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$SU(N)$ gauge invariant currents and hadronization

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The fermion and total currents generated by a gauge field are derived in the framework of a self-consistent consideration of the Dirac and Yang-Mills equations. Under the condition of gauge invariance, the obtained currents are found to be expressible in terms of the massive vector field generated by the initial gauge field. The mass of this vector field depends strongly on the occupancy numbers of the fermion subsystem, whereas the arisen mass term holds the gauge invariance of the modified Lagrangian. We show that such a modified Lagrangian can be reduced to the pure gluodynamic Lagrangian containing a mass term. By breaking the initial $SU(N)$ symmetry, we derive the Lagrangian governing the dynamics of the massive scalar particles, which can be treated as the octet of the pseudoscalar mesons. The contribution of both the quark-gluon interaction and the self-interaction gluon field to the masses of the octet particles is considered. Provided that the hadronization of the confinement matter into the pion triplet occurs, the QCD coupling constant is evaluated in this case in the developed model.

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I. INTRODUCTION

Schwinger pointed out the importance of the gauge invariance of a fermion current in the $(1+1)$ quantum electrodynamics (QED_2) for the first time [1,2]. The calculated fermion current is found to be expressible in terms of some massive field. On the other hand, it was shown [3] that, in the non-Abelian case, a Fermi theory in $(1+1)$ dimensions is also equivalent to the local boson theory.

The two-dimensional physics has been further developed in papers [4–6] which are particularly devoted to the derivation of the spectra of observable mesons [4–6] and baryons [4]. In terms of the QCD_2 action integral, it is shown [4–6] that the meson mass is approximately proportional to the square root of the number of colors and flavors. The systematic presentation of the two-dimensional quantum field theory is given in Ref. [7].

Another application of the $(1+1)$ physics [1–3] is the idea of the so-called flux tube model [8–10] in QCD which was put forward to describe the production of observable hadrons in the e^+e^- annihilations, $p-p$, and $A-A$ collisions at high energies [8,11–14]. Despite the obvious successes of the QCD_2 concept in treating experimental results, the problem of how to describe the states of observable particles in the realistic QCD_4 dynamics has still been one of the most important problems in strong interaction physics. There has also been, concurrently, a natural development of the QCD_2 models [1–6,8–14].

The dynamics of particles in QCD_4 is vastly more complicated than the QCD_2 case because of an increasing number of dimensions of the studied problem. In this way, it is reasonable to think that, as in the QCD_2 case, interaction of fermions with a gauge field can be expressed in terms of the massive gauge field. However, in calculating the total mass of a gauge field, which is important in developing the hadronization model, we have to take into account another source of the gauge field mass: the self-interacting non-Abelian field.

The problem of a gluon mass was repeatedly raised [15–36], mainly in the context of studying the low momentum properties of a gluon propagator in pure Yang-Mills theory. The earliest considerations of this problem concern deriving the gauge glueball spectrum in a variational approximation [15,16], and discovering the Gribov copies [17]. The latter [17] particularly leads to the necessity of a cutoff in the path integral in the configuration space and, as a consequence, results in an arising of the so-called Gribov mass. Based on the localization idea [21–23], the results [15–17] were further developed in papers [18–20], where the dynamic gluon mass was found to be aside from the Gribov one [17]. The last investigations of the gluon propagator at low momentum [24–36], which also include the lattice studies [24,27,32,35], show that the inverse gluon propagator always has a nonvanishing asymptote at small momentum, which is regardless of analytical properties of the propagator studied in various models. Such a behavior of the gluon propagator is reasonably interpreted as a gluon mass.

In the present paper a self-consistent solution of the Dirac equation in a non-Abelian gauge field is obtained in the absence of the additional restrictions [37–39] to the field structure, which are the consideration of the Dirac equation either in the field of the non-Abelian plane wave [38,39], or in the framework of the two-dimensional QCD [37]. On the basis of the derived formal solution of the Dirac equation, the fermion and total currents are obtained in an explicit form. The derived currents are found to be expressible in terms of the massive vector field, generated in the result of interaction between fermions and gauge fields. The obtained currents appear to be gauge invariant, and this satisfies the continuity equation. In this way, the derived contribution to the mass is sufficiently dependent on the occupancy numbers of the fermion subsystem. The obtained mass term holds the gauge invariance of the new Lagrangian, which, it has been found, can be reduced to the pure Yang-Mills Lagrangian with a mass term.

Based on the violation of the initial $SU(N)$ symmetry, the Lagrangian governing the octet of the massive pseudoscalar fields is derived in the QCD_4 case. We consider and study in detail the contributions of both the quark-gluon interaction and the self-interacting non-Abelian fields to the masses of the octet particles. In the approximation of the hadronization of the confinement matter to the pion triplet, the QCD coupling constant is estimated.

The paper is organized as follows. The second section contains the statement of a problem and the solution of the Dirac equation in an external $SU(N)$ gauge field. The fermion and total currents are considered in Sec. III. The gauge invariant pure Yang-Mills Lagrangian containing a mass term is obtained in Sec. IV. The Lagrangian governing the octet of the pseudoscalar mesons is considered in Sec. V. The last two sections are the discussion of the obtained results and the Conclusion. Some detail steps of the derivations are presented in the Appendixes.

II. SOLUTION OF THE DIRAC EQUATION

The Lagrangian governing the fermions interacting with an $SU(N)$ gauge field is [40]

$$\mathcal{L} = \left\{ \sum_f \left[\frac{1}{2} [\bar{\Psi}_f \gamma^\mu D_\mu \Psi_f - \bar{\Psi}_f m_f \Psi_f] - \frac{1}{2} [\bar{\Psi}_f \gamma^\mu \overleftarrow{D}_\mu \Psi_f + \bar{\Psi}_f m_f \Psi_f] \right] - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \right\}, \quad (1)$$

where A^a_μ and Ψ_f are the gauge and fermion fields in the Minkowski $(3+1)$-dimensional space-time with coordinates $x \equiv x^\mu = (x^0, \boldsymbol{x}) = (x^0, x^1, x^2, x^3)$, m_f is a fermion mass, g is the coupling constant, and f denotes a quark flavor. In Eq. (1) we introduce

$$D_\mu = i\partial_\mu + gT_a A^a_\mu, \qquad \overleftarrow{D}_\mu = i\overleftarrow{\partial}_\mu - gT_a A^a_\mu$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a_{bc} A^b_\mu A^c_\nu, \quad (2)$$

where γ^ν's are the standard Dirac matrices, T_a's are the infinitesimal operators satisfying the standard commutative relations and the normalization condition [40], and $a, b, c = 1 \ldots N^2 - 1$ are the $SU(N)$ group indices; $\partial_\mu = (\partial/\partial t, \nabla)$.

