial evidence in its favor, especially for supersymmetric theories. At present, the list of loop-level cases where duality-satisfying forms of the amplitude are known to hold includes:

(i) Up to four loops for four-point $\mathcal{N}=4$ super-Yang-Mills [5,13] in a form valid in D dimensions;
(ii) up to two loops for five external gluons in $\mathcal{N}=4$ super-Yang-Mills theory [45];
(iii) up to seven points for one-loop amplitudes in $\mathcal{N}=4$ super-Yang-Mills theory [46];
(iv) up to two loops for four-point identical-helicity pure Yang-Mills amplitudes [5];
(v) through n points for one-loop all-plus-, or single-minus-helicity pure Yang-Mills amplitudes [18];
(vi) through four loops for a two-point (Sudakov) form factor in $\mathcal{N}=4$ super-Yang-Mills theory [47];
(vii) one-loop four-point amplitudes in Yang-Mills theories with less than maximally supersymmetric amplitudes [17].

In this paper, we add the nonsupersymmetric pure Yang-Mills one-loop four-point amplitude in D dimensions to this list. Besides direct constructions, we note that the duality also appears to be consistent with loop-level infrared properties of both gauge and gravity theories [16].

Another significant aspect of the duality is the ease with which gravity loop integrands can be obtained from gauge-theory ones, once the duality is made manifest [4,5]. One simply replaces the color factor with a kinematic numerator from a second gauge theory,

$$c_i \to \tilde{n}_i. \tag{2.4}$$

This immediately gives the double-copy form of gravity amplitudes,

$$\mathcal{M}_m^{L\text{-loop}} = i^{L+1}\left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_{\mathcal{S}_m}\sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{\tilde{n}_j n_j}{\prod_{\alpha_j} p_{\alpha_j}^2}, \tag{2.5}$$

where $\tilde{n}_j$ and n_j are gauge-theory numerator factors. Only one of the two sets of numerators needs to satisfy the duality (2.2) [5,8] in order for the double-copy form (2.5) to be valid. The double-copy formalism has been studied at loop level in some detail in a variety of cases [5,13,14,16,23,24,45,46].

III. CONSTRUCTION OF DUALITY-SATISFYING INTEGRANDS

We now describe the construction of a duality-satisfying representation of the one-loop four-point amplitude in pure

Yang-Mills. Since we want the form to be valid in all dimensions and for all $D-2$ gluon states, we use formal polarizations instead of helicity states. This complicates the expression for the amplitude, but has the advantage that it allows us to straightforwardly study the amplitude and its gravity double copy in various dimensions. In this section, we also present a form of the two-loop pure Yang-Mills identical-helicity amplitude given in Ref. [19] that satisfies BCJ duality after some rearrangement and addition of diagrams that integrate to zero. In Sec. IV B, we use this amplitude to show that although four-graviton amplitudes are ultraviolet finite in $D=4$ at one loop, they diverge at two loops, in accordance with expectations.

A. One loop

For a one-loop n-point amplitude, the duality (2.2) can be used to express kinematic numerators of any diagram directly in terms of n-gon numerators. In particular, for the four-point case we have two basic relations determining triangle and bubble contributions from box numerators as illustrated in Fig. 2,

$$\begin{aligned} n_{12(34);p} &= n_{1234;p} - n_{1243;p}, \\ n_{(12)(34);p} &= n_{12(34);p} - n_{21(34);p}. \end{aligned} \tag{3.1}$$

The labels 1,2,3,4 refer to the momenta and states of each external leg, while the label p denotes the loop momentum of the leg indicated in Fig. 2. (The parentheses in the subscript of the numerators indicate which external legs are pinched off to form a tree attached to the loop.) Note that in the figure the momentum of each internal leg of each diagram is the same as in the other two diagrams *except* for the single internal leg that differs between the diagrams. In general, the bubble and triangle contributions are nonvanishing; indeed, this explicitly holds for the BCJ representation of the one-loop four-point amplitude of pure Yang-Mills theory that we construct.

Besides the diagrams in Fig. 2, there are diagrams with a bubble on an external leg and diagrams with a tadpole, as

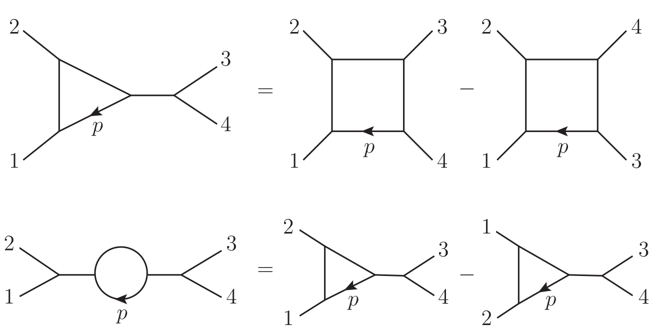

FIG. 2. The Jacobi relations determining either color or kinematic numerators of the four-point diagrams containing either a triangle or internal bubble.

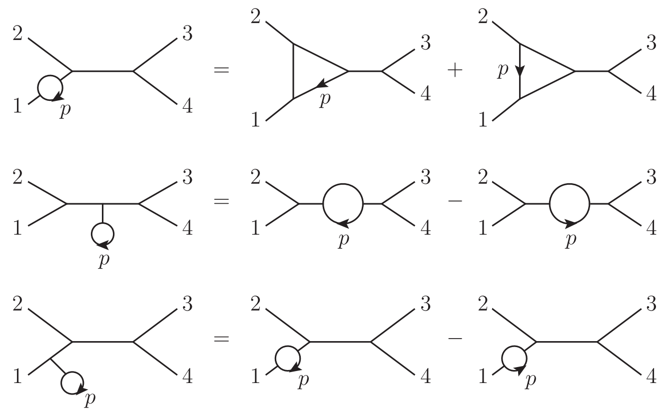

FIG. 3. The color or kinematic Jacobi relations involving a bubble on an external leg or a tadpole. These diagrams have vanishing contribution to the integrated amplitude.

shown in Fig. 3. The duality also determines the numerators of these diagrams via

$$\begin{aligned} n_{1(234);p} &= n_{12(34);p} + n_{1(43)2;p}, \\ n_{(1234);p} &= n_{(12)(34);p} - n_{(12)(34);-p}, \\ n_{(\hat{1}234);p} &= n_{1(234);p} - n_{1(234);-p}, \end{aligned} \tag{3.2}$$

corresponding respectively to the three relations in Fig. 3. [On the final line in Eq. (3.2), the hat marks leg 1 as the location where the tadpole is attached.] We use these equations to impose the auxiliary constraint that the tadpole numerators determined by BCJ duality vanish identically and that all terms in the bubble-on-external-leg diagrams integrate to zero as they do for Feynman diagrams. Thus, these diagrams are not necessary for determining the integrated amplitudes (though in $D=4$ the bubble-on-external-leg diagrams do affect the Yang-Mills ultraviolet divergence).

Once we impose the BCJ conditions, the amplitude is entirely specified by the box numerators. Our task is then to find an expression for the box numerators such that we obtain the correct amplitude. It is useful to impose a few auxiliary constraints to help simplify the one-loop construction:

(1) The box diagrams should have no more than four powers of loop momenta in the pure Yang-Mills case, matching the usual power count of Feynman-gauge Feynman diagrams.
(2) Each numerator written in terms of formal polarization vectors respects the symmetries of the diagrams. In particular, this condition implies that once a box diagram with one ordering of external legs is specified, the other orderings are obtained simply by relabeling.
(3) The numerators of tadpole diagrams vanish prior to integration.

(4) All terms in the bubble-on-external-leg diagrams integrate to zero, as they do for Feynman diagrams.

While it is not necessary to impose these conditions, they greatly simplify the construction. They ensure that the type of terms that appear in the ansatz are similar to those of ordinary Feynman-gauge Feynman diagrams, avoiding unnecessarily complicated terms. (Using generalized gauge invariance, one can always introduce arbitrarily complicated terms into amplitudes, which cancel at the end.)

The first three conditions simplify the construction by restricting the number of terms that appear. The purpose of the fourth auxiliary constraint is a bit more subtle. While bubble-on-external-leg Feynman diagrams are well defined in the on-shell limit, the freedom to reassign terms used in the construction of BCJ numerators can introduce ill-defined terms into such diagrams. As a simple example, consider the effect of the term $(k_1+k_2)^2\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1\varepsilon_3\cdot\varepsilon_4$ when added to the numerator of the first diagram of Fig. 3 (with k_i and ε_i external momenta and polarizations). Even after integration, this contribution to the diagram is ill-defined because of the on-shell intermediate propagator. Such singular contributions would need to be regularized by an appropriate off-shell continuation to ensure that the introduced singularities cancel properly against singularities of other diagrams. While in principle we can introduce such a regulator, it is best to avoid this complication altogether. The fourth condition ensures that we can treat the bubble-on-external-leg contributions in the same way as for Feynman diagrams. In particular, with the constraint imposed, the bubble-on-external-leg contributions match the Feynman-diagram property that they are proportional to $(k_i^2)^{(D-4)/2}$, after accounting for the intermediate on-shell propagator, and hence vanish in $D>4$, for k_i on shell. We note that even with the fourth constraint, near $D=4$ we encounter the same subtlety encountered with Feynman diagrams: Although bubble-on-external-leg contributions are set to zero in dimensional regularization, they can carry ultraviolet divergences. Such ultraviolet divergences cancel against infrared ones leaving a vanishing result for on-shell bubble-on-external-leg diagrams. The net effect is that in gauge theory, we need to account for such contributions to obtain the correct ultraviolet divergences. In contrast, in gravity even near $D=4$ there are neither infrared nor ultraviolet divergences hiding in the bubble-on-external-leg contributions because an extra two powers of numerator momenta give rise to an additional vanishing.

We start the construction with an ansatz containing all possible products of $\varepsilon_i\cdot\varepsilon_j$, $p\cdot\varepsilon_i$, $k_i\cdot\varepsilon_j$, $p\cdot k_i$, $p\cdot p$, s, and t, where the k_i are three independent external momenta, p is the loop momentum, ε_i are external polarization vectors, and

$$s=(k_1+k_2)^2,\qquad t=(k_2+k_3)^2 \tag{3.3}$$

are the usual Mandelstam invariants. By dimensional analysis, each numerator term must contain four momenta in addition to being linear in all four ε_i's. We also set $k_i\cdot\varepsilon_i=0$ and impose momentum conservation with $k_4=-k_1-k_2-k_3$ and $k_1\cdot\varepsilon_4=-k_2\cdot\varepsilon_4-k_3\cdot\varepsilon_4$. This yields 468 terms, each with a coefficient to be determined.

Our first constraint on the coefficients comes from demanding that the box numerator obey the rotation and reflection symmetries of the box diagram. This leaves us with 81 free coefficients. An ansatz for the full amplitude is then obtained by using the duality relations (3.1) and (3.2) to determine numerators for all other diagrams.

The next step is to determine coefficients in the ansatz by matching to the unitarity cuts of the amplitude. It is convenient to use a color-ordered form of the amplitude [48] for this matching. The seven diagrams contributing to the color-ordered amplitude, that is the coefficient of the color trace $N_c\mathrm{Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}]$, are shown in Fig. 4. The other color-ordered amplitudes are simple relabelings of this one. For the one-loop four-point amplitude, the s- and t-channel unitarity cuts shown in Fig. 5 are sufficient to determine this color-ordered amplitude up to terms that integrate to zero. One straightforward means for determining the cuts is to construct the amplitude in Feynman gauge and then take its unitarity cuts at the integrand level prior to

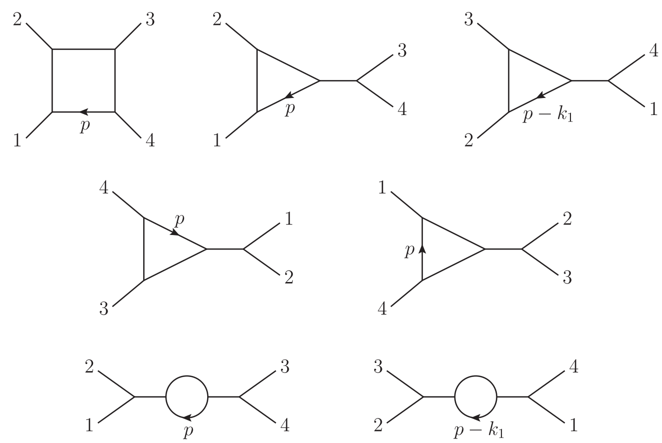

FIG. 4. The seven diagrams for the color-ordered amplitude with ordering (1,2,3,4).

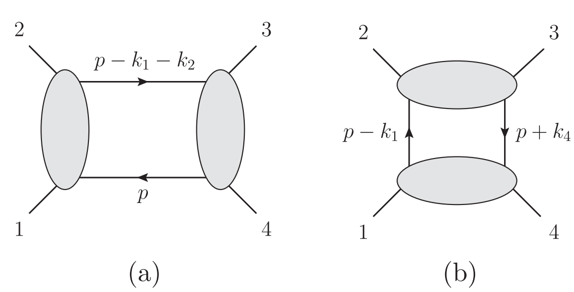

FIG. 5. The (a) s-channel and (b) t-channel unitarity cuts used to determine the amplitude. The exposed intermediate legs are on shell.

integration. This automatically gives us an expression for the cuts valid in D dimensions without any spurious denominators (such as light-cone denominators from physical state projectors). This matching procedure nontrivially rearranges the amplitude so that BCJ duality is manifest. After matching the cuts, we also impose the fourth auxiliary condition to tame the bubble-on-external-leg contributions. Finally we impose that the tadpole numerators vanish. Including all the auxiliary constraints with these conditions, we can solve for all but five free coefficients. Because the s- and t-channel unitarity cuts are independent of these parameters, the integrated amplitude should not depend on them.

