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Dimensional flow in the kappa-deformed spacetime

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We derive the modified diffusion equations defined on kappa spacetime and, using these, investigate the change in the spectral dimension of kappa spacetime with the probe scale. These deformed diffusion equations are derived by applying Wick's rotation to the κ-deformed Schrödinger equations obtained from different choices of Klein-Gordon equations in the κ-deformed spacetime. Using the solutions of these equations, obtained by perturbative method, we calculate the spectral dimension for different choices of the generalized Laplacian and analyze the dimensional flow in the κ spacetime. In the limit of commutative spacetime, we recover the well-known equality of spectral dimension and topological dimension. We show that the higher-derivative term in the deformed diffusion equations makes the spectral dimension unbounded (from below) at high energies. We show that the finite mass of the probe results in the spectral dimension becoming infinitely negative at low energies also. In all these cases, we have analyzed the effect of finite size of the probe on the spectral dimension.

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I. INTRODUCTION

Combining the principles of quantum mechanics and general relativity is known to result in spacetime uncertainties [1]. This leads to fuzziness of the spacetime at extremely short distances. This fuzziness can change the effective dimension of the spacetime at high energies [2–4]. Various approaches like string theory, loop gravity, and causal dynamical triangulation have been developed to unravel the nature of spacetime at extremely short distances. All these approaches have a common trait—they predict dimensional reduction [5–9]. Construction and analysis of a diffusion equation compatible with these approaches is a possible way to study the dimensional flow. Spectral dimension turns out to be an important tool for investigating the nature of spacetime at microscopic scales [5].

Noncommutative geometry is a possible way to capture the spacetime uncertainties and, thus, study the spacetime structure at Planck scale. κ spacetime is an example of a Lie algebraic–type noncommutative spacetime whose coordinates satisfy

$$[\hat{x}_0, \hat{x}_i] = ia\hat{x}_i, \qquad [\hat{x}_i, \hat{x}_j] = 0. \tag{1}$$

The significance of this spacetime to quantum gravity comes from the fact that it appears naturally in the low-energy limit of loop gravity [10] as well as in the context of doubly special relativity theories [11]. Analysis of diffusion on this space shows that the effective dimension is different from the topological dimension [12–16].

In this paper, we investigate the dimensional flow in the κ-deformed spacetime using the solution of the deformed diffusion equations. These κ-diffusion equations are constructed by a Wick's rotation of κ-deformed Schrödinger equations obtained as the nonrelativistic limit of the well-studied κ-deformed Klein-Gordon equations [17–21].

It is well known that the Schrödinger equation and diffusion equation are related by a Wick's rotation [22]. The time-dependent Schrödinger equation for a free particle is

$$i\hbar\frac{\partial}{\partial t}\phi(x,t) = -\frac{\hbar^2}{2\mu}\nabla^2\phi(x,t), \tag{2}$$

where μ is the particle's reduced mass, ∇^2 is the Laplacian, and ϕ is the wave function, and is mapped to

$$\frac{\partial}{\partial t}\phi = k\nabla^2\phi, \tag{3}$$

under the map $t \to -it$. By redefining $kt = \sigma$, one reexpresses the above equation as the standard diffusion equation.

The analysis of the effective dimension of the spacetime and its dependence on the probe scale is studied using a diffusion process [5]. In this approach, one investigate the behavior of a nonrelativistic particle undergoing diffusion in the space whose dimension is under study. This equation is solved by imposing the delta function initial condition which takes into account the point particle nature of the probe.

The motion of the nonrelativistic particle in a diffusion process in d-dimensional spacetime is governed by the diffusion equation

$$\frac{\partial}{\partial\sigma}U(x,y;\sigma) = \mathcal{L}U(x,y;\sigma), \tag{4}$$

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where σ is the diffusion time, $\mathcal{L}$ is the generalized Laplacian in the given space (of $d-1$ dimensions) and its solution $U(x, y; \sigma)$ is the probability density of diffusion from x to y during the diffusion time σ. Using the solution of the diffusion equation, one finds the return probability as

$$P(\sigma) = \frac{\int d^n x \sqrt{\det g_{\mu\nu}} U(x, x; \sigma)}{\int d^n x \sqrt{\det g_{\mu\nu}}}. \tag{5}$$