The Lagrangian (1) leads to the Dirac equation

$$\{i\gamma^\mu(\partial_\mu - ig \cdot A^a_\mu(x) T_a) - m_f\} \Psi_f(x) = 0. \quad (3)$$

The solution of Eq. (3) can be formally written in the operator form as follows:

$$\Psi_f(x) = \{T_{l(x_0;x)} \exp\} \left\{ igT_a \int dx^\mu A^a_\mu \right\} \psi_f(x), \quad (4)$$

where the symbol $\{T_{l(x_0;x)} \exp\}$ means that the integration is to be carried out along the line from the point x_0 to the point x, such that the factors in exponent expansion are chronologically ordered from x_0 to x. In this way, $\psi_f(x)$ obeys the free Dirac equation

$$\{i\gamma^\mu \partial_\mu - m_f\} \psi_f(x) = 0. \quad (5)$$

The general solution of Eq. (5) can be presented as a superposition of the Dirac plane waves

$$\psi_{\sigma c f}(x) = u_\sigma(P) \frac{\exp(-iP_\mu x^\mu)}{\sqrt{2\varepsilon(\vec{p})}} v_{c,f}(x_0),$$
$$\varepsilon^2(\vec{p}) = \vec{p}^2 + m_f^2, \qquad P \equiv P^\mu = (\varepsilon(\vec{p}), \vec{p}), \quad (6)$$

where P^μ is the 4-momentum and $u_\sigma(P)$ are the free Dirac bispinors, normalized by the doubled mass ($\bar{u}u = 2m_f$). The symbol $v_{c,f}(x_0)$ is a vector in the charged space, which also depends on a point in the Minkowski space-time. We take $v_{c,f}(x_0)$ to be normalized by the standard condition

$$(v^\dagger)_{c'f'}(x_0) v^{cf}(x_0) = \delta^c_{c'} \delta^f_{f'}. \quad (7)$$

Here, σ and c denote the spin and color variables.

Let us consider the operator

$$\hat{O} = \{T_{l(x_0;x)} \exp\} \left\{ igT_a \int dx^\mu A^a_\mu \right\}, \quad (8)$$

which is in Eq. (4). It acts on the vector $v_{c,f}(x_0)$ and carries out the parallel shift of this vector along the geodesic line from the point x_0 to the point x in the Minkowski space-time.

Taking into account Eqs. (4) and (6), we can write the general solution of Eq. (3) as follows:

$$\Psi_{c,f}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \sum_{\sigma\lambda} [u_\sigma(P) a_f(P, \sigma, \lambda, c) \theta(P^0) + u_\sigma(-P) a_f(-P, \sigma, \lambda, c) \theta(-P^0)] \frac{\exp(-iP_\mu x^\mu)}{\sqrt{2\varepsilon(\vec{p})}} \times \{T_{l(x_0;x)} \exp\} \left\{ igT_a \int dx^\mu A^a_\mu \right\} v_{c,f}(x_0), \quad (9)$$

where summation with respect to λ means summing over all of the possible trajectories of a fermion which connect the points x_0 and x in the Minkowski space-time. The subscribes c and f enumerate the colors and the flavor states, respectively; $\theta(z)$ is the unit step function. The coefficients $a_f(P,\sigma,\lambda,c)$ are related to particles or antiparticles and satisfy the standard commutative relations for the Fermi operators under the field quantization

$$[a_f(P,\sigma,\lambda,c),a^\dagger_{f'}(P',\sigma',\lambda',c')]_+ = (2\pi)^3\delta(\vec{p}-\vec{p}')\delta_{\sigma\sigma'}\delta_{cc'}\delta_{ff'}\delta_{\lambda\lambda'}. \tag{10}$$

The δ symbol with respect to the "variable" λ eliminates interception of the particle trajectories, which is the direct consequence of the superposition principle.

III. FERMION AND TOTAL CURRENTS

The initial Lagrangian (1) implies the self-consistency with respect to the interacting fermions and gauge fields, which leads to a very complicated structure of the currents generated by this Lagrangian. However, it is reasonable to expect some simplification in the calculations of fermion currents in external fields, owing to the derived solution (9) of the Dirac equation, since any current is the convolution of Ψ functions. Moreover, the fermion current in an external field is in the explicit form in the Lagrangian (1) that allows us to modify the initial Lagrangian to the form which appears to be more valid in order to study the hadronization process.

Let us calculate the fermion current generated by the external non-Abelian field. In deriving such a current, we follow Schwinger's idea consisting of the consideration of J^μ_a as a limit:

$$J^\mu_a(x) = g\sum_f\{\bar{\Psi}_f(x)\gamma^\mu T_a\Psi_f(x')\}, \qquad x'\to x. \tag{11}$$

Since the current is proportional to the fermion Green's function [1,2], which has the first order pole at $x=x'$, the terms containing $(x-x')$ in the expansion of $\bar{\Psi}_f(x)\Psi_f(x')$ in a series with respect to $(x-x')$ also give nonzero contribution into the calculated current [2].

We note that, because of the trace operation with respect to the color variable in Eq. (11), the current $J^\mu_a(x)$ contains the factor

$$(T_{l(x;x')}\exp)\left\{igT_a\int_x^{x'}A^a_\mu dx^\mu\right\}. \tag{12}$$

Then, expanding the operator exponent in the last equation as a series with respect to $(x'-x)\to 0$, we get

$$\begin{aligned}(T_{l(x;x')}\exp)&\left\{igT_a\int_x^{x'}A^a_\mu dx^\mu\right\}\\ &= 1+igT_a(x'-x)^\mu A^a_\mu(\xi)\\ &\quad+\frac{i}{2}gT_a(x'-x)^\mu(x'-x)^\nu\partial_\nu A^a_\mu(\xi)\\ &\quad-g^2(T_aT_b)(\tilde{x}'-\tilde{x})^\mu(x'-x)^\nu A^a_\mu(\tilde{\xi})A^b_\nu(\xi)\theta(\tilde{\xi}-\xi)\\ &\quad+\cdots,\end{aligned} \tag{13}$$

where $\tilde{\xi}\in[\tilde{x},\tilde{x}']$, and $\xi\in[x,x'];x'\to x$.