Using the shorthand notation,

$$\begin{aligned} &p_1 = p, \qquad p_2 = p - k_1, \qquad p_3 = p - k_1 - k_2, \\ &p_4 = p + k_4, \qquad \mathcal{E}_{ij} = \varepsilon_i \cdot \varepsilon_j, \\ &\mathcal{P}_{ij} = p_i \cdot \varepsilon_j, \qquad \mathcal{K}_{ij} = k_i \cdot \varepsilon_j, \end{aligned} \tag{3.4}$$

and setting the free parameters to zero for simplicity, the box numerator is

$$\begin{aligned} n_{1234;p} = -i\Bigg[&\frac{D_s-2}{8}\mathcal{E}_{14}\mathcal{E}_{23}p_1^2p_3^2 + \frac{D_s-2}{24}\mathcal{E}_{13}\mathcal{E}_{24}p_1^2p_3^2 - \frac{D_s-2}{24}\mathcal{E}_{12}\mathcal{E}_{34}p_1^2p_3^2 - \frac{2}{3}\mathcal{E}_{14}\mathcal{E}_{23}p_3^2 s \\ &- \frac{2}{3}\mathcal{E}_{13}\mathcal{E}_{24}p_2^2 s + \frac{2}{3}\mathcal{E}_{12}\mathcal{E}_{34}p_2^2 s + \frac{2}{3}\mathcal{E}_{14}\mathcal{E}_{23}p_2^2 s + \frac{1}{2}\mathcal{E}_{14}\mathcal{E}_{23}s^2 + 2\mathcal{E}_{23}\mathcal{K}_{24}\mathcal{K}_{41}p_3^2 \\ &+ \frac{D_s-74}{24}\mathcal{E}_{13}\mathcal{K}_{12}\mathcal{K}_{34}p_3^2 + \frac{D_s-74}{24}\mathcal{E}_{24}\mathcal{K}_{23}\mathcal{K}_{41}p_3^2 - \frac{D_s-26}{3}\mathcal{E}_{12}\mathcal{K}_{13}\mathcal{K}_{34}p_3^2 \\ &- \frac{D_s-26}{6}\mathcal{E}_{34}\mathcal{K}_{41}\mathcal{K}_{42}p_3^2 - \frac{D_s-26}{2}\mathcal{E}_{12}\mathcal{K}_{23}\mathcal{K}_{34}p_3^2 - \frac{D_s-26}{2}\mathcal{E}_{34}\mathcal{K}_{12}\mathcal{K}_{41}p_3^2 \\ &+ \frac{D_s-26}{12}\mathcal{E}_{34}\mathcal{K}_{31}\mathcal{K}_{42}p_3^2 + \frac{5(D_s-26)}{24}\mathcal{E}_{24}\mathcal{K}_{13}\mathcal{K}_{41}p_3^2 - \frac{D_s-26}{8}\mathcal{E}_{24}\mathcal{K}_{13}\mathcal{K}_{31}p_3^2 \\ &- \frac{11(D_s-26)}{24}\mathcal{E}_{24}\mathcal{K}_{23}\mathcal{K}_{31}p_3^2 - \frac{D_s-26}{24}\mathcal{E}_{34}\mathcal{K}_{12}\mathcal{K}_{31}p_3^2 + \frac{D_s-30}{2}\mathcal{E}_{13}\mathcal{K}_{12}\mathcal{K}_{24}p_3^2 \\ &- \frac{D_s-14}{6}\mathcal{E}_{13}\mathcal{K}_{34}\mathcal{K}_{42}p_3^2 + \frac{D_s-38}{6}\mathcal{E}_{13}\mathcal{K}_{24}\mathcal{K}_{42}p_3^2 - \frac{5(D_s-26)}{24}\mathcal{E}_{12}\mathcal{K}_{13}\mathcal{K}_{24}p_3^2 \\ &- \frac{11(D_s-26)}{24}\mathcal{E}_{12}\mathcal{K}_{23}\mathcal{K}_{24}p_3^2 + \frac{13D_s-290}{24}\mathcal{E}_{14}\mathcal{K}_{23}\mathcal{K}_{42}p_3^2 + (D_s-24)\mathcal{E}_{14}\mathcal{K}_{12}\mathcal{K}_{23}p_3^2 \\ &+ \frac{11(D_s-26)}{24}\mathcal{E}_{14}\mathcal{K}_{13}\mathcal{K}_{42}p_3^2 + \frac{11(D_s-26)}{24}\mathcal{E}_{14}\mathcal{K}_{12}\mathcal{K}_{13}p_3^2 - \frac{D_s-26}{12}\mathcal{E}_{23}\mathcal{K}_{24}\mathcal{K}_{31}p_3^2 \\ &- \frac{D_s-26}{12}\mathcal{E}_{23}\mathcal{K}_{31}\mathcal{K}_{34}p_3^2 - \frac{D_s-50}{12}\mathcal{E}_{23}\mathcal{K}_{34}\mathcal{K}_{41}p_3^2 - 4\mathcal{E}_{14}\mathcal{K}_{12}\mathcal{K}_{23}s - 2\mathcal{E}_{23}\mathcal{K}_{24}\mathcal{K}_{31}s \\ &- 2\mathcal{E}_{23}\mathcal{K}_{24}\mathcal{K}_{41}s - 2\mathcal{E}_{12}\mathcal{K}_{23}\mathcal{K}_{24}s - 2\mathcal{E}_{14}\mathcal{K}_{12}\mathcal{K}_{13}s - 2\mathcal{E}_{12}\mathcal{K}_{23}\mathcal{K}_{34}s \\ &+ \frac{7D_s-230}{12}\mathcal{E}_{23}\mathcal{K}_{31}\mathcal{P}_{44}p_3^2 + \frac{7D_s-230}{24}\mathcal{E}_{23}\mathcal{K}_{34}\mathcal{P}_{11}p_3^2 + \frac{7D_s-230}{24}\mathcal{E}_{23}\mathcal{K}_{41}\mathcal{P}_{44}p_3^2 \\ &+ \frac{7D_s-230}{24}\mathcal{E}_{13}\mathcal{K}_{34}\mathcal{P}_{22}p_3^2 + \frac{7D_s-230}{24}\mathcal{E}_{24}\mathcal{K}_{41}\mathcal{P}_{33}p_3^2 - \frac{7(D_s-26)}{24}\mathcal{E}_{24}\mathcal{K}_{13}\mathcal{P}_{11}p_3^2 \\ &+ \frac{7(D_s-26)}{24}\mathcal{E}_{12}\mathcal{K}_{13}\mathcal{P}_{44}p_3^2 - \frac{7D_s-230}{24}\mathcal{E}_{23}\mathcal{K}_{24}\mathcal{P}_{11}p_3^2 - \frac{7D_s-230}{24}\mathcal{E}_{12}\mathcal{K}_{24}\mathcal{P}_{33}p_3^2 \\ &- \frac{7D_s-230}{24}\mathcal{E}_{34}\mathcal{K}_{42}\mathcal{P}_{11}p_3^2 - \frac{11D_s-238}{24}\mathcal{E}_{13}\mathcal{K}_{12}\mathcal{P}_{44}p_3^2 - \frac{11D_s-238}{24}\mathcal{E}_{24}\mathcal{K}_{23}\mathcal{P}_{11}p_3^2 \\ &+ 2\mathcal{E}_{12}\mathcal{K}_{23}\mathcal{P}_{44}p_3^2 + 2\mathcal{E}_{34}\mathcal{K}_{12}\mathcal{P}_{11}p_3^2 - \frac{D_s-14}{6}\mathcal{E}_{13}\mathcal{K}_{42}\mathcal{P}_{44}p_3^2 - \frac{3(D_s-26)}{8}\mathcal{E}_{34}\mathcal{K}_{31}\mathcal{P}_{22}p_3^2 \\ &- \frac{3(D_s-26)}{8}\mathcal{E}_{24}\mathcal{K}_{31}\mathcal{P}_{33}p_3^2 - \frac{2(D_s-29)}{3}\mathcal{E}_{34}\mathcal{K}_{41}\mathcal{P}_{22}p_3^2 - \frac{2(D_s-29)}{3}\mathcal{E}_{12}\mathcal{K}_{34}\mathcal{P}_{33}p_3^2 \\ &+ \frac{13D_s-290}{24}\mathcal{E}_{14}\mathcal{K}_{42}\mathcal{P}_{33}p_3^2 + \frac{13D_s-290}{24}\mathcal{E}_{14}\mathcal{K}_{12}\mathcal{P}_{33}p_3^2 + \frac{13D_s-290}{24}\mathcal{E}_{14}\mathcal{K}_{23}\mathcal{P}_{22}p_3^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{2(D_s-29)}{3}\mathcal{E}_{13}\mathcal{K}_{24}\mathcal{P}_{22}p_3^2-2\mathcal{E}_{14}\mathcal{K}_{42}\mathcal{P}_{33}s-2\mathcal{E}_{34}\mathcal{K}_{41}\mathcal{P}_{22}s-2\mathcal{E}_{14}\mathcal{K}_{12}\mathcal{P}_{33}s\\
&-2\mathcal{E}_{12}\mathcal{K}_{24}\mathcal{P}_{33}s-2\mathcal{E}_{12}\mathcal{K}_{34}\mathcal{P}_{33}s-2\mathcal{E}_{14}\mathcal{K}_{23}\mathcal{P}_{22}s+2\mathcal{E}_{13}\mathcal{K}_{34}\mathcal{P}_{22}s+2\mathcal{E}_{24}\mathcal{K}_{41}\mathcal{P}_{33}s\\
&+2\mathcal{E}_{13}\mathcal{K}_{24}\mathcal{P}_{22}s-(D_s-2)\mathcal{E}_{23}\mathcal{P}_{11}\mathcal{P}_{44}p_3^2-\frac{D_s-2}{6}\mathcal{E}_{13}\mathcal{P}_{22}\mathcal{P}_{44}p_3^2-\frac{D_s-2}{6}\mathcal{E}_{24}\mathcal{P}_{33}\mathcal{P}_{11}p_3^2\\
&+\frac{D_s-2}{6}\mathcal{E}_{12}\mathcal{P}_{33}\mathcal{P}_{44}p_3^2+\frac{D_s-2}{6}\mathcal{E}_{34}\mathcal{P}_{11}\mathcal{P}_{22}p_3^2-4\mathcal{E}_{34}\mathcal{P}_{11}\mathcal{P}_{22}s+2\mathcal{E}_{13}\mathcal{P}_{22}\mathcal{P}_{44}s\\
&+2\mathcal{E}_{24}\mathcal{P}_{33}\mathcal{P}_{11}s+4\mathcal{K}_{12}\mathcal{K}_{13}\mathcal{K}_{24}\mathcal{K}_{31}+4\mathcal{K}_{12}\mathcal{K}_{23}\mathcal{K}_{24}\mathcal{K}_{31}+2\mathcal{K}_{12}\mathcal{K}_{13}\mathcal{K}_{31}\mathcal{K}_{34}\\
&+4\mathcal{K}_{12}\mathcal{K}_{23}\mathcal{K}_{31}\mathcal{K}_{34}+\mathcal{K}_{13}\mathcal{K}_{24}\mathcal{K}_{31}\mathcal{K}_{42}+2\mathcal{K}_{12}\mathcal{K}_{23}\mathcal{K}_{34}\mathcal{K}_{41}-4\mathcal{K}_{12}\mathcal{K}_{24}\mathcal{K}_{41}\mathcal{P}_{33}\\
&+4\mathcal{K}_{31}\mathcal{K}_{34}\mathcal{K}_{42}\mathcal{P}_{33}+4\mathcal{K}_{24}\mathcal{K}_{41}\mathcal{K}_{42}\mathcal{P}_{33}+4\mathcal{K}_{34}\mathcal{K}_{41}\mathcal{K}_{42}\mathcal{P}_{33}+4\mathcal{K}_{24}\mathcal{K}_{31}\mathcal{K}_{42}\mathcal{P}_{33}\\
&-8\mathcal{K}_{34}\mathcal{K}_{41}\mathcal{P}_{22}\mathcal{P}_{33}-8\mathcal{K}_{24}\mathcal{K}_{41}\mathcal{P}_{22}\mathcal{P}_{33}+4\mathcal{K}_{24}\mathcal{K}_{42}\mathcal{P}_{11}\mathcal{P}_{33}\\
&+(D_s-2)\mathcal{P}_{11}\mathcal{P}_{22}\mathcal{P}_{33}\mathcal{P}_{44}\Bigg]+\text{cyclic},
\end{aligned}
\tag{3.5}
$$

where D_s is a state-counting parameter, so that D_s-2 is the number of gluon states circulating in the loop. The notation "+ cyclic" indicates that one should include the three additional cyclic permutations of indices, giving a total of four permutations (1,2,3,4), (2,3,4,1), (3,4,1,2), (4,1,2,3) of all variables $\varepsilon_i, k_i, p_i, s, t$. Plain-text, computer-readable versions of the full expressions for the numerators, including also gluino- and scalar-loop contributions, can be found online [49]. In Eq. (3.5), we have written the expression for the box numerator in a different form than that available online in order to exhibit the cyclic symmetry.

We have explicitly checked that after reducing the pure Yang-Mills amplitude to an integral basis, the expression is free of arbitrary parameters and in $D=4$ matches the known expression for the amplitude in Ref. [50], after accounting for the fact that the expression in that paper is renormalized. The reduction for four-dimensional external states was carried out by expanding the external polarizations in terms of the external momenta plus a dual vector [51].