The logarithmic derivative of return probability $P_g(\sigma)$ gives us the spectral dimension of the corresponding $d-1$ dimensional space, i.e.,

$$D_s = -2 \frac{\partial \ln P(\sigma)}{\partial \ln \sigma}. \tag{6}$$

In the approach of studying the dimensional flow of spacetime [5,12–16], one uses the nonrelativistic diffusion equation given in Eq. (4) but replaces the d-1-dimensional Laplacian $\mathcal{L}$ with the Euclideanized Beltrami-Laplace operator defined on the concerned spacetime. One also interprets σ as the fictitious diffusion time. The quantum gravity effects do modify the Beltrami-Laplace operator, which is typically the kinetic part of the deformed field theory, defined on the spacetime under study. An equivalent approach, using the momentum space representation of the kinetic part of the deformed field theory (which is essentially the deformed energy-momentum relation) has also been used to study the spectral dimension of various models [23].

There have been attempts to study possible generalizations of the diffusion process described by Eq. (4), which includes changes in the Beltrami-Laplace operator, modification in the initial conditions as well as the modification of diffusion operator $\frac{\partial}{\partial \sigma}$ [24], in order to capture possible quantum gravity effects. Modification of diffusion equation to address the nontrivial scaling behavior of spacetime was analyzed and it was also shown that the diffusion equation do get modified by introducing diffusion in nonlinear time as well as by incorporating a nontrivial source term [24]. Natural generalization of diffusion equation involving fractional derivatives (in spatial coordinate as well as in diffusion time) was also introduced and discussed [24].

The spectral dimension of κ-Minkowski was studied in [13] and the fractal nature of spacetime with quantum group symmetry was exhibited for the case of Wick's rotated κ-Minkowski space. The Casimir of the κ-Poincare algebra was used to calculate the trace of the heat kernel in Wick's rotated κ-Minkowski space. The numerical evaluation of the resulting expression showed that the spectral dimension change from 4 to 3 with the probe scale. A study on similar lines is reported in [14]. Here, the spectral dimension was studied using three possible forms of the κ-deformed Laplacians in the momentum space. The Laplacians conceived from the Casimir of the bicovariant differential calculus displayed the dimensional reduction as the time change from 4 at low energies to 3 at high energies. For the Laplacian associated with the bi-crossproduct Casimir, spectral dimension varies from 4 to 6 with energy. A model compatible with the notion of relative locality [25–28] gives spectral dimension that goes to infinity as one move to UV regime [14].

The spectral dimension of κ spacetime using the κ-deformed diffusion equation was studied in our earlier work [16]. Using a mapping of noncommutative coordinates to commutative coordinates and their derivatives, we constructed the diffusion equation in κ spacetime from the Casimir of the undeformed κ-Poincare algebra [17]. Keeping terms up to second order in the deformation parameter a, we solved the diffusion equation perturbatively. The spectral dimension derived from this solution showed a length scale dependence. For a four-dimensional spacetime, we found that the spectral dimension decrease and become negative as we probe at higher energies.

In this paper, we construct possible modifications to the heat equation given in Eq. (4) due to κ deformation and study its implication on the scale dependence of the spacetime dimension. For this, we exploit the mapping between the Schrödinger equation and heat equation discussed above. Thus, we start from the well-studied κ-deformed Klein-Gordon equations written using the Beltrami-Laplace operator in commutative spacetime and derive its nonrelativistic limit. From the κ-deformed Schrödinger equation thus obtained, we construct the deformed heat equation by a Wick's rotation (by implementing the map $t \to -it$). Note that the Wick's rotation is applied to the theory written in the commutative spacetime and all the effects of noncommutativity are included through the deformation parameter a-dependent terms.[1] Note that the deformation parameter a is unaffected by the Wick's rotation. This allows us to investigate two related issues: (i) the spectral dimension of κ-deformed space and (ii) its scaling with the energy dimensional flow of the full κ-deformed spacetime. The first problem is studied by evaluating the spectral dimension of κ space using the κ-deformed heat equation derived from the κ-deformed Schrödinger equation. Here we use the $(d-1)$ dimensional Laplacian constructed as the space derivative part of the nonrelativistic limit of the κ-deformed Klein-Gordon equation, for $\mathcal{L}$ in Eq. (4). For investigating the second problem, we take the κ-deformed heat equation obtained by the Wick's rotation of the κ-deformed Schrödinger equation and replace the Laplacian with the Euclideaniced Beltrami-Laplace operator defined in

[1] The Wick's rotation in noncommutative spacetime is a nontrivial issue and has been analyzed in detail, particularly for the case of moyal spacetime in [29–31]. It was shown in [29] that the naive Wick's rotation will lead to the theory being nonunitary, and a consistent way to map the noncommutative theory from the Euclidean to the Minkowski signature was obtained in [29–32].

the d-dimensional κ-deformed spacetime, for $\mathcal{L}$ in Eq. (4). We have carried out this study by different choices of κ-deformed Klein-Gordon equations.