Let us take the limits $(\tilde{x}'-\tilde{x})\to 0$ and $(x'-x)\to 0$, such that

$$\frac{(\tilde{x}'-\tilde{x})}{(x'-x)}\to 0. \tag{14}$$

Then, the last term in the expansion in Eq. (13) is equal to zero. Substituting $(T_{l(x;x')}\exp)\{igT_a\int_x^{x'}A^a_\mu dx^\mu\}$ into Eq. (11), we obtain, for $(x'-x)\to 0$,

$$\begin{aligned}J^\mu_a = g^2\int\frac{d^4P}{(2\pi)^3}\sum_{f,c,\lambda\sigma}&\Bigg\{n_f(P,\sigma,\lambda,c)P^\mu\\ &\times\left(-\frac{\partial}{\partial P^\nu}\exp\left(-iP(x'-x)\right)\right)\\ &\times\frac{[\delta(P^0+\varepsilon(\vec{p}))+\delta(P^0-\varepsilon(\vec{p}))]}{\varepsilon(\vec{p})}\\ &\times\left(A^\nu_a(\xi)+\frac{1}{2}(x'-x)_\beta\partial^\nu A^\beta_a(\xi)\right)\Bigg\},\end{aligned} \tag{15}$$

where $n_f(P,\sigma,\lambda,c)$ denotes the occupancy numbers of fermion states:

$$n_f(P,\sigma,\lambda,c) = \langle a^\dagger_f(P,\sigma,\lambda,c)a_f(P,\sigma,\lambda,c)\rangle, \tag{16}$$

where the angle brackets denote averaging over all of the possible sets of the quantum numbers determining the fermion states.

The occupancy numbers $n_f(P,\sigma,\lambda,c)$ should be physically treated as the average number of the fermions with the quantum numbers (P,σ,c,f), which propagate along the fixed trajectory identified by the number λ. Thereat, the trajectories appear to be not interferant owing to the commutative relations (10), which relates to the superposition principle for fermion fields.

In obtaining the last equation, we have successively calculated a trace, introduced the additional integration with respect to the zeroth component of the vector P^μ (see Appendix A). We note that on taking the partial derivative $\partial^2\equiv\partial^2_{(x)}$ on $(x'-x)^\beta\partial_{\nu(x)}A^b_\beta(\xi)$, we get [37]

$$\begin{aligned}\partial^2_{(x)}\lim_{x'\to x}\{(x'-x)^\beta\partial_{\nu(x)}A^b_\beta(\xi)\} &= \lim_{x'\to x}\partial^2_{(x)}\{(x'-x)^\beta\partial_{\nu(x)}A^b_\beta(x)\} \\ &= \lim_{x'\to x}\{-2\partial^\beta_{(x)}\partial_{\nu(x)}A^b_\beta(x) \\ &\quad + (x'-x)^\beta\partial_{\beta(x)}\partial^\beta_{(x)}\{\partial_{\nu(x)}A^b_\beta(x)\}\}. \end{aligned} \tag{17}$$

Upon taking the limit $x'\to x$, the second term vanishes. Then, formally dividing both parts of Eq. (17) by ∂^2, we have, at $x'\to x$,

$$\lim_{x'\to x}((x'-x)^\lambda\partial_\nu A^b_\lambda(\xi)) = -2\frac{\partial^\beta\partial_\nu}{\partial^2}A^b_\beta(x). \tag{18}$$

Integrating partially with respect to P^μ in Eq. (15), we derive (see Appendix A)

$$J^\mu_a(x) = M^2(N_c,N_f)\left(A^\mu_a(x) - \frac{\partial_\lambda\partial^\mu}{\partial^2}A^\lambda_a(x)\right), \tag{19}$$

where N_c and N_f are the number of colors and flavors, respectively. The operator of the inverse squared derivative in Eq. (19) should be treated in terms of Eq. (17). In the momentum representation, this operator corresponds to division by the off-shell squared 4-momentum.

The factor $M^2(N_c,N_f)$ in Eq. (19), which has the dimension of the squared mass, is equal to [see Eq. (A6) in Appendix A]

$$\begin{aligned}M^2(N_c,N_f) &\equiv M^2 \\ &= \frac{g^2}{8}\int\frac{d^4P}{(2\pi)^3}\sum_{f,c,\lambda\sigma}\frac{\partial}{\partial P^\nu} \\ &\quad\times\left\{n_f(P,\sigma,\lambda,c)\frac{P^\nu[\delta(P^0+\varepsilon(\vec{p}))+\delta(P^0-\varepsilon(\vec{p}))]}{\varepsilon(\vec{p})}\right\}.\end{aligned} \tag{20}$$

The obtained mass M can be physically interpreted as the mass of a fermion field which is carried by a gauge field due to interaction between the fermion and gluon fields. We also note that M depends explicitly on the number of colors and flavors. Provided that the occupancy numbers $n_f(P,\sigma,\lambda,c)$ are approximately independent on N_c and N_f, we obtain

$$M^2(N_c,N_f) \propto N_f N_c, \tag{21}$$

which is in good agreement with the results obtained earlier in Refs. [4–6].

We add the current, induced by the self-interaction of a gauge field [40], to the $J^\mu_a(x)$ given by Eq. (19). Then, the total fermion current $I^\mu_a(x)$ generated by the non-Abelian external field is

$$I^\mu_a(x) = M^2\left(A^\mu_a(x) - \frac{\partial_\lambda\partial^\mu}{\partial^2}A^\lambda_a(x)\right) - gf^c_{ab}A^b_\nu F^{\mu\nu}_c. \tag{22}$$

The obtained current (22) is gauge invariant, and it obviously satisfies the continuity equation

$$\partial_\mu I^\mu_a(x) = 0. \tag{23}$$

We should note that the currents which are given by Eq. (19) and derived in paper [2] appear to be very similar. However, such a similarity is only formal since the current Eq. (19) depends strongly on the mass of a vector field, which is governed by the occupancy numbers of the fermion subsystem.