As another simple cross-check, we have extracted the ultraviolet divergences in $D=6,8$ and compared them to the known forms. In $D=6,8$, with our fourth auxiliary constraint there are no ultraviolet contributions from bubbles on external legs. This allows us to directly extract the ultraviolet divergences by introducing a mass regulator and then expanding in small external momenta using the methods of Ref. [52]. We find complete agreement with both earlier evaluations in Ref. [53]. We have also compared this to an extraction of the ultraviolet divergences directly using dimensional regularization without introducing an additional mass regulator and again find agreement.

B. Two loops

We now turn to two loops. As we shall discuss in Sec. IV, the four-graviton amplitude in the double-copy theory is ultraviolet finite at one loop. To test whether this continues at two loops, we need the two-loop amplitude. As it turns out, the identical-helicity amplitude is sufficient for our purposes because the divergence comes from an R^3 operator whose coefficient is fixed by this amplitude. We therefore now turn to finding a form of the two-loop identical-helicity amplitude where BCJ duality is manifest. It would be interesting to obtain a general two-loop construction valid for all states in D dimensions, but we do not do so here.

The identical-helicity pure Yang-Mills amplitude has previously been constructed in Ref. [19]. There the amplitude is given in the following representation:

$$
\mathcal{A}_4^{(2)}(1^+,2^+,3^+,4^+)=g^6\frac{1}{4}\sum_{\mathcal{S}_4}[C^{\mathrm{P}}_{1234}A^{\mathrm{P}'}_{1234}+C^{\mathrm{NP}}_{12;34}A^{\mathrm{NP}}_{12;34}],
\tag{3.6}
$$

where the sum runs over all 24 permutations of the external legs. We will describe the all-plus-helicity case; the all-negative-helicity case follows from parity conjugation. The prefactor of $1/4$ accounts for the overcount due to symmetries of the diagrams. C^{P}_{1234} and $C^{\mathrm{NP}}_{12;34}$ are the color factors obtained from the planar double-box and nonplanar double-box diagrams shown in Figs. 6(a) and 6(b), respectively, by dressing each vertex with an $\tilde{f}^{abc}$ and summing over the contracted color indices. $A^{\mathrm{P}'}_{1234}$ and $A^{\mathrm{NP}}_{12;34}$ are then the associated partial amplitudes. These partial amplitudes are [19]

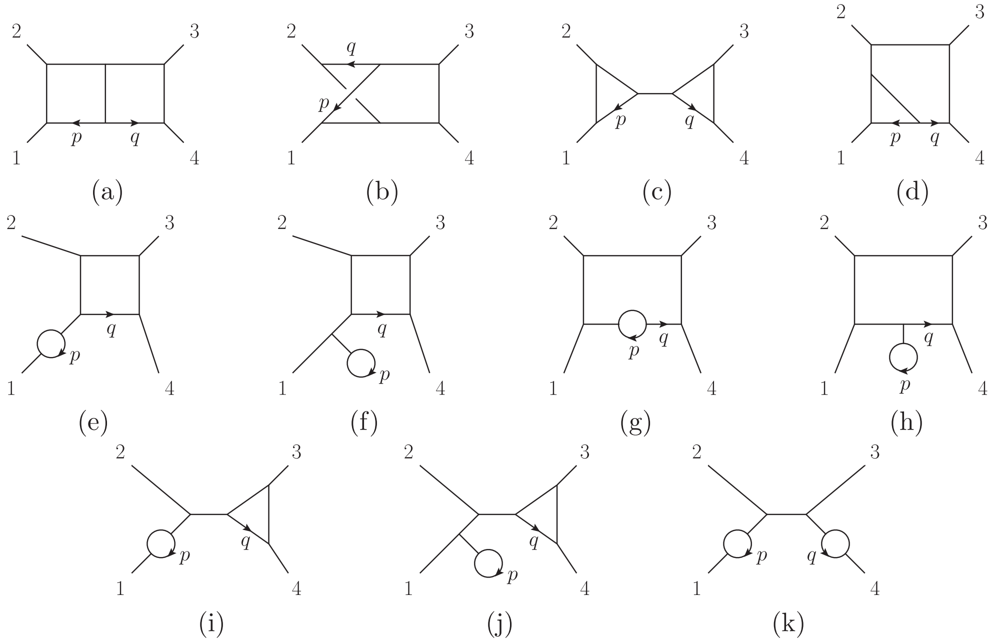

FIG. 6. The diagrams needed to describe an integrand for the identical-helicity amplitude where the duality between color and kinematics is manifest. When integrated diagrams (d)–(k) vanish. The (a) planar double-box, (b) nonplanar double-box, and (c) double-triangle integrals are the only nonvanishing ones under integration.

$$
\begin{aligned}
A^{\mathrm{P}'}_{1234} &= i\mathcal{T}\bigg\{ s\mathcal{I}^{\mathrm{P}}_4[(D_s-2)(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2)+16((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2)](s,t) \\
&\quad + 4(D_s-2)\mathcal{I}^{\text{bow-tie}}_4[(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q)](s) + \frac{(D_s-2)^2}{s}\mathcal{I}^{\text{bow-tie}}_4[\lambda_p^2\lambda_q^2((p+q)^2+s)](s,t)\bigg\}, \\
A^{\mathrm{NP}}_{12;34} &= i\mathcal{T}s\mathcal{I}^{\mathrm{NP}}_4[(D_s-2)(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2)+16((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2)](s,t), \qquad (3.7)
\end{aligned}
$$

where the permutation-invariant kinematic prefactor is given by

$$
\mathcal{T} \equiv \frac{[12][34]}{\langle 12\rangle\langle 34\rangle}, \qquad (3.8)
$$

where the angle and square brackets are standard spinor inner products. For the all-negative-helicity case, the angle and square products should be swapped. The planar double-box [Fig. 6(a)], nonplanar double-box [Fig. 6(b)], and bow-tie integrals (Fig. 7) are

$$
\begin{aligned}
\mathcal{I}^{\mathrm{P}}_4[\mathcal{P}(\lambda_i,p,q,k_i)](s,t) &\equiv \int \frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\frac{\mathcal{P}(\lambda_i,p,q,k_i)}{p^2q^2(p+q)^2(p-k_1)^2(p-k_1-k_2)^2(q-k_4)^2(q-k_3-k_4)^2}, \\
\mathcal{I}^{\mathrm{NP}}_4[\mathcal{P}(\lambda_i,p,q,k_i)](s,t) &\equiv \int \frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\frac{\mathcal{P}(\lambda_i,p,q,k_i)}{p^2q^2(p+q)^2(p-k_1)^2(q-k_2)^2(p+q+k_3)^2(p+q+k_3+k_4)^2}, \\
\mathcal{I}^{\text{bow-tie}}_4[\mathcal{P}(\lambda_i,p,q,k_i)](s,t) &\equiv \int \frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\frac{\mathcal{P}(\lambda_i,p,q,k_i)}{p^2q^2(p-k_1)^2(p-k_1-k_2)^2(q-k_4)^2(q-k_3-k_4)^2}, \qquad (3.9)
\end{aligned}
$$

where λ_p, λ_q, and λ_{p+q} represent the (-2ϵ)-dimensional components of loop momenta p, q, and $(p+q)$.

Reference [5] notes that a representation where the numerators satisfy the BCJ duality can be obtained directly from the representation of the amplitude given in Ref. [19]. Here we describe this in more detail, including additional diagrams that integrate to zero and are undetectable in ordinary unitarity cuts, but are needed to make the duality manifest.

We begin with a rearranged form of the identical-helicity amplitude,

$$\mathcal{A}_4^{(2)}(1^+,2^+,3^+,4^+) = g^6 \sum_{\mathcal{S}_4}\left[\frac{1}{4}C^{\rm P}_{1234}A^{\rm P}_{1234} + \frac{1}{4}C^{\rm NP}_{12;34}A^{\rm NP}_{12;34} + \frac{1}{8}C^{\rm DT}_{1234}A^{\rm DT}_{1234}\right]. \tag{3.10}$$

$C^{\rm DT}_{1234}$ is the color factor obtained from the stretched bow-tie or double-triangle diagram in Fig. 6(c). $A^{\rm NP}_{12;34}$ is given in Eq. (3.7), while $A^{\rm P}_{1234}$ and $A^{\rm DT}_{1234}$ are

$$\begin{aligned} A^{\rm P}_{1234} &= i\mathcal{T}\mathcal{I}^{\rm P}_4\bigg[\frac{(D_s-2)^2}{2}(p+q)^2\lambda_p^2\lambda_q^2 + 16s((\lambda_p\cdot\lambda_q)^2 - \lambda_p^2\lambda_q^2) \\ &\quad + (D_s-2)\left(s(\lambda_p^2\lambda_q^2 + \lambda_p^2\lambda_{p+q}^2 + \lambda_q^2\lambda_{p+q}^2\right) + 4(p+q)^2(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q))\bigg](s,t), \\ A^{\rm DT}_{1234} &= i\mathcal{T}\mathcal{I}^{\rm DT}_4\bigg[\frac{(D_s-2)^2}{2}(4p\cdot q + 2(p-q)\cdot(k_1+k_2) - s)\lambda_p^2\lambda_q^2 \\ &\quad + 8(D_s-2)(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q)(p^2+q^2-(p-q)\cdot(k_1+k_2)+s)\bigg](s,t). \end{aligned} \tag{3.11}$$

The double-triangle integral displayed in Fig. 6(c) is simply

$$\mathcal{I}^{\rm DT}_4[\mathcal{P}(\lambda_i,p,q,k_i)](s,t) = \frac{1}{s}\mathcal{I}^{\text{bow-tie}}_4[\mathcal{P}(\lambda_i,p,q,k_i)](s,t), \tag{3.12}$$

so that all integrals in the new representation of the amplitude are given by trivalent graphs.

This form of the amplitude differs from Eq. (3.6) by absorbing the bow-tie contribution depicted in Fig. 7 into both the planar double box in Fig. 6(a) and the double triangle in Fig. 6(c). When moving terms into the double box Fig. 6(a), we must multiply by a factor of $(p+q)^2$ in the numerator to cancel the central propagator, while in the double triangle Fig. 6(c), we must multiply by a factor of s. In this rearrangement we have also included terms that integrate to zero. In particular, the second term in the double-triangle contribution in Eq. (3.11) proportional to $(\lambda_p\cdot\lambda_q)$ integrates to zero and does not contribute to the integrated amplitude. We are therefore free to drop it. We can also modify the first term in the double-triangle integral into the form appearing in Ref. [19] by using the fact that the substitution

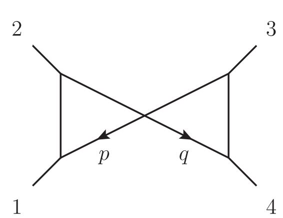

FIG. 7. The bow-tie integral appearing in the identical-helicity pure Yang-Mills amplitude.

$$(4p\cdot q + 2(p-q)\cdot(k_1+k_2) - s) \to 2(p+q)^2 + s \tag{3.13}$$

does not alter the value of the integrated amplitude: All terms that are proportional to p^2, q^2, $(p-k_1-k_2)^2$, and $(q-k_3-k_4)^2$ yield scale-free integrals that integrate to zero. Finally, to see the equivalence of the two representations, we note that the double triangle Fig. 6(c) has a different color factor from that of the planar double box Fig. 6(a). However, we can convert the double-triangle Fig. 6(c) color factor to the double-box Fig. 6(a) color factor via the color Jacobi identity $C^{\rm DT}_{1234} = C^{\rm P}_{1234} - C^{\rm P}_{2134}$. This matches the color assignment used in Ref. [19]. Although not manifest, the kinematic numerator reflects the antisymmetry of the Jacobi relations so that the additional terms picked up by $A^{\rm P}_{1234}$ and $A^{\rm P}_{2134}$ are simply related by relabelings. Thus, after integration our representation in Eq. (3.10) is equivalent to the one in Eq. (3.6), which comes from Ref. [19].