In the first case, thus, we study the spectral dimension of the spatial part of the κ-deformed spacetime. Since the spatial coordinates of the κ spacetime commute among themselves and it is the time coordinate which do not commute with the space-coordinates, this approach is appropriate to study how the space dimension of the κ spacetime changes as the probe scale is changed due to the noncommutativity between time and space coordinates. We see that the effect of noncommutativity is to introduce higher spatial derivative terms as well as terms involving both spatial and temporal derivatives in the deformed diffusion equation. The role of these terms on the spectral dimension is brought out here.

In the second case, we take the κ-deformed heat equation as the starting point of the analysis and replace the Laplacian $\mathcal{L}$ in Eq. (4) by the Euclideanized Beltrami-Laplace operator. Thus, here the noncommutativity shows itself in two ways—by introducing the higher-derivative terms in the deformed heat equation and also through the additional terms appearing in the deformed Baltrami-Laplace operator. Here also we do the analysis for different choices of the Baltrami-Laplace operator.

Organization of this paper is as follows. In the second section, we set up the diffusion equation using the deformed Klein-Gordon equation. We start with the Klein-Gordon equation in κ-Minkowski spacetime written in terms of commuting coordinates and all the effects of noncommutativity are contained in the a- (deformation parameter) dependent terms. By taking the nonrelativistic limit of this theory written in terms of the commutative variables and applying Wick's rotation, we derive the diffusion equation in the κ-deformed Euclidean space, valid up to first nonvanishing terms in the deformation parameter a. We then solve this diffusion equation perturbatively and use this solution to calculate the spectral dimension. We have also analyzed the change in the spectral dimension due to extended nature of the probe. In the next subsection, we start with a different choice of generalized κ-deformed Klein-Gordon equation and arrive at the κ-deformed diffusion equation. The spectral dimension is calculated using its solution and dimensional flow is analyzed. In Sec. III, we replace the Laplacian in the modified diffusion equation with the two different choices of Beltrami-Laplace operator and use this diffusion equation to calculate the spectral dimension. The analysis of the results and summary is presented in the last section.

II. κ-DEFORMED DIFFUSION EQUATION AND SPECTRAL DIMENSION

In this section, we derive the κ-deformed diffusion equations starting from two possible choices of κ-deformed Klein-Gordon equations. The diffusion equation is related to the Schrödinger equation under the mapping $t \to -it$ and we use this map to derive the deformed diffusion equation. By replacing t with $-it$ in the κ-deformed Schrödinger equations, derived by taking the nonrelativistic limit of the κ-deformed Klein-Gordon equation, we obtain the κ-deformed diffusion equations. Using perturbative method, we obtain its solution valid up to second order in the deformation parameter. From this solution, we calculate the return probability which is a measure of finding a particle back at the starting point after a finite time gap. Using this, we calculate the spectral dimension.

A. Diffusion equation from the κ-deformed Klein-Gordon equation $(D_\mu D^\mu - m^2)\phi = 0$

Here we derive the deformed diffusion equation from the nonrelativistic limit of κ-deformed Klein-Gordon equation $(D_\mu D^\mu - m^2)\phi = 0$. Consider an n-dimensional κ-deformed Minkowski space with signature $(-++\cdots+)$. The generalized Klein-Gordon equation on κ-deformed spacetime [20] is

$$\Box\left(1 + \frac{a^2}{4}\Box\right)\phi = \frac{m^2c^2}{\hbar^2}\phi, \tag{7}$$

where

$$\Box = \nabla_{n-1}^2 \frac{e^{-A}}{\varphi^2} + \partial_0^2 \frac{2(1-\cosh A)}{A^2}. \tag{8}$$

Here $\nabla_{n-1}^2 = \Sigma_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}$, $A = -ia\partial_0$, and we choose $\varphi = e^{-\frac{A}{2}}$. We expand Eq. (7) in terms of the deformation parameter and obtain the equation valid up to second order in a as

$$\left(\nabla_{n-1}^2 - \partial_0^2 + \frac{a^2}{4}\nabla_{n-1}^4 - \frac{a^2}{2}\nabla_{n-1}^2\partial_0^2 + \frac{a^2}{3}\partial_0^4\right)\phi = \frac{m^2c^2}{\hbar^2}\phi. \tag{9}$$