IV. GLUODYNAMICS LAGRANGIAN

Let us substitute the fermion field given by Eq. (9) into the Lagrangian (1). Such a modification of the Lagrangian results in taking into account only such trajectories of fermions which are governed by the Dirac equation, rather than all of the possible ones which are contained in the Lagrangian (1). Thus, provided that $\Psi(x)$ is in the form given by Eq. (9), the Lagrangian (1) can be rewritten as follows:

$$\begin{aligned}\mathcal{L} &= \sum_f\left[\frac{1}{2}[\bar\Psi_f(i\gamma^\mu\partial_\mu)\Psi_f - \bar\Psi_f m_f\Psi_f] - \frac{1}{2}[\bar\Psi_f(i\gamma^\mu\overleftarrow{\partial}_\mu)\Psi_f \right.\\ &\quad\left. + \bar\Psi_f m\Psi_f]\right] + M^2A^\mu_a(x)\left(A^a_\mu(x) - \frac{\partial^\lambda\partial_\mu}{\partial^2}A^a_\lambda(x)\right) \\ &\quad - \frac{1}{4}(\partial^\mu A^\nu_a(x) - \partial^\nu A^\mu_a(x) + gf^{bc}_a A^\mu_b A^\nu_c)(\partial_\mu A^a_\nu(x) \\ &\quad - \partial_\nu A^a_\mu(x) + gf^a_{bc}A^b_\mu A^c_\nu).\end{aligned} \tag{24}$$

In the last equation, we have picked out the interaction term and used the fact that this term is a product of the current (19) and gauge field $A^a_\nu(x)$. The Lagrangian (24) is gauge invariant despite the mass term, but it is strongly nonlocal due to the factor ∂^{-2}.

As has been already mentioned, the factor M in Eq. (24) is the mass of a fermion field which is carried by a gauge field because of the interaction between fermions and gauge fields. Therefore, it is reasonable to expect that the term $\mathcal{L}_k$, corresponding to the kinetic energy of the fermion subsystem, can be gotten rid of in Lagrangian (24) in the developed consideration. With

Eqs. (9), (10), and (16) in mind, we have for this term (see Appendix B)

$$\mathcal{L}_k = (-\partial^{\mu(x)})\left\{\lim_{x'\to x}\int 2\frac{d^4P}{(2\pi)^3}\times\sum_{\sigma\lambda,f,c}[n_f(P,\sigma,\lambda,c)\delta(P^\mu P_\mu - m^2)]\exp(-iP_\mu(x^\mu - x'^\mu)(\partial_{\mu(x)})\left\{igT_a\int_{x'}^{x}dx^\mu A^a_\mu\right\}\right\}. \quad (25)$$

Since Eq. (25) is a divergency of some function, the contribution of $\mathcal{L}_k$ into the total Lagrangian $\mathcal{L}$ can be omitted.

As a result, we have

$$\mathcal{L} = M^2 A^\mu_a(x)\left(A^a_\mu(x) - \frac{\partial^\lambda\partial_\mu}{\partial^2}A^a_\lambda(x)\right) - \frac{1}{4}(\partial^\mu A^\nu_a(x) - \partial^\nu A^\mu_a(x) + gf_a^{bc}A^\mu_b A^\nu_c)(\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + gf^a_{bc}A^b_\mu A^c_\nu). \quad (26)$$

The derived Lagrangian governs the pure gluodymanics so that all of the fermion subsystem is found to be incorporated into Lagrangian (26) by means of the mass M. In this way, the obtained Lagrangian has still been gauge invariant, remaining sufficiently nonlocal due to the operator ∂^{-2}.

V. LAGRANGIAN OF SCALAR PARTICLES

The main aim of this section is derive the Lagrangian governing observable particles. Therefore, we need to eliminate the unobservable degrees of freedom, which always are in the gauge invariant Lagrangians. To do it we fix a gauge, and we take the Lorenz one because of its relativistic invariance:

$$\partial_\mu A^\mu_a = 0. \quad (27)$$

Then, Lagrangian (26) takes the form

$$\mathcal{L} = M^2 A^\mu_a(x)A^a_\mu(x) - \frac{1}{4}(\partial^\mu A^\nu_a(x) - \partial^\nu A^\mu_a(x) + gf_a^{bc}A^\mu_b A^\nu_c)(\partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + gf^a_{bc}A^b_\mu A^c_\nu), \quad (28)$$

where the mass M is given by Eq. (20). We note that the obtained Lagrangian has lost its gauge invariance because of the condition (27), but it becomes local in comparison with the Lagrangian given by Eq. (26). The mass M in the Lagrangian is not the total mass of a vector field. This is only the mass generated by the interaction between fermions and gauge fields, whereas there is another source of the gauge field mass, which is that of the self-interacting non-Abelian gauge fields.

In the case of the $SU(3)$ symmetry, Lagrangian (28) contains eight independent fields. Therefore, it is reasonable to relate them with the octet of the pseudosalar mesons, arising in the result of the hadronization of the confinement matter. This is a specific confinement situation since there are only gluons in Lagrangian (28). As for the quark subsystem, the information about it appears to be incorporated into this Lagrangian by means of the mass term.

To derive the scalar particle Lagrangian, we primarily have to go from vector fields to scalar ones. We carry it out by separating the variables in $A^\mu_a(x)$ which correspond to the Minkowski and representation spaces. Let us assume that the hadronization occurs when the $SU(3)$ symmetry appears to be spontaneously broken, so that the fields $A^\mu_a(x)$ take the form

$$A^\mu_a(x) = \mathfrak{a}^\mu_a + e^\mu\varphi_a(x), \quad (29)$$

where $\varphi_a(x)$ are scalar functions, whereas the constant vectors $\mathfrak{a}^\mu_a$ are assumed to be orthogonal to both the unit vector e_μ and the scalar field gradients $\partial_\mu\varphi_a(x)$:

$$\mathfrak{a}^\mu_a e_\mu = 0, \qquad \mathfrak{a}^a_\mu\partial^\mu\varphi_b(x) = 0. \quad (30)$$

In this way, the unit vector e_μ is taken to be normalized by a relation:

$$e_\mu e^\mu = -1. \quad (31)$$

We note that Lorentz gauge (27) results in the additional orthogonality condition

$$e^\mu\partial_\mu\varphi_a(x) = 0. \quad (32)$$

The fields $A^\mu_a(x)$ governed by Eqs. (29)–(32) indicate that, physically, the scalar fields $\varphi^a(x)$ can only propagate along the direction in the Minkowski space-time which is perpendicular to the plane fixed by the orthogonal vectors e^μ and $\mathfrak{a}^\mu_a$. Since these planes are different for different a's, the fields $\varphi_a(x)$ are independent in terms of their evolution in the Minkowski space-time. We should also note that such a kinematic restriction in the field propagation leads to the arising of the additional mass of the field $\varphi_a(x)$, as will be shown below.