The integrand in Eq. (3.10) satisfies BCJ duality once we include additional contributions that integrate to zero. To find the full form, we consider Jacobi relations (2.2) around each internal propagator of the planar double box, the nonplanar double box, and the double triangle, as well as all resultant integrals that arise from these Jacobi relations. Duality relations where all three numerators are nonvanishing are depicted in Fig. 8. The need for additional nonvanishing numerators depicted in Figs. 6(d)–6(m) arises from these dual-Jacobi relations. Other sample Jacobi relations where one of the numerators vanishes are shown in Fig. 9. Up to relabelings, there are in total 16 such relations involving two nonvanishing numerators and one vanishing numerator. A fully duality-satisfying form is given by the numerators,

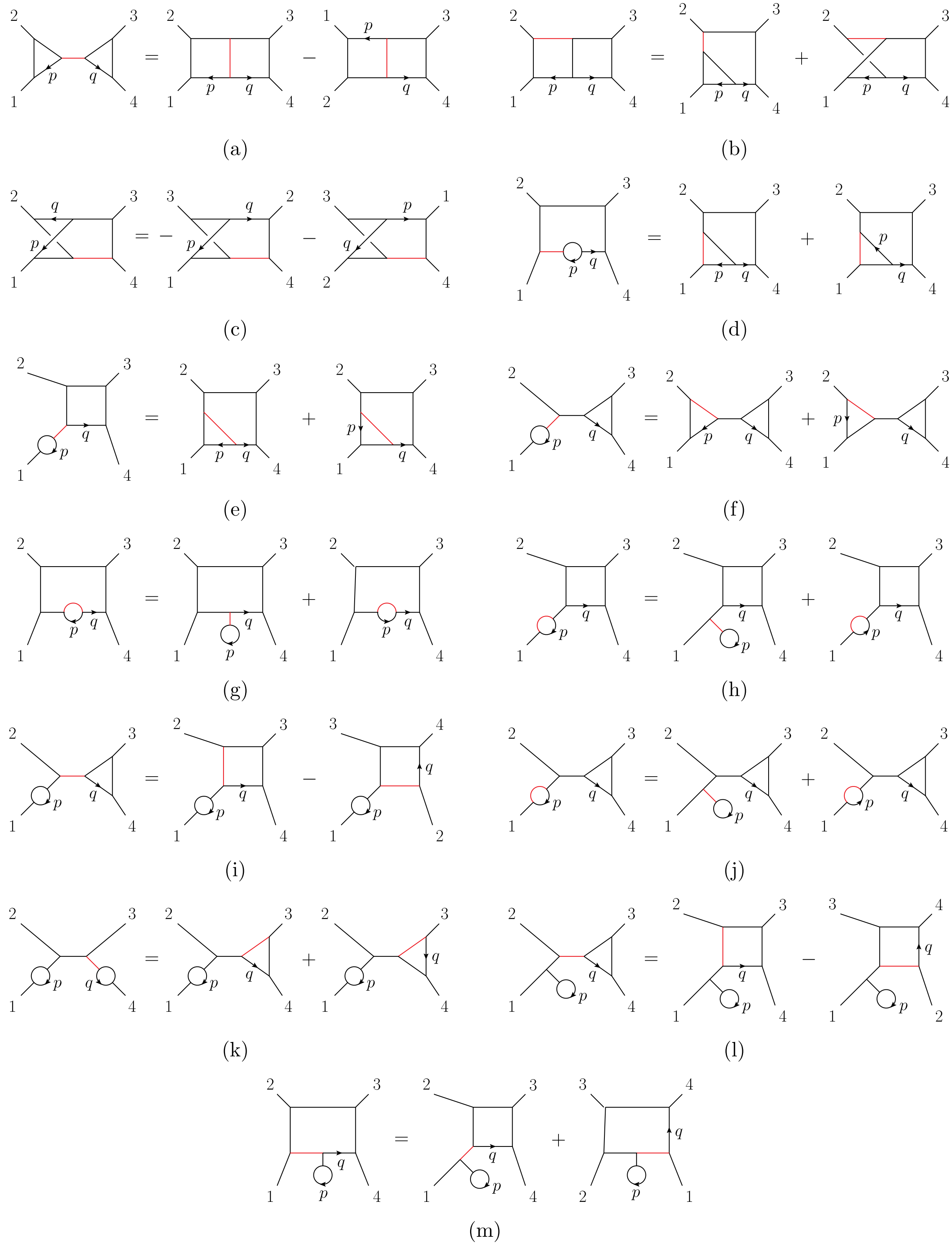

FIG. 8 (color online). The nontrivial duality relations (a)–(m) satisfied by the numerators of the identical-helicity two-loop amplitude. The shaded (red) leg marks the central leg of the applied Jacobi identity.

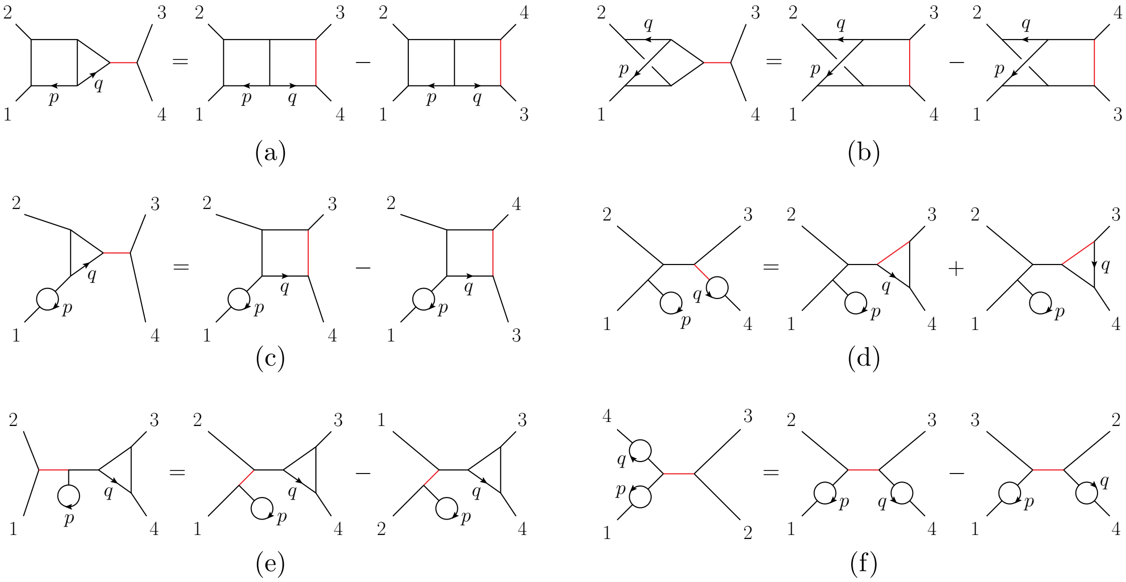

FIG. 9 (color online). Sample duality relations (a)–(f) involving graphs with vanishing numerators. In each relation, the leftmost diagram has a vanishing numerator. The shaded (red) leg marks the central leg of the applied dual-Jacobi identity.

$$
\begin{aligned}
\mathcal{P}^{\mathrm{P}}(\lambda_i,p,q,k_i) &= \frac{(D_s-2)^2}{2}(p+q)^2\lambda_p^2\lambda_q^2 + 16s((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2)\\
&\quad + (D_s-2)(s(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2)+4(p+q)^2(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q)),\\
\mathcal{P}^{\mathrm{NP}}(\lambda_i,p,q,k_i) &= (D_s-2)s(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2)+16s((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2),\\
\mathcal{P}^{\mathrm{DT}}(\lambda_i,p,q,k_i) &= \frac{(D_s-2)^2}{2}(4p\cdot q+2(p-q)\cdot(k_1+k_2)-s)\lambda_p^2\lambda_q^2\\
&\quad + 8(D_s-2)(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q)(p^2+q^2-(p-q)\cdot(k_1+k_2)+s),\\
\mathcal{P}^{(\mathrm{d})}(\lambda_i,p,q,k_i) &= \frac{(D_s-2)^2}{2}(p+q)^2\lambda_p^2\lambda_q^2+4(D_s-2)(p+q)^2(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q),\\
\mathcal{P}^{(\mathrm{e})}(\lambda_i,p,q,k_i) &= (D_s-2)^2(p^2+q^2-(p-q)\cdot k_1)\lambda_p^2\lambda_q^2\\
&\quad + 8(D_s-2)(2p\cdot q+(p-q)\cdot k_1)(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q)\\
\mathcal{P}^{(\mathrm{f})}(\lambda_i,p,q,k_i) &= -2(D_s-2)^2(p\cdot k_1)\lambda_p^2\lambda_q^2-16(D_s-2)(q\cdot k_1)(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q)\\
\mathcal{P}^{(\mathrm{g})}(\lambda_i,p,q,k_i) &= \frac{(D_s-2)^2}{2}((p+q)^2\lambda_p^2+p^2\lambda_{p+q}^2)\lambda_q^2\\
&\quad + 4(D_s-2)((p+q)^2(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q)-p^2(\lambda_q^2+\lambda_{p+q}^2)(\lambda_q\cdot\lambda_{p+q})),\\
\mathcal{P}^{(\mathrm{h})}(\lambda_i,p,q,k_i) &= 2(D_s-2)^2((p\cdot q)\lambda_p^2+p^2(\lambda_p\cdot\lambda_q))\lambda_q^2-8(D_s-2)(3p^2\lambda_q^2-q^2(\lambda_p^2+\lambda_q^2))(\lambda_p\cdot\lambda_q),\\
\mathcal{P}^{(\mathrm{i})}(\lambda_i,p,q,k_i) &= -\frac{(D_s-2)^2}{2}(4q\cdot k_2+s)\lambda_p^2\lambda_q^2-4(D_s-2)(4p\cdot k_2-s)(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q),\\
\mathcal{P}^{(\mathrm{j})}(\lambda_i,p,q,k_i) &= 8(D_s-2)s(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q),\\
\mathcal{P}^{(\mathrm{k})}(\lambda_i,p,q,k_i) &= (D_s-2)^2t\lambda_p^2\lambda_q^2, \qquad (3.14)
\end{aligned}
$$

where each $\mathcal{P}^x$ is the numerator of an integral $\mathcal{I}_4^x[\mathcal{P}^x(\lambda_i,p,q,k_i)](s,t)$ corresponding to diagram x depicted in Fig. 6. In contrast to the one-loop case, the duality-satisfying amplitudes do contain tadpole diagrams with nonvanishing numerators.

Although BCJ duality gives us a set of well-defined numerators for all diagrams, those diagrams with on-shell or vanishing intermediate propagators are ill-defined. However, all such ill-defined diagrams give vanishing contributions after integration. They also do not contribute to the standard two- and three-particle cuts. In more detail, using the numerators from Eq. (3.14), Figs. 6(d)–6(k) contain scale-free integrals that vanish after integration. Note that Figs. 6(e), 6(f), and 6(h)–6(k) are ill-defined. Figures 6(f), 6(h), and 6(j) contain a tadpole subdiagram. We set these to zero, just as they are set to zero in Feynman diagrams since the tadpole integral is scale free in dimensional regularization. Figures 6(e), 6(i), and 6(k) are also ill-defined for on-shell external legs because of the propagator carrying an on-shell momentum. With Feynman diagrams, this is normally dealt with by taking the legs off shell; in principle, we can also define an off-shell continuation, although it is nontrivial to do so consistently in our case. However, such ill-defined bubble-on-external-leg contributions again vanish in dimensional regularization, since the integrals are also scale free. In the gauge-theory case, although vanishing, these integrals can potentially contain ultraviolet divergences that cancel completely against infrared divergences. However, in the gravity case, which we are interested in here, the integrals are suppressed by an additional power of the on-shell invariant $k_i^2 = 0$ and therefore lead to ultraviolet divergences with zero coefficient. Figures 6(d) and 6(g) may appear to have nonvanishing cut contributions, but inverse propagators in the numerator cancel propagators, again leaving scale-free integrals that vanish.

In summary, the two-loop four-point all-plus-helicity pure Yang-Mills amplitude in a duality-satisfying representation is given by

$$\mathcal{A}_4^{(2)}(1^+,2^+,3^+,4^+) = g^6 \sum_{\mathcal{S}_4} \sum_{x\in\{\text{diagrams}\}} \frac{1}{S^x} C^x_{1234} A^x_{1234}, \tag{3.15}$$

where x labels diagrams in Fig. 6 with nonvanishing numerators. S^x is the symmetry factor of diagram x, while C^x_{1234} is the color factor. The partial amplitudes are given by

$$A^x_{1234} = i\mathcal{T}\mathcal{I}^x_4[\mathcal{P}^x(\lambda_i, p, q, k_i)](s,t), \tag{3.16}$$

where all diagrams except for those in Figs. 6(a)–6(c) integrate to zero in gauge theory. In Sec. IV B, we will use the double-copy relation (2.5) on these numerators to study the two-loop ultraviolet behavior of gravity coupled to a dilaton and an antisymmetric tensor.

IV. ULTRAVIOLET PROPERTIES OF GRAVITY

We now turn to the ultraviolet properties of the gravity double-copy theory consisting of a graviton, dilaton, and antisymmetric tensor, from the perspective of the double-copy formalism. The theory generated by taking the double copy of pure Yang-Mills corresponds to the low-energy effective Lagrangian of the bosonic part of string theory [54],

$$\mathcal{L} = \sqrt{-g}\left(\frac{2}{\kappa^2}R + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{6}e^{-2\kappa\phi/\sqrt{D-2}}H_{\mu\nu\rho}H^{\mu\nu\rho}\right), \tag{4.1}$$

where $H_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} + \partial_\nu A_{\rho\mu} + \partial_\rho A_{\mu\nu}$, and $A_{\mu\nu} = -A_{\nu\mu}$ is the rank-two antisymmetric tensor field.

Pure Einstein gravity is one-loop finite in four dimensions [32]. However, when coupled to a scalar (dilaton) [32] or to a rank-two antisymmetric tensor [34], the theory is divergent. We find that the double-copy theory coupled to both a dilaton and an antisymmetric tensor is also divergent, although for all these theories the four-point amplitudes with at least one external graviton are finite, as expected from simple counterterm arguments. We will show that the cancellation no longer holds at two loops, and the theory has an R^3 counterterm, in much the same way as it does for pure Einstein gravity [55]. In six dimensions, pure Einstein gravity is ultraviolet divergent at one loop [35]. We find the same to be true in our double-copy theory, and we find a divergence in eight dimensions as well. We will give the explicit form of the divergences for these cases. In carrying out these computations we use the four-dimensional helicity scheme [56]. It would be interesting to compare our results to ones obtained using the standard dimensional-regularization scheme, used in, for example, Ref. [55].

A. One loop

1. Four dimensions

In four dimensions, there is no one-loop four-point divergence when one external leg is a graviton [32,34] because the potential independent counterterms for such divergences vanish on shell or can be eliminated by the equations of motion. Using the double-copy formula (2.5), we have explicitly confirmed finiteness in one-loop four-point amplitudes containing at least one external graviton, with the remaining legs either gravitons, dilatons, or antisymmetric tensors. We obtain the gravity numerator from the double-copy formula (2.5) by taking the two Yang-Mills numerators, $\tilde{n}_i$ and n_i, to be equal to the BCJ form of the Yang-Mills numerator (3.5). As an interesting cross-check, we have obtained an asymmetric representation of the gravity amplitudes by taking the $\tilde{n}_i$ to be the numerators that satisfy BCJ duality and the n_i to be numerators obtained by gauge-theory Feynman rules in Feynman gauge, similar to the procedure used recently for half-maximal supergravity [23,24]. By generalized gauge invariance [5,8], this should be equivalent to the symmetric construction. Indeed, we find identical results for the ultraviolet divergences.

