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Effective field theory for spacetime symmetry breaking

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We discuss the effective field theory for spacetime symmetry breaking from the local symmetry point of view. By gauging spacetime symmetries, the identification of Nambu–Goldstone (NG) fields and the construction of the effective action are performed based on the breaking pattern of diffeomorphism, local Lorentz, and (an)isotropic Weyl symmetries as well as the internal symmetries including possible central extensions in nonrelativistic systems. Such a local picture distinguishes, e.g., whether the symmetry breaking condensations have spins and provides a correct identification of the physical NG fields, while the standard coset construction based on global symmetry breaking does not. We illustrate that the local picture becomes important in particular when we take into account massive modes associated with symmetry breaking, the masses of which are not necessarily high. We also revisit the coset construction for spacetime symmetry breaking. Based on the relation between the Maurer–Cartan one form and connections for spacetime symmetries, we classify the physical meanings of the inverse-Higgs constraints by the coordinate dimension of broken symmetries. Inverse Higgs constraints for spacetime symmetries with a higher dimension remove the redundant NG fields, whereas those for dimensionless symmetries can be further classified by the local symmetry breaking pattern.

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I. INTRODUCTION

Symmetry and its spontaneous breaking play an important role in various areas of physics. In particular, the low-energy effective field theory (EFT) based on the underlying symmetry structures provides a powerful framework for understanding the low-energy dynamics in the symmetry broken phase [1].

For internal symmetry breaking in Lorentz invariant systems, the EFT based on coset construction was established in the 1960s [2,3]. When a global symmetry group G is broken to a residual symmetry group H, the corresponding Nambu–Goldstone (NG) fields $\pi(x)$ are introduced as the coordinates of the coset space G/H, and the general effective action can be constructed from the Maurer–Cartan one form,

$$J_\mu dx^\mu = \Omega^{-1}\partial_\mu \Omega dx^\mu \qquad \text{with} \qquad \Omega(x) = e^{\pi(x)} \in G/H. \tag{1.1}$$

Such a coset construction was also extended to spacetime symmetry breaking [4,5] accompanied by the inverse-Higgs constraints [6] and has been applied to various systems (see, e.g., Refs. [7–19] for recent discussions). Although the coset construction captures certain aspects of spacetime symmetry breaking, its understanding seems incomplete compared to the internal symmetry case and as a result has generated a lot of recent research activities [7–29]. It would then be helpful to revisit the issue of spacetime symmetry breaking based on an alternative approach, providing a complementary perspective to the coset construction.

For this purpose, let us first revisit the identification of NG fields for spacetime symmetry breaking. As in standard textbooks, symmetry breaking structures are classified by the type of order parameters, and their local transformations generate the corresponding NG fields (we refer to this as the *local picture*). In Lorentz invariant systems, since only the condensation of scalar fields is allowed, we need not pay much attention to the type of order parameters. However, when Lorentz symmetry is broken or does not exist, the type of order parameters becomes more important. For example, when the order parameter is a non-Abelian charge density, there appear NG modes with a quadratic dispersion different from that in Lorentz invariant systems [30–36]. In addition, if the charge density and the other order parameter that break the same symmetry coexist, some massive modes associated with the symmetry breaking appear [11,37–39].[1]

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[1]In this paper we use the words "NG fields" to denote fields which transform nonlinearly under broken symmetries. In general, the NG fields can contain massive modes as well as massless modes. We refer to the massless modes in NG fields as the NG modes in particular.

TABLE I. Embedding of relativistic spacetime symmetries. In relativistic systems, spacetime symmetries can be classified into isometric and conformal transformations, by requiring spacetime isotropy. They can then be embedded into diffeomorphism, local Lorentz, and isotropic Weyl transformations.

Relativistic symmetry	Diffeomorphism	Local Lorentz	Isotropic Weyl
Translation	✓		
Isometry	✓	✓	
Conformal	✓	✓	✓

TABLE II. Embedding of nonrelativistic spacetime symmetries. One difference from the relativistic case is that nonrelativistic spacetime symmetries should preserve the spatial slicing. The corresponding coordinate transformations are then foliation preserving diffeomorphisms. Another difference is that nonrelativistic systems admit central extensions of spacetime symmetry algebras. Correspondingly, we included the internal $U(1)$ gauge symmetry in the above table. See Appendix A for details.

Nonrelativistic symmetry	Foliation preserving	Local rotation	(An)isotropic Weyl	Internal $U(1)$
Translation	✓			
Galilean	✓	✓		✓
Schrödinger	✓	✓	✓	✓
Galilean conformal	✓	✓	✓	

For spacetime symmetry breaking, the standard coset construction based on global symmetry (referred to as the *global picture*) does not distinguish the types of order parameters. As is well-known in the case of conformal symmetry breaking [5,6], a naive counting of broken spacetime symmetries based on the global picture contains redundant fields and causes a wrong counting of NG modes (see, e.g., Refs. [20,24,25]). The inverse-Higgs constraints are introduced to compensate such a mismatch of NG mode counting. As discussed in Refs. [11,13,14], the inverse-Higgs constraints eliminate not only the redundant fields but also the massive modes. Thus, to identify the physical NG fields, we should take into account the massive modes associated with the symmetry breaking in addition to the massless modes. Such massive modes often play an important role, e.g., the smectic-A phase of liquid crystals near the smectic-nematic phase transition, in which the rotation modes are massive [40]. In this paper, we would like to construct the effective action including these modes based on the local picture.

To proceed in this direction, it is convenient to recall the relation between the coset construction and gauge symmetry breaking for internal symmetry. When a gauge symmetry is broken, the NG fields are eaten by the gauge fields, and the dynamics is captured by the unitary gauge action for the massive gauge boson A_μ. Since the gauge boson mass is given by $m \sim gv$ with the gauge coupling g and the order parameter v, the unitary gauge is not adequate to discuss the global symmetry limit $g \to 0$, which corresponds to the singular massless limit. Rather, it is convenient to introduce NG fields by the Stückelberg method as

$$A_\mu \to A'_\mu = \Omega^{-1} A_\mu \Omega + \Omega^{-1}\partial_\mu \Omega \quad \text{with} \quad \Omega(x) \in G/H, \tag{1.2}$$

where G and H are the original and residual symmetry groups, respectively, and $\Omega(x)$ describes the NG fields. In this picture, we can take the global symmetry limit smoothly to obtain the same effective action constructed from the Maurer–Cartan one form (1.1). As this discussion suggests, the unitary gauge is convenient for constructing the general effective action. Indeed, it is standard to begin with the unitary gauge in the construction of the dilaton effective action and the effective action for inflation [41]. Based on this observation, we apply the following recipe of the effective action construction to spacetime symmetry breaking in this paper:

(1) gauge the (broken) global symmetry;
(2) write down the unitary gauge effective action;
(3) introduce NG fields by the Stückelberg method, and decouple the gauge sector.

Our starting point is that any spacetime symmetry can be locally generated by Poincaré transformations and (an) isotropic rescalings. Correspondingly, we can embed any spacetime symmetry transformation into diffeomorphisms (diffs), local Lorentz transformations, and (an)isotropic Weyl transformations (see Tables I and II for concrete embedding of global spacetime symmetry). We then would like to gauge the original global symmetry to local ones. First, diffeomorphism invariance and local Lorentz invariance can be realized by introducing the curved spacetime action with the metric $g_{\mu\nu}$ and the vierbein e_μ^m.[2] On the other hand, there are two typical ways to realize isotropic Weyl invariance: Weyl gauging and Ricci gauging. In general, we can gauge the Weyl symmetry by introducing a gauge field W_μ and defining the covariant derivatives appropriately

[2]We use Greek letters for the curved spacetime indices and Latin letters for the (local) Minkowski indices.

(Weyl gauging), whereas we can introduce a Weyl invariant curved space action if the original system is conformal (Ricci gauging). The anisotropic Weyl symmetry can be also gauged in a similar way. These procedures for spacetime symmetry allow us to gauge all the global symmetries together with the internal symmetries.

Once gauging global symmetry, we identify the broken local symmetry from the condensation pattern

$$\langle\Phi^A(x)\rangle = \bar{\Phi}^A(x) \tag{1.3}$$

and construct the effective action based on symmetry breaking structures. Here and in what follows, we use capital Latin letters for internal symmetry indices, and the spin indices are implicit unless otherwise stated. When the condensation is spacetime dependent, diffeomorphism invariance is broken. On the other hand, local Lorentz invariance, (an)isotropic Weyl invariance, and internal gauge invariance are broken when the condensation has the Lorentz charge (spin), scaling dimension, and internal charge, respectively. If the symmetry breaking pattern is given, it is straightforward to take the unitary gauge and construct the effective action following the recipe. We will first apply our approach to some concrete examples to illustrate the importance of the local viewpoint of spacetime symmetry breaking. We will then revisit the coset construction from such a local perspective. One important difference from the EFT for the internal symmetry breaking case in Lorentz invariant systems is that the EFT constructed from the unitary gauge contains not only massless modes but also massive modes associated with spacetime symmetry breaking. These massive modes transform nonlinearly under the broken symmetries; i.e., they are NG fields.

The organization of this paper is as follows. In Sec. II, we explain our basic strategy in more detail. After reviewing the EFT for internal symmetry breaking, we discuss how global spacetime symmetry can be gauged. We then summarize how to construct the effective action based on the local symmetry breaking pattern. In Sec. III, we apply our approach to codimension-1 branes to illustrate the difference between the global and the local pictures of spacetime symmetry breaking. In the global picture, one may characterize the branes by the spontaneous breaking of translation and Lorentz invariance. In the local picture, on the other hand, such a symmetry breaking pattern can be further classified by the spin of the condensation forming the branes. We see that the spectra of massive modes associated with symmetry breaking depend on the spin of the condensation and the mass of massive modes is not necessarily high. If the masses are small compared with the typical energies of the system, the modes play a role as low-energy degrees of freedom. Therefore, the local picture becomes important in following the dynamics of such massive modes appropriately. In Sec. IV, we discuss a system with one-dimensional periodic modulation, i.e., a system in which the condensation is periodic in one direction, by applying the effective action constructed in Sec. III. We find that the dispersion relations of NG modes for the broken diffeomorphism are constrained by the minimum energy condition, in contrast to the codimension-1 brane case. In Sec. V, such a discussion is extended to the breaking of a mixture of spacetime and internal symmetries. In Sec. VI, we revisit the coset construction from the local symmetry picture. We first show that the parametrization of NG fields in the coset construction is closely related to the local symmetry picture. We then discuss the relation between the Maurer–Cartan one form and the connections for spacetime symmetries. We finally classify the physical meanings of the inverse-Higgs constraints based on the coordinate dimension of the broken symmetries. In Sec. VII, we make a brief comment on applications to gravitational systems. The final section is devoted to a summary. Details on the nonrelativistic case are summarized in Appendix A. A derivation of the Nambu-Goto action based on our EFT approach is presented in Appendix B.

II. BASIC STRATEGY

In this section we outline our basic strategy to construct the effective action for symmetry breaking including ones that involve spacetime symmetry breaking. In Sec. II A, we first review the relation between the coset construction and gauge symmetry breaking for internal symmetry and explain how the local picture can be used to construct the effective action for global symmetry breaking. To extend this discussion to spacetime symmetry, in Secs. II B and II C, we discuss how global spacetime symmetries can be embedded into local ones. We then present our recipe for the effective action construction in Sec. II D.

A. EFT for internal symmetry breaking

Let us first review how the coset construction for internal symmetry breaking [2,3] can be reproduced from the effective theory for gauge symmetry breaking. Suppose that a global symmetry group G is spontaneously broken to a subgroup H and the coset space G/H satisfying

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad \text{with} \quad [\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}, \tag{2.1}$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively. $\mathfrak{m}$ represent the broken generators. In the coset construction, we introduce representatives of the coset space G/H as $\Omega = e^{\pi}$ with $\pi \in \mathfrak{m}$, of which the left G transformation is given by

$$\Omega(\pi) \to \Omega(\pi') = g\Omega(\pi)h^{-1}(\pi, g), \tag{2.2}$$

where $g \in G$ and $h(\pi, g) \in H$. In general, $h^{-1}(\pi, g)$ depends on both g and π. The transformation from π to π' is nonlinear, so that it is called the nonlinear realization. If g is the element of the unbroken symmetry H, $g{=}h{\in}H$, π linearly transforms: $\pi'(x) = h\pi(x)h^{-1}$. Here, it is useful to introduce the Maurer–Cartan one form to construct the effective Lagrangian, which is defined as

$$J_\mu dx^\mu \equiv \Omega^{-1}\partial_\mu \Omega dx^\mu. \tag{2.3}$$

If we decompose J_μ into the broken component $J_\mu^{\mathfrak{m}} \in \mathfrak{m}$ and the unbroken component $J_\mu^{\mathfrak{h}} \in \mathfrak{h}$ as $J_\mu = J_\mu^{\mathfrak{m}} + J_\mu^{\mathfrak{h}}$, each component transforms as

$$J_\mu^{\mathfrak{m}} \to h J_\mu^{\mathfrak{m}} h^{-1}, \qquad J_\mu^{\mathfrak{h}} \to h J_\mu^{\mathfrak{h}} h^{-1} + h\partial_\mu h^{-1}, \tag{2.4}$$

under G transformation. Here, note that the broken component $J_\mu^{\mathfrak{m}}$ transforms covariantly. In general, $\mathfrak{m}$ is reducible under H transformation, and it can be decomposed into direct sums, $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \cdots \oplus \mathfrak{m}_N$. At the leading order in the derivative expansion, the effective Lagrangian for Lorentz invariant systems is given by[3]

$$\mathcal{L} = \sum_{a=1}^{N} F_a^2 \mathrm{tr}[J_\mu^{\mathfrak{m}_a} J^{\mathfrak{m}_a \mu}], \tag{2.5}$$

where the trace is defined in a G-invariant way. $J_\mu^{\mathfrak{m}_a}$ and F_a^2 are the components of the Maurer–Cartan one form and the decay constant for each irreducible sector, respectively. By construction, this Lagrangian is invariant under G transformation.

We next move on to the effective action construction for gauge symmetry breaking. In this case, it is convenient to take the unitary gauge, where the NG fields are eaten by the gauge field A_μ and do not fluctuate. The general effective action can then be constructed only from the massive gauge field A_μ in an H gauge invariant way. In relativistic systems, the effective Lagrangian takes the form

$$\mathcal{L} = \mathrm{tr}\left[\frac{1}{2g^2} F^{\mu\nu} F_{\mu\nu} + v_a^2 A_\mu^{\mathfrak{m}_a} A^{\mathfrak{m}_a \mu} + \cdots\right], \tag{2.6}$$

where g and v_a are the gauge coupling and the order parameters for symmetry breaking, respectively, and $A_\mu^{\mathfrak{m}} \in \mathfrak{m}$ is the gauge field in the broken symmetry sector. Note that, since we are considering the unitary gauge, Eq. (2.6) is not invariant under G gauge transformation. Because the gauge boson mass is given by $m \sim g v_a$, the global symmetry limit $g \to 0$ for fixed v_a corresponds to the singular massless limit, so that the unitary gauge is not appropriate to discuss the global symmetry limit. To take the global symmetry limit, it is convenient to introduce the NG fields $\pi(x) \in \mathfrak{m}$ by performing a field-dependent gauge transformation (the Stückelberg method):

$$A_\mu \to \Omega^{-1} A_\mu \Omega + \Omega^{-1}\partial_\mu \Omega \quad \text{with} \quad \Omega = e^{\pi(x)}. \tag{2.7}$$

The G gauge invariance can be recovered by assigning a nonlinear transformation rule on the NG fields $\pi(x)$, and we can take the global symmetry limit $g \to 0$ smoothly in this picture. Since the gauge sector decouples from the NG fields in the global symmetry limit, the effective action for the NG fields can be obtained by the replacements:

$$v_a^2 \to F_a^2, \tag{2.8}$$

$$A_\mu \to J_\mu = \Omega^{-1}\partial_\mu \Omega. \tag{2.9}$$

The latter is nothing but the Maurar–Cartan one form. As this discussion suggests, the unitary gauge is useful to construct the general ingredients needed to obtain the effective action for global symmetry breaking. Note that Wess–Zumino terms in the coset construction are reproduced by Chern–Simons terms in the unitary gauge action.

B. Local properties of spacetime symmetries

In the previous subsection, we saw that the unitary gauge action for gauge symmetry breaking can be used to construct the general effective action for global symmetry breaking. We now would like to extend such a discussion to spacetime symmetry breaking. For this purpose, let us recall the local properties of (infinitesimal) spacetime symmetry transformations in this subsection.[4] Any spacetime symmetry transformation has an associated coordinate transformation

$$x^\mu \to x'^\mu = x^\mu - \epsilon^\mu(x), \tag{2.10}$$

and its local properties around a point $x^\mu = x_*^\mu$ can be read off by expanding the parameter $\epsilon^\mu(x)$ covariantly as

$$\epsilon^\mu(x) = \epsilon^\mu(x_*) + (x^\nu - x_*^\nu)\nabla_\nu \epsilon^\mu(x_*) + \mathcal{O}((x - x_*)^2). \tag{2.11}$$

The first term is the zeroth order in $x - x_*$ and describes translations of the coordinate system. On the other hand, deformations of the coordinate system are encoded in the second term (the linear order in $x - x_*$), which can be decomposed as

$$\begin{aligned} &\nabla_\mu \epsilon^\nu = \delta_\mu^\nu \lambda + s_\mu{}^\nu + \omega_\mu{}^\nu \quad \text{with} \quad s_\mu{}^\mu = 0, \\ &s_{\mu\nu} = s_{\nu\mu}, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}. \end{aligned} \tag{2.12}$$

The trace part λ and the symmetric traceless part $s_{\mu\nu}$ are local isotropic rescalings (dilatations) and local anisotropic rescalings, respectively. The antisymmetric part $\omega_{\mu\nu}$ corresponds to local Lorentz transformations. Any spacetime symmetry can therefore be locally decomposed into Poincaré transformations and isotropic/anisotropic rescalings. Correspondingly, the symmetry transformations of local fields are specified by their Lorentz charges and isotropic/anisotropic scaling dimensions.

[3]For simplicity, we do not consider matter fields in this paper.

[4]Though we concentrate on infinitesimal transformations for simplicity, extension to the finite case is straightforward.

As is suggested by Eqs. (2.11) and (2.12), we can embed global spacetime symmetry transformations into diffeomorphisms, local Lorentz transformations, and local isotropic/anisotropic Weyl transformations. For simplicity, let us consider the case of relativistic systems in this section (see Appendix A for an extension to the nonrelativistic case). In relativistic systems, any spacetime symmetry transformation can be locally decomposed into the Poincaré part and the dilatation part because anisotropic rescalings are incompatible with the Lorentz symmetry:

$$\nabla_\mu \epsilon^\nu = \delta_\mu^\nu \lambda + \omega_\mu{}^\nu \quad \text{with} \quad \omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{2.13}$$

Note that we have conformal transformations for general functions $\lambda(x)$ and isometric transformations for $\lambda = 0$ because the metric field transforms as

$$\delta g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = 2 g_{\mu\nu} \lambda. \tag{2.14}$$

The transformation rules of local fields are then determined by their spin and scaling dimension. When a field $\Phi(x)$ follows a representation Σ_{mn} of the Lorentz algebra and has a scaling dimension Δ_Φ, its symmetry transformation is given by

$$\Phi(x) \to \Phi'(x) = \Phi(x) + \Delta_\Phi \lambda(x)\Phi(x) + \frac{1}{2}\omega^{mn}(x)\Sigma_{mn}\Phi(x) + \epsilon^\mu(x)\nabla_\mu \Phi(x). \tag{2.15}$$

Here, the curved spacetime indices (Greek letters) and the local Lorentz indices (Latin letters) are converted by the vierbein e_μ^m as $\omega^{mn} = e_\mu^m e_\nu^n \omega^{\mu\nu}$. The covariant derivative $\nabla_\mu \Phi$ is defined in terms of the spin connection S_μ^{mn} as

$$\nabla_\mu \Phi = \partial_\mu \Phi + \frac{1}{2} S_\mu^{mn} \Sigma_{mn} \Phi \quad \text{with} \quad S_\mu^{mn} = e_\nu^m \partial_\mu e^{\nu n} + e_\lambda^m \Gamma_{\mu\nu}^\lambda e^{\nu n}, \tag{2.16}$$

with the Christoffel symbols $\Gamma_{\mu\nu}^\lambda$ defined by

$$\Gamma_{\mu\nu}^\lambda \equiv \frac{g^{\lambda\rho}}{2}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \tag{2.17}$$

To identify the transformation (2.15) as local symmetries, it is convenient to rewrite it in the form,

$$\Phi'(x) = \Phi(x) + \Delta_\Phi \lambda(x)\Phi(x) + \frac{1}{2}(\omega^{mn}(x) + \epsilon^\mu(x) S_\mu^{mn}(x))\Sigma_{mn}\Phi(x) + \epsilon^\mu(x)\partial_\mu \Phi(x). \tag{2.18}$$

We then notice that the latter three terms can be thought of as local Weyl transformations, local Lorentz transformations, and diffeomorphisms, respectively. Since the transformation rule of Φ, $g_{\mu\nu}$, and e_μ^m under each local transformation is given by

$$\text{local Weyl:}\ \delta\Phi = \Delta_\Phi \tilde{\lambda}\Phi, \qquad \delta g_{\mu\nu} = -2\tilde{\lambda} g_{\mu\nu}, \qquad \delta e_\mu^m = -\tilde{\lambda} e_\mu^m, \tag{2.19}$$

$$\text{local Lorentz:}\ \delta\Phi = \frac{1}{2}\tilde{\omega}^{mn}\Sigma_{mn}\Phi, \qquad \delta g_{\mu\nu} = 0, \qquad \delta e_\mu^m = \tilde{\omega}^m{}_n e_\mu^n, \tag{2.20}$$

$$\text{diffeomorphism:}\ \delta\Phi = \tilde{\epsilon}^\mu \partial_\mu \Phi, \qquad \delta g_{\mu\nu} = \nabla_\mu \tilde{\epsilon}_\nu + \nabla_\nu \tilde{\epsilon}_\mu, \qquad \delta e_\mu^m = \nabla_\mu \tilde{\epsilon}^m - \tilde{\epsilon}^\nu S_\nu^m{}_n e_\mu^n, \tag{2.21}$$

we can reproduce the transformation (2.18) by the parameter choice

$$\tilde{\lambda} = \lambda, \qquad \tilde{\omega}^{mn} = \omega^{mn} + \epsilon^\mu S_\mu^{mn}, \qquad \tilde{\epsilon}^\mu = \epsilon^\mu. \tag{2.22}$$

Note that the metric $g_{\mu\nu}$ and the vierbein e_μ^m are invariant under the original global transformation (2.22), although they are not invariant under general local ones. Any global spacetime symmetry in relativistic systems can therefore be embedded into local Weyl transformations, local Lorentz transformations, and diffeomorphisms.