Substituting Eq. (29) into Lagrangian (28) and taking into account Eqs. (27) and (30)–(32), we derive, after direct calculations,

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\varphi^a(x))(\partial_\mu\varphi_a(x)) - \frac{1}{2}(\mathcal{M}^2)^a_b\varphi_a(x)\varphi^b(x), \quad (33)$$

where $(\mathcal{M}^2)^a_b$ is the matrix of the squared masses which is given by the formula

$$(\mathcal{M}^2)^a_b = 2M^2(N_c, N_f)\delta^a_b - g^2 \mathfrak{a}^\mu_c \mathfrak{a}^{c'}_\mu f_b{}^{cs} f^a{}_{c's}. \quad (34)$$

We note that Lagrangian (33) has still been $SU(3)$ invariant. Let us follow Gell-Mann [41] and take the conservation of the isospin T and strangeness S, rather than supporting the exact $SU(3)$ symmetry, under hadronization into the octet of the colorless mesons. This means breaking the initial symmetry $SU(3)$ up to the $SU_{S=0}(2) \bigotimes SU_{S=1}(2) \bigotimes U(1)$ one. The new symmetry implies that these eight pseudoscalar mesons, which are $\pi^\pm, \pi^0, K^\pm, K^0, \tilde{K}^0, \eta$, including antiparticle (K^- and $\tilde{K}^0$), can be placed into the strangenessless pion triple, two kaon doublets at $S = \pm 1$, where $S = -1$ corresponds to antiparticles, and an η meson having zeroth isospin and strangeness. Moreover, such a symmetry violation affects the mass $M = M(N_c, N_f)$ since its value depends explicitly on the number of colors N_c and flavors N_f.

Therefore, let us establish the relation of these pseudoscalar mesons to the fields φ_a by means of the complex subscribing $a \Rightarrow (T, S)$:

$$\varphi_a(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \varphi_3(x) \\ \varphi_4(x) \\ \varphi_5(x) \\ \varphi_6(x) \\ \varphi_7(x) \\ \varphi_8(x) \end{pmatrix} \equiv \begin{pmatrix} \varphi_{\pi^+}(x) \\ \varphi_{\pi^-}(x) \\ \varphi_{\pi^0}(x) \\ \varphi_{K^+}(x) \\ \varphi_{K^-}(x) \\ \varphi_{K^0}(x) \\ \varphi_{\tilde{K}^0}(x) \\ \varphi_\eta(x) \end{pmatrix} \equiv \begin{pmatrix} \varphi_{(1,0)}(x) \\ \varphi_{(1,0)}(x) \\ \varphi_{(1,0)}(x) \\ \varphi_{(1/2,1)}(x) \\ \varphi_{(1/2,-1)}(x) \\ \varphi_{(1/2,1)}(x) \\ \varphi_{(1/2,-1)}(x) \\ \varphi_{(0,0)}(x) \end{pmatrix}, \quad (35)$$

where T and S are the isospin and the strangeness, respectively.

Such defined fields φ_a do not correspond to the observable mesons since the mass term in Lagrangian (33) has not diagonalized yet.

Based on the structure of the vector φ_a given by Eq. (35), which follows from the $SU_{S=0}(2) \otimes SU_{S=1}(2) \otimes U(1)$ symmetry, we have for the mass matrix

$$\mathcal{M}^2 = \begin{pmatrix} 2M_1^2+m_1^2 & -\mu_1^2 & -\mu_1^2 & 0 & 0 & 0 & 0 & 0 \\ -\mu_1^2 & 2M_1^2+m_1^2 & -\mu_1^2 & 0 & 0 & 0 & 0 & 0 \\ -\mu_1^2 & -\mu_1^2 & 2M_1^2+m_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2M_2^2+m_2^2 & 0 & -\mu_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2M_2^2+m_2^2 & 0 & -\mu_2^2 & 0 \\ 0 & 0 & 0 & -\mu_2^2 & 0 & 2M_2^2+m_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_2^2 & 0 & 2M_2^2+m_2^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2M_3^2+m_3^2 \end{pmatrix}. \quad (36)$$

In the last formula we introduce

$$\begin{aligned} &m_1^2 = -g^2 \mathfrak{a}^\mu_c \mathfrak{a}^{c'}_\mu f_1^{cs} f^1{}_{c's} = -g^2 \mathfrak{a}^\mu_c \mathfrak{a}^{c'}_\mu f_2{}^{cs} f^2{}_{c's} = -g^2 \mathfrak{a}^\mu_c \mathfrak{a}^{c'}_\mu f_3^{cs} f^3{}_{c's}; \qquad m_3^2 = -g^2 \mathfrak{a}^\mu_c \mathfrak{a}^{c'}_\mu f_8{}^{cs} f^8{}_{c's}; \\ &m_2^2 = -g^2 \mathfrak{a}^\mu_c \mathfrak{a}^{c'}_\mu f_4{}^{cs} f^4{}_{c's} = -g^2 \mathfrak{a}^\mu_c \mathfrak{a}^{c'}_\mu f_5{}^{cs} f^5{}_{c's} = -g^2 \mathfrak{a}^\mu_c \mathfrak{a}^{c'}_\mu f_6{}^{cs} f^6{}_{c's} = -g^2 \mathfrak{a}^\mu_c \mathfrak{a}^{c'}_\mu f_7{}^{cs} f^7{}_{c's}; \\ &M_1 = M(N_c, N_f = 2); \qquad M_2 = M(N_c, N_f = 3), \qquad M_3 = M(N_c, N_f = 2). \end{aligned} \quad (37)$$

As for $\mu_{1,2}$, they are given by the corresponding nondiagonal elements in Eq. (34).

We note that matrix (36) has a block structure. Therefore, the left upper (3×3) block and the (4×4) block, which is next to it below, can be independently diagonalized by the (3×3) and (4×4) unitarian matrices (see Appendix C).

Upon carrying out such a transformation (see Appendix C), we go to such a new basis $\Phi_a(x)$ that the diagonalized Lagrangian of the meson octet takes the form

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \Phi^a(x))(\partial_\mu \Phi_a(x)) - \frac{1}{2}(m^2_{\rm oct})^a_b \Phi_a(x)\Phi^b(x), \quad (38)$$

where the Φ_a's are the components of the octet, which are listed in the following order: $\pi^+, \pi^-, \pi^0, K^+K^-, K^0, \tilde{K}^0, \eta^0$ (see also Appendix C), whereas the mass matrix $(m^2_{\rm oct})^a_b$ consists only of the diagonal elements, which are the squared masses of the octet particle [see also Eq. (C3) in Appendix C]:

$$(m^2_{\rm oct})^a_b = {\rm diag}(m_{\pi^+}; m_{\pi^-}; m_{\pi^0}; m_{K^+}; m_{K^-}; m_{K^0}; m_{\tilde{K}^0}; m_\eta). \tag{39}$$

Lagrangian (38) governs the octet of the massive scalar particles, while the elements of the mass matrix $(m^2_{\rm oct})^a_b$ appear to be generated by both the quark-gluon interaction and the self-interacting gauge fields.