To evaluate the ultraviolet divergences, we expand in small external momenta to reduce to logarithmically divergent integrals [52]. We then simplify tensor integrals composed of loop momenta in the numerators by using

Lorentz invariance, which implies that the integrals must be linear combinations of products of metric tensors $\eta^{\mu\nu}$. (See Ref. [13] for a recent discussion of evaluating tensor vacuum integrals.) With the insertion of a massive infrared regulator, we finally integrate simple one-loop integrals to find the potential ultraviolet divergence. Due to our auxiliary conditions, contributions from bubbles on external legs vanish, as they would for ordinary gravity Feynman diagrams. We therefore obtain our entire result from box, triangle, and bubble-on-internal-leg diagrams.

For completeness we have also computed the divergences directly in dimensional regularization without introducing a mass regulator, using techniques similar to those for two loops in Appendix A. After subtracting the infrared divergence as computed in Appendix B, we find complete agreement with our result found using the massive infrared regulator.

We obtain an expression for the divergence in terms of formal polarization vectors. By taking linear combinations of the product of polarization vectors from each copy of Yang-Mills, we can project onto the graviton, dilaton, and antisymmetric tensor states. In $D = 4$ this is conveniently implemented by using spinor helicity [57]. Graviton polarization tensors correspond to the "left" and "right" copies of Yang-Mills according to $\varepsilon^{h+}_{\mu\nu} \to \varepsilon^{+}_{\mathrm{L}\mu}\varepsilon^{+}_{\mathrm{R}\nu}$ and $\varepsilon^{h-}_{\mu\nu} \to \varepsilon^{-}_{\mathrm{L}\mu}\varepsilon^{-}_{\mathrm{R}\nu}$. For the dilaton and antisymmetric tensor, we symmetrize and antisymmetrize in opposite-helicity configurations according to $\varepsilon^{\phi}_{\mu\nu} \to \frac{1}{\sqrt{2}}(\varepsilon^{+}_{\mathrm{L}\mu}\varepsilon^{-}_{\mathrm{R}\nu} + \varepsilon^{-}_{\mathrm{L}\mu}\varepsilon^{+}_{\mathrm{R}\nu})$ and $\varepsilon^{A}_{\mu\nu} \to \frac{1}{\sqrt{2}}(\varepsilon^{+}_{\mathrm{L}\mu}\varepsilon^{-}_{\mathrm{R}\nu} - \varepsilon^{-}_{\mathrm{L}\mu}\varepsilon^{+}_{\mathrm{R}\nu})$. By substituting the explicit polarizations, we find that all configurations where at least a single leg is a graviton are free of ultraviolet divergences,

$$\mathcal{M}^{(1)}(1^h, 2, 3, 4)|_{\mathrm{div}} = 0, \tag{4.2}$$

where leg 1 is either a positive- or negative-helicity graviton, and the other three states are unspecified.

We however find divergences for the cases with no external gravitons. For the four-dilaton amplitude, we find

$$\mathcal{M}^{(1)}(1^\phi, 2^\phi, 3^\phi, 4^\phi)|_{\mathrm{div}} = \frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^4 \frac{i}{(4\pi)^2} \frac{1132 - 92D_s + 3D_s^2}{120}(s^2 + t^2 + u^2), \tag{4.3}$$

corresponding to the operator,

$$\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^4 \frac{1}{(4\pi)^4} \frac{1132 - 92D_s + 3D_s^2}{240}(D_\mu\phi D^\mu\phi)^2. \tag{4.4}$$

This result is similar to the one obtained long ago by 't Hooft and Veltman [32]. However, in our case we have an antisymmetric tensor which can circulate in the loop, altering the numerical coefficient. We note that the operator in Ref. [32] looks different than above, but it can be written in a similar way through use of the field equations of motion.

The amplitude with four antisymmetric tensors is also one-loop divergent in four dimensions. In four dimensions, the antisymmetric tensor is dual to a scalar field, so we expect the divergence to be the same as that for dilatons. Indeed, the divergence in the four-antisymmetric-tensor amplitude for a theory with an antisymmetric tensor coupled to gravity is equal to that of the four-dilaton amplitude in a theory of a dilaton coupled to gravity [34]. In congruence, we find the divergence for four external antisymmetric tensors to also be given by the same expression as the four-dilaton divergence (4.3),

$$\mathcal{M}^{(1)}(1^A, 2^A, 3^A, 4^A)|_{\mathrm{div}} = \mathcal{M}^{(1)}(1^\phi, 2^\phi, 3^\phi, 4^\phi)|_{\mathrm{div}}. \tag{4.5}$$

In terms of the antisymmetric tensor fields, the divergence is generated by the operator,

$$\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^4 \frac{1}{(4\pi)^4} \frac{1132 - 92D_s + 3D_s^2}{2160}(H_{\mu\nu\rho}H^{\mu\nu\rho})^2. \tag{4.6}$$

The counterterm that cancels the divergence is given by the negative of this operator.

In addition to the above divergences, there is also a divergence in the $D = 4$, $\phi\phi AA$ amplitude. This divergence is given by

$$\begin{aligned}\mathcal{M}^{(1)}&(1^\phi, 2^\phi, 3^A, 4^A)|_{\mathrm{div}} \\ &= \frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^4 \frac{i}{(4\pi)^4}\left(\frac{1116 - 76D_s - D_s^2}{120}s^2 \right.\\ &\quad \left. + \frac{-1124 + 84D_s - D_s^2}{120}(t^2 + u^2)\right),\end{aligned} \tag{4.7}$$

which corresponds to the operator,

$$\begin{aligned}\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^4 \frac{1}{(4\pi)^4}&\left(\frac{1124 - 84D_s + D_s^2}{60}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}D^\mu\phi D^\nu\phi\right.\\ &\left. - \frac{1132 - 92D_s + 3D_s^2}{360}H_{\mu\nu\rho}H^{\mu\nu\rho}D_\sigma\phi D^\sigma\phi\right).\end{aligned} \tag{4.8}$$

2. Six dimensions

In six dimensions for external gravitons, the only independent invariant operator at one loop [58] is

$$R_{\alpha\beta\mu\nu}R^{\mu\nu\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}. \tag{4.9}$$

This corresponds to the known $D=6$ one-loop divergence of pure Einstein gravity given in Ref. [35]. We have computed the coefficient of the $D=6$ divergence for the double-copy theory of a graviton coupled to a dilaton and an antisymmetric tensor. In this case, the divergence is given by the operator,

$$-\frac{1}{\epsilon}\frac{1}{(4\pi)^3}\frac{(D_s-2)^2}{30240}R_{\alpha\beta\mu\nu}R^{\mu\nu\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}. \tag{4.10}$$

Appropriate powers of the coupling are generated by expanding the metric around flat space, $g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu}$. Although we do not include the explicit forms of the counterterms here, we have also found divergences for the following amplitudes (as well as their permutations and parity conjugates) involving external dilatons and antisymmetric tensors, where we restrict the external states to four dimensions:

$$\begin{aligned}&\mathcal{M}^{(1)}(1^\phi,2^+,3^+,4^+), \qquad \mathcal{M}^{(1)}(1^\phi,2^\phi,3^+,4^+),\\ &\mathcal{M}^{(1)}(1^\phi,2^\phi,3^\phi,4^+), \qquad \mathcal{M}^{(1)}(1^\phi,2^\phi,3^\phi,4^\phi),\\ &\mathcal{M}^{(1)}(1^A,2^A,3^+,4^+), \qquad \mathcal{M}^{(1)}(1^A,2^A,3^\phi,4^+),\\ &\mathcal{M}^{(1)}(1^A,2^A,3^\phi,4^\phi).\end{aligned} \tag{4.11}$$

3. Eight dimensions

In eight dimensions, there are seven linearly independent R^4 operators [59]:

$$\begin{aligned}T_1&=(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2,\\ T_2&=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\lambda}R_{\gamma\delta\kappa}{}^{\sigma}R^{\gamma\delta\kappa\lambda},\\ T_3&=R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\gamma}R^{\lambda\gamma}{}_{\delta\kappa}R^{\rho\sigma\delta\kappa},\\ T_4&=R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\gamma}R^{\rho\lambda}{}_{\delta\kappa}R^{\sigma\gamma\delta\kappa},\\ T_5&=R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\gamma}R^{\rho}{}_{\delta}{}^{\lambda}{}_{\kappa}R^{\sigma\delta\gamma\kappa},\\ T_6&=R_{\mu\nu\rho\sigma}R^{\mu}{}_{\lambda}{}^{\rho}{}_{\gamma}R^{\lambda}{}_{\delta}{}^{\gamma}{}_{\kappa}R^{\nu\delta\sigma\kappa},\\ T_7&=R_{\mu\nu\rho\sigma}R^{\mu}{}_{\lambda}{}^{\rho}{}_{\gamma}R^{\lambda}{}_{\delta}{}^{\nu}{}_{\kappa}R^{\gamma\delta\sigma\kappa}.\end{aligned} \tag{4.12}$$

On shell, the combination

$$U=-\frac{T_1}{16}+T_2-\frac{T_3}{8}-T_4+2T_5-T_6+2T_7 \tag{4.13}$$

is a total derivative, so only six of the T_i are independent on shell. In terms of these operators, the divergence for gravity coupled to a dilaton and an antisymmetric tensor at one loop in $D=8$ is

$$\begin{aligned}&\frac{1}{\epsilon}\frac{1}{(4\pi)^4}\frac{1}{1814400}[(4274-899D_s+11D_s^2)T_1\\ &-40(466-103D_s-2D_s^2)T_2\\ &-2(1886+319D_s-D_s^2)T_3\\ &-180(1034+D_s)T_4+16(1196+34D_s-D_s^2)T_6\\ &+64(12454+71D_s+D_s^2)T_7+cU],\end{aligned} \tag{4.14}$$

where c is a free parameter multiplying the total derivative (4.13).

We have also found that the following four-point amplitudes involving dilatons and antisymmetric tensors diverge in $D=8$:

$$\begin{aligned}&\mathcal{M}^{(1)}(1^\phi,2^\phi,3^+,4^+), \qquad \mathcal{M}^{(1)}(1^\phi,2^\phi,3^+,4^-),\\ &\mathcal{M}^{(1)}(1^\phi,2^\phi,3^\phi,4^\phi), \qquad \mathcal{M}^{(1)}(1^A,2^A,3^+,4^+),\\ &\mathcal{M}^{(1)}(1^A,2^A,3^+,4^-), \qquad \mathcal{M}^{(1)}(1^A,2^A,3^\phi,4^+),\\ &\mathcal{M}^{(1)}(1^A,2^A,3^\phi,4^\phi), \qquad \mathcal{M}^{(1)}(1^A,2^A,3^A,4^A),\end{aligned} \tag{4.15}$$

where we have again chosen the external states to be four dimensional. The other configurations are finite.

B. Ultraviolet properties of gravity at two loops in four dimensions

Pure Einstein gravity in $D=4$ is one-loop finite, but it does diverge at two loops [55]. This suggests that the two-loop four-graviton amplitude, including also the dilaton and antisymmetric tensor, should diverge as well. For external gravitons, the only independent operator is the same R^3 operator for one loop in six dimensions (4.9). Our aim is to find its coefficient.

The R^3 operator generates a nonvanishing four-point amplitude for identical-helicity gravitons, illustrated in Fig. 10. This means that we can determine the coefficient of this operator by computing the four-graviton all-plus-helicity amplitude. Fortunately, as we discussed in Sec. III, we have the BCJ form of the required all-plus-helicity Yang-Mills amplitude. Applying the double-copy formula (2.5) to the Yang-Mills amplitude in Eq. (3.15) immediately gives us the corresponding gravity integrand, simply by squaring the numerators. Figures 6(d)–6(k) integrate to zero in gravity just as they did in Yang-Mills. In addition, as

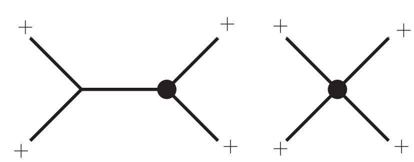

FIG. 10. The R^3 operator diagrams that contribute to the all-plus-helicity four-graviton amplitude. The solid dot represents vertices generated by the R^3 operator.