C. Gauging spacetime symmetry

In the previous subsection, we discussed that any spacetime symmetry transformation in relativistic systems is locally generated by Poincaré and Weyl transformations. Isometric transformations can be embedded in diffeomorphisms and local Lorentz transformations, whereas conformal transformations require local Weyl transformations as well. Since NG fields correspond to local transformations of the order parameters for broken symmetries, we would like to construct the effective action from this local symmetry point of view. For this purpose, let us summarize how we can gauge global spacetime symmetry to those local symmetries.

When the system is isometry invariant before symmetry breaking, we gauge the Poincaré symmetry by introducing the curved spacetime action with the metric $g_{\mu\nu}(x)$ and the vierbein $e_\mu^m(x)$. For example, an action in a nongravitational system on Minkowski space,

$$S[\Phi] = \int d^4x \mathcal{L}[\Phi, \partial_m \Phi], \tag{2.23}$$

can be reformulated as

$$S[\Phi] \to S[\Phi, g_{\mu\nu}, e^m_\mu] = \int d^4x \sqrt{-g} \mathcal{L}[\Phi, e^\mu_m \nabla_\mu \Phi], \tag{2.24}$$

where the covariant derivative ∇_μ is given by Eq. (2.16). From the viewpoint of the curved space action (2.24), the original nongravitational system can be reproduced by taking the metric $g_{\mu\nu}$ and the vierbein e^m_μ as the Minkowski ones with the gauge choice,

$$g_{\mu\nu} = \eta_{\mu\nu}, \qquad e^m_\mu = \delta^m_\mu. \tag{2.25}$$

The original global Poincaré symmetry can be also understood as the residual symmetries under the gauge conditions (2.25). The same story holds for nongravitational systems on curved spacetimes.

On the other hand, there are two typical ways to gauge the Weyl symmetry: Weyl gauging and Ricci gauging (see, e.g., Ref. [42]). When the system is conformal, we can introduce a local Weyl invariant curved spacetime action, essentially because the local Weyl invariance is equivalent to the traceless condition of the energy-momentum tensor. Such a procedure is called the Ricci gauging, and we need not introduce additional fields in this case. When the Ricci gauging is not applicable, we need to introduce a gauge field W_μ for Weyl symmetry and the covariant derivative D_μ defined by

$$D_\mu \Phi = \nabla_\mu \Phi + (\Delta_\Phi \delta^\nu_\mu - \Sigma_\mu{}^\nu) W_\nu \Phi, \tag{2.26}$$

where $\Sigma_\mu{}^\nu = e^m_\mu \Sigma_m{}^n e^\nu_n$ and the local Weyl transformation rule is given by[5]

$$\begin{aligned} \Phi \to \Phi' = e^{\Delta_\Phi \lambda} \Phi, \qquad & g_{\mu\nu} \to g'_{\mu\nu} = e^{2\lambda} g_{\mu\nu}, \\ e^m_\mu \to e'^m_\mu = e^\lambda e^m_\mu, \qquad & W_\mu \to W'_\mu = W_\mu - \partial_\mu \lambda. \end{aligned} \tag{2.28}$$

If the curved spacetime action is global Weyl invariant, a local Weyl invariant action can be obtained by replacing the covariant derivative ∇_μ with the Weyl covariant derivative D_μ. For example, Eq. (2.24) is reformulated as

$$S[\Phi, g_{\mu\nu}, e^m_\mu, W_\mu] = \int d^4x \sqrt{-g} \mathcal{L}[\Phi, e^\mu_m D_\mu \Phi]. \tag{2.29}$$

[5]Note that the gauge field W^μ with an upper spacetime index transforms as

$$W^\mu = g^{\mu\nu} W_\nu \to W'^\mu = e^{-2\lambda} g^{\mu\nu} (W_\nu - \partial_\mu \lambda) = e^{-2\lambda} (W^\mu - \partial^\mu \lambda). \tag{2.27}$$

Note that the original action (2.23) can be reproduced by imposing the gauge condition

$$g_{\mu\nu} = \eta_{\mu\nu}, \qquad e^m_\mu = \delta^m_\mu, \qquad W_\mu = 0, \tag{2.30}$$

and symmetries of the action are reduced to the original global ones. Also, while the first two conditions in Eq. (2.30) are always invariant under the original global symmetries, the condition $W_\mu = 0$ is not necessarily invariant when the original system is conformal. Indeed, it is not invariant under the special conformal transformation. Correspondingly, the Weyl gauge field W_μ appears in a particular combination in the action. For example, the action of a massless free scalar ϕ can be gauged as

$$\begin{aligned} & -\frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\nu} (\partial_\mu + W_\mu) \phi (\partial_\nu + W_\nu) \phi \\ & \quad = -\frac{1}{2} \int d^4x \sqrt{-g} [(\partial_\mu \phi)^2 - (\nabla_\mu W^\mu - W^2) \phi^2], \end{aligned} \tag{2.31}$$

where W_μ appears in a special conformal invariant combination $\nabla_\mu W^\mu - W^2$.

D. EFT recipe

As we have discussed, all the global symmetries in relativistic systems can be embedded into diffeomorphisms, local Lorentz symmetries, local Weyl symmetry, and internal gauge symmetries. We can also gauge the global symmetry by the use of the procedures in the previous subsection and the standard internal gauging. Similar discussions hold for nonrelativistic systems accompanied by local anisotropic Weyl symmetries and internal symmetry associated with the possible central extension, as we illustrate in Appendix A. We now extend the discussion in Sec. II A for internal symmetry to spacetime symmetry. First, the symmetry breaking patterns are classified by the condensation patterns:

$$\langle \Phi^A(x) \rangle = \bar{\Phi}^A(x). \tag{2.32}$$

When the condensation is spacetime dependent, diffeomorphism invariance is broken. On the other hand, local Lorentz invariance, local isotropic/anisotropic Weyl invariance, and internal gauge invariance are broken when the condensation has a Lorentz charge (spin), scaling dimension, and internal charge, respectively. Once we identify the symmetry breaking pattern, we can construct the effective action based on the following recipe just as in the case of internal symmetry breaking:

(1) gauge the (broken) global symmetry;
(2) write down the unitary gauge effective action;
(3) introduce NG fields by the Stückelberg method, and decouple the gauge sector.

TABLE III. Broken symmetries, condensation patterns, and gauge fields.

Diffeomorphism	Local Lorentz	Local Weyl	Internal gauge
Spacetime dependence	Spin	Scaling dimension	Internal charge
Metric $g_{\mu\nu}$	Vierbein e_μ^m	Weyl gauge field W_μ	Gauge field A_μ

The first step can be performed by introducing gauge fields based on the procedure in Sec. II C (see also Table III). We then take the unitary gauge, where the NG fields do not fluctuate. Using the dynamical degrees of freedom in the unitary gauge, we construct the general unitary gauge effective action invariant under the residual symmetries. Finally, we perform the Stückelberg method to introduce the NG fields and restore the full gauge symmetry. By decoupling the gauge sector, we obtain the effective action for the NG fields. In the following sections, we apply this recipe to concrete examples for spacetime symmetry breaking.

We emphasize that the condensation pattern rather than the breaking pattern of global symmetries plays an important role in identifying the NG fields, unlike the case for internal symmetry breaking in Lorentz invariant systems. The breaking pattern of global symmetries itself cannot distinguish the breaking of diffeomorphism, local Lorentz, and Weyl symmetries. As will be seen in the next section, this difference becomes important when we discuss the massive modes originating from the symmetry breaking, although the existence of massless modes does not depend on the condensation pattern.

III. CODIMENSION-1 BRANE

In this section, we apply our approach to codimension-1 branes on the Minkowski space to illustrate the difference between the global and the local picture of spacetime symmetry breaking. In the global picture, one may characterize the branes by the spontaneous breaking of the translation and Lorentz invariance. In the local picture, on the other hand, such a symmetry breaking pattern can be further classified by the spin of the condensation forming the brane (see also Fig. 1):

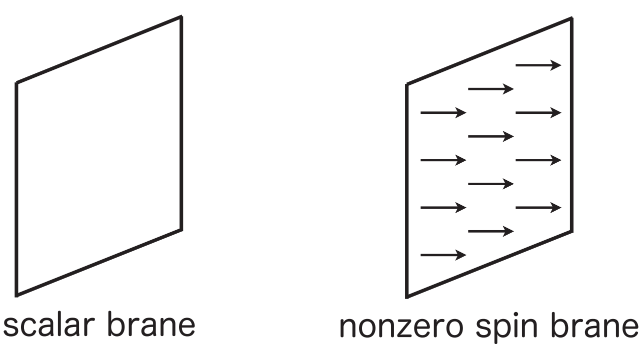

FIG. 1. Scalar vs nonzero spin branes. While the diffeomorphism symmetry is broken for both types of branes, the local Lorentz symmetry is broken only for the nonzero spin case.

(1) Scalar brane:
When a scalar field forms a codimension-1 brane, the only broken symmetry is the diffeomorphism invariance in the z-direction orthogonal to the brane. In particular, the local Lorentz symmetry is not broken although the global one is.

(2) Nonzero spin branes:
When a nonzero spin field forms a codimension-1 brane and the condensation is aligned to the z-direction, the local Lorentz invariance associated with the z-direction is broken as well as the z-diffeomorphism invariance.

Since those two cases are classified into the same category in the global picture, the local picture is necessary to distinguish them. In the rest of this section, we discuss in which situation the difference becomes important, if we take into account the massive modes associated with symmetry breaking.

In Sec. III A, we first perform the tree-level analysis of NG fields around scalar brane backgrounds, to illustrate our strategy in the previous section concretely. In Secs. III B and III C, we construct the general effective action for the diffeomorphism symmetry breaking and apply it to single scalar brane backgrounds. In Secs. III D and III E, we include local Lorentz symmetry breaking in the effective action construction and apply it to single nonzero spin brane backgrounds. For single brane backgrounds, it turns out that the dynamics in the low-energy limit results in the same action regardless of the spin of the field that condenses. However, we find that the degeneracy is resolved beyond the low-energy limit and the resolving scale is not necessarily high. We see that the effective action based on the local picture can be used to investigate such an intermediate scale.

Though discussion in this section is only for single brane backgrounds, the effective action constructed in Secs. III B and III D is applicable to more general setups. In Sec. IV, we discuss periodic modulation and clarify the difference from the single brane case.

A. Real scalar field model for scalar brane

To illustrate our strategy, let us begin with a real scalar field model,

$$S = \int d^4x \left[-\frac{1}{2}\partial_m\phi\partial^m\phi - V(\phi) \right] \quad \text{with}$$

$$V = \frac{g^2}{2}(\phi^2 - v^2)^2, \tag{3.1}$$

and perform the tree-level analysis of NG fields around domain-wall configurations. Here, g and v are constant parameters, and the potential $V(\phi)$ has two minima at $\phi = \pm v$. The equation of motion

$$\Box\phi - V'(\phi) = 0 \tag{3.2}$$

has the following domain-wall solution with the boundary conditions $\phi(z = \pm\infty) = \pm v$:

$$\phi(x) = \bar{\phi}(z) = v \tanh \beta z, \tag{3.3}$$

where $\beta = gv$ characterizes the thickness of the brane. Note that there exists a one-parameter family of domain-wall solutions $\phi(x) = \bar{\phi}(z - z_0)$ parametrized by the brane position z_0 because of the translation invariance. The domain-wall configuration Eq. (3.3) breaks the translation invariance, and the corresponding NG field $\pi(x)$ can be obtained by promoting z_0 to a field as[6]

$$\phi(x) = \bar{\phi}(z + \pi(x)). \tag{3.4}$$

The action for the NG field is then given by

$$\begin{aligned} S &= \int d^4x \left[-\frac{1}{2}\partial_m\bar{\phi}(z+\pi)\partial^m\bar{\phi}(z+\pi) - V(\bar{\phi}(z+\pi))\right] \\ &= \int d^4x \left[-\frac{\bar{\phi}'(z+\pi)^2}{2}\partial_m(z+\pi)\partial^m(z+\pi) - V(\bar{\phi}(z+\pi))\right]. \end{aligned} \tag{3.5}$$

Using the integrated version of the equation of motion,[7]

$$\bar{\phi}'' - V'(\bar{\phi}) = 0 \leftrightarrow \frac{1}{2}\bar{\phi}'^2 - V(\bar{\phi}) = 0, \tag{3.6}$$

we can further reduce the action (3.5) to the form

$$\begin{aligned} S &= \int d^4x \left[-\frac{\bar{\phi}'(z+\pi)^2}{2}(\partial_m(z+\pi)\partial^m(z+\pi) + 1)\right] \\ &= -\frac{1}{2}\int d^4x \bar{\phi}'(z+\pi)^2 \partial_m\pi\partial^m\pi, \end{aligned} \tag{3.7}$$

where we dropped total derivative terms at the second equality.

Let us then reproduce the action (3.7) along the line of our strategy. Following the EFT recipe in the previous section, we first gauge the translation symmetry to the diffeomorphism symmetry by introducing the curved coordinate action

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]. \tag{3.8}$$

We next consider fluctuations around the domain-wall background (3.3). Since z-coordinate transformations of $\bar{\phi}(z)$ generate fluctuations of $\phi(x)$, we can take the unitary gauge $\phi(x) = \bar{\phi}(z)$ at least as long as fluctuations are small. In other words, we can choose a coordinate frame such that the constant-ϕ slices coincide with the constant-z slices. In this coordinate frame, the action is given by

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{zz}\bar{\phi}'(z)^2 - V(\bar{\phi})\right] \\ &= -\frac{1}{2}\int d^4x\sqrt{-g}\bar{\phi}'(z)^2(1 + g^{zz}), \end{aligned} \tag{3.9}$$

where we used Eq. (3.6) in the second equality. Note that the action (3.9) enjoys only the $(2+1)$-dimensional diffeomorphism symmetry along the t, x, y-directions

$$x^\mu \to x'^\mu = x^\mu - \epsilon^\mu(x) \quad \text{with} \quad \epsilon^z = 0 \tag{3.10}$$

and the NG field is eaten by the metric $g_{\mu\nu}$ in this gauge. We then restore the z-diffeomorphism invariance by the field-dependent coordinate transformation (the Stückelberg method)

$$z \to \tilde{z} \quad \text{with} \quad \tilde{z} + \tilde{\pi}(\tilde{x}) = z. \tag{3.11}$$

After the transformation, the action (3.9) takes the form

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\bar{\phi}'(z+\pi)^2(1 + g^{\mu\nu}\partial_\mu(z+\pi)\partial_\nu(z+\pi)), \tag{3.12}$$

where we dropped the tilde for simplicity. The action (3.12) is now invariant under the full diffeomorphism symmetry by assigning the following nonlinear transformation rule on the NG field π:

$$\pi(x) \to \pi'(x') = \pi(x) + \epsilon^z(x) \quad \text{with} \quad x'^\mu = x^\mu - \epsilon^\mu(x). \tag{3.13}$$

[6]Our parametrization of the NG field $\pi(x)$ corresponds to the transformation parameter ϵ in Eq. (2.15).

[7]In general, the integrated equation of motion takes the form $\frac{1}{2}\bar{\phi}'^2 - V(\bar{\phi}) = \text{constant}$. However, the constant term vanishes for the potential (3.1) and the solution (3.3) to obtain the second equation in Eq. (3.6).

Finally, we remove gauge degrees of freedom by taking the Minkowski coordinate. Since we are working on the Minkowski space, the full diffeomorphism invariance, nonlinearly realized by the NG fields, allows us to set the metric field as $g_{\mu\nu} = \eta_{\mu\nu}$. The action (3.12) is then reduced to Eq. (3.7).

In this subsection, we illustrated our approach by performing the tree-level analysis of NG fields around domain-wall backgrounds in the model (3.1). As we have seen, the introduction of the curved coordinate action (3.9) allows us to impose the unitary gauge condition $\phi(x) = \bar{\phi}(z)$, which breaks the z-diffeomorphism invariance. The scalar $\phi(x)$ is then eaten by the metric field $g_{\mu\nu}$. By performing the Stückelberg method and removing the gauge degrees of freedom, we obtained the action for NG fields. More generally, the action (3.1) can be modified with higher derivative terms due to quantum corrections for example. In the next subsection, we construct the general effective action for NG fields by introducing the general unitary gauge action consistent with the symmetry.

B. Effective action for z-diffeomorphism symmetry breaking

We then construct the general effective action for the z-diffeomorphism symmetry breaking, by introducing the general unitary gauge action consistent with the symmetry. Just as in the previous real scalar model, let us introduce the metric field and work in the general coordinate system. We can then impose the unitary gauge condition, which prohibits fluctuations of the NG field and breaks the z-diffeomorphism symmetry. In such a unitary gauge, the dynamical degrees of freedom are the metric field $g_{\mu\nu}$ only (if there are no additional matter degrees of freedom), and there remains the $(2+1)$-dimensional diffeomorphism symmetry. Schematically, we write this unitary gauge setup as

$$g_{\mu\nu}(x) + (2+1)\text{-dim diffs.} \tag{3.14}$$

The general effective action is then constructed from the metric field in a $(2+1)$-dimensional diffeomorphism invariant way. This setup is essentially the same as the one in single-field inflation [41]. Following the results there, ingredients of the unitary gauge effective action are given by

$$\begin{aligned}&\text{scalar functions of } z,\ g_{\mu\nu},\ R_{\mu\nu\rho\sigma},\\ &\text{and their covariant derivatives.}\end{aligned} \tag{3.15}$$

The lowest few terms of the expansion in fluctuations around the Minkowski metric, $g_{\mu\nu} - \eta_{\mu\nu}$, and derivatives are given by

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}[\alpha_1(z) + \alpha_2(z)g^{zz} + \alpha_3(z)(g^{zz}-1)^2]. \tag{3.16}$$

Here, $\alpha_i(z)$'s are scalar functions of z, which depend on the details of the microscopic theory. Note that g^{zz} arises from $\partial_\mu\phi\partial^\mu\phi = \bar{\phi}'^2 g^{zz}$ in the previous real scalar model. One may then identify α_3 with higher derivative interactions in the real scalar model.

We next introduce the NG field π for the z-diffeomorphism by the Stückelberg method. Just as we did in the previous subsection, we perform a field-dependent coordinate transformation (3.11). Practically, this transformation can be realized by replacing a function $f(z)$ with $f(z+\pi)$ [41], where we dropped the tilde for simplicity. Correspondingly, the unitary gauge action (3.16) is transformed as

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}[\alpha_1(z+\pi) + \alpha_2(z+\pi)(g^{zz} + 2\partial^z\pi + \partial_\mu\pi\partial^\mu\pi) + \alpha_3(z+\pi)(2\partial^z\pi + \partial_\mu\pi\partial^\mu\pi)^2], \tag{3.17}$$

where we used

$$\begin{aligned}g^{zz} = g^{\mu\nu}\delta^z_\mu\delta^z_\nu &\to g^{\mu\nu}\partial_\mu(z+\pi)\partial_\nu(z+\pi)\\ &= g^{zz} + 2\partial^z\pi + \partial_\mu\pi\partial^\mu\pi.\end{aligned} \tag{3.18}$$

The full diffeomorphism symmetry is now restored accompanied by the nonlinearly transformation rule (3.13). We then would like to remove the gauge degrees of freedom and construct the effective action for the NG field π only. Since we are working on the Minkowski space, the full diffeomorphism invariance, nonlinearly realized by the NG fields, allows us to set the metric field as $g_{\mu\nu} = \eta_{\mu\nu}$. In this Minkowski coordinate system, the ingredients (3.15) are given by

$$\begin{aligned}&\text{scalar functions of } z+\pi,\ g_{\mu\nu} = \eta_{\mu\nu},\ R_{\mu\nu\rho\sigma} = 0,\\ &\text{and their derivatives,}\end{aligned} \tag{3.19}$$

and the effective action (3.17) can be expressed as

$$S = -\frac{1}{2}\int d^4x[\alpha_1(z+\pi) + \alpha_2(z+\pi)(1 + 2\partial_z\pi + \partial_m\pi\partial^m\pi) + \alpha_3(z+\pi)(2\partial_z\pi + \partial_m\pi\partial^m\pi)^2], \tag{3.20}$$

where Latin indices indicate that we use the Minkowski coordinate.