We next construct κ-deformed Schrödinger equation by taking the nonrelativistic limit of the κ-deformed Klein-Gordon equation. Note that Eq. (9) is written completely in the commutative spacetime. This allows us to use the well-known calculation scheme to obtain the nonrelativistic limit [33]. Thus, we start with the ansatz wave function ϕ where one separates out the rest mass dependence, and we further use the fact that in the nonrelativistic limit, kinetic energy is very small compared to rest mass energy. So we start with the ansatz

$$\phi(x,t) = \varphi(x,t)e^{-i\frac{mc^2}{\hbar}t} \tag{10}$$

in Eq. (9). Here x is a point in the $(n-1)$ space. Effectively, this ansatz allows us to extract a term containing the rest mass m. In the nonrelativistic limit, the kinetic energy (KE)

is small compared to rest mass energy, i.e., $KE \ll mc^2$ and hence we have

$$\left| i\hbar \frac{\partial \varphi}{\partial t} \right| \ll mc^2 \varphi. \tag{11}$$

Substituting Eq. (10) in Eq. (9) and after using the fact that KE is much smaller than the rest mass energy [stated in Eq. (11)], we get the κ-deformed Schrödinger equation as

$$\nabla_{n-1}^2 \varphi + i\frac{2m}{\hbar}\frac{\partial \varphi}{\partial t} + \frac{a^2}{4}\nabla_{n-1}^4 \varphi + ia^2 \frac{m}{\hbar}\frac{\partial}{\partial t}\nabla_{n-1}^2 \varphi + \frac{a^2}{2}\frac{m^2c^2}{\hbar^2}\nabla_{n-1}^2 \varphi + ia^2\frac{4m^3c^2}{3\hbar^3}\frac{\partial \varphi}{\partial t} + a^2 \frac{m^4c^4}{3\hbar^4}\varphi = 0. \tag{12}$$

By changing t to $-it$ in the above, we get the deformed diffusion equation as

$$\nabla_{n-1}^2 \varphi - \frac{2m}{\hbar}\frac{\partial \varphi}{\partial t} + \frac{a^2}{4}\nabla_{n-1}^4 \varphi - a^2 \frac{m}{\hbar}\frac{\partial}{\partial t}\nabla_{n-1}^2 \varphi + \frac{a^2}{2}\frac{m^2c^2}{\hbar^2}\nabla_{n-1}^2 \varphi - a^2\frac{4m^3c^2}{3\hbar^3}\frac{\partial \varphi}{\partial t} + a^2 \frac{m^4c^4}{3\hbar^4}\varphi = 0. \tag{13}$$

Redefining $kt = \sigma$ with $k = \frac{\hbar}{2m}$ and after some rearrangements, we obtain κ-deformed diffusion equation as

$$\frac{\partial \varphi}{\partial \sigma} = \nabla_{n-1}^2 \varphi + \frac{a^2c^2}{8k^2}\nabla_{n-1}^2 \varphi + \frac{a^2}{4}\nabla_{n-1}^4 \varphi - \frac{a^2}{2}\frac{\partial}{\partial \sigma}\nabla_{n-1}^2 \varphi - \frac{a^2c^2}{6k^2}\frac{\partial \varphi}{\partial \sigma} + \frac{a^2c^4}{48k^2}\varphi, \tag{14}$$

where $\nabla_{n-1}^2 = \Sigma_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}$. In the above equation φ is a function of x and σ. It is clear from Eq. (14) that, in the commutative limit ($a \to 0$), we obtain the usual diffusion equation. Note that in deriving the κ-deformed diffusion equation from the κ-deformed Schrödinger equation, we only replace t with $-it$ and absorb the $\frac{\hbar}{2m}$ factor into the diffusion scale σ. The κ-deformation parameter a does not get any modification under this mapping.

Note that the deformed diffusion equation has the higher-order spatial derivative (∇_{n-1}^4) and terms involving products of temporal and spatial derivatives, i.e., $\frac{\partial}{\partial \sigma}\nabla_{n-1}^2$. But there are no higher-derivative terms with respect to (scaled) time (σ). These features would turn out to be significant in the calculation of spectral dimension of κ-deformed spacetime.