In the tensor representation of the $SU(3)$ group, the meson octet has the form [41]

$$\hat{P} = \sum_0^8 T_a \Phi^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi^3 + \frac{\Phi^8}{\sqrt{3}} & \Phi^1 - i\Phi^2 & \Phi^4 - i\Phi^5 \\ \Phi^1 + i\Phi^2 & -\Phi^3 + \frac{\Phi^8}{\sqrt{3}} & \Phi^6 - i\Phi^7 \\ \Phi^4 + i\Phi^5 & \Phi^6 - i\Phi^7 & -\frac{2\Phi^8}{\sqrt{3}} \end{pmatrix} \equiv \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & K^0 \\ K^- & \tilde{K}^0 & -\frac{2\eta^0}{\sqrt{6}} \end{pmatrix}, \tag{40}$$

where the symbols $\pi^+, \pi^-, \pi^0, K^+K^-, K^0, \tilde{K}^0, \eta^0$ represent the corresponding fields.

VI. DISCUSSION

Since the matrix $(m^2_{\rm oct})^a_b$ is diagonal, Lagrangian (38) consists of three noninterferant terms, which allows us to study independently the mass matrix in the Lagrangian given by Eq. (33). Let us start a consideration from the term governing the lightest mesons, pions, since they are the particles which appear to be most often created in high energy processes such as p-p, p-A and A-A collisions. Comparing Eq. (36) with Eq. (C2) and using [42], we obtain, after a direct calculation,

$$\sqrt{2M_1^2 + m_1^2} = \sqrt{\frac{2m^2_{\pi^\pm} + m^2_{\pi^0}}{3}} \simeq 1.37 \times 10^2 \text{ MeV}; \qquad \mu_1 = \sqrt{\frac{m^2_{\pi^\pm} - m^2_{\pi^0}}{3}} \simeq 21.4 \text{ MeV}. \tag{41}$$

The last formula shows that the nondiagonal elements in matrix (36), which are in the upper left block, are sufficiently less than the diagonal ones.

Although the equations in formula (41) do not allow us to derive M_1 and m_1 independently, they enable us to estimate the value of the coupling constant g. To do it, we need to make some additional assumptions, and to concretize the conditions under which pions are produced. First, let

$$m_1 \simeq \mu_1. \tag{42}$$

Furthermore, we assume that the pions are only produced in the result of the hadronization of the confinement matter at the temperature T_c (see, for example [12,43]). We also suggest that the matter consists of the gluons whose dynamics is governed by Lagrangian (28). Moreover, we neglect the creation of the heavy mesons belonging to this octet.

Since the temperature T_c is of the order of some hundreds of MeV, the lightest quarks, which are only in a pion, can be considered ultrarelativistic particles. Then, in the absence of the quark condensate [44], such quarks are approximately distributed according to the Fermi-Dirac formula with the zeroth chemical potential:

$$n_f(P, \sigma, \lambda, c) \simeq \frac{1}{1 + \exp(\varepsilon(\vec{p})/T)}. \tag{43}$$

In the case of ultrarelativistic particles, the leading contribution to the integral in Eq. (20) gives a differentiation of P^ν and $\varepsilon(\vec{p})$. Then, taking into account the degeneracy with respect to the colors and flavors of the quarks, we derive from Eq. (20)

$$\frac{m^2_{\pi^0} - m_1^2}{2} = M_1^2 = 9g^2 \cdot \int \frac{d^3p}{p(2\pi)^3} \frac{1}{1 + \exp(p/T)} = \frac{9g^2T^2}{2\pi^2} \int_0^\infty \frac{xdx}{1 + \exp(x)} = \frac{3g^2T^2}{8}. \tag{44}$$

The last formula establishes the relationship between the mass of the observable particle, temperature, and coupling constant, which allows us to get information about the states of the confinement matter under the chosen assumption. Provided that $m_{\pi^0} = 135$ MeV [42], we obtain $gT_c \simeq 154$ MeV. Taking T_c in the interval from 200 to 300 MeV, we find that the coupling constant g varies from 0.51 to 0.76.

We note that the matrix element μ_1 is directly connected with the transverse (or magnetic) gluon mass $m_{\rm mag}$ because of the transverseness relation (30). On the other hand, this magnetic mass is [45]

$$m_{\rm mag} \simeq \frac{7.2 \cdot g^2 T}{4\pi}. \tag{45}$$

Owing to the block structure of matrix (36) and provided that the pion dominance is as assumed above, we can approximately set $M_1 \simeq \sqrt{3} m_{\rm mag}$. Then, using Eqs. (44) and (45), we obtain

$$g \simeq \frac{\pi\sqrt{2}}{7.2} \simeq 0.61. \tag{46}$$

The obtained result appears to be inside the range of variation of the coupling constant which was already derived in the previous estimation.

Furthermore, we analyze the kaon part of Lagrangian (38). Comparing Eq. (36) with Eq. (C2), we find that the nondiagonal elements μ_2 in matrix (36) are

$$\mu_2 = \sqrt{\frac{m^2_{\bar{K}^0} - m^2_{K^\pm}}{2}} \simeq 44.5 \text{ MeV} \ll m_{K^\pm}, \qquad (47)$$

which is found to be very similar to the situation that took place above, when the relationship between the diagonal and nondiagonal matrix elements in Eq. (36) were studied in the pion case.

We should also note that, since the masses $m_\pi, m_{K^\pm}, m_\eta$ are in the hierarchy $m_{\pi^\pm} < m_{K^\pm} < m_\eta$, whereas $M_1 = M_3$, the mass m_3 should be more than m_1 by three times, at least.

VII. CONCLUSION

The self-consistent dynamics of fermion and boson fields in the gauge $SU(N)$ model is considered beyond the perturbative approximation. The formal solution of the Dirac equation, which allows us to calculate the bilinear convolutions of fermion fields in the explicit form, is derived. On the basis of the obtained solution, the fermion and total currents in the $SU(N)$ gauge model in the $(3+1)$ Minkowski space-time are derived. The obtained currents are gauge invariant, and they satisfy the continuity equation. In this way, the transformation properties of the currents are entirely determined by the gauge field which appears to be massive, which is similar to the results obtained in Ref. [2] in the specific case of the two-dimensional space-time. We show that in the absence of the quark condensate [44], the derived mass depends on the number of colors, as $\sqrt{N_c}$, that coincides with the results obtained earlier [4–6,37,39].