So far, we have not taken into account the background equation of motion. Indeed, Eq. (3.20) contains linear order terms in π:

$$S = -\frac{1}{2}\int d^4x[\alpha_1(z) + \alpha_2(z) + (\alpha_1'(z) + \alpha_2'(z))\pi + 2\alpha_2(z)\partial_z\pi + \mathcal{O}(\pi^2)]. \tag{3.21}$$

To remove such tadpole terms, we impose the background equation of motion $\alpha_1'(z) = \alpha_2'(z)$. In the following, we use $\alpha_1 = \alpha_2$ because the constant shift of α_1 does not change the action for π. Then, we obtain the action,

$$S = -\frac{1}{2}\int d^4x[\alpha_1(z+\pi)\partial_m\pi\partial^m\pi + \alpha_3(z+\pi)(2\partial_z\pi(x) + \partial_m\pi\partial^m\pi)^2] - \frac{1}{2}\int d^4x\frac{d}{dz}A_1(z+\pi), \tag{3.22}$$

where $A_1(z) = 2\int^z dz'\alpha_1(z')$ and the second term is a total derivative. Note that the derivative with respect to z in the last term of Eq. (3.22) acts not only explicitly on z but also on $\pi(z)$, i.e., $dA_1(z+\pi)/dz = A_1'(z+\pi)(1+\partial_z\pi)$. Up to the second order in π, the bulk action (the first term) can be expanded as

$$S_{\text{bulk}} = -\frac{1}{2}\int d^4x\alpha_1(z)[\partial_{\hat{m}}\pi\partial^{\hat{m}}\pi + c_z^2(z)(\partial_z\pi)^2] \quad \text{with} \quad c_z^2(z) = 1 + 4\frac{\alpha_3(z)}{\alpha_1(z)}, \tag{3.23}$$

where $\hat{m} = t, x, y$ and c_z can be interpreted as the propagating speed in the z-direction. On the other hand, the total derivative term can be expanded as

$$S_{\text{t.d.}} = -\frac{1}{2}\int d^4x\frac{d}{dz}[A_1(z) + 2\alpha_1(z)\pi + \alpha_1'(z)\pi^2 + \mathcal{O}(\pi^3)], \tag{3.24}$$

where we note that there can arise linear order terms in π from the total derivative term. We will revisit its physical meaning in the next section.

C. Physical spectra for single scalar domain wall

We next take a close look at the effective action (3.22) and discuss physical spectra for a single scalar brane case. For simplicity, let us consider the case $\alpha_3(z) = 0$ in this subsection.[8] Then, the brane profile is characterized by the free function $\alpha_1(z)$ in the effective action. Generically, $\alpha_1(z)$ is related to the order parameter $\bar{\phi}'(z)$ as

$$\alpha_1(z) \sim \bar{\phi}'(z)^2 \tag{3.25}$$

because the g^{zz} operator in the unitary gauge action typically arises from

$$g^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} = \bar{\phi}'^2 g^{zz}. \tag{3.26}$$

To illustrate the physical spectra, it is convenient to take the well-studied domain-wall profile obtained in Sec. III A,

$$\bar{\phi}(z) = v\tanh\beta z, \tag{3.27}$$

and the corresponding function $\alpha_1(z)$ of the form

$$\alpha_1(z) = \bar{\phi}'(z)^2 = \frac{\beta^2 v^2}{\cosh^4\beta z}. \tag{3.28}$$

Here, the constants v and β characterize the domain-wall profile, and in particular, β specifies the thickness of the brane. We then determine the physical spectra for the NG field π. From the bulk action (3.23), the linear order equation of motion follows as[9]

$$\pi'' - 4\beta\tanh\beta z\pi' + \partial_\perp^2\pi = 0 \tag{3.29}$$

in the coordinate space. Here, the prime denotes the derivative with respect to z, and $\partial_\perp^2 \equiv \partial_{\hat{m}}\partial^{\hat{m}} = -\partial_t^2 + \partial_x^2 + \partial_y^2$. By the Fourier transformation in the $x^{\hat{m}}$ directions along the brane, we rewrite it as

$$\pi_{k_\perp}'' - 4\beta\tanh\beta z\pi_{k_\perp}' - k_\perp^2\pi_{k_\perp} = 0 \quad \text{with} \quad \pi_{k_\perp}(z) = \int d^3x_\perp\pi(x^{\hat{m}}, z)e^{-ix^{\hat{m}}k_{\perp\hat{m}}}, \tag{3.30}$$

the linearly independent solutions of which are given by

$$\pi_{k_\perp} = 1, \qquad 12\beta z + 8\sinh 2\beta z + \sinh 4\beta z \qquad \text{for } k_\perp^2 = 0, \tag{3.31}$$

and

[8]The assumption here is just for simplicity, and our result should hold for more general setups qualitatively.

[9]Notice that the linear term in the total derivative term (3.24) vanishes because $\alpha_1(z) = 0$ outside the domain wall $|z| \gg 1/\beta$, where the z-diffeomorphism invariance is unbroken. We will revisit this point in Sec. IV.

$$\pi_{k_\perp} = \exp\left(\pm 2\beta z\sqrt{1+\frac{k_\perp^2}{4\beta^2}}\right)\left[\left(1+\frac{k_\perp^2}{6\beta^2}\right)\cosh 2\beta z \mp \sqrt{1+\frac{k_\perp^2}{4\beta^2}}\sinh 2\beta z + \frac{k_\perp^2}{6\beta^2}\right] \quad \text{for } k_\perp^2 \neq 0. \tag{3.32}$$

We notice that only the constant mode, $\pi_{k_\perp} = 1$, has a finite value throughout the space, whereas the other modes diverge outside the brane. Since the π field corresponds to the translational transformation parameter, the constant mode, $\pi_{k_\perp} = 1$, generates a shift of brane position without changing the brane profile and can be interpreted as the standard gapless NG mode propagating along the brane. It is also convenient to express the solutions in terms of the canonically normalized field $\pi^c_{k_\perp} = \alpha_1^{1/2}\pi_{k_\perp}$:

$$\pi^c_{k_\perp} = \frac{\beta v}{(1+\cosh 2\beta z)/2}, \qquad \beta v\frac{12\beta z + 8\sinh 2\beta z + \sinh 4\beta z}{(1+\cosh 2\beta z)/2} \quad \text{for } k_\perp^2 = 0, \tag{3.33}$$

and

$$\pi^c_{k_\perp} = \beta v \exp\left(\pm 2\beta z\sqrt{1+\frac{k_\perp^2}{4\beta^2}}\right)\frac{\left(1+\frac{k_\perp^2}{6\beta^2}\right)\cosh 2\beta z \mp \sqrt{1+\frac{k_\perp^2}{4\beta^2}}\sinh 2\beta z + \frac{k_\perp^2}{6\beta^2}}{(1+\cosh 2\beta z)/2} \quad \text{for } k_\perp^2 \neq 0. \tag{3.34}$$

This normalization provides how the energy of each mode distributes in the z-direction. For example, it is clear that the energy of the gapless NG mode localizes on the brane. We also notice that the solutions in Eq. (3.34) have a finite energy density for $-k_\perp^2 > 4\beta^2$ as well as the first solution in Eq. (3.33). More concretely, the two modes in Eq. (3.34) behave like massive modes with the mass 2β outside the brane $|z| \gg 1/\beta$,

$$\pi^c_{k_\perp} \sim \exp(\pm i k_z z) \quad \text{with} \quad k_\perp^2 + k_z^2 = -4\beta^2. \tag{3.35}$$

Also, gauge transformation parameters corresponding to the two modes diverge outside the brane $|z| \gg 1/\beta$, as is suggested by Eq. (3.32). We therefore interpret them as bulk propagations of the original scalar field $\phi(z)$, rather than standard NG modes. In Appendix B, we show that the low-energy effective action after integrating out those gapped modes is nothing but the Nambu–Goto action.

To summarize, there exist two types of physical modes around the single scalar brane background: the standard massless NG mode localizing on the brane and the massive modes propagating in the bulk direction. In particular, only the standard localized NG mode is relevant in the low-energy scale $E \ll \beta$, and the standard coset construction takes into account these degrees of freedom only. Conversely, if β is much smaller than a typical scale of excitation energy, the massive modes are not negligible, and they should be included in the low-energy effective theory.

D. Inclusion of local Lorentz symmetry breaking

We then discuss the case when a nonzero spin field has a space-dependent condensation. To illustrate the degrees of freedom and residual symmetries in the unitary gauge, let us consider a (spacelike) vector A_m on the Minkowski space as a concrete example. Suppose that a vector field A_m has a space-dependent condensation of the form

$$\langle A_m(x)\rangle = \delta^3_m v(z). \tag{3.36}$$

Here and in what follows, we use integers 0, 1, 2, 3 to denote the t-, x-, y-, and z-directions of the local Lorentz index. Since $A_m(x)$ has a Lorentz charge, the local Lorentz symmetry is broken as well as the z-diffeomorphism invariance. Following the EFT recipe, we then introduce the vierbein e^m_μ to gauge the Lorentz symmetry. Schematically, we write the degrees of freedom and symmetries after introducing the vierbein as

$$\begin{aligned}&A_m(x),\ e^m_\mu(x) + (3+1)\text{-dim diffs},\\ &(3+1)\text{-dim local Lorentz}.\end{aligned} \tag{3.37}$$

To take the unitary gauge, it is convenient to note the decomposition

$$A_m(x) = \Lambda_m{}^3(x)v(z+\pi(x)) \quad \text{with} \quad \Lambda_m{}^n(x) \in SO(3,1), \tag{3.38}$$

where $\Lambda_m{}^3(x)$ specifies the direction of A_m and corresponds to the NG field for the local Lorentz symmetry. On the other hand, $\pi(x)$ specifies the amplitude of A_m and corresponds to the NG field for the z-diffeomorphism. Using the local Lorentz and diffeomorphism invariance, we can remove those NG fields to set $\Lambda_m{}^3 = \delta^3_m$ and $\pi = 0$, at least as long as the fluctuations are small. In such a unitary gauge, the only dynamical degrees of freedom are the vierbein e^m_μ, and the residual symmetries are the $(2+1)$-dimensional local

Lorentz and diffeomorphism invariance along the t, x, and y-directions. Schematically, we write this setup as

$$e^m_\mu(x) + (2+1)\text{-dim diffs}, \qquad (2+1)\text{-dim local Lorentz}. \tag{3.39}$$

We then construct the effective action based on these degrees of freedom and residual symmetries. Schematically, let us decompose the effective action into the three types of contributions as

$$S = S_P + S_L + S_{PL}. \tag{3.40}$$

Here, S_P is the effective action (3.16) and breaks the diffeomorphism invariance only:

$$S_P = -\frac{1}{2}\int d^4x\sqrt{-g}[\alpha_1(z) + \alpha_2(z)g^{zz} + \alpha_3(z)(g^{zz}-1)^2]. \tag{3.41}$$

On the other hand, S_L and S_{PL} break the local Lorentz invariance, and in particular, S_L represents terms that may exist even if the diffeomorphism is unbroken (the following β_i's are constants when the diffeomorphism is unbroken). At the lowest order with respect to fluctuations and derivatives, they are given by[10]

$$S_L = \int d^4x\sqrt{-g}\Big[-\frac{\beta_1(z)}{4}(\nabla_\mu e^3_\nu - \nabla_\nu e^3_\mu)^2 - \frac{\beta_2(z)}{2}(\nabla^\mu e^3_\mu)^2 - \frac{\beta_3(z)}{2}(e^\nu_3\nabla_\nu e^3_\mu)^2\Big], \tag{3.42}$$

$$S_{PL} = \int d^4x\sqrt{-g}\gamma_1(z)(e^\mu_3 n_\mu - 1), \tag{3.43}$$

where $\beta_i(z)$ and $\gamma_1(z)$ are scalar functions depending on z and the unit vector $n_\mu = \delta^z_\mu/\sqrt{g^{zz}}$ breaks the z-diffeomorphism invariance explicitly. We next introduce the NG fields by the Stückelberg method and decouple the gauge degrees of freedom. As in Sec. III B, we first introduce the NG fields π for the z-diffeomorphism by the field-dependent gauge transformation (3.11). Similarly, we introduce NG fields $\xi_{\hat{m}}$'s for the local Lorentz transformation in the 3-$\hat{m}$ plane ($\hat{m} = 0, 1, 2$) as

$$e^\mu_m(x) \to \tilde{e}^\mu_m(x) = \Lambda_m{}^n(x)e^\mu_n(x) \quad \text{with} \quad \Lambda_m{}^n(x) = (\exp[\xi^{\hat{\ell}}(x)\Sigma_{\hat{\ell}3}])_m{}^n \in SO(3,1), \tag{3.44}$$

where Σ_{mn}'s are generators of $SO(3,1)$. In particular, $e^\mu_3(x)$ is transformed as

$$e^\mu_3 \to \tilde{e}^\mu_3 = \Lambda_3{}^m e^\mu_m = \left[\delta^m_3\left(1 - \frac{1}{2}\xi_{\hat{m}}\xi^{\hat{m}}\right) + \delta^m_{\hat{m}}\xi^{\hat{m}} + \mathcal{O}(\xi^3)\right]e^\mu_m. \tag{3.45}$$

Since the full diffeomorphism and local Lorentz invariance can be restored by assigning nonlinear transformation rules on the NG fields π and $\xi_{\hat{m}}$, we can set $e^m_\mu = \delta^m_\mu$ using the full gauge degrees of freedom. After these procedures, S_L and S_{PL} take the form

$$\begin{aligned} S_L &= \int d^4x\Big[-\frac{\beta_1(z+\pi)}{4}(\partial_m\Lambda_{3n} - \partial_n\Lambda_{3m})^2 - \frac{\beta_2(z+\pi)}{2}(\partial_m\Lambda_3{}^m)^2 - \frac{\beta_3(z+\pi)}{2}(\Lambda_3{}^n\partial_n\Lambda^3_m)^2\Big] \\ &= \int d^4x\Big[-\frac{\beta_1(z)}{4}(\partial_{\hat{m}}\xi_{\hat{n}} - \partial_{\hat{n}}\xi_{\hat{m}})^2 - \frac{\beta_2(z)}{2}(\partial^{\hat{m}}\xi_{\hat{m}})^2 - \frac{\beta_1(z)+\beta_3(z)}{2}(\partial_z\xi_{\hat{m}})^2 + \cdots\Big], \end{aligned} \tag{3.46}$$

$$\begin{aligned} S_{PL} &= \int d^4x\gamma_1(z+\pi)\left[\Lambda_3{}^m\frac{\delta^z_m + \partial_m\pi}{\sqrt{1+2\partial_z\pi + \partial_n\pi\partial^n\pi}} - 1\right] \\ &= \int d^4x\Big[-\frac{\gamma_1(z)}{2}(\xi_{\hat{m}} - \partial_{\hat{m}}\pi)^2 + \cdots\Big], \end{aligned} \tag{3.47}$$

where the dots stand for the cubic and higher orders in perturbations. Here, it should be noticed that S_L contains the kinetic terms for $\xi_{\hat{m}}$, whereas S_{PL} contains the mass term for $\xi_{\hat{m}}$ and mixing interactions between $\xi_{\hat{m}}$ and π. Also note that Eqs. (3.46) and (3.47) do not contain linear order terms. The background equation of motion therefore requires $\alpha_1'(z) = \alpha_2'(z)$ so that the tadpole terms in S_P vanish. The effective action for NG modes is then given by Eqs. (3.22)–(3.24), (3.46), and (3.47).

E. Qualitative features of nonzero spin branes

We then apply the obtained effective action to single brane backgrounds. For simplicity, let us concentrate on the case $\alpha_3 = \beta_2 = \beta_3 = 0$ and consider the following second order action[11]:

[10]Note that there are some ambiguities in the expression of the effective action. For example, we can use $e^z_3 - 1$ instead of $e^\mu_3 n_\mu - 1$ to define S_{PL}. However, the difference between the two descriptions can be absorbed by the redefinition of α_i's in (3.41), and it turns out that the generic effective action at the lowest order is given by (3.41), (3.42), and (3.43).

[11]Although we set several parameters to be zero for simplicity, our results should hold for more general setups qualitatively. Also note that the β_2 coupling induces kinetic terms with a wrong sign and such terms are prohibited by the stability of the background, though it is not prohibited only from the symmetry point of view.

$$S = \int d^4x \left[-\frac{\alpha_1(z)}{2} \partial_m \pi \partial^m \pi - \frac{\beta_1(z)}{4} [(\partial_{\hat{m}} \xi_{\hat{n}} - \partial_{\hat{n}} \xi_{\hat{m}})^2 + 2(\partial_z \xi_{\hat{m}})^2] - \frac{\gamma_1(z)}{2} (\xi_{\hat{m}} - \partial_{\hat{m}} \pi)^2 \right]. \tag{3.48}$$

Among the three functions $\alpha_1(z)$, $\beta_1(z)$, and $\gamma_1(z)$ characterizing the brane profile, $\alpha_1(z)$ and $\gamma_1(z)$ are associated with the breaking of z-diffeomorphism invariance. These two functions therefore have support on the brane and vanish outside, just as $\alpha_1(z)$ for a single scalar brane. On the other hand, $\beta_1(z)$ does not necessarily vanish outside the brane and has a nonzero value as long as the local Lorentz symmetry is broken. More concretely, it is convenient to introduce a function $v(z)$ parametrizing the local Lorentz symmetry breaking, just as we did in Eq. (3.36) for vector condensation. In terms of $v(z)$, the three functions are typically given by

$$\alpha_1(z) \sim \gamma_1(z) \sim v'(z)^2, \qquad \beta_1(z) \sim v(z)^2, \tag{3.49}$$

where, for simplicity, we assumed that $v(z)$ is the only field breaking the z-diffeomorphism invariance. One important point is that there exist several types of β_1 profiles even for single brane backgrounds. In the rest of this subsection, we discuss how the low-energy dynamics depends on the β_1 profile using two typical examples depicted in Fig. 2.

In the first case (the left figure), local Lorentz symmetry is broken only on the brane. A typical $v(z)$ profile is given by

$$v(z) = \frac{v_0}{\cosh \beta z}, \tag{3.50}$$

and the functions $\alpha_1(z)$, $\beta_1(z)$, and $\gamma_1(z)$ vanish outside the brane $|z| \gg 1/\beta$. Similarly to the discussion in Sec. III C, the NG modes π and $\xi_{\hat{m}}$ cannot have a z-dependence at the energy scale $E \ll \beta$, because their kinetic terms vanish outside the brane. The action (3.48) can then be reduced effectively to the three-dimensional one,

$$S = \int d^3x \left[-\frac{A}{2} \partial_{\hat{m}} \pi \partial^{\hat{m}} \pi - \frac{B}{4} (\partial_{\hat{m}} \xi_{\hat{n}} - \partial_{\hat{n}} \xi_{\hat{m}})^2 - \frac{C}{2} (\xi_{\hat{m}} - \partial_{\hat{m}} \pi)^2 \right], \tag{3.51}$$

where A, B, and C are constants defined by

$$A = \int_{-\infty}^{\infty} dz \alpha_1(z), \qquad B = \int_{-\infty}^{\infty} dz \beta_1(z),$$
$$C = \int_{-\infty}^{\infty} dz \gamma_1(z). \tag{3.52}$$

In terms of the normalization factor v_0 and the thickness $1/\beta$, these parameters can be estimated as

$$A \sim v_0^2 \beta, \qquad B \sim \frac{v_0^2}{\beta}, \qquad C \sim v_0^2 \beta. \tag{3.53}$$

Since $\xi_{\hat{m}}$ acquires a mass $m \sim \beta$, the dynamics at the energy scale $E \ll \beta$ is governed by the NG mode π for the z-diffeomorphism. In particular, we can integrate out the $\xi_{\hat{m}}$ field as

$$\xi_{\hat{m}} = \partial_{\hat{m}} \pi + \mathcal{O}(E^2/\beta^2). \tag{3.54}$$

The low-energy dynamics is then reduced to the same one as the scalar brane. As we revisit in Sec. VI D 2, Eq. (3.54) corresponds to the inverse-Higgs constraint in the standard coset construction.