was mentioned in Sec. III B, the second term of the double-triangle in Eq. (3.11) also integrates to zero; in fact, due to the simple identity,

$$-(p-q)\cdot(k_1+k_2)+s=\frac{1}{2}(p-k_1-k_2)^2+\frac{1}{2}(q+k_1+k_2)^2-\frac{1}{2}p^2-\frac{1}{2}q^2, \tag{4.16}$$

all such terms will integrate to zero because the inverse propagators lead to scale-free integrals. Thus, the four-graviton all-plus-helicity amplitude is given by

$$\mathcal{M}^{(2)}(1^+,2^+,3^+,4^+)=\left(\frac{\kappa}{2}\right)^6\sum_{\mathcal{S}_4}\left[\frac{1}{4}M^{\mathrm{P}}_{1234}+\frac{1}{4}M^{\mathrm{NP}}_{12;34}+\frac{1}{8}M^{\mathrm{DT}}_{1234}\right], \tag{4.17}$$

where

$$\begin{aligned}
M^{\mathrm{P}}_{1234}&=i\mathcal{T}^2\mathcal{I}^{\mathrm{P}}_4\Bigg[\Bigg(\frac{(D_s-2)^2}{2}(p+q)^2\lambda_p^2\lambda_q^2+16s((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2)\\
&\quad+(D_s-2)(s(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2)+4(p+q)^2(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q))\Bigg)^2\Bigg](s,t)\\
&=i\mathcal{T}^2\Bigg\{\mathcal{I}^{\mathrm{P}}_4[((D_s-2)s(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2)+16s((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2))^2](s,t)\\
&\quad+\mathcal{I}^{\text{bow-tie}}_4\Bigg[2\Bigg(4(D_s-2)(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q)+\frac{(D_s-2)^2}{2}\lambda_p^2\lambda_q^2\Bigg)\\
&\quad\times((D_s-2)s(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2)+16s((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2))\Bigg](s,t)\\
&\quad+\mathcal{I}^{\text{bow-tie}}_4\Bigg[(p+q)^2\Bigg(\frac{(D_s-2)^2}{2}\lambda_p^2\lambda_q^2+4(D_s-2)(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q)\Bigg)^2\Bigg](s,t)\Bigg\},\\
M^{\mathrm{NP}}_{12;34}&=i\mathcal{T}^2s^2\mathcal{I}^{\mathrm{NP}}_4[((D_s-2)(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2)+16((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2))^2](s,t),\\
M^{\mathrm{DT}}_{1234}&=i\mathcal{T}^2\mathcal{I}^{\mathrm{DT}}_4\Bigg[\Bigg(\frac{(D_s-2)^2}{2}(4p\cdot q+2(p-q)\cdot(k_1+k_2)-s)\lambda_p^2\lambda_q^2\Bigg)^2\Bigg](s,t)\\
&=i\mathcal{T}^2\frac{1}{s}\mathcal{I}^{\text{bow-tie}}_4\Bigg[\Bigg(\frac{(D_s-2)^2}{2}(4p\cdot q+2(p-q)\cdot(k_1+k_2)-s)\lambda_p^2\lambda_q^2\Bigg)^2\Bigg](s,t).
\end{aligned} \tag{4.18}$$

We have explicitly confirmed that s-, t-, and u-channel unitarity cuts are satisfied. We did so numerically keeping the internal states in integer dimensions $D=6$ and $D=8$.

To obtain the ultraviolet divergences, we integrate the amplitudes in dimensional regularization. We carry out the extraction of the ultraviolet divergences in two ways. In the first approach we simply use dimensional regularization and then subtract the known infrared divergences, leaving only the ultraviolet ones. In the second approach we introduce a mass regulator to separate the ultraviolet singularities from the infrared divergences, as carried out in Appendix C. Either method yields the same result. In fact, the second method also shows that the vanishing integrals that we dropped, including Figs. 6(d)–6(k) and the second term of the double-triangle in Eq. (3.11), are not ultraviolet divergent.

The dimensionally regularized integrals are performed in Appendix A. Equation (A10) gives the planar double-box integrals, Eq. (A17) gives the nonplanar double-box integrals, and Eq. (A21) gives the bow-tie integrals. The infrared divergence from Appendix B is

$$\begin{aligned}
\mathcal{M}^{(2)}(1^+,2^+,3^+,4^+)|_{\mathrm{IR\,div}}&=-\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^6\frac{i}{(4\pi)^4}\mathcal{T}^2\frac{(D_s-2)^2}{120}(s^2+t^2+u^2)\\
&\quad\times\left[s\log\left(\frac{-s}{\mu^2}\right)+t\log\left(\frac{-t}{\mu^2}\right)+u\log\left(\frac{-u}{\mu^2}\right)\right].
\end{aligned} \tag{4.19}$$

We insert the divergent parts of the integrals evaluated using dimensional regularization into Eq. (4.18), then insert these results into Eq. (4.17) and perform the permutation sum. Finally we subtract the infrared divergence and arrive at the two-loop ultraviolet divergence of gravity coupled to a dilaton and an antisymmetric tensor for four external positive-helicity gravitons:

$$\mathcal{M}^{(2)}(1^+,2^+,3^+,4^+)|_{\mathrm{UV\,div}} = \frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^4}\mathcal{T}^2 \frac{(2D_s^4 - 136D_s^3 + 2883D_s^2 - 35164D_s + 103052)stu}{10800}. \quad (4.20)$$

For our second method, we evaluate the ultraviolet divergences of the required integrals by going to vacuum integrals and using a massive infrared regulator, sidestepping the need to subtract the infrared divergence. The ultraviolet divergences of the individual integrals are calculated in Appendix C. After permutations, the contributions of the planar double-box, nonplanar double-box, and double-triangle components are

$$\begin{aligned}
\mathcal{M}^{\mathrm{P}}(1^+,2^+,3^+,4^+)|_{\mathrm{UV\,div}} &= -\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^4}\mathcal{T}^2 \frac{(2D_s^3 - 63D_s^2 + 588D_s - 1420)stu}{180},\\
\mathcal{M}^{\mathrm{NP}}(1^+,2^+,3^+,4^+)|_{\mathrm{UV\,div}} &= -\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^4}\mathcal{T}^2 \frac{(21D_s^2 - 4D_s - 396)stu}{240},\\
\mathcal{M}^{\mathrm{DT}}(1^+,2^+,3^+,4^+)|_{\mathrm{UV\,div}} &= \frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^4}\mathcal{T}^2 \frac{(D_s - 2)^4 stu}{5400}.
\end{aligned} \quad (4.21)$$

Summing these contributions, we find complete agreement with Eq. (4.20).

We can reexpress the two-loop divergence in terms of the operator that generates it. By matching the amplitude generated by the diagrams with an R^3 vertex shown in Fig. 10 to the divergence in Eq. (4.20), we find that the operator

$$-\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^4} \frac{2D_s^4 - 136D_s^3 + 2883D_s^2 - 35164D_s + 103052}{648000} R_{\alpha\beta\mu\nu}R^{\mu\nu\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta} \quad (4.22)$$

generates the bare two-loop divergence for gravity coupled to a dilaton and an antisymmetric tensor. As will be discussed in Ref. [60] there are also additional evanescent counterterm contributions.

V. CONCLUSION

In this paper we constructed a representation of the one-loop four-point amplitude of pure Yang-Mills theory explicitly exhibiting the duality between color and kinematics. This construction is the first nonsupersymmetric example at loop level valid in any dimension with no restriction on the external states. The cost of this generality is relatively complicated expressions in terms of formal polarization vectors.

The duality between color and kinematics and its associated gravity double-copy structure has proven useful for unraveling ultraviolet properties in various dimensions [5,13,23,24,31]. Using the one-loop four-point pure Yang-Mills amplitude with the duality manifest, we obtained the integrand for the corresponding amplitude in a theory of a graviton, dilaton, and antisymmetric tensor. In $D = 4$, we found that one-loop four-point amplitudes with one or more external gravitons are ultraviolet finite, while amplitudes involving only external dilatons or antisymmetric tensor fields diverge. This result is similar to those of earlier studies involving gravity coupled either to a scalar, an antisymmetric tensor, or other matter and is in line with simple counterterm arguments [32–34]. We gave the explicit form, including numerical coefficients, for all four-point divergences in this theory. Since our construction is valid in any dimension, we also investigated the ultraviolet properties of the double-copy theory in higher dimensions. In particular, we showed that in $D = 6, 8$ the one-loop four-graviton amplitudes diverge, as expected, and gave the explicit form of these divergences including their numerical coefficients.

In order to investigate whether the observed $D = 4$ ultraviolet finiteness of the amplitudes with one or more external gravitons continues beyond one loop, we also computed the coefficient of the potential two-loop R^3 divergences. This was greatly simplified by the observation that the coefficient of the divergence can be determined from the identical-helicity four-graviton configuration. The required gravity amplitude was then easily constructed via the double-copy property, by first finding a representation of the pure Yang-Mills amplitude that satisfies the duality. The existence of such a representation has already been noted in Ref. [5]. Here we provided the explicit representation, including diagrams that integrate to zero not present in the original form of the two-loop identical-helicity amplitude given in Ref. [19]. We found that the two-loop amplitude with external gravitons is indeed divergent and that the R^3 counterterm has a nonzero coefficient. This is not surprising given that pure Einstein gravity diverges at two loops [55]. Our paper definitively shows that, as one might have expected, the double-copy property by itself cannot render a gravity theory ultraviolet finite. For ultraviolet finiteness, an additional mechanism such as supersymmetry is needed. Further progress in clarifying the ultraviolet structure of gravity theories will undoubtedly rely on new multiloop calculations to guide theoretical developments. We expect that the duality between color and kinematics will continue to play an important role in this.

ACKNOWLEDGMENTS

We thank J. J. M. Carrasco, H.-H. Chi, L. Dixon, H. Elvang, H. Johansson, S. Naculich, R. Roiban, E. Serna Campillo, and A. Tseytlin for helpful discussions. We also thank D. O'Connell for important comments on an earlier version of this paper. We thank R. Roiban for cross-checking our one-loop integral reductions and L. Dixon for cross checking our two-loop integrals. This research was supported by the U.S. Department of Energy under Contracts No. DE-FG03-91ER40662 and No. DE-SC0007859. The work of S. D. was supported by a U.S. Department of Energy Graduate Student Fellowship under Contract No. DE-SC0008279.

Note added in proof.—A subsequent analysis has revealed that at two loops there are evanescent counterterms not accounted for in our two-loop analysis [60]. The appearance of such counterterms is unexpected because there are no corresponding one-loop divergences in $D = 4$ that can act as subdivergences. The net effect is that the numerical coefficient of the divergence in Eq. (4.22) should be interpreted as a bare result without counterterm or subdivergence subtractions. Consequently, the coefficient will be modified, although the conclusion that there is a divergence is unaltered. This surprising phenomenon, as well as the counterterm subtraction terms, will be described in Ref. [60].

APPENDIX A: TWO-LOOP DIMENSIONALLY REGULARIZED INTEGRALS

In this appendix, we explicitly compute the divergent parts of dimensionally regularized two-loop integrals in $D = 4 - 2\epsilon$, appearing in Sec. IV B. In general, both ultraviolet and infrared divergences appear as poles in ϵ so we must subtract the infrared ones in order to obtain the ultraviolet ones.

We start with the planar double-box integral displayed in Fig. 6(a), following the discussion in Sec. 4 of Ref. [61],

$$\mathcal{I}_4^{\rm P}[\mathcal{P}(\lambda_i, p, q, k_i)](s,t) \equiv \int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{\mathcal{P}(\lambda_i, p, q, k_i)}{p^2 q^2 (p+q)^2 (p-k_1)^2 (p-k_1-k_2)^2 (q-k_4)^2 (q-k_3-k_4)^2}. \tag{A1}$$

Using Schwinger parameters, we rewrite the planar double-box integral with constant numerator as

$$\mathcal{I}_4^{\rm P}[1](s,t) = \frac{1}{(4\pi)^D} \prod_{i=1}^{7} \int_0^\infty dt_i [\Delta_{\rm P}(T)]^{-\frac{D}{2}} \exp\left[-\frac{Q_{\rm P}(s,t,t_i)}{\Delta_{\rm P}(T)}\right], \tag{A2}$$

where

$$\Delta_{\rm P}(T) = (T_p T_q + T_p T_{pq} + T_q T_{pq}), \tag{A3}$$

and

$$\begin{aligned} T_p &= t_3 + t_4 + t_5, \\ T_q &= t_1 + t_2 + t_7, \\ T_{pq} &= t_6. \end{aligned} \tag{A4}$$

T_p, T_q, and T_{pq} are sums of Schwinger parameters corresponding to propagators with loop momenta p, q, and $p+q$, respectively. We also have

$$Q_{\rm P}(s,t,t_i) = -s(t_1 t_2 T_p + t_3 t_4 T_q + t_6 (t_1 + t_3)(t_2 + t_4)) - t t_5 t_6 t_7. \tag{A5}$$

To account for factors of λ_p^2, λ_q^2, and λ_{p+q}^2 in the numerator, we take derivatives on the (-2ϵ)-dimensional part of the (Wick-rotated) integral:

$$\int d\lambda_p^{-2\epsilon} d\lambda_q^{-2\epsilon} \exp\left[-T_p \lambda_p^2 - T_q \lambda_q^2 - T_{pq} \lambda_{p+q}^2\right] \propto [\Delta_{\rm P}(T)]^{\epsilon}, \tag{A6}$$

with respect to T_p, T_q, and T_{pq}. This introduces additional factors to be inserted in the integrand in Eq. (A2). For example,

$$
\begin{aligned}
(\lambda_p^2)^4 &\to -\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)\left(\frac{T_q+T_{pq}}{\Delta_{\rm P}(T)}\right)^4,\\
(\lambda_p^2)^3\lambda_q^2 &\to \epsilon^2(1-\epsilon)(2-\epsilon)\frac{(T_q+T_{pq})^2}{\Delta_{\rm P}(T)^3}-\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)\frac{(T_q+T_{pq})^2T_{pq}^2}{\Delta_{\rm P}(T)^4},\\
(\lambda_p^2)^2\lambda_q^2\lambda_{p+q}^2 &\to \epsilon^2(1-\epsilon)^2\frac{1}{\Delta_{\rm P}(T)^2}+\epsilon(1-\epsilon)(2-\epsilon)\frac{\epsilon(T_q^2+T_{pq}^2)+2T_qT_{pq}}{\Delta_{\rm P}(T)^3}-\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)\frac{T_q^2T_{pq}^2}{\Delta_{\rm P}(T)^4}. \qquad (A7)
\end{aligned}
$$