B. Spectral dimension

To find the heat kernel $\varphi(x, y; \sigma)$ of the κ-deformed diffusion equation obtained in Eq. (14), we express the solution as a perturbative series in a as

$$\varphi = \varphi_0 + a\varphi_1 + a^2\varphi_2. \tag{15}$$

We note that the dimension of the terms satisfy the relations, $[\varphi_1] = \frac{1}{L}[\varphi_0]$ and $[\varphi_2] = \frac{1}{L^2}[\varphi_0]$.

Using Eq. (15) in Eq. (14) and equating the terms of same order in a, we solve the above equation. The zeroth-order terms in a lead to

$$\frac{\partial}{\partial \sigma}\varphi_0(x, y; \sigma) = \nabla_{n-1}^2 \varphi_0(x, y; \sigma). \tag{16}$$

The Laplacian ∇_{n-1}^2 is with respect to x coordinates and will act on the x dependence of the heat kernel. The solution to this equation is given by

$$\varphi_0(x, y; \sigma) = \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i - y_i)^2}{4\sigma}}. \tag{17}$$

Next, equating the first-order terms in a gives us the equation

$$\frac{\partial}{\partial \sigma}\varphi_1(x, y; \sigma) = \nabla_{n-1}^2 \varphi_1(x, y; \sigma). \tag{18}$$

Note that here too, ∇_{n-1}^2 is the Laplacian with respect to x coordinates (and this notation is used in the remaining part of this paper) and this will act on the first argument of φ_1, namely x.

The solution $\varphi_1(x, y; \sigma)$ satisfying the above equation also have the same form as $\varphi_0(x, y; \sigma)$ since both satisfy the same heat equation. Thus, we get

$$\varphi_1(x, y; \sigma) = \frac{\alpha}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i - y_i)^2}{4\sigma}}, \tag{19}$$

where the constant α has dimension of L^{-1}. Now by equating the second-order terms in a in Eq. (14), we find

$$\frac{\partial \varphi_2}{\partial \sigma} = \nabla_{n-1}^2 \varphi_2 + \frac{c^2}{8k^2}\nabla_{n-1}^2 \varphi_0 + \frac{1}{4}\nabla_{n-1}^4 \varphi_0 - \frac{1}{2}\frac{\partial}{\partial \sigma}\nabla_{n-1}^2 \varphi_0 - \frac{c^2}{6k^2}\frac{\partial \varphi_0}{\partial \sigma} + \frac{c^4}{48k^2}\varphi_0. \tag{20}$$

Substituting the solution for φ_0 from Eq. (17) in the above equation and after straightforward manipulations, we get

$$\frac{\partial \varphi_2}{\partial \sigma} = \nabla_{n-1}^2 \varphi_2 + \left[\frac{c^4}{48k^4} + \frac{c^2}{48k^2}\frac{(n-1)}{\sigma} - \frac{(n^2-1)}{16\sigma^2} - \frac{c^2}{96k^2}\frac{\Sigma_{i=1}^{n-1}(x_i - y_i)^2}{\sigma^2} + \frac{(n+1)}{16\sigma^3}\Sigma_{i=1}^{n-1}(x_i - y_i)^2 - \frac{1}{64\sigma^4}(\Sigma_{i=1}^{n-1}(x_i - y_i)^2)^2\right]\frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i - y_i)^2}{4\sigma}}. \tag{21}$$

The above equation is of the generic form:

$$\frac{\partial}{\partial\sigma}\varphi_2(X,\sigma) = \nabla^2_{n-1}\varphi_2(X,\sigma) + f(X,\sigma). \tag{22}$$

For a given initial condition, $\varphi_2(X,0) = g(X)$, the solution to above equation can be written as [34]

$$\varphi_2(X,\sigma) = \int_{R^{n-1}} \Phi(X-X',\sigma)g(X')dX' + \int_0^\sigma \int_{R^{n-1}} \Phi(X-X',\sigma-s)f(X',s)dX'ds, \tag{23}$$

where

$$\Phi(X,\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{(X_1^2+X_2^2+\cdots+X_{n-1}^2)}{4\sigma}}. \tag{24}$$