We next consider the second case (the right figure), where local Lorentz symmetry is broken also outside the brane and a typical $v(z)$ profile is given by

$$v(z) = \bar{v}(1 + \delta \tanh \beta z). \tag{3.55}$$

Since the functions $\alpha_1(z)$ and $\gamma_1(z)$ localize on the brane, these two contributions can be reduced to the three-dimensional ones at the energy scale $E \ll \beta$,

$$S = \int d^4x \left[-\frac{\beta_1(z)}{4} [(\partial_{\hat{m}} \xi_{\hat{n}} - \partial_{\hat{n}} \xi_{\hat{m}})^2 + 2(\partial_z \xi_{\hat{m}})^2] \right] + \int d^3x \left[-\frac{A}{2} \partial_{\hat{m}} \pi \partial^{\hat{m}} \pi - \frac{C}{2} (\xi_{\hat{m}} - \partial_{\hat{m}} \pi)^2 \right], \tag{3.56}$$

where A and C can be estimated as

$$A = \int_{-\infty}^{\infty} dz \alpha_1(z) \sim \bar{v}^2 \delta^2 \beta, \qquad C = \int_{-\infty}^{\infty} dz \gamma_1(z) \sim \bar{v}^2 \delta^2 \beta. \tag{3.57}$$

Let us then consider the parameter region $\delta \ll 1$ in particular,

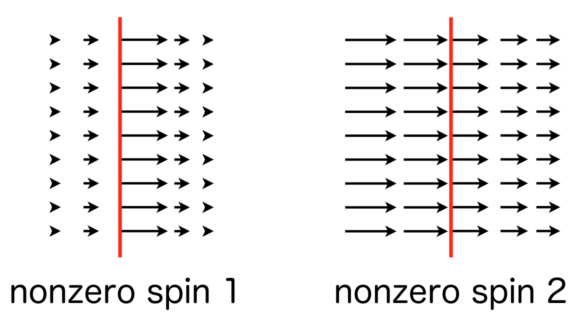

FIG. 2 (color online). Two examples for nonzero spin branes. In the first example (left figure), both of the diffeomorphism and local Lorentz symmetries are broken only on the brane. On the other hand, in the second example (right figure), the local Lorentz symmetry is broken in the whole spacetime.

$$S \sim -\bar{v}^2 \left\{ \int d^4x[(\partial_{\hat{m}}\xi_{\hat{n}} - \partial_{\hat{n}}\xi_{\hat{m}})^2 + 2(\partial_z\xi_{\hat{m}})^2] + \delta^2\beta \int d^3x[\partial_{\hat{m}}\pi\partial^{\hat{m}}\pi + (\xi_{\hat{m}} - \partial_{\hat{m}}\pi)^2] \right\}, \tag{3.58}$$

where we dropped some numerical coefficients for simplicity. An interesting point is that there exists a hierarchy in the energy scale. First, at the low-energy limit $E \ll \delta\beta(\ll \beta)$, the equations of motion inside the brane are of the form

$$\xi_{\hat{m}} = \partial_{\hat{m}}\pi + \mathcal{O}(E^2/(\delta^2\beta^2)\cdot\xi),$$
$$\partial_{\hat{m}}^2\pi = 2\partial^{\hat{m}}(\xi_{\hat{m}} - \partial_{\hat{m}}\pi) = \mathcal{O}(E^3/(\delta^2\beta^2)\cdot\xi), \tag{3.59}$$

which results in the equation of motion $\partial_{\hat{m}}^2\pi = 0$ for a massless field on the brane. Just as the first case, the equation of motion for $\xi_{\hat{m}}$ (inside the brane) corresponds to the inverse-Higgs constraint, and the dynamics of π is reduced to the same one as the scalar brane case. On the other hand, at the energy scale $\delta\beta \ll E \ll \beta$, the action (3.58) can be further approximated as

$$S \sim -\bar{v}^2 \left\{ \int d^4x[(\partial_{\hat{m}}\xi_{\hat{n}} - \partial_{\hat{n}}\xi_{\hat{m}})^2 + 2(\partial_z\xi_{\hat{m}})^2] + \delta^2\beta \int d^3x(\partial_{\hat{m}}\pi\partial^{\hat{m}}\pi - \xi^{\hat{m}}\partial_{\hat{m}}\pi) \right\}, \tag{3.60}$$

where $\xi_{\hat{m}}$ and π can be thought of as massless fields propagating in the fourth dimension and localizing on the brane, respectively. The bulk field $\xi_{\hat{m}}$ and the localized field π then interact with each other by the $\xi^{\hat{m}}\partial_{\hat{m}}\pi$ interaction. The dynamics at this energy scale is different from both of the single scalar brane case and the first example for single nonzero spin branes.

In Fig. 3, we summarize the qualitative features of three types of single brane backgrounds discussed in this section: one for a single scalar brane and two for a single nonzero spin brane. In the low-energy limit, the dynamics of π (after integrating out $\xi_{\hat{m}}$) in each setup results in the same one, which can be captured by the standard coset construction based on the global symmetry picture. However, this degeneracy is resolved beyond the low-energy limit, and the resolving energy scale is not necessarily high compared with the scale β of the brane thickness. Such a scale can be in our interests, and we need to specify the symmetry breaking pattern based on the local symmetry picture to investigate such an intermediate scale. This is one point of our paper.

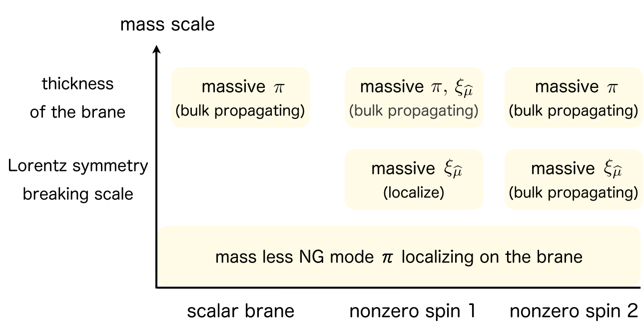

FIG. 3 (color online). Physical spectra for three types of branes. While the massless spectrum is the same between the three, the massive spectrum depends on symmetry breaking patterns in the local picture.

IV. ONE-DIMENSIONAL PERIODIC MODULATION

As we discussed in the previous section, the effective action for diffeomorphism symmetry breaking contains free functions of coordinates, and their profiles are directly related to the breaking pattern. For example, in the single brane case, the z-diffeomorphism invariance is broken only on the brane, and the functions α_i's in Eq. (3.20) are localized on it. In this section, we discuss one-dimensional periodic modulation, i.e., a system in which the condensation is periodic in one direction, by changing the profile of those functions. As depicted in Fig. 4, such symmetry breaking patterns are realized in condensed matter systems such as the smectic-A phase of liquid crystals [40] and the Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) phase of the superconductor [43,44]. With these types of applications in mind, we extend our analysis to nonrelativistic systems in Minkowski space and discuss generic features in the dispersion relations of NG modes in the presence of one-dimensional periodic modulation.

Let us first extend the previous effective action to nonrelativistic systems. There are several possibilities of generalization to nonrelativistic systems. Here, we assume that the system has spacetime translations and spatial rotational symmetries. In particular, we do not consider Galilean symmetry for simplicity.[12] The rest of the discussions is completely parallel to the relativistic case, and it is straightforward to perform the construction of the effective action. The effective action for the z-diffeomorphism symmetry breaking is then given by

$$S_P = \frac{1}{2}\int d^4x \left[\tilde{\alpha}_1(z+\pi)\dot{\pi}^2 - \alpha_1(z+\pi)(\partial_i\pi)^2 - \alpha_3(z+\pi)(2\partial_z\pi(x) + (\partial_i\pi)^2)^2 - \frac{d}{dz}A_1(z+\pi)\right], \tag{4.1}$$

[12]When the system originally enjoys Galilean symmetry, some modifications may be required. See also Appendix A. Under this simplification, the only difference from the relativistic case is that we consider spatial diffeomorphism, instead of the full diffeomorphism, to construct the effective action in the unitary gauge.

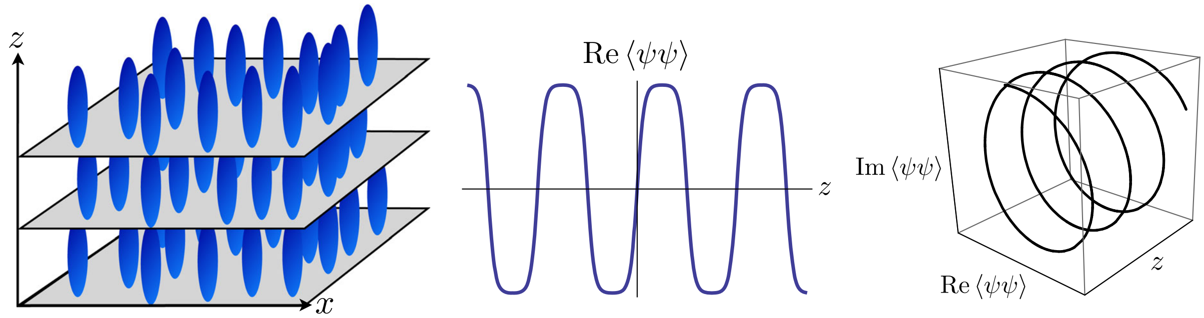

FIG. 4 (color online). Examples of one-dimensional periodic modulation. In the smectic-A phase (the left figure), the layer structure breaks the diffeomorphism symmetry on the whole spacetime. In the FFLO phase of superconductor (the center and right figures), the chiral condensation arises in an inhomogeneous way.

where π is the NG field for the z-diffeomorphism and $A_1(z) = 2\int^z dz'\alpha_1(z')$. In contrast to the relativistic case, the functions $\tilde{\alpha}_1$ and α_1 in front of the temporal and spatial kinetic terms are independent. Up to second order in π, the bulk contribution (the first three terms) and the total derivative contribution (the last term) can be expanded as

$$S_{P,\text{bulk}} = \frac{1}{2}\int d^4x[\tilde{\alpha}_1(z)\dot{\pi}^2 - \alpha_1(z)(\partial_i\pi)^2 - 4\alpha_3(z)(\partial_z\pi)^2], \tag{4.2}$$

$$S_{P,\text{t.d.}} = -\frac{1}{2}\int d^4x\frac{d}{dz}[A_1(z) + 2\alpha_1(z)\pi + \alpha_1'(z)\pi^2], \tag{4.3}$$

where we note that $S_{P,\text{t.d.}}$ contains a linear order term if $\alpha_1 \neq 0$ at the boundary. This feature will be important in the following discussions.

We then discuss systems with one-dimensional periodic modulation based on this effective action. Suppose that the condensation is periodic in the z-direction and is characterized by a discrete symmetry $z \to z' = z + a$ with a being the periodicity. Since the functions $\tilde{\alpha}_1$ and α_i are periodic because of the residual discrete symmetry, they generically have support on the whole spacetime. In particular, $\alpha_1(z)$, if it exists, does not vanish at the boundary and leads to a linear order term in Eq. (4.3), which is in sharp contrast to the single brane case. Let us take a closer look at this linear order term and discuss its implications for the dispersion relations of NG modes. First, this linear order term is not relevant as long as the NG field π decays at spatial infinity $\lim_{z\to\pm\infty}\pi(x) = 0$ since it is a total derivative. It is essentially because we impose the background (bulk) equations of motion in the effective action construction, which guarantee the stability of backgrounds under small perturbations with a fixed boundary condition. However, the boundary linear term becomes relevant if we consider a configuration with $\lim_{z\to\pm\infty}\pi(x) \neq 0$ as this implies the existence of configurations with a lower energy, just as tadpoles in the bulk action. For example, let us consider a configuration of the form

$$\pi(x) = \epsilon z \quad (\epsilon\text{: constant}), \tag{4.4}$$

which corresponds to a rescaling $z \to z' = (1-\epsilon)z$. The energy contributions $E_{\text{t.d.}}$ from Eq. (4.2) and E_{bulk} from Eq. (4.3) for this configuration are given by

$$E_{\text{t.d.}} \sim \alpha_1\epsilon V, \qquad E_{\text{bulk}} \sim (\alpha_1 + 4\alpha_3)\epsilon^2 V, \tag{4.5}$$

where V is the spatial volume. It then turns out that there exists a low-energy direction along a small negative ϵ. We therefore conclude that the α_1 term is prohibited when the background energy is a local minimum in the configuration space and the effective action at the lowest order derivative is

$$\begin{aligned} S_P &= \frac{1}{2}\int d^4x[\tilde{\alpha}_1(z+\pi)\dot{\pi}^2 \\ &\quad - \alpha_3(z+\pi)(2\partial_z\pi(x) + (\partial_i\pi)^2)^2] \\ &= \frac{1}{2}\int d^4x[\tilde{\alpha}_1(z)\dot{\pi}^2 - 4\alpha_3(z)(\partial_z\pi)^2 + \mathcal{O}(\pi^3)]. \end{aligned} \tag{4.6}$$

The corresponding dispersion relation for the NG mode π is

$$\omega^2 = c_1 k_z^2, \tag{4.7}$$

where ω is the energy, k_z is the momentum in the first Brillouin zone, $|k_z| \leq \pi/a$, and c_1 is the coefficient depending on $\tilde{\alpha}_1$ and α_3. It should be emphasized that the momentum $k_\perp^2 = k_x^2 + k_y^2$ in the x- and y-directions does not appear in the dispersion relation Eq. (4.7) up to this order [45]. By including higher order terms in the effective action, we can explicitly show that higher order derivative corrections to the dispersion relations are schematically

$$\omega^2 \sim c_1 k_z^2 + c_2 k_z k_\perp^2 + c_3 k_\perp^4 \tag{4.8}$$

up to the second order in k_z and $k_\perp^2$, where c_i are constants.[13]

It would be also interesting to illustrate that such higher order correction terms can arise after integrating out massive NG fields for local rotations. The effective action for such a symmetry breaking pattern can be easily obtained by extending the construction in the previous section to nonrelativistic systems. Here, we again consider that the system has spacetime translations and spatial rotational symmetries. The second order action for NG fields is then given by

$$S_L = \int d^4x \left[\frac{\tilde{\beta}_1 + \beta_1}{2} (\partial_t \xi_{\hat{i}})^2 - \frac{\beta_1}{4} (\partial_{\hat{i}} \xi_{\hat{j}} - \partial_{\hat{j}} \xi_{\hat{i}})^2 - \frac{\beta_2}{2} (\partial_{\hat{i}} \xi_{\hat{i}})^2 - \frac{\beta_1 + \beta_3}{2} (\partial_z \xi_{\hat{i}})^2 \right], \tag{4.9}$$

$$S_{PL} = \int d^4x \left[-\frac{\gamma_1}{2} (\xi_{\hat{i}} - \partial_{\hat{i}} \pi)^2 + \frac{\gamma_1}{2} \dot{\pi}^2 + \cdots \right], \tag{4.10}$$

where $\hat{i} = x, y$ and $\xi_{\hat{i}}$'s are NG fields for rotations in the $\hat{i} - z$ plane. Also we assume that $\tilde{\beta}_1$, β_i's, and γ_1 are constants for simplicity, though they have z-dependence in general. Just as in the relativistic case, S_{PL} contains a mass term of $\xi_{\hat{i}}$ and mixing interactions between π and $\xi_{\hat{i}}$. In the low-energy limit, the equation of motion for $\xi_{\hat{i}}$ is reduced to

$$\xi_{\hat{i}} = \partial_{\hat{i}} \pi. \tag{4.11}$$

Substituting it to Eq. (4.9), we obtain the effective interaction of the form

$$S_{\rm eff} = \int d^4x \left[\frac{\tilde{\beta}_1 + \beta_1}{2} (\partial_t \partial_{\hat{i}} \pi)^2 - \frac{\beta_2}{2} (\partial_{\hat{i}}^2 \pi)^2 - \frac{\beta_1 + \beta_3}{2} (\partial_z \partial_{\hat{i}} \pi)^2 + \frac{\gamma}{2} \dot{\pi}^2 \right], \tag{4.12}$$

which gives the corrections to the coefficients of k_z^2 and $k_\perp^4$ in the dispersion relation.

To summarize, in the systems with one-dimensional periodic modulation, the (locally) minimum energy condition constrains the dispersion relations of NG modes for the broken diffeomorphism as in Eq. (4.8). In particular, the massive NG fields for local rotations can for example induce higher derivative corrections in the dispersion relations.

We note that such discussions on dispersion relations suggest that the one-dimensional order can be realized only at zero temperature. It is because the finite temperature effect breaks the order parameter in the thermodynamic limit; the contribution of the thermal fluctuation of the NG mode to the order parameter is proportional to

$$\langle \pi^2(x) \rangle = T \int \frac{d^2k_\perp dk_z}{(2\pi)^3} \frac{1}{k_z^2 + c^2 k_\perp^4} = \frac{T}{4\pi c} \ln \frac{\Lambda}{\mu}, \tag{4.13}$$

where $c = \beta_2/(4\alpha_3)$ and we introduced UV and IR cutoffs, Λ and μ. At $\mu \to 0$, this correction is logarithmically divergent; it leads to the vanishing order parameter [45]. A typical example is a smectic-A phase of liquid crystals, in which the order parameter vanishes, and the quasi-long range order appears [46]. Also note that if there exists an external field that explicitly breaks rotation symmetry such as a magnetic field, the term $k_\perp^2$ appears in the dispersion relation of π. As the result, the fluctuation of π to the order parameter is suppressed, and the order parameter remains finite [45].

V. MIXTURE OF INTERNAL AND SPACETIME SYMMETRIES

Our approach to construct the effective action is applicable not only to spacetime symmetry breaking but also to the breaking of a mixture of spacetime and internal symmetries. In this section, as a simplest example, we discuss the case when a global internal $U(1)$ symmetry and a translation symmetry are broken to the diagonal Abelian symmetry.

A. Complex scalar field model

Let us begin with a complex scalar field model and illustrate concretely the degrees of freedom and residual symmetries in the unitary gauge. Suppose that a complex scalar field follows the symmetry transformation rule

$$U(1): \phi(x) \to \phi'(x) = e^{i\lambda} \phi(x), \quad \text{translation}: \phi(x) \to \phi'(x) = \phi(x + \epsilon), \tag{5.1}$$

where the transformation parameters λ and ϵ^μ are constants. When it has a background condensation

$$\langle \phi(x) \rangle = r_0 e^{iut} \quad (r_0 \quad \text{and} \quad u: \text{real constants}), \tag{5.2}$$

the internal $U(1)$ and time-translational symmetries are broken to the diagonal one,

$$\phi(x) \to \phi'(x) = e^{i\lambda} \phi(x + \epsilon) \quad \text{with} \quad \lambda = -u\epsilon^0 . \epsilon^i = 0. \tag{5.3}$$

We then reinterpret this symmetry breaking from the local symmetry point of view. For this purpose, let us gauge the internal $U(1)$ and translation symmetry by introducing a gauge field A_μ for the internal $U(1)$ symmetry and the metric field $g_{\mu\nu}$. Their $U(1)$ gauge transformations are

[13]Note that the counting of the scaling dimension changes when the dispersion relation is anisotropic. For Eq. (4.8), k_z and $k_\perp^2$ have the same scaling dimension, so that the terms displayed there are the lowest order in the derivative expansion.

$$WA_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu\lambda(x),$$
$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = g_{\mu\nu}(x), \tag{5.4}$$

and their diffeomorphism transformations are

$$A_\mu(x) \to A'_\mu(x) = \frac{\partial(x^\nu + \epsilon^\nu)}{\partial x^\mu} A_\nu(x+\epsilon),$$
$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = \frac{\partial(x^\rho + \epsilon^\rho)}{\partial x^\mu}\frac{\partial(x^\sigma + \epsilon^\sigma)}{\partial x^\nu} g_{\rho\sigma}(x+\epsilon). \tag{5.5}$$

Also the transformation rule of the scalar field ϕ is given by replacing the transformation parameters λ and ϵ^μ in Eq. (5.1) by local ones $\lambda(x)$ and $\epsilon^\mu(x)$. Correspondingly, the unbroken diagonal symmetry (5.3) is gauged as

$$\phi(x) \to \phi'(x) = e^{i\lambda(x+\epsilon(x))}\phi(x+\epsilon(x)) \quad \text{with}$$
$$\lambda(x) = -u\epsilon^0(x), \qquad \epsilon^i = 0, \tag{5.6}$$

which is realized by performing the time diffeomorphism $\epsilon^0(x)$ after the internal $U(1)$ gauge transformation $\lambda(x)$. The background (5.2) is then characterized by the symmetry breaking of the internal $U(1)$ gauge and diffeomorphism symmetries to the diagonal gauge symmetry (5.6) and spatial diffeomorphism symmetries.

We next discuss the degrees of freedom and the residual symmetries in the unitary gauge. First, the setup before taking the unitary gauge can be schematically written as

$$\phi(x),\ A_\mu(x),\ g_{\mu\nu}(x) + \text{internal}\, U(1)\, \text{gauge},\ \text{diffs}. \tag{5.7}$$

To take the unitary gauge, it is convenient to note the decomposition

$$\phi(x) = (r_0 + \sigma(x))e^{i(ut+\pi(x))}, \tag{5.8}$$

where $\pi(x)$ and $\sigma(x)$ are real scalar fields. Since internal $U(1)$ gauge transformations and time diffeomorphisms generate local shifts of $\pi(x)$, it can be identified with the NG field, and we can impose the unitary gauge condition $\pi(x) = 0$. Just as the background (5.2), the gauge condition $\pi(x) = 0$ is invariant under the diagonal gauge transformations (5.6) and spatial diffeomorphisms. Schematically, the dynamical degrees of freedom and the residual symmetries in the unitary gauge are given by

$$\sigma(x),\ A_\mu(x),\ g_{\mu\nu}(x) + \text{diagonal gauge},\ \text{spatial diffs}. \tag{5.9}$$

Note that $\sigma(x)$ is interpreted as a matter field, which cannot be absorbed by gauge transformations and is generically massive.

B. Construction of effective action

In the previous subsection, we illustrated the unitary gauge setup using a complex scalar field model. More generally, the dynamical degrees of freedom and the residual symmetries in the unitary gauge for this type of symmetry breaking are given by the minimal setup

$$A_\mu(x),\ g_{\mu\nu}(x) + \text{diagonal gauge},\ \text{spatial diffs} \tag{5.10}$$

and possibly with additional matter fields such as $\sigma(x)$ in Eq. (5.9). In this subsection, we construct the effective action for the minimal setup (5.10) concretely.

First, just as the case of diffeomorphism symmetry breaking, the ingredients of the effective action covariant under spatial diffeomorphisms are given by

$$\text{scalar functions of } t,\ A_\mu(x),\ g_{\mu\nu},\ R_{\mu\nu\rho\sigma},$$
$$\text{and their covariant derivatives.} \tag{5.11}$$

The general unitary gauge action is then constructed from these ingredients in an invariant way under the diagonal gauge transformation. To write down such an effective action, it is convenient to note the diagonal gauge transformation of the gauge field,

$$A_\mu(x) \to A'_\mu(x) = \frac{\partial\tilde{x}^\nu}{\partial x^\mu}A_\nu(\tilde{x}) + u\partial_\mu\epsilon(x) \quad \text{with}$$
$$\tilde{x}^\mu = x^\mu + \delta^\mu_0\epsilon(x). \tag{5.12}$$

Since the time coordinate t is invariant under the diagonal transformation

$$t \to t = \tilde{t} - \epsilon(x), \tag{5.13}$$

we can find the following combination covariant under the diagonal transformation:

$$u\partial_\mu t + A_\mu(x) \to u\left[\frac{\partial\tilde{t}}{\partial x^\mu} - \partial_\mu\epsilon(x)\right] + \frac{\partial\tilde{x}^\nu}{\partial x^\mu}A_\nu(\tilde{x}) + u\partial_\mu\epsilon(x)$$
$$= \frac{\partial\tilde{x}^\nu}{\partial x^\mu}\left[u\frac{\partial\tilde{t}}{\partial\tilde{x}^\nu} + A_\nu(\tilde{x})\right]. \tag{5.14}$$

Note that other ingredients such as the metric field $g_{\mu\nu}$ are also covariant. We therefore conclude that the time coordinate t and the gauge field A_μ can appear only in the form $u\partial_\mu t + A_\mu = u\delta^0_\mu + A_\mu$ and therefore ingredients of the unitary gauge effective action are

$$u\delta^0_\mu + A_\mu,\ g_{\mu\nu},\ R_{\mu\nu\rho\sigma},\ \text{and their covariant derivatives.} \tag{5.15}$$

The lowest few terms in fluctuations and derivatives are then given by

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}[\alpha(u^2 g^{00} + 2uA^0 + A_\mu A^\mu) + \beta(u^2(1+g^{00}) + 2uA^0 + A_\mu A^\mu)^2], \tag{5.16}$$

where α and β are constants independent of t. Also note that we used $g^{\mu\nu}(u\delta^0_\mu + A_\mu)(u\delta^0_\nu + A_\nu) = u^2 g^{00} + 2uA^0 + A_\mu A^\mu$.