We account for extra factors of $\Delta_{\rm P}^a(T)$ by shifting the dimension $D \to D-2a$. Following Smirnov [62], we change six of the seven Schwinger parameters to Feynman parameters with the delta-function constraint $\sum_{i\neq 6}\alpha_i = 1$:

$$
\mathcal{I}_4^{\rm P}[\mathcal{P}(\lambda_p,\lambda_q)](s,t) = \frac{\Gamma[7-D+\gamma]}{(4\pi)^D}\int_0^\infty d\alpha_6\prod_{i\neq 6}\int_0^1 d\alpha_i\,\delta\left(1-\sum_{i\neq 6}\alpha_i\right)\frac{[\Delta_{\rm P}(T)]^{7-\frac{3D}{2}+\gamma}}{[Q_{\rm P}(s,t,\alpha_i)]^{7-D+\gamma}}D(\alpha_i), \qquad (A8)
$$

where $D(\alpha_i)$ represents the extra factors in one term of Eq. (A7), with $t_i \to \alpha_i$. The parameter γ counts the factors of α_i in $D(\alpha_i)$ and can take on values 0, 2, and 4 for the integrals under consideration here. Next we perform a change of variables that imposes the delta-function constraint [62]:

$$
\begin{aligned}
&\alpha_1 = \beta_1\xi_3, \qquad \alpha_2 = (1-\xi_5)(1-\xi_4), \qquad \alpha_3 = \beta_2\xi_1, \qquad \alpha_4 = \xi_5(1-\xi_2),\\
&\alpha_5 = \beta_2(1-\xi_1), \qquad \alpha_7 = \beta_1(1-\xi_3), \qquad \beta_1 = (1-\xi_5)\xi_4, \qquad \beta_2 = \xi_5\xi_2. \qquad (A9)
\end{aligned}
$$

We then integrate these parameters to obtain a Mellin-Barnes representation, which we again integrate. Finally we arrive at the dimensionally regularized results of our required planar double-box integrals:

$$
\begin{aligned}
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^4](s,t) &= \mathcal{I}'^{\rm P} - \frac{1}{(4\pi)^4}\frac{s+2t}{360\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[(\lambda_{p+q}^2)^4](s,t) &= 2\mathcal{I}'^{\rm P} - \frac{1}{(4\pi)^4}\frac{29s+4t}{180\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^3\lambda_q^2](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{480\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^3\lambda_{p+q}^2](s,t) &= \mathcal{I}'^{\rm P} + \frac{1}{(4\pi)^4}\frac{s-t}{360\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^2(\lambda_q^2)^2](s,t) &= \mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^2(\lambda_{p+q}^2)^2](s,t) &= \mathcal{I}'^{\rm P} - \frac{1}{(4\pi)^4}\frac{s+2t}{720\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^2\lambda_q^2\lambda_{p+q}^2](s,t) &= \frac{1}{(4\pi)^4}\frac{s}{720\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[\lambda_p^2\lambda_q^2(\lambda_{p+q}^2)^2](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{240\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[\lambda_p^2(\lambda_{p+q}^2)^3](s,t) &= \mathcal{I}'^{\rm P} - \frac{1}{(4\pi)^4}\frac{5s+t}{180\epsilon}+\mathcal{O}(\epsilon^0), \qquad (A10)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}'^{\rm P} \equiv \frac{1}{(4\pi)^4}\Bigg[&\frac{1}{840s\epsilon^2}(2s^2+st+2t^2)(-s)^{-2\epsilon}e^{-2\epsilon\gamma_{\rm E}}+\frac{1}{88200su^4\epsilon}(4s^6+753s^5t+4306s^4t^2+9144s^3t^3-315\pi^2s^3t^3\\
&+9381s^2t^4+4813st^5+1019t^6)+\frac{t^3(11s^2+7st+2t^2)}{840su^3\epsilon}\log\left(\frac{t}{s}\right)-\frac{s^2t^3}{280u^4\epsilon}\log^2\left(\frac{t}{s}\right)\Bigg]+\mathcal{O}(\epsilon^0). \qquad (A11)
\end{aligned}
$$

All integrals above are symmetric under $\lambda_p \leftrightarrow \lambda_q$.

Next we look at the nonplanar double-box integrals:

$$\mathcal{I}_4^{\rm NP}[\mathcal{P}(\lambda_i,p,q,k_i)](s,t)\equiv\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\frac{\mathcal{P}(\lambda_i,p,q,k_i)}{p^2q^2(p+q)^2(p-k_1)^2(q-k_2)^2(p+q+k_3)^2(p+q+k_3+k_4)^2},\qquad\text{(A12)}$$

whose evaluation follows that of the planar double-box integrals quite closely. $\Delta_{\rm NP}(T)$ takes the same form as $\Delta_{\rm P}(T)$ in Eq. (A3), except that

$$T_p=t_1+t_2,\qquad T_q=t_3+t_4,\qquad T_{pq}=t_5+t_6+t_7.\qquad\text{(A13)}$$

We then also have

$$Q_{\rm NP}(s,t,u,t_i)=-s(t_1t_3t_5+t_2t_4t_7+t_5t_7(T_p+T_q))-tt_2t_3t_6-ut_1t_4t_6.\qquad\text{(A14)}$$

In this case, we find it advantageous to only change the four Schwinger parameters associated with T_p and T_q to Feynman parameters, resulting in

$$\mathcal{I}_4^{\rm NP}[\mathcal{P}(\lambda_p,\lambda_q)]=\frac{\Gamma[7-D+\gamma]}{(4\pi)^D}\prod_{i=5}^{7}\int_0^\infty d\alpha_i\prod_{j=1}^{4}\int_0^1 d\alpha_j\delta\left(1-\sum_{i=1}^{4}\alpha_i\right)\frac{[\Delta_{\rm NP}(T)]^{7-\frac{3D}{2}+\gamma}}{[Q_{\rm NP}(s,t,u,\alpha_i)]^{7-D+\gamma}}D(\alpha_i).\qquad\text{(A15)}$$

We impose the delta-function constraint via further redefinition:

$$\alpha_1=\xi_3(1-\xi_1),\qquad\alpha_2=\xi_3\xi_1,\qquad\alpha_3=(1-\xi_3)(1-\xi_2),\qquad\alpha_4=(1-\xi_3)\xi_2.\qquad\text{(A16)}$$

Once again we can straightforwardly integrate the parameters and use the Mellin-Barnes representation to evaluate our required nonplanar double-box integrals:

$$\begin{aligned}
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^4](s,t)&=\mathcal{I}'^{\rm NP}-\frac{1}{(4\pi)^4}\frac{215s^2+342st+342t^2}{50400s\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[(\lambda_{p+q}^2)^4](s,t)&=\frac{1}{(4\pi)^4}\frac{s}{80\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^3\lambda_q^2](s,t)&=\mathcal{I}'^{\rm NP}-\frac{1}{(4\pi)^4}\frac{215s^2+342st+342t^2}{50400s\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^3\lambda_{p+q}^2](s,t)&=\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^2(\lambda_q^2)^2](s,t)&=\mathcal{I}'^{\rm NP}-\frac{1}{(4\pi)^4}\frac{230s^2+171st+171t^2}{25200s\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^2(\lambda_{p+q}^2)^2](s,t)&=\frac{1}{(4\pi)^4}\frac{s}{160\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^2\lambda_q^2\lambda_{p+q}^2](s,t)&=\frac{1}{(4\pi)^4}\frac{s}{1440\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[\lambda_p^2\lambda_q^2(\lambda_{p+q}^2)^2](s,t)&=\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[\lambda_p^2(\lambda_{p+q}^2)^3](s,t)&=\frac{1}{(4\pi)^4}\frac{s}{160\epsilon}+\mathcal{O}(\epsilon^0),
\end{aligned}\qquad\text{(A17)}$$

where

$$\mathcal{I}'^{\rm NP} \equiv \frac{1}{(4\pi)^4}\left[\frac{1}{840 s\epsilon^2}(2t^2+tu+2u^2)(-s)^{-\epsilon}(-t)^{-\epsilon}e^{-2\epsilon\gamma_{\rm E}} + \frac{1}{352800 s^5\epsilon}(5581u^6+25188u^5t+51783u^4t^2+64352u^3t^3 \right.$$
$$\left. -1260\pi^2u^3t^3+51783u^2t^4+25188ut^5+5581t^6) + \frac{u^3(11t^2+7tu+2u^2)}{840s^4\epsilon}\log\left(\frac{u}{t}\right) - \frac{t^3u^3}{280s^5\epsilon}\log^2\left(\frac{u}{t}\right)\right]$$
$$+\mathcal{O}(\epsilon^0). \tag{A18}$$

As with the planar results, the above are valid under the exchange $\lambda_p \leftrightarrow \lambda_q$.

Finally we evaluate the bow-tie integrals:

$$\mathcal{I}_4^{\text{bow-tie}}[\mathcal{P}(\lambda_i,p,q,k_i)](s) \equiv \int \frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\frac{\mathcal{P}(\lambda_i,p,q,k_i)}{p^2q^2(p-k_1)^2(p-k_1-k_2)^2(q-k_4)^2(q-k_3-k_4)^2}. \tag{A19}$$

The bow-tie integrals are relatively simple because they are products of two one-loop integrals. Similar techniques involving Schwinger parameters and Mellin-Barnes representations can be used on each one-loop integral. Since bubbles with a massless leg vanish in dimensional regularization, the replacement $(p+q)^2 \to 2p\cdot q$ is valid in the numerator. We also use the tensor reduction $(\lambda_p\cdot\lambda_q)^2 \to \lambda_p^2\lambda_q^2/(-2\epsilon)$. For the bow-tie integrals appearing in Eq. (4.18), this tensor reduction is the only source of an ultraviolet divergence. When evaluating the bow-tie contributions then, we expose $(\lambda_p\cdot\lambda_q)^2$ factors through the substitutions,

$$\lambda_{p+q}^2 \to \lambda_p^2+\lambda_q^2+2(\lambda_p\cdot\lambda_q), \qquad (p+q)^2 \to (2p_{(4)}\cdot q_{(4)}) - 2(\lambda_p\cdot\lambda_q). \tag{A20}$$

Only terms containing a $(\lambda_p\cdot\lambda_q)^2$ are ultraviolet divergent; there are no terms with $(\lambda_p\cdot\lambda_q)^4$ or higher powers of $(\lambda_p\cdot\lambda_q)$. The relevant bow-tie integrals are then given by

$$\mathcal{I}_4^{\text{bow-tie}}[(\lambda_p^2)^2(\lambda_p\cdot\lambda_q)^2](s) = \frac{1}{(4\pi)^4}\frac{s^2}{720\epsilon}+\mathcal{O}(\epsilon^0),$$
$$\mathcal{I}_4^{\text{bow-tie}}[\lambda_p^2\lambda_q^2(\lambda_p\cdot\lambda_q)^2](s) = \frac{1}{(4\pi)^4}\frac{s^2}{1152\epsilon}+\mathcal{O}(\epsilon^0),$$
$$\mathcal{I}_4^{\text{bow-tie}}[(\lambda_p^2)^2\lambda_q^2(\lambda_p\cdot\lambda_q)^2](s) = \frac{1}{(4\pi)^4}\frac{s^3}{8640\epsilon}+\mathcal{O}(\epsilon^0),$$
$$\mathcal{I}_4^{\text{bow-tie}}[(\lambda_p^2)^2(\lambda_q^2)^2(\lambda_p\cdot\lambda_q)^2](s) = \frac{1}{(4\pi)^4}\frac{s^4}{64800\epsilon}+\mathcal{O}(\epsilon^0),$$
$$\mathcal{I}_4^{\text{bow-tie}}[(\lambda_p^2)^2(\lambda_p\cdot\lambda_q)^2(2p_{(4)}\cdot q_{(4)})](s,t) = -\frac{1}{(4\pi)^4}\frac{s^2(10s-t)}{15120\epsilon}+\mathcal{O}(\epsilon^0),$$
$$\mathcal{I}_4^{\text{bow-tie}}[\lambda_p^2\lambda_q^2(\lambda_p\cdot\lambda_q)^2(2p_{(4)}\cdot q_{(4)})](s,t) = -\frac{1}{(4\pi)^4}\frac{s^2(12s-t)}{28800\epsilon}+\mathcal{O}(\epsilon^0). \tag{A21}$$

These are also symmetric under the exchange $\lambda_p \leftrightarrow \lambda_q$.

APPENDIX B: TWO-LOOP INFRARED DIVERGENCE

In this appendix we obtain the two-loop infrared divergence for the four-point all-plus-helicity graviton amplitude in the theory of gravity coupled to a dilaton and an antisymmetric tensor using dimensional regularization in $D=4-2\epsilon$. We subtract the infrared divergence from the total divergence to obtain the ultraviolet divergence. Infrared divergences in gravity can be obtained by exponentiating the divergence found at the one-loop order [16,63,64]. In the cases where there is a divergence at one loop, the infrared singularities are "one-loop exact"; however, in the all-plus-helicity gravitons case, the first divergence occurs at two loops. Nevertheless, the same principles apply. More specifically we are concerned with the exponentiation of the gravitational soft function, which describes the effects of soft graviton exchange between external particles.