By using the derived current, we obtain the Lagrangian governing the pure gluodynamics. The derived Yang-Mills Lagrangian also contains the mass term generated by the interaction of fermions with gauge fields.

In the QCD case, by breaking the initial $SU(3)$ symmetry in the gluodynamics Lagrangian, we derive the Lagrangian governing the octet of the pseudoscalar mesons. The contribution of the quark-gluon interaction and the self-interacting gluon field to the meson masses is derived. Provided that only the hadronization of the confinement gluon matter into the pion triplet occurs, we evaluate the QCD coupling constant, which appears to be in the interval from 0.51 to 0.76.

We note in conclusion that the proposed violation of the $SU(3)$ symmetry, which is governed by Eqs. (29)–(32), should be considered as a physical ansatz. Although such symmetry breaking allows us to obtain the reasonable results concerning the hadronization into the meson octet, the developed model of the hadronization demands additional verification. In particular, it concerns generalization of the obtained result to the hadronization of the confinement matter into the mesons which are beyond the considered octet.

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APPENDIX A MASS CALCULATION

We substitute the wave function given by Eq. (9) into formula (11). Taking into account Eqs. (13) and (14), we derive

$$J^\mu_a(x) = \frac{ig^2}{2}\int \frac{d^3\vec{p}}{(2\pi)^3}\int \frac{d^3\vec{p}\,'}{(2\pi)^3} \sum_{\sigma,\sigma',\lambda\lambda',c,c'f,f'} [\bar{u}_\sigma(P)\gamma^\mu u_\sigma(P')]\langle a^\dagger_f(P,\sigma,\lambda,c)a_{f'}(P',\sigma',\lambda',c')\rangle$$
$$\times \frac{\exp(+iP_\mu x^\mu - iP'_\mu x'^\mu)}{\sqrt{2\varepsilon(\vec{p})2\varepsilon(\vec{p}\,')}}(\theta(P^0)+\theta(-P^0))\cdot(x'-x)_\beta\left(A^\beta_a(\xi) + \frac{1}{2}(x'-x)_\nu\partial^\beta A^\nu_a(\xi)\right), \qquad x'\to x. \qquad (A1)$$

The correlator $\langle a^\dagger_f(P,\sigma,\lambda,c)a_{f'}(P',\sigma',\lambda',c')\rangle$ can be expressed in terms of the occupancy numbers of fermions $n_f(P,\sigma,\lambda,c)$:

$$\langle a^\dagger_f(P,\sigma,\lambda,c)a_{f'}(P',\sigma',\lambda',c')\rangle = (2\pi)^3 n_f(P,\sigma,\lambda,c)\delta(\vec{p}-\vec{p}\,')\delta_{\sigma\sigma'}\delta_{\lambda\lambda'}\delta_{cc'}\delta_{ff'}. \qquad (A2)$$

To proceed further, we substitute (A2) into (A1) and introduce the additional integration with respect to P^0. As a result, we obtain

$$J_a^\mu(x) = \frac{ig^2}{2}\int \frac{d^4P}{(2\pi)^3}\sum_{\sigma,\lambda,c,f} P^\mu n_f(P,\sigma,\lambda,c)[\delta(P^0-\varepsilon(\vec{p}))+\delta(P^0+\varepsilon(\vec{p}))]\frac{\exp(iP_\mu(x^\mu-x'^\mu))}{\varepsilon(\vec{p})}$$
$$\times (x'-x)_\beta\left(A_a^\beta(\xi)+\frac{1}{2}(x'-x)_\nu\partial^\beta A_a^\nu(\xi)\right)$$
$$= \frac{g^2}{2}\int \frac{d^4P}{(2\pi)^3}\sum_{\sigma,\lambda,c,f} P^\mu n_f(P,\sigma,\lambda,c)\frac{[\delta(P^0-\varepsilon(\vec{p}))+\delta(P^0+\varepsilon(\vec{p}))]}{\varepsilon(\vec{p})}\left(-\frac{\partial}{\partial P^\beta}\right)\exp(iP_\mu(x^\mu-x'^\mu))$$
$$\times\left(A_a^\beta(\xi)+\frac{1}{2}(x'-x)_\nu\partial^\beta A_a^\nu(\xi)\right),\qquad x'\to x. \tag{A3}$$

Taking into account Eq. (18), we obtain, after integration by parts,

$$J_a^\mu(x) = \frac{g^2}{2}\int \frac{d^4P}{(2\pi)^3}\sum_{\sigma,\lambda,c,f}\left(\frac{\partial}{\partial P^\beta}\right)\left(P^\mu n_f(P,\sigma,\lambda,c)\frac{[\delta(P^0-\varepsilon(\vec{p}))+\delta(P^0+\varepsilon(\vec{p}))]}{\varepsilon(\vec{p})}\right)\left(A_a^\beta(\xi)+\frac{1}{2}(x'-x)_\nu\partial^\beta A_a^\nu(\xi)\right). \tag{A4}$$

Since the integrand in Eq. (A4) is the tensor in the Minkowski space-time, it has to be proportional to the metric tensor $g^{\mu\beta}$:

$$\int \frac{d^4P}{(2\pi)^3}\sum_{\sigma,\lambda,c,f}\left(\frac{\partial}{\partial P^\beta}\right)\left(P^\mu n_f(P,\sigma,\lambda,c)\frac{[\delta(P^0-\varepsilon(\vec{p}))+\delta(P^0+\varepsilon(\vec{p}))]}{\varepsilon(\vec{p})}\right) = Ag_\beta^\mu, \tag{A5}$$

where A is a constant. Calculating the convolutions with $g^{\mu\beta}$ in the left- and right-hand sides in Eq. (A5), we get

$$A = \frac{1}{4}\int \frac{d^4P}{(2\pi)^3}\sum_{\sigma,\lambda,c,f}\left(\frac{\partial}{\partial P^\nu}\right)\left(P^\nu n_f(P,\sigma,\lambda,c)\frac{[\delta(P^0-\varepsilon(\vec{p}))+\delta(P^0+\varepsilon(\vec{p}))]}{\varepsilon(\vec{p})}\right). \tag{A6}$$

The last formula allows us to rewrite the mass factor M^2 in Eq. (19) in the form given by Eq. (20).