We next introduce the effective action for NG field $\pi(x)$ by performing the Stückelberg method. For example, we can introduce the NG field by a field-dependent $U(1)$ gauge transformation,[14]

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\pi(x). \tag{5.17}$$

After this transformation, the ingredients (5.15) are given by

$$u\delta^0_\mu + \partial_\mu\pi + A_\mu,\ g_{\mu\nu},\ R_{\mu\nu\rho\sigma},\ \text{and their covariant derivatives.} \tag{5.18}$$

Note that the internal $U(1)$ gauge and diffeomorphism invariance can be recovered by assigning the following nonlinear transformation rule to the NG field:

$$U(1){:}\ \pi(x) \to \pi'(x) = \pi(x) + \lambda(x),$$
$$\text{diffs}{:}\ \pi(x) \to \pi'(x) = \pi(x + \epsilon(x)) + u\epsilon^0(x). \tag{5.19}$$

We can now set that $A_\mu = 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$ using the nonlinearly realized full gauge symmetries. The ingredients of the action for the NG field are then

$$u\delta^0_\mu + \partial_\mu\pi,\ \eta_{\mu\nu},\ \text{and their derivatives.} \tag{5.20}$$

Also the effective action corresponding to the unitary gauge action (5.16) is given by

$$S = -\frac{1}{2}\int d^4x[\alpha(-u^2 - 2u\partial_0\pi + \partial_\mu\pi\partial^\mu\pi) + \beta(-2u\partial_0\pi + \partial_\mu\pi\partial^\mu\pi)^2]$$
$$= \frac{\alpha}{2}\int d^4x\left[c_s^{-2}(\dot\pi^2 - c_s^2(\partial_i\pi)^2) + (1 - c_s^{-2})\left(\frac{1}{u}\dot\pi\partial_\mu\pi\partial^\mu\pi - \frac{1}{4u^2}(\partial_\mu\pi\partial^\mu\pi)^2\right)\right], \tag{5.21}$$

where we dropped temporal total derivatives and the constant term. The propagating speed c_s of the NG field $\pi(x)$ is given by $c_s^{-2} = 1 - \frac{4\beta u^2}{\alpha}$. Note that the obtained effective action takes a similar form as that for time-diffeomorphism symmetry breaking. The only difference is that the coefficients are constant instead of functions of time. In particular, because of this, there are no linear order terms in Eq. (5.21), and the background equation of motion is satisfied from the beginning. Also note that linear order terms in temporal total derivatives, dropped in the second line of Eq. (5.21), are not problematic in contrast to those in spatial total derivatives. It is because temporal total derivatives affect only initial conditions and they are not relevant once we specify the initial conditions.

C. Comments on the breaking of spatial translation

Before closing this section, we would like to make a comment on the case when spatial translation, rather than time translation, is broken in a mixed way with a global internal $U(1)$ symmetry. One typical condensation pattern for such symmetry breaking is given by

$$\langle\phi(x)\rangle = r_0 e^{iuz} \quad (r_0 \ \text{and}\ \ u{:}\ \text{real constants}) \tag{5.22}$$

in the complex scalar model of Sec. V A. This kind of symmetry breaking is discussed in the context of dense QCD matter [47–58] for example. The effective action for this symmetry breaking can be constructed in a parallel way to the construction in Sec. V B. If we work in the nonrelativistic system and assume that Galilean symmetry does not exist from the beginning, the effective action is

$$S = \frac{1}{2}\int d^4x[\tilde\alpha\dot\pi^2 - \alpha(u^2 + 2u\partial_z\pi + (\partial_i\pi)^2) - \beta(2u\partial_z\pi + (\partial_i\pi)^2)^2]. \tag{5.23}$$

The important point is that the spatial total derivative term in the action (5.23) contains a linear order term. Just as we discussed in Sec. IV, such linear order terms in spatial total derivatives are prohibited by the requirement that the background energy is at a local minimum in the configuration space. We then have $\alpha = 0$, and the dispersion relation of the NG mode $\pi(x)$ is schematically given by

$$\omega^2 \sim c'_1 k_z^2 + c'_2 k_z k_\perp^2 + c'_3 k_\perp^4 \tag{5.24}$$

[14]In the present symmetry breaking pattern, there are several ways to introduce the NG field because both of the internal $U(1)$ gauge and time-diffeomorphism symmetries are broken. For example, we can perform a field-dependent time diffeomorphism instead of the $U(1)$ gauge transformation. However, the resulting effective action is equivalent up to field redefinition of the NG field.

up to the second order in k_z and $k_\perp^2$, where c_i' are coefficients, and the last two terms on the right-hand side come from higher derivative terms in the effective action. For example, the NG mode in a model of the FFLO phase indeed accommodates this type of dispersion relations [59–61]. We will revisit this issue via the symmetry arguments of the present paper elsewhere [62].

VI. COSET CONSTRUCTION REVISITED

In this section, we revisit the coset construction for spacetime symmetry breaking, based on our discussion. In the first two subsections, we introduce a nonlinear realization for broken spacetime symmetries [4,5] and show that the parametrization of NG fields in a nonlinear realization is closely related to the local symmetry picture. In Sec. VI C, we summarize the general ingredients of the effective action for spacetime symmetry breaking. In particular, we discuss the relation between the Maurer–Cartan one form and connections for spacetime symmetries. We also comment on the difference from the internal symmetry case. In Sec. VI D, we revisit the role of the inverse-Higgs constraints, focusing on the relation to our approach based on the local picture. We classify the physical meaning of the inverse-Higgs constraints based on the coordinate dimension of broken symmetries.

Throughout this section, for simplicity, we concentrate on symmetry breaking in the case of the Minkowski space and assume that the system originally enjoys translation symmetries in all directions.

A. Local decomposition of spacetime symmetries

In the first two subsections we introduce nonlinear realization for broken spacetime symmetries and discuss its properties. For this purpose, it is convenient to consider a local field $\Phi(x)$ that belongs to a linear irreducible representation of spacetime and internal symmetries, which can be related to a field $\Phi(0)$ at the origin by a translation:

$$\Phi(x) = \Omega_P(x)\Phi(0) \quad \text{with} \quad \Omega_P(x) = e^{x^m P_m}, \tag{6.1}$$

where P_m is the translation generator. Correspondingly, we can relate symmetry transformations of $\Phi(x)$ to those of $\Phi(0)$. For example, we can rewrite the special conformal transformations of $\Phi(x)$ as[15]

[15]We define the symmetry generators such that

$$\begin{aligned}
&[D, P_m] = -P_m, \quad [D, K_m] = K_m,\\
&[K_m, P_n] = 2(\eta_{mn} D + L_{mn}),\\
&[L_{mn}, P_\ell] = -\eta_{m\ell} P_n + \eta_{n\ell} P_m, \;\; [L_{mn}, K_\ell] = -\eta_{m\ell} K_n + \eta_{n\ell} K_m,\\
&[L_{mn}, L_{rs}] = -\eta_{mr} L_{ns} + 3 \text{ terms}
\end{aligned} \tag{6.2}$$

and other commutators vanish.

$$\begin{aligned}
b^m K_m \Phi(x) = \Omega_P(x)[(-2x^n x^\ell b_\ell + x^2 b^n) P_n + 2b^m x^n L_{mn}\\
+ 2b_m x^m D + b^m K_m]\Phi(0),
\end{aligned} \tag{6.3}$$

where L_{mn}, D, and K_m are generators of Lorentz transformations, dilatations, and special conformal transformations, respectively. Since the origin $x=0$ is invariant under Lorentz transformations, dilatations, and special conformal transformations, the last three terms in the brackets act linearly on $\Phi(0)$. Moreover, when Φ is a primary field, the special conformal generator K_m acts trivially on $\Phi(0)$,

$$\begin{aligned}
b^m K_m \Phi(x) = \Omega_P(x)[(-2x^n x^\ell b_\ell + x^2 b^n) P_n + 2b^m x^n L_{mn}\\
+ 2b_m x^m D]\Phi(0).
\end{aligned} \tag{6.4}$$

It is then natural to identify the last two terms in the brackets as local Lorentz and local Weyl transformations at the point x. More explicitly, one may rewrite (6.4) as

$$\begin{aligned}
b^m K_m \Phi(x) &= (-2x^n x^\ell b_\ell + x^2 b^n) P_n \Phi(x)\\
&\quad + \Omega_P(x)[2b^m x^n L_{mn} + 2b_m x^m D]\Omega_P^{-1}(x)\Phi(x)\\
&= [1 + \Omega_P(\tilde{x})[2b^m \tilde{x}^n L_{mn} + 2b_m \tilde{x}^m D]\\
&\quad \times \Omega_P^{-1}(\tilde{x})]\Phi(\tilde{x}) - \Phi(x) + \mathcal{O}(b^2),
\end{aligned} \tag{6.5}$$

where $\tilde{x}^n = x^n - 2x^n x^\ell b_\ell + x^2 b^n + \mathcal{O}(b^2)$. The expression (6.5) corresponds to the local decomposition of spacetime symmetries discussed in Sec. II: the first term is the Lorentz transformation and dilatation around the point x^m. Note that the transformation $x \to \tilde{x}$ is identified with an inverse transformation of $x \to x'^n = x^n + 2x^n x^\ell b_\ell - x^2 b^n + \mathcal{O}(b^2)$ if we use the notation in Sec. II.

More generally, the decomposition (6.1) allows us to express arbitrary spacetime symmetry transformations in terms of diffeomorphisms, local Lorentz transformations, and local (an)isotropic Weyl transformations, just as we did in Sec. II. Suppose that the spacetime and internal symmetry algebra contains symmetry generators with the coordinate dimension $n \geq 0$ as well as the translation symmetry generators $\mathfrak{g}_P$. Let us also introduce the Lie algebra $\mathfrak{g}_n$ of symmetry generators with the coordinate dimension n, which satisfies the commutation relations

$$[\mathfrak{g}_m, \mathfrak{g}_n] = \mathfrak{g}_{m+n}. \tag{6.6}$$

When the space and time coordinates have the same scaling dimension, $\mathfrak{g}_n$'s are schematically represented in the coordinate space as

$$\sim x^{m_1} x^{m_2} \ldots x^{m_{n+1}} \partial_m \quad \text{and} \quad \sim x^{m_1} x^{m_2} \ldots x^{m_n} T_a, \tag{6.7}$$

and $\mathfrak{g}_P = \mathfrak{g}_{-1}$ in particular. Here, T_a is a generator of Lie algebra. Note that when the internal symmetry belongs to an Abelian group the spacetime and internal symmetry may mix.[16] Just as we did for infinitesimal transformations above, we can rewrite any spacetime symmetry transformation g in the form

$$g\Phi(x) = \Omega_0(x'^{-1}(x); g)\Omega_1(x'^{-1}(x); g)\ldots\Phi(x'^{-1}(x)), \tag{6.8}$$

where $x'^{-1}(x)$ is the inverse function of $x'(x)$ associated with the coordinate transformation, $x \to x' = x'(x)$. $\Omega_n(x; g)$ is the element of the Lie group G with $\mathfrak{g}_n$ around the point x. In general, $\Omega_n(x; g)$ depends on both x and g. When Φ is a primary field, $\Omega_n(x; g)$ with $n \geq 1$ acts on $\Phi(x)$ trivially, so that we have

$$g\Phi(x') = \Omega_0(x; g)\Phi(x). \tag{6.9}$$

We can then identify $\Omega_0(x; g)$ as the local Lorentz, local (an)isotropic Weyl, and internal transformations. In this way, the expression (6.1) provides the local decomposition of spacetime symmetries.

B. Nonlinear realization

We then introduce a nonlinear realization for broken spacetime and internal symmetries [4,5] and discuss its relation to the local decomposition in the previous subsection. Suppose that an original global symmetry group G is broken to a subgroup H, where G and H include both of internal and spacetime symmetries. To construct the effective action, it is convenient to decompose the symmetry generators as

$$\mathfrak{g} = \mathfrak{g}_P \oplus \hat{\mathfrak{g}} = \mathfrak{g}_P \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 + \cdots, \tag{6.10}$$

where $\mathfrak{g}_P$ and $\hat{\mathfrak{g}}$ are for translation and nontranslational symmetry generators. The nontranslational part $\hat{\mathfrak{g}}$ is made from subalgebras of $\mathfrak{g}_n$ of spacetime and internal symmetry generators with the coordinate dimension n. We further decompose them into the residual symmetry parts $\mathfrak{h}$'s and the broken symmetry parts $\mathfrak{m}$'s as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \qquad \mathfrak{g}_P = \mathfrak{h}_P \oplus \mathfrak{m}_P, \qquad \hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \hat{\mathfrak{m}},$$
$$\mathfrak{g}_n = \mathfrak{h}_n \oplus \mathfrak{m}_n (n \geq 0). \tag{6.11}$$

In contrast to the internal symmetry case, Eq. (2.1), we assume that

$$[\hat{\mathfrak{h}}, \mathfrak{g}_P \oplus \hat{\mathfrak{m}}] = \mathfrak{g}_P \oplus \hat{\mathfrak{m}} \tag{6.12}$$

rather than $[\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}$ in the following.[17] In this case, we need to employ all the translation generators in addition to broken ones for parametrizing the coordinate of the coset space. Correspondingly, we use representatives of the coset $G/\hat{H}$ rather than G/H to realize the original symmetry group G [4,5],

$$\Omega = \Omega_P \Omega_0 \Omega_1 \ldots \quad \text{with} \quad \Omega_P = e^{Y^m(\bar{x})P_m},$$
$$\Omega_n = e^{\pi_n(\bar{x})} (n \geq 0), \tag{6.13}$$

where $\pi_n(\bar{x}) \in \mathfrak{m}_n$ are NG fields for broken nontranslational symmetries and $x^m(\bar{x})$'s are the Minkowski coordinates. We note that $\bar{x}^\mu$'s are not the Minkowski coordinates but rather the unitary gauge coordinates, as we will see.[18] One useful choice of the unitary coordinate is[19]

$$Y^{\hat{m}}(\bar{x}) = \bar{x}^{\hat{m}}, \qquad Y^a(\bar{x}) = \bar{x}^a + \pi^a(\bar{x}), \tag{6.14}$$

where the indices $\hat{m}$ and a denote directions with and without translation invariance, respectively, and π^a's are NG fields for broken translation symmetries. Under a global left G transformation, the representative transforms as

$$\Omega(Y, \pi) \to \Omega(Y', \pi') = g\Omega(Y, \pi)h^{-1}(\pi, g). \tag{6.15}$$

For the translation $x \to x' + a$, the NG fields transform as $Y'^m(\bar{x}) = Y^m(\bar{x}) - a^m$, and $\pi_n{}'(\bar{x}) = \pi_n(\bar{x})$. In general, the NG fields π'_n transform nonlinearly. Here, we notice that the expression (6.13) takes a similar form as the local decomposition (6.8) of spacetime symmetries. Indeed, from the global left G transformation property of Ω, it turns out that NG fields π_n are identified with transformation parameters for $\mathfrak{g}_n$ transformations around the point $Y^m(\bar{x})$. In particular, $\pi_0(\bar{x})$ should be understood as NG fields for local Lorentz and local (an) isotropic Weyl transformations, rather than those for global ones. Also, π_n's with $n \geq 1$ correspond to redundant NG fields because primary fields at a point $Y^m(\bar{x})$ are invariant under the $\mathfrak{g}_n$ transformations around the same point $Y^m(\bar{x})$ for $n \geq 1$.

Such identification can be also understood from the Maurer–Cartan one form

[16] A typical example is the Galilean boost generator, which is expressed as $t\partial_m - x_m m T_0$. Here, m is the mass of the particle, and T_0 is the Abelian generator corresponding to the particle number. See Appendix A for details.

[17] In most of the symmetry breaking patterns in our interests, $[\hat{\mathfrak{h}}, \hat{\mathfrak{m}} \oplus \mathfrak{g}_P] = \hat{\mathfrak{m}} \oplus \mathfrak{g}_P$, but $[\mathfrak{h}, \mathfrak{m}] \neq \mathfrak{m}$. For example, when rotation symmetries are broken, we have $[\mathfrak{h}_P, \mathfrak{m}] = \mathfrak{h}_P$.

[18] In this section, we employ the bar symbol for the unitary gauge coordinates to distinguish those with Minkowski's.

[19] If we are not interested in dynamics in $\bar{x}^a$-directions, we set $Y^m(\bar{x}) = (\bar{x}^{\hat{m}}, \pi^a)$ and define NG fields as $\bar{x}^a$-independent fields. Though such a simplification is often performed in the literature, we keep $\bar{x}^a$-dependence for generality.

$$J_\mu = \Omega^{-1}\partial_\mu\Omega = \hat{\Omega}^{-1}(\partial_\mu Y^m P_m)\hat{\Omega} + \hat{\Omega}^{-1}\partial_\mu\hat{\Omega} \quad \text{with} \quad \hat{\Omega} = \Omega_0\Omega_1\ldots, \tag{6.16}$$

where we used $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] = \hat{\mathfrak{g}}$. In the coset construction, its $\mathfrak{g}_p$-component is used as the vierbein,

$$e_\mu^m = [J_\mu]_{P_m} = [\hat{\Omega}^{-1}(\partial_\mu Y^n P_n)\hat{\Omega}]_{P_m} = [\Omega_0^{-1}(\partial_\mu Y^n P_n)\Omega_0]_{P_m}, \tag{6.17}$$

where $[A]_{P_m}$ denotes the P_m-component of A and we used $[\mathfrak{g}_m, \mathfrak{g}_n] = \mathfrak{g}_{m+n}$ at the last equality. We notice that the vierbein (6.17) depends on Y^m and Ω_0 only, and it is independent of Ω_n's with $n \geq 1$. Also, Ω_0 just transforms the vierbein without changing the Minkowski coordinate Y^m. Such properties are again consistent with the interpretation that Ω_0 represents NG fields for local symmetry transformations and Ω_n's with $n \geq 1$ do not generate physical degrees of freedom.

To summarize, the representative of the coset space (6.13) is closely related to the local symmetry picture, rather than the global one. In particular, all the NG fields can be described by Y^m, Ω_0 only, and their identification should be based on the local picture.

C. Ingredients of the effective action

We next take a closer look at the ingredients of the effective action based on the identification of NG fields in the previous subsection. For simplicity, we concentrate on the relativistic case in the following. Suppose that local fields Φ^A have background condensations,

$$\langle \Phi^A(x) \rangle = \bar{\Phi}^A(x), \tag{6.18}$$

where A denotes both the internal and local Lorentz indices and $\bar{\Phi}^A$'s are spacetime dependent in general. Just as we usually do for internal symmetry breaking (see, e.g., Ref. [63]), let us decompose Φ^A into the NG field part and the matter part $\tilde{\Phi}^A$ as

$$\Phi^A(x) = \Omega_P(\pi)[\Omega_0(x,\pi_0)]^A{}_B\tilde{\Phi}^B(x), \tag{6.19}$$

where $\Omega_0(x,\pi_0) = \Omega_{\text{int}}\Omega_L\Omega_D$ with Ω_{int}, Ω_L, and Ω_D being representatives for broken internal, local Lorentz, and local Weyl symmetries, respectively.[20] Note that Ω_n's with $n \geq 1$ do not appear here because they are redundant NG fields and do not transform Φ^A's. It is also useful to introduce the unitary gauge coordinate $\bar{x}^\mu(x) = x^\mu + \pi^\mu(x)$ and rewrite Eq. (6.19) to

$$\Phi^A(x) = [\Omega_0(\bar{x},\pi_0)]^A{}_B\tilde{\Phi}^B(\bar{x}). \tag{6.20}$$

From Eq. (6.9), $\Phi'^A(x)$ transforms under G as

$$\begin{aligned}\Phi'^A(x') = g\Phi^A(x') &= [\Omega_0(\bar{x}',g)\Omega_0(\bar{x}',\pi_0)]^A{}_B\tilde{\Phi}^B(\bar{x}') \\ &= [\Omega_0(\bar{x}',\pi_0'(\pi,g))]^A{}_B\tilde{\Phi}'^B(\bar{x}')\end{aligned} \tag{6.21}$$

with $\bar{x}'(x') = \bar{x}(x)$, $\tilde{\Phi}'^A(\bar{x}') = [h(\pi,g)]^A{}_B\tilde{\Phi}^B(\bar{x})$, which follow the same transformation rule as those of the coset construction (6.15). In this unitary gauge coordinate, the Minkowski coordinate is expressed by the inverse function $Y^m(\bar{x})$ such that $Y^m(\bar{x}(x)) = x^m$. Then, the vierbein e_μ^m is given by

$$e_\mu^m = \partial_\mu Y^m(\bar{x}). \tag{6.22}$$

Just as the internal case, the Maurer–Cartan type one form arises from the derivative of Φ^A as

$$\begin{aligned}\partial_\mu\Phi^A = [\Omega_{\text{int}}\Omega_L\Omega_D]^A{}_B[(\Omega_{\text{int}}^{-1}\partial_\mu\Omega_{\text{int}} \\ + \Omega_L^{-1}\partial_\mu\Omega_L + \Omega_D^{-1}\partial_\mu\Omega_D)^B{}_C\tilde{\Phi}^C + \partial_\mu\tilde{\Phi}^B],\end{aligned} \tag{6.23}$$

which suggests that the three terms in the parentheses play the role of connections. Indeed, if we introduce the internal gauge field A_μ and the Weyl gauge field W_μ, the inverse local transformation $(\Omega_{\text{int}}\Omega_L\Omega_D)^{-1}$ maps the configuration $A_\mu = W_\mu = 0$ to the configuration

$$A_\mu = \Omega_{\text{int}}^{-1}\partial_\mu\Omega_{\text{int}}, \qquad W_\mu = \Omega_D^{-1}\partial_\mu\Omega_D. \tag{6.24}$$

Similarly, the vierbein (6.22) and the corresponding spin connection $S_\mu = \frac{1}{2}S_\mu^{mn}L_{mn}$ are mapped to

$$\begin{aligned}e_\mu^m &= [\Omega_D^{-1}\Omega_L^{-1}(\partial_\mu Y^m P_m)\Omega_L\Omega_D]_{P_m}, \\ S_\mu &= \Omega_L^{-1}\partial_\mu\Omega_L + \frac{1}{2}(e_\mu^m e_\nu^n - e_\mu^n e_\nu^m)W^\nu L_{mn}.\end{aligned} \tag{6.25}$$

Note that, via those identifications, Eq. (6.23) can be reduced to the Weyl covariant derivative (2.26) as

$$\partial_\mu\Phi^A = [\Omega_{\text{int}}\Omega_L\Omega_D]^A{}_B D_\mu\tilde{\Phi}^B. \tag{6.26}$$

Also note that the vierbein coincides with the P_m-component (6.17) of the Maurer–Cartan one form. In this way, the Maurer–Cartan one form (6.16) with $\Omega_n = 1$ $(n \geq 1)$ can be identified with connections and the vierbein.