Using the initial condition $\varphi_2(X,0) = \delta^{n-1}(X)$, we obtain the first term φ_{21} in Eq. (23) as

$$\varphi_{21}(x,y;\sigma) = \frac{\beta}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}, \tag{25}$$

where β has dimension L^{-2}. The second term on the rhs of Eq. (23), φ_{22}, is calculated as

$$\begin{aligned}\varphi_{22}(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}\Bigg[&\left(\frac{c^4}{48k^4} - \frac{n^2-1}{16\sigma^2} + \frac{c^2}{48k^2}\frac{(n-1)}{\sigma}\right)(\sigma-\epsilon)\\ &-\left(\frac{c^2}{96k^2\sigma^2} + \frac{1}{64\sigma^4}\Sigma_{i=1}^{n-1}(x_i-y_i)^2 - \frac{(n+1)}{16\sigma^3}\right)\Sigma_{i=1}^{n-1}(x_i-y_i)^2(\sigma-\epsilon)\\ &-\left(\frac{c^2}{24k^2}\frac{1}{\sigma\sqrt{\sigma\pi}} + \frac{1}{8\sigma^3\sqrt{\sigma\pi}}\Sigma_{i=1}^{n-1}(x_i-y_i)^2\right)\Sigma_{i=1}^{n-1}(x_i-y_i)(\sigma\tan^{-1}q - \epsilon q)\\ &-\frac{1}{2\sigma^3\pi}\left[(x_1-y_1)\Sigma_{i=2}^{n-1}(x_i-y_i) + (x_2-y_2)\Sigma_{i=3}^{n-1}(x_i-y_i) + \cdots + (x_{n-2}-y_{n-2})(x_{n-1}-y_{n-1})\right]A\\ &+\frac{1}{4\sigma^2\sqrt{\sigma\pi}}\Sigma_{i=1}^{n-1}(x_i-y_i)((5n+2)\sigma\tan^{-1}q - (4n+2)\sigma q - nq\epsilon)\Bigg],\end{aligned} \tag{26}$$

where $q = \sqrt{\frac{\sigma}{\epsilon} - 1}$ and $A = \sigma\ln(\sigma/\epsilon) - \sigma + \epsilon)$. Using Eqs. (17), (19), (25), and (26) in Eq. (15), we find the heat kernel valid up to second order in a. Using the definition [Eq. (5)] of return probability, we obtain $P(\sigma)$ (in the limit $\epsilon \to 0$) as

$$P(\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}}\left[1 + a\alpha + a^2\beta + a^2\frac{c^4}{48k^4}\sigma - a^2\frac{(n^2-1)}{16\sigma} + a^2\frac{c^2}{48k^2}(n-1)\right]. \tag{27}$$

The spectral dimension is found by taking the logarithmic derivative of the above return probability. Thus, we get

$$D_s = (n-1) - \frac{a^2}{8}\frac{(n^2-1)}{\sigma} - \frac{a^2c^4}{24k^4}\sigma. \tag{28}$$

From the above expression, we see that apart from the usual $(n-1)$ term, we have two additional terms and both of them are of second order in a. They arise due to the noncommutative nature of κ spacetime. One term is dependent on the topological dimension n we started with and the other term is independent of the initial dimension. Note that the diffusion scale σ appears in the n- dependent correction as inverse, whereas in the second correction term, it appears linearly. In the commutative limit, we see that the spectral dimension is the same as the topological dimension, i.e., $D_s = n-1$.

For $n = 4$ with $a = c = k = 1$, we obtain $D_s = 3 - \frac{\sigma}{24} - \frac{15}{8\sigma}$. From this, we see that in the limit $\sigma \to 0$, the spectral dimension $D_s \to -\infty$. As σ increases the spectral dimension also increases and reaches a maximum value $D_s \sim 2.44$ for $\sigma \sim 6.7$. As we go further, the spectral dimension starts decreasing [see Fig. 1].

In general, for $n = 4$ and with $c = k = 1$, we get an inequality for σ, $\frac{36}{a^2} - \sqrt{\frac{1296}{a^4} - 45} < \sigma < \frac{36}{a^2} + \sqrt{\frac{1296}{a^4} - 45}$, where the spectral dimension becomes positive and it takes the negative value outside this range. The condition on the deformation parameter, $a^2 < \frac{72\sigma}{45+\sigma^2}$, implies that the spectral dimension is positive.