APPENDIX B TRANSFORMATION OF A KINEMATIC TERM OF THE LAGRANGIAN

Let us transform the kinematic part of Lagrangian (24) which corresponds to the fermion fields. The direct calculations give

$$\mathcal{L}_k = \sum_f[\bar{\Psi}_f(x)(i\gamma^\mu\partial_\mu)\Psi_f(x)-\bar{\Psi}_f(x)m_f\Psi_f(x)] = \lim_{x'\to x}\sum_f[\bar{\Psi}_f(x')(i\gamma^\mu\partial_{\mu(x)})\Psi_f(x)-\bar{\Psi}_f(x')m\Psi_f(x)]$$
$$= \lim_{x'\to x}\int 2\frac{d^4P}{(2\pi)^3}\sum_{\sigma\lambda,f,c}[n_f(P,\sigma,\lambda,c)\delta(P^\mu P_\mu-m^2)]((iP^\mu\partial_{\mu(x)})-m_f^2)\exp(-iP_\mu(x^\mu-x'^\mu))\left\{igT_a\int_{x'}^x dx^\mu A_\mu^a\right\}$$
$$= \lim_{x'\to x}\int 2\frac{d^4P}{(2\pi)^3}\sum_{\sigma\lambda,f,c}[n_f(P,\sigma,\lambda,c)\delta(P^\mu P_\mu-m^2)]\exp(-iP_\mu(x^\mu-x'^\mu))(iP^\mu\partial_{\mu(x)})\left\{igT_a\int_{x'}^x dx^\mu A_\mu^a\right\}$$
$$= \lim_{x'\to x}\int 2\frac{d^4P}{(2\pi)^3}\sum_{\sigma\lambda,f,c}[n_f(P,\sigma,\lambda,c)\delta(P^\mu P_\mu-m^2)](-\partial^{\mu(x)}\exp(-iP_\mu(x^\mu-x'^\mu)))(\partial_{\mu(x)})\left\{igT_a\int_{x'}^x dx^\mu A_\mu^a\right\}$$
$$= (-\partial^{\mu(x)})\left\{\lim_{x'\to x}\int 2\frac{d^4P}{(2\pi)^3}\sum_{\sigma\lambda,f,c}[n_f(P,\sigma,\lambda,c)\delta(P^\mu P_\mu-m^2)]\exp(-iP_\mu(x^\mu-x'^\mu)(\partial_{\mu(x)})\left\{igT_a\int_{x'}^x dx^\mu A_\mu^a\right\}\right\}$$
$$+\lim_{x'\to x}\int 2\frac{d^4P}{(2\pi)^3}\sum_{\sigma\lambda,f,c}[n_f(P,\sigma,\lambda,c)\delta(P^\mu P_\mu-m^2)]\exp(-iP_\mu(x^\mu-x'^\mu)(\partial^{\mu(x)}\partial_{\mu(x)})\left\{igT_a\int_{x'}^x dx^\mu A_\mu^a\right\}. \tag{B1}$$

Expanding the exponent in the last line of Eq. (B1), as has already been done in Eq. (13), we derive that this term is equal to zero. Then, the kinematic part of Lagrangian (24), which corresponds to the fermion subsystem, is equal to

$$\mathcal{L}_k = (-\partial^{\mu(x)})\left\{\lim_{x'\to x}\int 2\frac{d^4P}{(2\pi)^3}\sum_{\sigma\lambda,f,c}[n_f(P,\sigma,\lambda,c)\delta(P^\mu P_\mu - m^2)]\exp(-iP_\mu(x^\mu - x'^\mu)(\partial_{\mu(x)})\left\{igT_a\int_{x'}^{x} dx^\mu A_\mu^a\right\}\right\}. \tag{B2}$$

APPENDIX C DIAGONALIZATION OF THE MASS MATRIX

The direct calculations show that matrix (36) can be transformed into the diagonal form by means of the independent diagonalization of the left upper (3×3) block and the (4×4) block, which is next to it below. These blocks are diagonalized by the matrices

$$\hat{T}_{3\times 3} = \frac{1}{6}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \hat{T}_{4\times 4} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \tag{C1}$$

In the result of such a diagonalization, the mass matrix $\mathcal{M}^2$ given by Eq. (36) takes the form

$$2\cdot\begin{pmatrix} M_1^2+\frac{m_1^2+\mu_1^2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_1^2+\frac{m_1^2+\mu_1^2}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_1^2+\frac{m_1^2-2\mu_1^2}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_2^2+\frac{m_2^2-\mu_2^2}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_2^2+\frac{m_2^2-\mu_2^2}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_2^2+\frac{m_2^2+\mu_2^2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_2^2+\frac{m_2^2+\mu_2^2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_3^2+\frac{m_3^2}{2} \end{pmatrix}. \tag{C2}$$

Introducing the standard notations for the diagonal matrix elements, we have

$$(\mathcal{M}^2) \equiv (m_{\mathrm{oct}}^2)_b^a = \begin{pmatrix} m_{\pi^+} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{\pi^-} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{\pi^0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{K^+} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{K^-} & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & m_{K^0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_{\tilde{K}^0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_\eta \end{pmatrix}, \tag{C3}$$

where the corresponding diagonal matrix elements are the same in both matrices given by Eqs. (C2) and (C3).

The direct calculations show that matrix (C3) appears to be unchangeable if the bases $\varphi_a(x)$ and $\Phi_a(x)$ are related to one another by means of the unitarian transformations V and U, which are dictated by the following formula:

$$V\varphi_a(x) \equiv \begin{pmatrix} \hat{T}_{3\times3} & 0 & 0 \\ 0 & \hat{T}_{4\times4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi_{\pi^+}(x) \\ \varphi_{\pi^-}(x) \\ \varphi_{\pi^0}(x) \\ \varphi_{K^+}(x) \\ \varphi_{K^-}(x) \\ \varphi_{K^0}(x) \\ \varphi_{\tilde{K}^0}(x) \\ \varphi_{\eta}(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \pi^+(x) \\ \pi^-(x) \\ \pi^0(x) \\ K^+(x) \\ K^-(x) \\ K^0(x) \\ \tilde{K}^0(x) \\ \eta(x) \end{pmatrix} \equiv U\Phi_a(x), \qquad \text{(C4)}$$

where the matrices $\hat{T}_{3\times3}$ and $\hat{T}_{4\times4}$ are given by Eq. (C1).

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