Just as the internal symmetry case, the above decomposition provides ingredients of the effective action. To illustrate the difference from the internal symmetry case, let us consider constructing the effective action for NG fields without matter fields, the ingredients of which can be obtained by setting that $\tilde{\Phi}^A = \bar{\Phi}^A$. The original fields Φ^A's and their derivatives are then given by

[20] Though we keep Ω_{int}, Ω_L, and Ω_D in our discussions, we turn on NG fields only for broken symmetries. For example, if the Lorentz symmetry is not broken, we set $\Omega_L = 1$.

$$\Phi^A = [\Omega_{\rm int}\Omega_L\Omega_D]^A{}_B\bar{\Phi}^B, \tag{6.27}$$

$$\partial_\mu\Phi^A = [\Omega_{\rm int}\Omega_L\Omega_D]^A{}_B[(\Omega_{\rm int}^{-1}\partial_\mu\Omega_{\rm int} + \Omega_L^{-1}\partial_\mu\Omega_L + \Omega_D^{-1}\partial_\mu\Omega_D)^B{}_C\bar{\Phi}^C + \partial_\mu\bar{\Phi}^B]. \tag{6.28}$$

One difference from the internal symmetry case is that local Lorentz indices can be coupled to the translation generator P_m at the same time as they are representations of local Lorentz symmetry. For example, when the condensation has a local Lorentz index, $\bar{\Phi}^n$, there can be a coupling of the form

$$\mathrm{tr}[(e_\mu^m P_m)(\Phi^n P_n)], \tag{6.29}$$

of which the expression after the inverse local transformation $(\Omega_{\rm int}\Omega_L\Omega_D)^{-1}$ is given by

$$\mathrm{tr}[(e_\mu^m P_m)(\bar{\Phi}^n P_n)] = e_\mu^m\bar{\Phi}_m \quad \text{with} \quad e_\mu^m = [\Omega_D^{-1}\Omega_L^{-1}(\partial_\mu Y^m P_m)\Omega_L\Omega_D]_{P_m}, \tag{6.30}$$

where the trace for P_m is defined as $\mathrm{tr}[P_m P_n] = \eta_{mn}$. Another difference is that $\bar{\Phi}^A$ can be spacetime dependent. For example, when the condensation is inhomogeneous in the z-direction, $\bar{\Phi}^A(\bar{z})$, we obtain functions of $\bar{z}$ from terms without derivatives like

$$\Phi^A\Phi_A \to \bar{\Phi}^A(\bar{z})\bar{\Phi}_A(\bar{z}). \tag{6.31}$$

Similarly, the derivative $\partial_\mu\Phi^A$ leads to functions of $\bar{z}$ and their derivatives as well as the Maurer–Cartan-type one form

$$\partial_\mu\Phi^A = [\Omega_{\rm int}\Omega_L\Omega_D]^A{}_B[(\Omega_{\rm int}^{-1}\partial_\mu\Omega_{\rm int} + \Omega_L^{-1}\partial_\mu\Omega_L + \Omega_D^{-1}\partial_\mu\Omega_D)^B{}_C\bar{\Phi}^C + \delta_\mu^{\bar{z}}\partial_{\bar{z}}\bar{\Phi}^B]. \tag{6.32}$$

With those modifications to the internal symmetry case, the general effective action can be constructed from the one forms $\Omega_{\rm int}^{-1}\partial_\mu\Omega_{\rm int}$, $\Omega_L^{-1}\partial_\mu\Omega_L$, and $\Omega_D^{-1}\partial_\mu\Omega_D$; the vierbein e_μ^m; and functions of coordinates in the inhomogeneous directions. Note that the volume element also contains a NG field through the determinant of the vierbein. Since those one forms are related to the connections S_μ, W_μ, and A_μ, it is obvious that those ingredients are the same as the ones in the approach based on the local picture.

D. Inverse-Higgs constraint

In the coset construction for spacetime symmetry breaking, one imposes the so-called inverse-Higgs constraints to remove the redundant NG fields [5,6] and the massive degrees of freedom [11,13,14]. For a broken (global) symmetry generator $A \in \mathfrak{m}$, we compute its commutator with the translation generator P_m, which contains both the broken and unbroken symmetry generators in general:

$$[P_m, A] \sim B + C \quad \text{with} \quad B \in \mathfrak{m}, \qquad C \in \mathfrak{h}. \tag{6.33}$$

When the commutator contains broken symmetry generators, $B \neq 0$, we remove the NG field for A by imposing a certain constraint in a consistent way with the symmetry structure. Typically, we require that the B-component of the Maurer–Cartan one form is zero,

$$[\Omega^{-1}\partial_\mu\Omega]_B = 0, \tag{6.34}$$

which generically relates the NG field for A to a derivative of the NG field for B. The effective action is then constructed from the Maurer–Cartan one form with the condition (6.34) imposed. At the end of this section, we revisit the role of such inverse-Higgs constraints and redundant NG fields, focusing on their counterparts in the approach based on the local symmetry viewpoint. In particular, we show that its physical meaning is different between the case $A \in \mathfrak{g}_{(n)}$ with $n \geq 1$ and the case $A \in \mathfrak{g}_{(0)}$.

1. Redundant NG fields for special conformal symmetry

An illustrative example for the first case is the redundant NG fields for special conformal symmetry [5,6,20,64,65]. Suppose that the conformal symmetry group is broken to its subgroup. To perform the coset construction, let us first classify the symmetry generators by the coordinate dimension as

$$\mathfrak{g}_{-1} = \{P_m\}, \qquad \mathfrak{g}_0 = \{L_{mn}, D\}, \qquad \mathfrak{g}_1 = \{K_m\}. \tag{6.35}$$

Based on this classification, we introduce the representative of the coset space Ω as

$$\Omega = e^{Y^m P_m}\Omega_0\Omega_1 \quad \text{with} \quad \Omega_0 = \Omega_L\Omega_D, \qquad \Omega_1 = \Omega_K. \tag{6.36}$$

Here, $\Omega_K = e^{\chi^m K_m}$ describes NG fields for special conformal transformations, which should be interpreted as redundant ones as we have discussed. We then calculate the corresponding Maurer–Cartan one form $J_\mu = \Omega^{-1}\partial_\mu\Omega$. First, its $\mathfrak{g}_P$-component is the vierbein

$$e_\mu^m P_m = \Omega_0^{-1}\partial_\mu Y^m P_m\Omega_0. \tag{6.37}$$

On the other hand, the $\mathfrak{g}_0$-component is given by

$$\Omega_0^{-1}\partial_\mu\Omega_0 - [\chi^m K_m, e_\mu^n P_n] = \Omega_L^{-1}\partial_\mu\Omega_L + \Omega_D^{-1}\partial_\mu\Omega_D - 2\chi_m e_\mu^m D - 2(\chi^m e_\mu^n)L_{mn}, \tag{6.38}$$

which is reduced to the connections in (6.23) if we set $\chi^m = 0$. The $\mathfrak{g}_1$-component is given by

$$[\partial_\mu \chi^m + \chi^2 e_\mu^m + [\Omega_L^{-1}\partial_\mu \Omega_L]_{L_{mn}}\chi_n + ([\Omega_D^{-1}\partial_\mu \Omega_D]_D - 2\chi_n e_\mu^n)\chi^m]K_m, \quad (6.39)$$

which vanishes when $\chi^m = 0$. Using the relations (6.24) and (6.25), we can rearrange the $\mathfrak{g}_0$- and $\mathfrak{g}_1$-components in terms of the spin connection S_μ^{mn} and the Weyl gauge field W_μ as

$$[J_\mu]_{L_{mn}} = S_\mu^{mn} - (e_\mu^m W^n - e_\mu^n W^m) + 2(e_\mu^m \chi^n - e_\mu^n \chi^m), \quad (6.40)$$

$$[J_\mu]_D = W_\mu - 2\chi_\mu, \quad (6.41)$$

$$[J_\mu]_{K_m} = \nabla_\mu \chi^m + W^m \chi_\mu + (\chi^2 - W^\nu \chi_\nu)e_\mu^m + (W_\mu - 2\chi_\mu)\chi^m, \quad (6.42)$$

where $\nabla_\mu \chi^m = \partial_\mu \chi^m + S_\mu^{mn}\chi_n$, and the local Minkowski indices and the global coordinate indices are converted to each other by the vierbein (6.37) as $\chi_\mu = e_\mu^m \chi_m$ and $W^m = e_\mu^m W^\mu$.

We now discuss the role of inverse-Higgs constraints. Suppose that the special conformal symmetry is broken, and for simplicity, let us assume that the translation symmetry is unbroken. The commutator relevant to inverse-Higgs constraints is then

$$[P, K] \sim D + L. \quad (6.43)$$

Since at least one of the dilatation symmetry and the Lorentz symmetry is broken if the special conformal symmetry is broken, we remove the NG field χ^m for the special conformal transformation by imposing the inverse-Higgs constraint. This statement corresponds to the fact that χ^m is a redundant NG field and does not generate physical degrees of freedom [5,6,20]. We then take a closer look at the inverse-Higgs constraints in the following two cases.

(1) Broken dilatation and unbroken Lorentz:

Let us first consider the case when the dilatation symmetry is broken but the Lorentz symmetry is unbroken. In this case, the inverse-Higgs constraints [6] can be stated as

$$[J_\mu]_D = 0 \leftrightarrow \chi_\mu = \frac{1}{2}W_\mu. \quad (6.44)$$

Using this constraint, the L_{mn}-component (6.40) is reduced to the spin connection

$$[J_\mu]_{L_{mn}} = S_\mu^{mn}. \quad (6.45)$$

On the other hand, the K_m-component (6.42) becomes

$$[J_\mu]_{K_m} = e^{m\nu}\frac{1}{2}\left(\nabla_\mu W_\nu + W_\mu W_\nu - \frac{1}{2}g_{\mu\nu}W^2\right). \quad (6.46)$$

As discussed in Ref. [42], the Weyl transformations of the combination in the parentheses can be related to those of the Ricci tensor $R_{\mu\nu}$ as

$$\Delta\left[\nabla_\mu W_\nu + W_\mu W_\nu - \frac{1}{2}g_{\mu\nu}W_\rho W^\rho\right] = \Delta\left[\frac{1}{2-d}\left(R_{\mu\nu} - \frac{1}{2(d-1)}g_{\mu\nu}R\right)\right], \quad (6.47)$$

where Δ denotes Weyl transformations and d is the spacetime dimension. Since the metric constructed from the vierbein (6.37) is conformally flat, we can further rewrite (6.46) in terms of the Ricci tensor as

$$[J_\mu]_{K_m} = \frac{e^{m\nu}}{2(2-d)}\left(R_{\mu\nu} - \frac{1}{2(d-1)}g_{\mu\nu}R\right). \quad (6.48)$$

In this way, the K_m-component reproduces the Ricci tensor in the unitary gauge [9]. To summarize, the L_{mn}- and K_m- components of the Maurer–Cartan one form reproduce the spin connection and the Ricci tensor, and we have a vanishing D-component. In particular, the Weyl gauge field W_μ does not appear explicitly.

This is indeed consistent with the local symmetry picture. As we mentioned earlier, any conformal system on the Minkowski space can be reformulated in a local Weyl invariant way by introducing an appropriate curved spacetime action (the Ricci gauging). The unitary gauge effective action should then be written without using Weyl gauge fields explicitly.

(2) Broken Lorentz and broken dilatation:

We next consider the case when both the dilatation and the Lorentz symmetry are broken. For concreteness, let us assume that the Lorentz symmetry associated with the 3-direction, i.e., $L_{3\hat{n}} = -L_{\hat{n}3}$ with $\hat{n} \neq 3$, is broken. We now have two types of inverse-Higgs constraints:

$$[J_\mu]_{L_{mn}} = S_\mu^{mn} - (e_\mu^m W^n - e_\mu^n W^m) + 2(e_\mu^m \chi^n - e_\mu^n \chi^m) = 0,$$
$$[J_\mu]_D = W_\mu - 2\chi_\mu = 0. \quad (6.49)$$

Since the role of the inverse-Higgs constraints here is to remove redundant NG fields consistently, we do not have to impose both conditions. Indeed, the global transformation does not mix the two constraints, so that we can impose one of them alone. By imposing the second condition

$$[J_\mu]_D = 0 \leftrightarrow \chi_\mu = \frac{1}{2} W_\mu, \tag{6.50}$$

the other components of the Maurer–Cartan one form can be reduced to

$$[J_\mu]_{P_m} = e_\mu^m, \qquad [J_\mu]_{L_{mn}} = S_\mu^{mn},$$
$$[J_\mu]_{K_m} = \frac{e^{m\nu}}{2(2-D)}\left(R_{\mu\nu} - \frac{1}{2(D-1)} g_{\mu\nu} R\right). \tag{6.51}$$

Just as the first example, the inverse-Higgs constraint guarantees that the Weyl gauge field does not appear explicitly in the unitary gauge effective action.

To summarize, the role of inverse-Higgs constraints of this type is to remove redundant NG fields. In particular, in the relativistic case, they are closely related to whether the original system permits the Ricci gauging or not. Correspondingly, the inverse-Higgs constraints convert the Weyl gauge fields into the Ricci tensors, so that the obtained action does not contain Weyl gauge fields explicitly.

2. Single brane revisited

An illustrative example for the second case is the single brane, which we discussed in Sec. III. From the global symmetry point of view, all the examples there are characterized by the symmetry breaking from the $(1+3)$-dimensional Poincaré symmetry to the $(1+2)$-dimensional one. In the coset construction, we then introduce NG fields for both the broken translation and broken (global) Lorentz symmetries. The representative of the coset space and the nonzero components of the Maurer–Cartan one form are

$$\Omega = \Omega_P \Omega_L, \qquad [J_\mu]_{P_m} = e_\mu^m, \qquad [J_\mu]_{L_{mn}} = S_\mu^{mn}. \tag{6.52}$$

Notice that, accompanied by functions of z and matters, general ingredients of the effective action for nonzero spin branes in Sec. III D can be obtained. As we have discussed, the NG fields in the nonlinear realization are identified with the local symmetry transformation parameters, and they generate physical degrees of freedom only when the corresponding local symmetries are broken. Therefore, NG fields for the Lorentz symmetries are physical ones for the nonzero spin branes but redundant ones for the scalar branes in this construction. It is in a sharp contrast to the first case discussed in Sec. VI D 1, where NG fields for higher-dimensional generators are always redundant ones.

We next discuss the role of inverse-Higgs constraints. The relevant commutators here are those of broken Lorentz symmetry generators, and translation generators given by

$$[P_m, L_{3\hat{n}}] = \eta_{3m} P_{\hat{n}} - \eta_{\hat{n}m} P_3, \tag{6.53}$$

which contains the broken generator, P_3, on the right-hand side. This commutator suggests that the mass term of NG fields for Lorentz symmetries and their mixing interaction with NG fields for diffeomorphisms can be constructed from the P_3 component of the Maurer–Cartan one form [11,13,14]. In our construction, there are several options for the inverse-Higgs constraints,[21]

$$[J_\mu]_{P_3} = e_\mu^3 = n_\mu, \qquad [J_\mu]_{P_3} = e_\mu^3 = \delta_\mu^3, \tag{6.54}$$

where $n_\mu = \frac{\delta_\mu^z}{\sqrt{g^{zz}}}$ is a unit vector perpendicular to the brane. Both of the conditions are satisfied by the background configuration, $\pi = 0$, and are also consistent with the original symmetry. Also, the second condition is equivalent to a combination of the first one, $e_\mu^3 = n_\mu$, and $g^{zz} = 1$. Finally, let us illustrate their physical interpretations:

(1) Condition $e_\mu^3 = n_\mu$:
This condition provides three constraints that make three NG fields for local Lorentz symmetries freeze out. Indeed, it exactly coincides with the procedure in Sec. III E to integrate out the massive Lorentz NG fields, because the interaction (3.43) leads to the constraint $n^\mu e_\mu^3 = 1 \leftrightarrow e_\mu^3 = n_\mu$ in the low-energy limit. It should be noticed that the removed NG fields are physical massive ones for the nonzero spin brane case but redundant ones for the scalar brane case.

(2) Condition $g^{zz} = 1$:
It is nothing but the condition (B9) to remove the gapped modes in the diffeomorphism NG field. The resulting effective action then turns out to be the Nambu–Goto action for the gapless NG mode localizing on the brane. Note that the ambiguity in (6.54) corresponds to the choice whether we integrate out the gapped modes in the diffeomorphism NG field or not.

In this way, the inverse-Higgs constraints for nonzero spin/scalar branes remove massive/redundant NG fields for Lorentz symmetries and gapped modes in the diffeomorphism NG field.

It might be useful to note that the general effective action for diffeomorphism symmetry breaking in Sec. III B can be

[21] In particular the conditions are different from Eq. (6.34). It is because we chose the unitary gauge coordinate as Eq. (6.14) and the Maurer-Cartan one form does not vanish even if NG fields vanish, $\pi = 0$.

constructed without introducing redundant NG fields for Lorentz symmetries. Consider the following representative of the coset space and the corresponding Maurer–Cartan one form:

$$\Omega = \Omega_P, \qquad [J_\mu]_{P_m} = e_\mu^m. \tag{6.55}$$

Ingredients of the effective action are then this Maurer–Cartan one form, functions of the coordinate z, matter fields, and their covariant derivatives. It is obvious that those ingredients reproduce the general ingredients discussed in Sec. III B. We can then construct the effective action for scalar branes before integrating out gapped modes for example.

To summarize, the conventional inverse-Higgs constraints can be classified into the following three types by their physical meanings:

(1) When spacetime symmetries of the coordinate dimension $n \geq 1$, $\mathfrak{g}_n$ with $n \geq 1$, are broken, the role of inverse-Higgs constraints is to remove redundant NG fields. In particular, in the relativistic systems, it is closely related to whether the original system permits the Ricci gauging or not.

(2) When global spacetime symmetries of the coordinate dimension zero, $\mathfrak{g}_0$, are broken as well as translation symmetries, we introduce NG fields for $\mathfrak{g}_0$ in the coset construction. However, if the broken local symmetries are only diffeomorphisms, NG fields for $\mathfrak{g}_0$ are redundant ones, and the inverse-Higgs constraints remove them. Also, we do not necessarily have to introduce NG fields for $\mathfrak{g}_0$ in our construction, as long as we include gapped modes in the effective action.

(3) On the other hand, when local Lorentz or local (an) isotropic Weyl symmetries are broken (as well as diffeomorphism symmetries), the corresponding physical NG fields acquire a mass. Under certain conditions, the inverse-Higgs constraint can be identified with the procedure to take the low-energy limit and integrate out massive NG fields.

VII. APPLICATION TO GRAVITATIONAL SYSTEMS

Before closing this paper, we would like to make a brief comment on the applications of our formulation to gravitational systems. As we mentioned in the Introduction, the EFT approach for inflation [41] is based on the symmetry argument in the unitary gauge. In the unitary gauge, the relevant degrees of freedom in single-field inflation are the metric $g_{\mu\nu}$ only, and the residual symmetries are the time-dependent spacial diffeomorphisms. This setup is essentially the same as the scalar branes, and indeed our discussion in Sec. III B is parallel to that of Ref. [41]; the only differences between the two cases are the background spacetime and whether we decouple the gravity sector or not. By keeping the gravity sector without decoupling, we can apply our strategy in Sec. II to gravitational systems. We will apply our approach to gravitational systems such as inflationary models with different symmetry breaking patterns elsewhere.