We also investigate the effect of the extended nature of the probe on the spectral dimension. For this purpose, we consider the Gaussian distribution as our initial condition in solving Eq. (14); i.e., we take

$$\varphi(x,y;0) = \frac{1}{(4\pi a^2)^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4a^2}}, \tag{29}$$

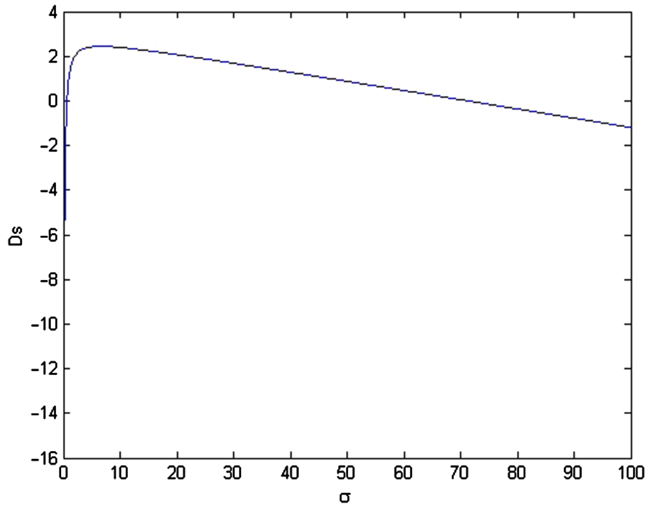

FIG. 1 (color online). Spectral dimension as a function of σ for $n = 4,\ a = 1,\ c = k = 1$.

instead of the delta function condition used to obtain φ_0, φ_1 and φ_2. Using this, we solve Eq. (14) and obtain the zeroth-order solution as

$$\varphi_0(x, y; \sigma) = \frac{1}{(4\pi(\sigma + a^2))^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i - y_i)^2}{4(\sigma + a^2)}}. \tag{30}$$

Keeping terms up to second order in a, we find

$$\varphi_0(x, y; \sigma) = \frac{1}{(4\pi\sigma)^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i - y_i)^2}{4\sigma}} \times \left(1 + \frac{a^2}{4\sigma^2}\Sigma_{i=1}^{n-1}(x_i - y_i)^2 - (n-1)\frac{a^2}{2\sigma}\right). \tag{31}$$

Similarly, we obtain φ_1 from Eq. (18), valid up to first order in a as

$$\varphi_1(x, y; \sigma) = \frac{\alpha}{(4\pi\sigma)^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i - y_i)^2}{4\sigma}}. \tag{32}$$

For φ_2 we need to consider only the zeroth-order terms in a, since the expression for φ contains $a^2\varphi_2$ and, thus, the solution for φ_2 will be the same as we obtained in Eqs. (25) and (26).

Using this, we calculate the spectral dimension as

$$D_s = (n-1) - \frac{a^2 c^4}{24k^4}\sigma - \frac{a^2}{8}\frac{(n^2-1)}{\sigma} - \frac{a^2}{\sigma}(n-1). \tag{33}$$

By comparing with Eq. (28), we note that we have an extra term $-\frac{a^2}{\sigma}(n-1)$, which is due to the extended nature of the probe. Further, we note that the dimensional flow has the same general behavior as the one obtained with point particle probe in Eq. (28). Here again, we note that there are terms with σ^{-1} dependence and one term with linear dependence on σ, the diffusion scale. The finite size effect of the test particle introduces a correction which is proportional to the inverse power of σ.

C. Diffusion equation for $(\Box - m^2)\phi = 0$ and spectral dimension

Equation (7) and Eq. (8) show that both $D_\mu D^\mu$ and $\Box$ operator have the same commutative limit. Thus, the requirement of the correct commutative limit allows

$$\Box\phi = \frac{m^2 c^2}{\hbar^2}\phi \tag{34}$$

as a possible κ-deformed Klein-Gordon equation. Expanding this equation up to first nonvanishing terms in a, we find

$$\left(\nabla_{n-1}^2 - \partial_0^2 + \frac{a^2}{12}\partial_0^4\right)\phi = \frac{m^2 c^2}{\hbar^2}\phi. \tag{35}$$

Using Eq. (10) and Eq. (11) in the above, we obtain the nonrelativistic limit of Eq. (35) as

$$\nabla_{n-1}^2\varphi + i\frac{2m}{\hbar}\frac{\partial\varphi}{\partial t} + i\frac{a^2}{3}\frac{m^3 c^2}{\hbar^3}\frac{\partial\varphi}{\partial t} + \frac{a^2}{12}\frac{m^4 c^4}{\hbar^4}\varphi = 0. \tag{36}$$