Following the discussion of Ref. [64], a gravity scattering amplitude can be written as

$$\mathcal{M}_n = S_n \cdot H_n, \tag{B1}$$

where S_n is the infrared-divergent soft function and H_n is the infrared-finite hard function. Each quantity in Eq. (B1) can be written as a loop expansion in powers of $(\kappa/2)^2(4\pi e^{-\gamma_E})^\epsilon$:

$$\mathcal{M}_n = \sum_{L=0}^{\infty} \mathcal{M}_n^{(L)}, \qquad S_n = 1 + \sum_{L=1}^{\infty} S_n^{(L)}, \qquad H_n = \sum_{L=0}^{\infty} H_n^{(L)}. \tag{B2}$$

The soft function is given by the exponential of the lowest-order infrared divergence:

$$S_n = \exp\left[\frac{\sigma_n}{\epsilon}\right], \qquad \sigma_n = \left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^{2-\epsilon}} e^{-\gamma_E \epsilon} \sum_{j=1}^{n} \sum_{i<j} s_{ij} \log\left(\frac{-s_{ij}}{\mu^2}\right), \qquad s_{ij} = (k_i + k_j)^2. \tag{B3}$$

An L-loop amplitude can then be written as

$$\mathcal{M}_n^{(L)} = \sum_{l=0}^{L} \frac{1}{(L-l)!} \left[\frac{\sigma_n}{\epsilon}\right]^{L-l} H_n^{(l)}(\epsilon). \tag{B4}$$

For four-point amplitudes, we have

$$\sigma_4 = \left(\frac{\kappa}{2}\right)^2 \frac{2}{(4\pi)^{2-\epsilon}} e^{-\gamma_E \epsilon} \left[s \log\left(\frac{-s}{\mu^2}\right) + t \log\left(\frac{-t}{\mu^2}\right) + u \log\left(\frac{-u}{\mu^2}\right)\right], \tag{B5}$$

and the one-loop infrared divergence is given by

$$\mathcal{M}_4^{(1)}|_{\mathrm{IR\,div}} = \frac{\sigma_4}{\epsilon} \mathcal{M}_4^{(0)}. \tag{B6}$$

We used this to subtract the infrared divergence from our dimensionally regularized one-loop result in Sec. IVA 1 to isolate the ultraviolet divergence. The four-point two-loop infrared divergence is given by

$$\mathcal{M}_4^{(2)}|_{\mathrm{IR\,div}} = \frac{1}{2}\left[\frac{\sigma_4}{\epsilon}\right]^2 \mathcal{M}_4^{(0)} + \frac{\sigma_4}{\epsilon} H_4^{(1)}(\epsilon)\big|_{\mathrm{IR\,div}}. \tag{B7}$$

For the all-plus-helicity gravitons case, the tree amplitude $\mathcal{M}_4^{(0)}$ vanishes. The one-loop amplitude is therefore infrared finite and equal to the one-loop infrared-finite hard function. The one-loop amplitude can be computed using the double-copy procedure in Sec. IVA 1 and is given by [65]

$$\mathcal{M}^{(1)}(1^+, 2^+, 3^+, 4^+) = -\left(\frac{\kappa}{2}\right)^4 \frac{i}{(4\pi)^2} \left(\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\right)^2 \frac{(D_s-2)^2}{240} (s^2 + t^2 + u^2). \tag{B8}$$

The two-loop infrared divergence is then

$$\begin{aligned}\mathcal{M}^{(2)}(1^+, 2^+, 3^+, 4^+)|_{\mathrm{IR\,div}} = & -\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^4} \left(\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\right)^2 \frac{(D_s-2)^2}{120} (s^2 + t^2 + u^2) \\ & \times \left[s \log\left(\frac{-s}{\mu^2}\right) + t \log\left(\frac{-t}{\mu^2}\right) + u \log\left(\frac{-u}{\mu^2}\right)\right]. \end{aligned} \tag{B9}$$

APPENDIX C: TWO-LOOP ULTRAVIOLET DIVERGENCES FROM VACUUM INTEGRALS

In this appendix we compute the ultraviolet divergences of the integrals in Sec. IV B. The techniques are very similar to those used to study the one-loop ultraviolet properties of gravity in Sec. IV. However, before we can use them, we must deal with the (-2ϵ)-dimensional components λ_p, λ_q, and λ_{p+q} in the numerators of the integrals using the techniques in Sec. 4.1 of Ref. [61].

The effect of inserting factors of λ_p, λ_q, and λ_{p+q} into the planar and nonplanar double-box integrals is very similar to inserting factors of $v\cdot p$, $v\cdot q$, and $v\cdot(p+q)$, where

$$v^\mu \equiv \epsilon^\mu{}_{\nu_1\nu_2\nu_3} k_1^{\nu_1} k_2^{\nu_2} k_3^{\nu_3}. \tag{C1}$$

Example parameter insertions for factors of λ_i are given in Eq. (A7). For polynomials in $v\cdot p$ and $v\cdot q$, we have

$$\begin{aligned}
(v\cdot p)^8 &\to 105\left(\frac{stu}{8}\right)^4 \frac{(T_q+T_{pq})^4}{\Delta^4},\\
(v\cdot p)^6(v\cdot q)^2 &\to \left(\frac{stu}{8}\right)^4\left[15\frac{(T_q+T_{pq})^2}{\Delta^3}+105\frac{(T_q+T_{pq})^2T_{pq}^2}{\Delta^4}\right],\\
(v\cdot p)^4(v\cdot q)^4 &\to \left(\frac{stu}{8}\right)^4\left[9\frac{1}{\Delta^2}+90\frac{T_{pq}^2}{\Delta^3}+105\frac{T_{pq}^4}{\Delta^4}\right],\\
(v\cdot p)^4(v\cdot q)^2(v\cdot(p+q))^2 &\to \left(\frac{stu}{8}\right)^4\left[9\frac{1}{\Delta^2}+15\frac{3T_q^2+3T_{pq}^2-2(T_q+T_{pq})^2}{\Delta^3}+105\frac{T_q^2T_{pq}^2}{\Delta^4}\right].
\end{aligned}\tag{C2}$$

These are valid for both the planar and nonplanar double boxes provided the corresponding definitions for Δ, T_p, T_q, and T_{pq} given in Appendix A are used.

We can also relate polynomials in $v\cdot p$ and $v\cdot q$ to the λ_i. The four-dimensional component of the loop momenta p can be written as

$$p_{[4]}^\mu \equiv c_1^p k_1^\mu + c_2^p k_2^\mu + c_3^p k_3^\mu + c_v^p v^\mu, \tag{C3}$$

where

$$\begin{aligned}
c_1^p &= \frac{1}{2su}[-t(2p\cdot k_1)+u(2p\cdot k_2)+s(2p\cdot k_3)],\\
c_2^p &= \frac{1}{2st}[t(2p\cdot k_1)-u(2p\cdot k_2)+s(2p\cdot k_3)],\\
c_3^p &= \frac{1}{2tu}[t(2p\cdot k_1)+u(2p\cdot k_2)-s(2p\cdot k_3)],\\
c_v^p &= -\frac{4}{stu}\epsilon_{\mu\nu_1\nu_2\nu_3}p^\mu k_1^{\nu_1}k_2^{\nu_2}k_3^{\nu_3} = -\frac{4}{stu}v\cdot p.
\end{aligned}\tag{C4}$$

We therefore have

$$p^2+\lambda_p^2 = p_{[4]}\cdot p_{[4]} = sc_1^pc_2^p + tc_2^pc_3^p + uc_1^pc_3^p - \frac{1}{4}stu(c_v^p)^2, \tag{C5}$$

or

$$\lambda_p^2 = -\frac{4}{stu}(v\cdot p)^2 + \hat{\mathcal{P}}_p, \tag{C6}$$

where

$$\hat{\mathcal{P}}_p \equiv -p^2 + sc_1^p c_2^p + tc_2^p c_3^p + uc_1^p c_3^p. \tag{C7}$$

Similarly, we have

$$\lambda_q^2 = -\frac{4}{stu}(v\cdot q)^2 + \hat{\mathcal{P}}_q,$$
$$\lambda_{p+q}^2 = -\frac{4}{stu}(v\cdot (p+q))^2 + \hat{\mathcal{P}}_{pq}, \tag{C8}$$

where

$$\hat{\mathcal{P}}_q \equiv -q^2 + sc_1^q c_2^q + tc_2^q c_3^q + uc_1^q c_3^q,$$
$$\hat{\mathcal{P}}_{pq} \equiv -(p+q)^2 + s(c_1^p + c_1^q)(c_2^p + c_2^q) + t(c_2^p + c_2^q)(c_3^p + c_3^q) + u(c_1^p + c_1^q)(c_3^p + c_3^q). \tag{C9}$$

These relations, along with the parameter replacements in Eqs. (A7) and (C2), allow us to rewrite the integrals involving factors λ_i in terms of integrals involving tensor products between the loop momenta and the external momenta. For a general function $f(p\cdot k_i, q\cdot k_i)$, we have

$$\begin{aligned}
\int (\lambda_p^2)^4 f &= -\frac{\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)}{105}\left(\frac{8}{stu}\right)^4 \int (v\cdot p)^8 f \\
&= -\frac{16\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)}{(1-2\epsilon)(3-2\epsilon)(5-2\epsilon)(7-2\epsilon)}\int \hat{\mathcal{P}}_p^4 f, \\
\int (\lambda_p^2)^3 \lambda_q^2 f &= -\frac{16\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)}{(1-2\epsilon)(3-2\epsilon)(5-2\epsilon)(7-2\epsilon)}\int \hat{\mathcal{P}}_p^3 \hat{\mathcal{P}}_q f \\
&\quad + \frac{12\epsilon(1-\epsilon)(2-\epsilon)}{(3-2\epsilon)(5-2\epsilon)(7-2\epsilon)}\int \frac{\hat{\mathcal{P}}_p^2 f}{\Delta}, \\
\int (\lambda_p^2)^2 (\lambda_q^2)^2 f &= -\frac{16\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)}{(1-2\epsilon)(3-2\epsilon)(5-2\epsilon)(7-2\epsilon)}\int \hat{\mathcal{P}}_p^2 \hat{\mathcal{P}}_q^2 f \\
&\quad + \frac{16\epsilon(1-\epsilon)(2-\epsilon)}{(3-2\epsilon)(5-2\epsilon)(7-2\epsilon)}\int \frac{\hat{\mathcal{P}}_p \hat{\mathcal{P}}_q f}{\Delta} \\
&\quad - \frac{6\epsilon(1-\epsilon)}{(5-2\epsilon)(7-2\epsilon)}\int \frac{f}{\Delta^2}, \\
\int (\lambda_p^2)^2 \lambda_q^2 \lambda_{p+q}^2 f &= -\frac{16\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)}{(1-2\epsilon)(3-2\epsilon)(5-2\epsilon)(7-2\epsilon)}\int \hat{\mathcal{P}}_p^2 \hat{\mathcal{P}}_q \hat{\mathcal{P}}_{pq} f \\
&\quad + \frac{4\epsilon(1-\epsilon)(2-\epsilon)}{(3-2\epsilon)(5-2\epsilon)(7-2\epsilon)}\int \frac{\hat{\mathcal{P}}_p(\hat{\mathcal{P}}_p + 2\hat{\mathcal{P}}_q + 2\hat{\mathcal{P}}_{pq})f}{\Delta} \\
&\quad - \frac{6\epsilon(1-\epsilon)}{(5-2\epsilon)(7-2\epsilon)}\int \frac{f}{\Delta^2},
\end{aligned} \tag{C10}$$

where a factor $1/\Delta$ indicates that a shift in dimension of the integral should be made: $D \to D+2$, $\epsilon \to \epsilon - 1$ [ϵ's in prefactors in Eq. (C10) should *not* be shifted, however].

Once we have integrals in a form involving tensor products between the loop momenta and external momenta, we expand in small external momenta to reduce to logarithmically divergent integrals, just as we did in the one-loop case. This gives us vacuum integrals. We then reduce the tensors involving loop momenta using Lorentz covariance and insert an infrared mass regulator. By integrating we obtain the ultraviolet divergences. Since every prefactor in Eq. (C10) contains a factor of ϵ, to get the ultraviolet divergence, we only need the $1/\epsilon^2$ pole of the integrals on the right-hand side. These leading contributions have no dependence on the mass regulator, so we are unaffected by subdivergence issues due to the mass regulator. The ultraviolet divergences of the planar and nonplanar double-box integrals are then

$$
\begin{aligned}
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^4](s,t) &= \mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[(\lambda_{p+q}^2)^4](s,t) &= -\frac{1}{(4\pi)^4}\frac{14s+t}{90\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^3\lambda_q^2](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{480\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^3\lambda_{p+q}^2](s,t) &= \frac{1}{(4\pi)^4}\frac{2s+t}{360\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^2(\lambda_q^2)^2](s,t) &= \mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^2(\lambda_{p+q}^2)^2](s,t) &= \frac{1}{(4\pi)^4}\frac{s+2t}{720\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^2\lambda_q^2\lambda_{p+q}^2](s,t) &= \frac{1}{(4\pi)^4}\frac{s}{720\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[\lambda_p^2\lambda_q^2(\lambda_{p+q}^2)^2](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{240\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm P}[\lambda_p^2(\lambda_{p+q}^2)^3](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{40\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^4](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{80\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[(\lambda_{p+q}^2)^4](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{80\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^3\lambda_q^2](s,t) &= \mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^3\lambda_{p+q}^2](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{80\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^2(\lambda_q^2)^2](s,t) &= -\frac{1}{(4\pi)^4}\frac{7s}{1440\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^2(\lambda_{p+q}^2)^2](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{160\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^2\lambda_q^2\lambda_{p+q}^2](s,t) &= \frac{1}{(4\pi)^4}\frac{s}{1440\epsilon}+\mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[\lambda_p^2\lambda_q^2(\lambda_{p+q}^2)^2](s,t) &= \mathcal{O}(\epsilon^0),\\
\mathcal{I}_4^{\rm NP}[\lambda_p^2(\lambda_{p+q}^2)^3](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{160\epsilon}+\mathcal{O}(\epsilon^0).
\end{aligned}
\tag{C11}
$$

The bow-tie integrals do not contain infrared divergences, and their ultraviolet divergences were computed in Appendix A. Combining all the pieces then gives us the ultraviolet divergence in Eq. (4.20).

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