VIII. SUMMARY

In this paper, we discussed the EFT approach for spacetime symmetry breaking from the local symmetry point of view. The identification of NG fields and the construction of the effective action are based on the local picture of symmetry breaking, i.e., the breaking of diffeomorphism, local Lorentz, and (an)isotropic Weyl symmetries as well as the internal symmetries including possible central extensions in nonrelativistic systems. This picture distinguishes, e.g., whether the condensations have Lorentz charges (spins), while the standard coset construction based on the global symmetry breaking picture with the inverse-Higgs constraints does not. The distinction enable us to provide a correct identification of the physical NG fields because they are generated by local transformations of condensations.

To illustrate the difference between the global and local pictures of spacetime symmetry breaking, in Sec. III, we discussed codimension-1 branes, in which global translation and rotation symmetries are broken. In the global picture, the low-energy degrees of freedom are the NG field for the broken translation. In the local picture, these degrees of freedom correspond to the NG fields for the broken diffeomorphism. For scalar branes, both pictures give the same EFT. However, the situation is different for nonzero spin branes. In this case, the condensation has a spin, so that, in addition to the NG field for diffeomorphism breaking, there appear *massive* NG fields associated with local Lorentz symmetry breaking as the physical degrees of freedom, which nonlinearly transform under global broken symmetry. One might think such massive modes are irrelevant in the low-energy EFT. This is true for the EFT at the low-energy scale compared with the mass, and the EFT will be the same as that in the global picture. However, when the scale of order parameters for translation and rotation breaking have different scales, the mass could be smaller than other typical mass scales of the system, and thus the massive modes may become relevant as the low-energy degrees of freedom. For example, in cosmology, massive fields with the Hubble scale mass affect the cosmological perturbations (see, e.g., Refs. [66–68] for recent discussions), so that massive modes associated with symmetry breaking can be relevant when they have a mass less than or comparable to the Hubble scale.

In Secs. IV and V, we also discussed a system in which the condensation is periodic in one direction. We found that the dispersion relations of NG modes for the broken diffeomorphism are constrained by the minimum energy condition, in contrast to the codimension-1 brane case.

Such a property would be important, e.g., in the inhomogeneous chiral condensation phase [62].

In Sec. VI, we revisited the coset construction from the local symmetry point of view. It was pointed out that the inverse-Higgs constraints have two physical meanings [11,13,14]: removing redundant NG fields and massive fields. The standard coset construction does not distinguish these two. Based on the relation between the Maurer–Cartan one form and connections for spacetime symmetries, we classified these meanings of inverse-Higgs constraints by the coordinate dimension of broken symmetries. Inverse-Higgs constraints for spacetime symmetries with a higher dimension remove redundant NG fields, and in particular, those for the special conformal symmetry are closely related to the fact that the original system admits Ricci gauging. Those for dimensionless symmetries can be further classified by the local symmetry breaking pattern, just as the codimension-1 brane case in Sec. III.

Although we mainly focused on the relativistic case, it would be interesting to extend the discussion to the nonrelativistic case. It would be also interesting to include supersymmetry in our discussion. We defer such studies to future work.

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APPENDIX A: SPACETIME SYMMETRY IN NONRELATIVISTIC SYSTEMS

In this Appendix, we extend the discussion in Sec. II to nonrelativistic systems. After some geometrical preliminaries, we discuss local properties of nonrelativistic spacetime symmetries. We then summarize how they can be gauged and embedded into local symmetries.

1. Geometrical preliminaries

a. 3 + 1 decomposition

In nonrelativistic systems, there exists a particular time direction, and constant-time slices specify a spatial foliation structure. Correspondingly, spacetime symmetries in nonrelativistic systems should preserve the foliation structure. To discuss such systems and symmetries, it is convenient to introduce a timelike vector field n_μ perpendicular to the spatial slices normalized as

$$g^{\mu\nu} n_\mu n_\nu = -1. \tag{A1}$$

The induced metric $h_{\mu\nu}$ on the slices is then given by

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{A2}$$

We also introduce the projectors onto the temporal and spatial directions as

$$\text{temporal projector}: \; -n^\mu n_\nu, \qquad \text{spatial projector}: \; h^\mu_\nu. \tag{A3}$$

In the following, we often write the temporal component and spatial projection of a vector v^μ as

$$v_\parallel = n_\mu v^\mu, \qquad v^\mu_\perp = h^\mu_\nu v^\nu. \tag{A4}$$

b. Decomposition of local Lorentz indices

It is also convenient to decompose local Lorentz indices into the temporal and spatial directions in a similar way. Using the projectors,

$$\text{temporal projector}: \; \delta^m_0 \delta^0_n, \qquad \text{spatial projector}: \; \delta^m_n - \delta^m_0 \delta^0_n, \tag{A5}$$

we decompose the vierbein e^μ_m as

$$e^\mu_m = h^\mu_\nu e^\nu_n (\delta^n_m - \delta^n_0 \delta^0_m) + h^\mu_\nu e^\nu_0 \delta^0_m - n^\mu n_\nu e^\nu_0 \delta^0_m - n^\mu n_\nu e^\nu_n (\delta^n_m - \delta^n_0 \delta^0_m), \tag{A6}$$

where the second and the fourth terms mix the temporal/spatial coordinate indices and the spatial/temporal local Lorentz indices. Such terms can be eliminated by performing local Lorentz boost transformations such that the temporal directions of the global coordinate and the local Lorentz frame coincide with each other. Indeed, we can always impose the gauge condition

$$e^\mu_0 = n^\mu \tag{A7}$$

to obtain the vierbein of the form

$$e^\mu_m = \tilde{e}^\mu_m + n^\mu \delta^0_m \quad \text{with} \quad \tilde{e}^\mu_m = h^\mu_\nu e^\nu_n (\delta^n_m - \delta^n_0 \delta^0_m). \tag{A8}$$

Note that the gauge condition (A7) is invariant under diffeomorphisms and local rotations. In the rest of this Appendix, we impose the gauge condition (A7) and use the expression (A8) of the vierbein.

The spatial projection $\tilde{e}^\mu_m$ can then be identified with the spatial dreibein. First, its square reproduces the spatial induced metric:

$$\tilde{e}^\mu_m \tilde{e}^\nu_n \eta^{mn} = h^{\mu\nu}, \qquad \tilde{e}^\mu_m \tilde{e}^\nu_n h_{\mu\nu} = \tilde{e}^\mu_m \tilde{e}^\nu_n g_{\mu\nu} = \eta_{mn} + \delta^0_m \delta^0_n. \tag{A9}$$

Also the relations

$$\tilde{e}^m_\mu = (\delta^m_n - \delta^m_0 \delta^0_n) e^n_\mu = e^m_\nu h^\nu_\mu, \\ n^\mu \delta^0_m = -n^\mu n_\nu e^\nu_m = e^\mu_n \delta^n_0 \delta^0_m \tag{A10}$$

guarantee that the decomposition of coordinate indices and that of local Lorentz indices are consistent. More concretely, we can use the notation $v^m_\perp$ with the local Lorentz index consistently:

$$v^m_\perp = (\delta^m_n - \delta^m_0 \delta^0_n) v^n = e^m_\nu v^\nu_\perp = \tilde{e}^m_\nu v^\nu_\perp. \tag{A11}$$

The temporal projection is also consistent between the two:

$$v_\parallel = n_\mu v^\mu = n_\mu e^\mu_m v^m = v_0 = -v^0. \tag{A12}$$

2. Local properties of nonrelativistic spacetime symmetries

We now discuss local properties of nonrelativistic spacetime symmetries under some plausible assumptions on the foliation structure and symmetry transformations.

a. Nonrelativistic ansatz

When we take a nonrelativistic limit of relativistic systems, the time direction is typically identified with that in a rest frame of massive free particles. It would then be natural to assume that the timelike vector n_μ generates timelike geodesics and satisfies

$$n^\nu \nabla_\nu n^\mu = 0. \tag{A13}$$

As we mentioned earlier, spacetime symmetries in nonrelativistic systems should preserve the foliation structure. Coordinate transformations preserving the foliation structure (foliation preserving diffeomorphism transformations) can be defined as[22]

$$x^\mu \to x'^\mu = x^\mu - \epsilon^\mu(x) \quad \text{with} \\ h^\lambda_\mu \mathcal{L}_\epsilon n_\lambda \equiv h^\lambda_\mu (\epsilon^\nu \nabla_\nu n_\lambda + n_\nu \nabla_\lambda \epsilon^\nu) = 0, \tag{A16}$$

where $\mathcal{L}_\epsilon$ is the Lie derivative along ϵ^μ. From Eq. (A13), we obtain $h^\nu_\mu \partial_\nu \epsilon_\parallel = 0$, which guarantees that the time component of the transformation parameter is constant on each slice. In the rest of this Appendix, we assume that the timelike vector n_μ satisfies the geodesic assumption (A13) and the nonrelativistic spacetime symmetries satisfy the condition (A16).

b. Local decomposition

As we discussed in Sec. II B, local properties of spacetime symmetry are determined by the covariant derivative of the corresponding coordinate transformation parameter ϵ^μ. In nonrelativistic systems, it is convenient to decompose $\nabla_\mu \epsilon_\nu$ using the projectors (A3) as

$$\begin{aligned} \nabla_\mu \epsilon_\nu &= -\nabla_\mu (n_\nu \epsilon_\parallel) + \nabla_\mu \epsilon_{\perp\nu} \\ &= -K_{\mu\nu}\epsilon_\parallel - n_\nu \partial_\mu \epsilon_\parallel + \nabla_\mu \epsilon_{\perp\nu} \\ &= -K_{\mu\nu}\epsilon_\parallel + n_\mu n_\nu (n^\rho \partial_\rho \epsilon_\parallel) + \nabla_\mu \epsilon_{\perp\nu}, \end{aligned} \tag{A17}$$

where $K_{\mu\nu} = h^\rho_\mu \nabla_\rho n_\nu$ is the extrinsic curvature on the spatial slices, and we used the geodesic condition (A13) and the foliation preserving condition, $h^\nu_\mu \partial_\nu \epsilon_\parallel = 0$, at the second and the third equalities, respectively. The last term can be further decomposed as

$$\begin{aligned} \nabla_\mu \epsilon_{\perp\nu} &= h^\alpha_\mu h^\beta_\nu \nabla_\alpha \epsilon_{\perp\beta} - h^\alpha_\mu n_\nu n^\beta \nabla_\alpha \epsilon_{\perp\beta} - n_\mu n^\alpha \nabla_\alpha \epsilon_{\perp\nu} \\ &= h^\alpha_\mu h^\beta_\nu \nabla_\alpha \epsilon_{\perp\beta} + K_{\mu\rho}\epsilon^\rho_\perp n^\nu - n_\mu h^\rho_\nu n^\alpha \nabla_\alpha \epsilon_{\perp\rho}, \end{aligned} \tag{A18}$$

where we used $n_\alpha \epsilon^\alpha_\perp = 0$ and the geodesic condition (A13) at the second equality. We then have

$$\begin{aligned} \nabla_\mu \epsilon_\nu = {}& n_\mu n_\nu (n^\rho \partial_\rho \epsilon_\parallel) + (h^\alpha_\mu h^\beta_\nu \nabla_\alpha \epsilon_{\perp\beta} - K_{\mu\nu}\epsilon_\parallel) \\ &+ (K_{\mu\rho}\epsilon^\rho_\perp n_\nu - n_\mu h^\rho_\nu n^\alpha \nabla_\alpha \epsilon_{\perp\rho}), \end{aligned} \tag{A19}$$

where the first term represents local rescalings in the temporal direction. For later use, we define

$$\lambda_\parallel = -n^\mu \partial_\mu \epsilon_\parallel. \tag{A20}$$

The second term in Eq. (A19) describes deformations of spatial coordinates, and it can be decomposed as

$$h^\alpha_\mu h^\beta_\nu \nabla_\alpha \epsilon_{\perp\beta} - K_{\mu\nu}\epsilon_\parallel = \omega_{\perp\mu\nu} + s_{\perp\mu\nu} + \lambda_\perp h_{\mu\nu}, \tag{A21}$$

where the antisymmetric part $\omega_{\perp\mu\nu}$, the symmetric traceless part $s_{\perp\mu\nu}$, and the trace part $\lambda_\perp$ generate local rotations, anisotropic spatial rescalings, and isotropic spatial

[22]If we choose the coordinate system such that x^t coincides with the time direction, a concrete form of n_μ is given by

$$n_\mu = -\frac{\delta^t_\mu}{\sqrt{-g^{tt}}}. \tag{A14}$$

Correspondingly, the geodesic condition (A13) and the foliation preserving condition (A16) can be stated as

$$\partial_i g^{tt} = \partial_i \epsilon^t = 0. \tag{A15}$$

rescalings, respectively. Spatial isotropy requires that $s_{\perp\mu\nu} = 0$ to obtain

$$h^\alpha_\mu h^\beta_\nu \nabla_\alpha \epsilon_{\perp\beta} - K_{\mu\nu}\epsilon_\parallel = \omega_{\perp\mu\nu} + \lambda_\perp h_{\mu\nu}. \tag{A22}$$

The last term in Eq. (A19) mixes the temporal and spatial directions. If we introduce parameters $b^\pm_\mu$ as

$$b^\pm_\mu = -h_{\mu\nu} n^\rho \nabla_\rho \epsilon^\nu_\perp \pm K_{\mu\nu}\epsilon^\nu_\perp, \tag{A23}$$

we can rewrite the last term in Eq. (A19) as

$$K_{\mu\rho}\epsilon^\rho_\perp n_\nu - n_\mu h_{\nu\rho} n_\alpha \nabla^\alpha \epsilon^\rho_\perp = \frac{1}{2} n_\mu (b^+_\nu + b^-_\nu) + \frac{1}{2} n_\nu (b^+_\mu - b^-_\mu). \tag{A24}$$

Here, note that $n^\mu b^\pm_\mu = 0$. It should be also noted that when the extrinsic curvature is zero $b^+_\mu = b^-_\mu$ is the temporal derivative of $\epsilon^\mu_\perp$. As it suggests, $b^\pm_\mu$ can be thought of as local Galilei boosts.

To summarize, using the quantities introduced above, we can decompose $\nabla_\mu \epsilon_\nu$ for nonrelativistic spacetime symmetries as

$$\nabla_\mu \epsilon_\nu = \omega_{\perp\mu\nu} + \frac{1}{2} n_\mu (b^+_\nu + b^-_\nu) + \frac{1}{2} n_\nu (b^+_\mu - b^-_\mu) - \lambda_\parallel n_\mu n_\nu + \lambda_\perp h_{\mu\nu}, \tag{A25}$$

where $\omega_{\perp\mu\nu}$, $\lambda_\parallel$, and $\lambda_\perp$ describe local rotations, temporal rescalings, and spatial rescaling, respectively. The parameters $b^\pm_\mu$ are associated with local Galilei boosts.

c. Transformation rule of n^μ, $h^{\mu\nu}$, and $\tilde{e}^\mu_m$

To understand the physical interpretation of the above decomposition, it would be useful to note the transformation rule of the unit vector n^μ, the induced metric $h^{\mu\nu}$, and the spatial dreibein $\tilde{e}^\mu_m$ under infinitesimal foliation preserving diffeomorphisms. First, their general coordinate transformations are given by

$$\delta n^\mu = -n^\rho \nabla_\rho \epsilon^\mu + \epsilon^\rho \nabla_\rho n^\mu, \tag{A26}$$

$$\delta h^{\mu\nu} = -(h^{\mu\rho}\nabla_\rho \epsilon^\nu + h^{\nu\rho}\nabla_\rho \epsilon^\mu) + \epsilon^\rho \nabla_\rho h^{\mu\nu}, \tag{A27}$$

$$\delta \tilde{e}^\mu_m = -\tilde{e}^\nu_m \nabla_\nu \epsilon^\mu + \epsilon^\rho \partial_\rho \tilde{e}^\mu_m + \epsilon^\rho \Gamma^\mu_{\rho\nu} \tilde{e}^\nu_m. \tag{A28}$$

By using the geodesic condition (A13) and the foliation preserving condition (A16), they can be reduced to the form[23]

$$\delta n^\mu = -\lambda_\parallel n^\mu + b^\mu_+, \qquad \delta h^{\mu\nu} = -2\lambda_\perp h^{\mu\nu},$$
$$\delta \tilde{e}^\mu_m = \omega^{\mu}_{\perp\ \nu} \tilde{e}^\nu_m - \lambda_\perp \tilde{e}^\mu_m - \epsilon^\rho \tilde{S}_{\rho m}{}^n \tilde{e}^\mu_n, \tag{A29}$$

where we defined

$$\tilde{S}^{mn}_\mu = (\delta^m_r - \delta^m_0 \delta^0_r) S^{rs}_\mu (\delta^n_s - \delta^n_0 \delta^0_s) = \tilde{e}^m_\nu \partial_\mu \tilde{e}^{\nu n} + \tilde{e}^m_\lambda \Gamma^\lambda_{\mu\nu} \tilde{e}^{\nu n}. \tag{A30}$$

Note that the transformations of spatial quantities $h^{\mu\nu}$ and $\tilde{e}^\mu_m$ (with upper indices) depend only on the spatial components $\omega_{\perp\mu\nu}$ and $\lambda_\perp$. In particular, the spatial metric $h^{\mu\nu}$ (and h_{ij} also) is invariant under transformations with $\lambda_\perp = 0$. Such properties are consistent with the interpretation that $\omega_{\perp\mu\nu}$ and $\lambda_\perp$ generate local rotations and spatial rescalings.

3. Examples: Galilean, Schrödinger, and Galilean conformal symmetries

Before discussing the embedding of nonrelativistic spacetime symmetries into local ones, let us perform the local decomposition for concrete nonrelativistic spacetime symmetries in this subsection. As illustrative examples, we consider Galilean, Schrödinger, and Galilean conformal symmetries on the Minkowski space.

a. Galilean symmetry

Galilean symmetry is generated by translations P_μ, rotations J_{ij}, and Galilei boosts B_i. Their algebras can be obtained by taking the nonrelativistic limit of the Poincaré algebra, except for a possible central extension in the commutator of spatial translations and Galilei boosts,

$$[P_i, B_j] = -\delta_{ij} M, \tag{A31}$$

where the central charge M is associated with the mass energy and it can be identified with the internal $U(1)$ charge associated with the particle number conservation. As is suggested by the commutator (A31), the Galilei boost generates internal $U(1)$ transformations as well as the spacetime transformation. Using the notation in Sec. VI A, we can express the Galilei boost as

$$v^i B_i = t v^i \partial_i - (v \cdot x) M, \tag{A32}$$

where v^i is the transformation parameter. Since its spacetime transformation part takes the form

$$\epsilon^t = 0, \qquad \epsilon^i = v^i t, \tag{A33}$$

nonzero components in the decomposition (A25) are given by

$$b^i_+ = b^i_- = -v^i, \tag{A34}$$

[23]For notational simplicity, we use $b^\mu_\pm$ to denote $g^{\mu\nu} b^\pm_\nu$.

which is consistent with the observation that $b^\mu_\pm$'s are associated with local Galilei boosts. Note that local decompositions of other generators are the same as the relativistic case.

b. Schrödinger symmetry

We next consider the Schrödinger symmetry [69,70], which is generated by

$$\tilde{D} = 2t\partial_t + x^i\partial_i, \qquad \tilde{K} = t^2\partial_t + tx^i\partial_i - \frac{1}{2}x^2 M, \quad \text{(A35)}$$

and Galilean symmetry generators. Nonzero components in the decomposition (A25) for $\lambda\tilde{D}$ are

$$\frac{1}{2}\lambda_\parallel = \lambda_\perp = \lambda. \quad \text{(A36)}$$

On the other hand, those for $\mu\tilde{K}$ are

$$\frac{1}{2}\lambda_\parallel = \lambda_\perp = \mu t, \qquad b^i_+ = b^i_- = -\mu x^i. \quad \text{(A37)}$$

Here, λ and μ are transformation parameters. We notice that both of $\tilde{D}$ and $\tilde{K}$ have the rescaling components satisfying $\lambda_\parallel = 2\lambda_\perp$. In other words, the dynamical exponent is $z = 2$.

c. Galilean conformal symmetry

Finally, let us consider the Galilean conformal symmetry (see, e.g., Ref. [71] for references). For this purpose, it is convenient to introduce the extended Galilean conformal algebra generated by

$$L^{(n)} = (n+1)t^n x^i\partial_i + t^{n+1}\partial_t,$$
$$M^{(n)}_i = t^n\partial_i, \; J^{(n)}_{ij} = t^n(x^i\partial_j - x^j\partial_i), \quad \text{(A38)}$$

where n is an arbitrary integer. In terms of these operators, the Galilean conformal symmetry generators are given by $L^{(n)}$ and $M^{(n)}_i$ with $n = 0, \pm 1$, and $J^{(0)}_{ij}$. Using a function $\Lambda(t)$ of time, the coordinate transformation associated with $L^{(n)}$'s can be recast as

$$\epsilon^t = \Lambda(t), \qquad \epsilon^i = \Lambda'(t)x^i, \quad \text{(A39)}$$

and nonzero components in the decomposition (A25) are

$$\lambda_\parallel = \lambda_\perp = \Lambda'(t), \qquad b^i_+ = b^i_- = -\Lambda''(t)x^i. \quad \text{(A40)}$$

Note that the dynamical exponent is $z = 1$. On the other hand, coordinate transformations associated with $M^{(n)}_i$'s take the form

$$\epsilon^t = 0, \qquad \epsilon^i = B^i(t), \quad \text{(A41)}$$

and nonzero components are

$$b^i_+ = b^i_- = -B'^i(t), \quad \text{(A42)}$$

which can be thought of as a time-dependent generalization of Galilei boosts. Similarly, $J^{(n)}_{ij}$'s can be regarded as a time-dependent generalization of spatial rotations.

4. Embedding into local symmetries

As we have seen in the previous subsection, nonrelativistic spacetime symmetries generically have a particular dynamical exponent z, and the decomposition (A25) takes the form

$$\nabla^\mu\epsilon^\nu = \omega^{\mu\nu}_\perp + \frac{1}{2}n^\mu(b^\nu_+ + b^\nu_-) + \frac{1}{2}n^\nu(b^\mu_+ - b^\mu_-)$$
$$+ \lambda(-zn^\mu n^\nu + h^{\mu\nu}). \quad \text{(A43)}$$

Let us concentrate on such symmetries in the following. They also admit central extensions. In this subsection, we first discuss how nonrelativistic spacetime symmetries without central extensions can be embedded into local symmetries. We then extend discussions to the case with central extensions.

a. Without central extensions

Let us begin with the case without central extensions. In this case, the transformation rules of local fields are determined by their local rotation charge and scaling dimension. Suppose that a local field $\Phi(x)$ follows a representation $\tilde{\Sigma}_{mn}$ and has a scaling dimension Δ_Φ, where $\tilde{\Sigma}_{mn}$ is projected on to the spatial direction: $\tilde{\Sigma}_{0n} = \tilde{\Sigma}_{m0} = 0$. It is then transformed as[24]

$$\delta\Phi = \Delta_\Phi\lambda(x)\Phi + \frac{1}{2}\omega^{mn}_\perp(x)\tilde{\Sigma}_{mn}\Phi + \epsilon^\mu(x)\nabla_\mu\Phi, \quad \text{(A45)}$$

where $\omega^{mn}_\perp = \tilde{e}^m_\mu\tilde{e}^n_\nu\omega^{\mu\nu}_\perp$. The covariant derivative is defined by

$$\nabla_\mu\Phi = \partial_\mu\Phi + \frac{1}{2}S^{mn}_\mu\tilde{\Sigma}_{mn}\Phi = \partial_\mu\Phi + \frac{1}{2}\tilde{S}^{mn}_\mu\tilde{\Sigma}_{mn}\Phi, \quad \text{(A46)}$$

[24]Note that fields with coordinate indices can be decomposed into local fields following some representations of local rotations, by using the vierbein. For example, a gauge field A_μ can be decomposed as

$$A^\mu = -n^\mu A_\parallel + A^\mu_\perp = n^\mu A^0 + \tilde{e}^\mu_m A^m_\perp. \quad \text{(A44)}$$

Here, $A_\parallel = -A^0$ and $A^m_\perp$ are a scalar and a spatial vector, respectively, and their transformation rules follow from Eq. (A45). In this way, any local field can be expressed in terms of local fields with the transformation rule (A45), the timelike vector n_μ, the spatial induced metric $h_{\mu\nu}$, and the spatial dreibein $\tilde{e}^\mu_m$.

where $\tilde{S}_\mu^{mn}$ is given by Eq. (A30). Rewriting Eq. (A45) as

$$\delta\Phi = \Delta_\Phi \lambda(x)\Phi + \frac{1}{2}(\omega_\perp^{mn}(x) + \epsilon^\mu(x)\tilde{S}_\mu^{mn}(x))\tilde{\Sigma}_{mn}\Phi + \epsilon^\mu(x)\partial_\mu\Phi, \tag{A47}$$

we notice that the three terms can be thought of as anisotropic Weyl transformations, local rotations, and diffeomorphisms. Since the transformation rule of Φ under anisotropic Weyl transformations, local rotations, and diffeomorphisms is given by

$$\text{anisotropic Weyl: } \delta\Phi = \Delta_\Phi\tilde{\lambda}\Phi,$$
$$\text{local rotation: } \delta\Phi = \frac{1}{2}\tilde{\omega}_\perp^{mn}\tilde{\Sigma}_{mn}\Phi,$$
$$\text{diffs: } \delta\Phi = \tilde{\epsilon}^\mu\partial_\mu\Phi, \tag{A48}$$

the transformation (A47) can be reproduced by the parameter choice,

$$\tilde{\lambda} = \lambda, \qquad \tilde{\omega}_\perp^{mn} = \omega_\perp^{mn} + \epsilon^\mu\tilde{S}_\mu^{mn}, \qquad \tilde{\epsilon}^\mu = \epsilon^\mu, \tag{A49}$$

where $\tilde{\lambda}$, $\tilde{\omega}_\perp^{mn}$, and $\tilde{\epsilon}$ are transformation parameters of anisotropic Weyl transformations, local rotations, and diffeomorphisms, respectively. Similarly, the transformation rule of $h^{\mu\nu}$, $\tilde{e}_m^\mu$, and n^μ under local symmetries are given by

$$\text{anisotropic Weyl: } \delta h^{\mu\nu} = 2\tilde{\lambda}h^{\mu\nu}, \qquad \delta\tilde{e}_m^\mu = \tilde{\lambda}\tilde{e}_m^\mu,$$
$$\delta n^\mu = z\tilde{\lambda}n^\mu, \tag{A50}$$

$$\text{local rotation: } \delta h^{\mu\nu} = 0, \qquad \delta\tilde{e}_m^\mu = \tilde{\omega}_{\perp m}{}^n\tilde{e}_n^\mu, \qquad \delta n^\mu = 0 \tag{A51}$$

and Eqs. (A26)–(A28), where z is the dynamical exponent. It then turns out that the spatial induced metric $h^{\mu\nu}$ and the spatial dreibein $\tilde{e}_m^\mu$, (h_{ij}, $\tilde{e}_i^m$, and δn_μ also) are invariant under the (global) nonrelativistic spacetime symmetry transformation given by the parameter choice (A49). On the other hand, however, the timelike vector is not invariant and transforms as

$$\delta n^\mu = b_+^\mu. \tag{A52}$$

b. Central extension

We then consider the case with the central extension. In this case, spacetime symmetries can generate internal $U(1)$ transformations as well as spacetime ones, just as Galilei boosts do. Using the notation in Sec. VI A, let us write such spacetime symmetries as

$$\epsilon^\mu(x)\partial_\mu + \alpha(x)M, \tag{A53}$$

where M is the internal $U(1)$ generator and $\alpha(x)$ is the corresponding parameter. For example, the Galilei boost v^iB_i can be expressed as $\epsilon^t = 0, \epsilon^i = v^it$, and $\alpha = -v_ix^i$, as we illustrated in Sec. A. 3. When a local field Φ has an internal $U(1)$ charge im, the transformation rule (A45) is extended to

$$\delta\Phi = \lambda(x)\Delta_\phi\Phi + \frac{1}{2}\omega_\perp^{mn}(x)\tilde{\Sigma}_{mn}\Phi + \epsilon^\mu(x)\nabla_\mu\Phi + im\alpha(x)\Phi. \tag{A54}$$

Also, the internal $U(1)$ gauge field is transformed as

$$\delta A_\mu = A_\nu\nabla_\mu\epsilon^\nu + \epsilon^\rho\nabla_\rho A_\mu - \partial_\mu\alpha. \tag{A55}$$

Note that the transformation rule of the temporal component and the spatial projection of the gauge field is given by

$$\delta A_\parallel = z\lambda A_\parallel + \epsilon^\mu\partial_\mu A_\parallel - n^\mu\partial_\mu\alpha, \tag{A56}$$

$$\delta A_{\perp m} = \lambda A_{\perp m} + \omega_{\perp m}{}^n A_{\perp n} + \epsilon^\rho\nabla_\rho A_{\perp m} - \tilde{e}_m^\mu\partial_\mu\alpha, \tag{A57}$$

of which the dependence on λ, $\omega_{\perp m}{}^n$, and ϵ^μ is consistent with Eq. (A45). Since the internal $U(1)$ gauge transformations of Φ and A_μ are given by

$$\delta\Phi = im\tilde{\alpha}\Phi, \qquad \delta A_\mu = -\partial_\mu\tilde{\alpha}, \tag{A58}$$

the (global) nonrelativistic spacetime symmetry transformation can be reproduced by the parameter set $\tilde{\alpha} = \alpha$ and Eq. (A49), where $\tilde{\alpha}$ is the internal $U(1)$ gauge transformation parameter. Note that the transformation rule of $h^{\mu\nu}$, $\tilde{e}_m^\mu$, and n^μ is the same as the case without central extensions: Under the (global) nonrelativistic spacetime symmetry transformation, the spatial metric and dreibein are invariant, but the timelike vector transforms as (A52).

To summarize, the transformation rule (A54) of standard matter fields can be naturally reproduced by embedding global nonrelativistic spacetime symmetries into diffeomorphisms, local rotations, anisotropic Weyl symmetries, and internal $U(1)$ gauge symmetries associated with the central extension. Identification of symmetry breaking patterns and the corresponding NG fields should therefore be based on those local symmetries: When the condensation is spacetime dependent, diffeomorphism invariance is broken. The local rotation symmetry, the anisotropic Weyl symmetry, and the internal $U(1)$ symmetry are broken when the condensation has a rotation charge, scaling dimension, and internal $U(1)$ charge, respectively. The effective action construction can then be performed in a similar way to the relativistic case, by gauging those local symmetries. This is one point of this Appendix.

It should be also noted that the timelike vector n^μ and the internal $U(1)$ gauge field A_μ transform nonlinearly under nonrelativistic spacetime symmetries with nonvanishing b_μ^+

and $\partial_\mu \alpha$. This situation is similar to the Weyl gauge field W_μ in conformal systems. As we mentioned in Sec. II C, when we perform Weyl gauging in conformal field theories, the Weyl gauge field W_μ appears in a particular combination because it is not special conformal invariant by itself. In the next subsection, we will illustrate that a similar situation occurs for n^μ and A_μ in Galilei boost invariant systems.

5. Gauging nonrelativistic spacetime symmetries

We then summarize how global nonrelativistic spacetime symmetries can be gauged into local ones. First, the diffeomorphism symmetry, local rotation symmetry, and internal $U(1)$ gauge symmetry associated with the central extension can be realized by introducing covariant quantities n^μ, $h^{\mu\nu}$, and $\tilde{e}_m^\mu$ introduced in Appendix A. 1 and the gauge field A_μ. For example, the free fermion action,

$$S = \int d^4x \left[i\psi^* \partial_t \psi - \frac{1}{2m} |\partial_i \psi|^2 \right], \qquad \text{(A59)}$$

can be reformulated as

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} i n^\mu \psi^* (\overleftrightarrow{\nabla}_\mu + imA_\mu)\psi - \frac{1}{2m} h^{\mu\nu} (\nabla_\mu - imA_\mu)\psi^* (\nabla_\nu + imA_\nu)\psi \right], \qquad \text{(A60)}$$

where the covariant derivative ∇_μ is defined by Eq. (A46) and $\psi^* \overleftrightarrow{\nabla}_\mu \psi \equiv \psi^* \nabla_\mu \psi - (\nabla_\mu \psi^*)\psi$. This curved space action enjoys the full diffeomorphism symmetry, the local rotation symmetry, and the internal $U(1)$ gauge symmetry. Note that the original action (A59) can be reproduced by setting that

$$h^{\mu\nu} = \eta^{\mu\nu} + \delta_0^\mu \delta_0^\nu, \qquad n^\mu = \delta_0^\mu, \qquad A_\mu = 0. \qquad \text{(A61)}$$

As we mentioned in the pervious subsection, the above conditions are not invariant under the global symmetries with $b_\mu^+ \neq 0$, $\partial_\mu \alpha \neq 0$, or both. Indeed, under a finite Galilei boost,

$$\psi'(x) = e^{im\alpha(x)} \psi(x+\epsilon) \quad \text{with} \quad \epsilon^t = 0,$$
$$\epsilon^i = v^i t, \qquad \alpha = -v_i x^i - \frac{1}{2} v^2 t, \qquad \text{(A62)}$$

the timelike vector n^μ and the gauge field A_μ are transformed as

$$n'^\mu(x) = n^\mu(x+\epsilon) - \delta_i^\mu v^i n^t(x+\epsilon),$$
$$A'_\mu(x) = A_\mu(x+\epsilon) + \delta_\mu^t \left(v^i A_i(x+\epsilon) + \frac{1}{2} v^2 \right) + \delta_\mu^i v_i, \qquad \text{(A63)}$$

which breaks the conditions (A61). This situation is similar to the special conformal transformation of the Weyl gauge field W_μ. Similarly to the previous case, by rewriting the action (A60) as

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} i (n^\mu + h^{\mu\nu} A_\nu)(\psi^* \overleftrightarrow{\nabla}_\mu \psi) - \frac{m}{2} (2n^\mu A_\mu + h^{\mu\nu} A_\mu A_\nu)|\psi|^2 - \frac{1}{2m} h^{\mu\nu} \nabla_\mu \psi^* \nabla_\nu \psi \right], \qquad \text{(A64)}$$

we notice that n^μ and A_μ appear in the following combinations:

$$n^\mu + h^{\mu\nu} A_\nu, \qquad 2n^\mu A_\mu + h^{\mu\nu} A_\mu A_\nu, \qquad \text{(A65)}$$

which are Galilei boost invariant. Note that such combinations are known to be Milne boost invariant in the context of the Newton–Cartan geometry. See, e.g., Ref. [72,73] for details.

Finally, let us consider the anisotropic Weyl symmetry. Just as the Ricci gauging in relativistic systems, it is known to be possible to introduce anisotropic Weyl invariant curved space actions for some class of nonrelativistic conformal theories. If such a procedure cannot be performed, we need to introduce a gauge field W_μ just as the Weyl gauging in relativistic systems. If the curved space action is invariant under global anisotropic Weyl transformations, we can always introduce a local anisotropic Weyl invariant action by replacing the covariant derivative ∇_μ with the Weyl covariant derivative,[25]

$$D_\mu \Phi = \nabla_\mu \Phi + (\Delta_\Phi \delta_\mu^\nu - \tilde{\Sigma}_\mu{}^\nu) W_\nu \Phi, \qquad \text{(A66)}$$

where $\tilde{\Sigma}_\mu{}^\nu = \tilde{e}_\mu^m \Sigma_m{}^n \tilde{e}_n^\nu$ and the local anisotropic Weyl transformation rule is given by

$$\Phi \to \Phi' = e^{\Delta_\Phi \lambda} \Phi, \qquad n^\mu \to n'^\mu = e^{-z\lambda} n^\mu,$$
$$h^{\mu\nu} \to h'^{\mu\nu} = e^{-2\lambda} h_{\mu\nu},$$
$$\tilde{e}_m^\mu \to \tilde{e}'^\mu_m = e^{-\lambda} e_m^\mu, \qquad W_\mu \to W'_\mu = W_\mu - \partial_\mu \lambda. \qquad \text{(A67)}$$

In contrast to the relativistic case, it seems not well understood under what conditions Weyl gauging can be converted to Ricci gauging. It would be interesting to

[25] See, e.g., Ref. [74] for recent discussions on the gauging of the anisotropic Weyl symmetry.

investigate this issue by extending the discussion [42] in relativistic systems.

APPENDIX B: DERIVATION OF NAMBU–GOTO ACTION

In Sec. III C, we discussed that our effective action for a single scalar brane contains gapped modes in addition to gapless NG modes localizing on the brane. In this Appendix, we show that the low-energy effective action after integrating out massive modes is nothing but the Nambu–Goto action. As we have discussed, the unitary gauge action for z-diffeomorphism symmetry breaking takes the form

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}[\alpha_1(z)(1+g^{zz}) + \alpha_3(z)(\delta g^{zz})^2 + \mathcal{O}((\delta g^{zz})^3)] \quad \text{with} \quad \delta g^{zz} = g^{zz} - 1 \tag{B1}$$

at the lowest dimension. To discuss its relation to the Nabmu–Goto action, it is convenient to rewrite

$$S = -\frac{1}{2}\int d^4x\sqrt{-h}[2\alpha_1(z) + \tilde{\alpha}_3(\delta g^{zz})^2 + \mathcal{O}((\delta g^{zz})^3)], \tag{B2}$$

where h is the determinant of the induced metric, $h_{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\nu}}$, on the constant z surfaces. Here, we follow the convention in Sec. III, e.g., $\hat{\mu} = t, x, y$. We also introduced $\tilde{\alpha}_3 = \frac{1}{4}\alpha_1(z) + \alpha_3(z)$. In the following, we show that the integration of gapped modes provides a constraint $\delta g^{zz} = 0$ and the effective action is reduced to the Nambu–Goto action in the low-energy regime.

For this purpose, let us first write down the second order action for the NG field. In the unitary gauge coordinate, the NG field for the broken z-diffeomorphism is eaten by the metric field. The induced metric, $h_{\hat{\mu}\hat{\nu}}$, and the z-component, g^{zz}, are given by[26]

$$h_{\hat{\mu}\hat{\nu}}(x) = \eta_{\hat{\mu}\hat{\nu}} + \partial_{\hat{\mu}}\pi(x)\partial_{\hat{\nu}}\pi(x),$$
$$g^{zz} = 1 + 2\partial_z\pi + 3(\partial_z\pi)^2 + (\partial_{\hat{\mu}}\pi)^2 + \mathcal{O}(\pi^3), \tag{B3}$$

where note that $\partial_z\pi$ appears only in g^{zz}. The second order action then takes the form

$$S_2 = -\frac{1}{2}\int d^4x[\alpha_1(z)(\partial_{\hat{\mu}}\pi)^2 + 4\tilde{\alpha}_3(\partial_z\pi)^2] = -\frac{1}{2}\int d^4x\alpha_1\left[(\partial_{\hat{\mu}}\pi)^2 - \pi\left(\frac{4\tilde{\alpha}_3}{\alpha_1}\partial_z^2 + \frac{4\tilde{\alpha}_3{}'}{\alpha_1}\partial_z\right)\pi\right], \tag{B4}$$

[26]We define π in the unitary gauge coordinate such that $z_{\rm flat} = z - \pi(x)$, where $z_{\rm flat}$ is the flat space coordinate and z and x are the unitary gauge coordinates.

where we dropped total derivative terms. The physical spectrum is now determined by the eigenvalue problem of the operator, $\frac{\tilde{\alpha}_3}{\alpha_1}\partial_z^2 + \frac{\tilde{\alpha}_3{}'}{\alpha_1}\partial_z$. Note that our analysis in Sec. III C was for $\alpha_1 = 4\tilde{\alpha}_3 = \frac{\beta^2 v^2}{\cosh^4 \beta z}$ in particular. There, we had two types of physical modes: gapless modes localizing on the brane and gapped modes propagating in the bulk. Let us assume that such a qualitative feature holds generically in more general setups for a single domain wall. We then expand the NG field, π, by those modes as

$$\pi(x) = \pi_0(x_\perp) + \sum_\lambda\sum_{i=\pm}\pi_{\lambda_i}(x_\perp)u_{\lambda_i}(z), \tag{B5}$$

where $x_\perp$ stands for coordinates in the transverse directions, t, x, y, and $\sum_\lambda$ denotes both the sum and integral over $\lambda . u_{\lambda_i}$ $(i = \pm)$ stands for two eigenfunctions with the eigenvalue λ satisfying

$$\left(\frac{\tilde{\alpha}_3}{\alpha_1}\partial_z^2 + \frac{\tilde{\alpha}_3'}{\alpha_1}\partial_z\right)u_{\lambda_i} + \lambda u_{\lambda_i} = 0 \quad \text{and} \quad \int dz\alpha_1 u_{\lambda_+}u_{\lambda_-} = 0. \tag{B6}$$

The second order action is now reduced to the form

$$S = -\frac{1}{2}\int dz\alpha_1(z)\int d^3x_\perp(\partial_{\hat{\mu}}\pi_0)^2 - \frac{1}{2}\sum_\lambda\sum_{i=\pm}\int dz\alpha_1(z)(u_{\lambda_i}(z))^2 \times\int d^3x_\perp[(\partial_{\hat{\mu}}\pi_{\lambda_i})^2 + \lambda\pi_{\lambda_i}^2], \tag{B7}$$

where λ can be thought of as the mass squared in three dimensions. Note that such a mass term originates from the $\tilde{\alpha}_3$ term in (B2). Also, the linear equation of motion for gapped modes reduces to $\pi_{\lambda_\pm} = 0$ in the low-energy limit, $|k_{\hat{\mu}}|^2 \ll \lambda$.

Finally, we extend the previous discussion to the nonlinear level and derive the Nambu–Goto action. Just as the linear order discussions, the $\tilde{\alpha}_3$ term plays an important role,

$$-\frac{1}{2}\int d^4x\sqrt{-h}\tilde{\alpha}_3(\delta g^{zz})^2 = -\frac{1}{2}\int d^4x\sqrt{-h}\tilde{\alpha}_3(2\partial_z\pi + 3(\partial_z\pi)^2 + (\partial_{\hat{\mu}}\pi)^2 + \mathcal{O}(\pi^3))^2, \tag{B8}$$

where note that the factor, $\sqrt{-h}$, does not contain $\partial_z\pi$. As we have discussed, the mass term for the gapped modes, π_{λ_i}, arises from this interaction at the second order action level. If we include higher order terms, there appear mixing interactions between gapped and gapless modes. In the low-energy limit, the equation of motion for the gapped mode is then given by

$$2\partial_z\pi + 3(\partial_z\pi)^2 + (\partial_{\hat{\mu}}\pi)^2 + \mathcal{O}(\pi^3) = 0 \leftrightarrow g^{zz} = 1, \tag{B9}$$

so that the effective action for the gapless mode reduces to the Nambu–Goto type one,

$$S_{\text{eff}} = -\frac{1}{2}\int d^4x\sqrt{-h}\alpha_1(z) = -T\int d^3x\sqrt{-h} \quad \text{with} \quad T = \frac{1}{2}\int dz\alpha_1(z), \tag{B10}$$

where h contains gapless modes only and T can be identified with the brane tension.

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