After mapping t to $-it$ and redefining $kt = \sigma$ (where $k = \frac{\hbar}{2m}$), we reexpress the above equation as

$$\frac{\partial\varphi}{\partial\sigma} = \nabla_{n-1}^2\varphi - \frac{a^2 c^2}{24k^2}\frac{\partial\varphi}{\partial\sigma} + a^2\frac{c^4}{192k^4}\varphi. \tag{37}$$

Unlike Eq. (14), here we do not have higher-derivatives terms. We perturbatively solve this deformed diffusion equation using the series expansion of φ given in Eq. (15). The zeroth-order terms give

$$\frac{\partial}{\partial\sigma}\varphi_0(x, y; \sigma) = \nabla_{n-1}^2\varphi_0(x, y; \sigma), \tag{38}$$

whose solution is

$$\varphi_0(x, y; \sigma) = \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i - y_i)^2}{4\sigma}}. \tag{39}$$

Equating the first-order terms in a on both sides of Eq. (37) gives

$$\frac{\partial}{\partial\sigma}\varphi_1(x, y; \sigma) = \nabla_{n-1}^2\varphi_1(x, y; \sigma). \tag{40}$$

The solution to this equation is

$$\varphi_1(x,y;\sigma) = \frac{\alpha}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}. \tag{41}$$

Note that α has the dimension of inverse length. Next we collect the terms of having a^2 from both sides of Eq. (37) to get

$$\frac{\partial\varphi_2}{\partial\sigma} = \nabla_{n-1}^2\varphi_2 - \frac{c^2}{24k^2}\frac{\partial\varphi_0}{\partial\sigma} + \frac{c^4}{192k^4}\varphi_0. \tag{42}$$

Substituting for φ_0 from Eq. (39) in the above, reduces Eq. (42) to

$$\frac{\partial\varphi_2}{\partial\sigma} = \nabla_{n-1}^2\varphi_2 + \left[\frac{c^2}{48k^2}\frac{(n-1)}{\sigma} - \frac{c^2}{96k^2}\frac{1}{\sigma^2}\Sigma_{i=1}^{n-1}(x_i-y_i)^2 + \frac{c^4}{192k^4}\right]\frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}. \tag{43}$$

Using Eq. (23), we solve this differential equation. Then the first term of Eq. (23) will give φ_{21} as

$$\varphi_{21}(x,y;\sigma) = \frac{\beta}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}, \tag{44}$$

where β has dimension L^{-2}. The second term on rhs of Eq. (23), φ_{22} is evaluated as

$$\varphi_{22}(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}\left[\left(\frac{c^4}{192k^4} - \frac{c^2}{96k^2\sigma^2}\Sigma_{i=1}^{n-1}(x_i-y_i)^2 + \frac{c^2}{48k^2\sigma}(n-1)\right)(\sigma-\epsilon) - \frac{c^2}{24k^2}\frac{1}{\sigma\sqrt{\sigma\pi}}\Sigma_{i=1}^{n-1}(x_i-y_i)^2\left(\sigma\tan^{-1}\sqrt{\frac{\sigma}{\epsilon}-1} - \epsilon\sqrt{\frac{\sigma}{\epsilon}-1}\right)\right]. \tag{45}$$

Using Eqs. (39), (41), (44), and (45) in Eq. (15), we find the heat kernel valid up to second order in a. From this we calculate the return probability (in the limit $\epsilon \to 0$) as

$$P(\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}}\left[1 + a\alpha + a^2\beta + a^2\frac{c^4}{192k^4}\sigma + a^2\frac{c^2}{48k^2}(n-1)\right]. \tag{46}$$

Using this we find the spectral dimension to be

$$D_s = (n-1) - \frac{a^2c^4}{96k^4}\sigma. \tag{47}$$

The correction of the spectral dimension is of second order in a, and it is independent of the initial dimension. Thus, we see that the change in spectral dimension is the same for spacetimes of all dimensions. Here we see that the a-dependent correction to the spectral dimension is linear in the diffusion scale σ. Unlike the spectral dimension obtained in Eq. (28), there is no term involving σ^{-1} in Eq. (47).

In the commutative limit we have $D_s = n - 1$, same as the topological dimension. For $n = 4$, $k = c = 1$, it is easy to see from Fig. 2 that spectral dimension $D_s = 3$ exactly at $\sigma = 0$, and it starts decreasing as σ increases. For $\sigma = \frac{288}{a^2}$ the spectral dimension vanishes and it is negative for higher values of σ.

Now we want to see the change in spectral dimension due to the extended nature of the probe. We use the Gaussian function as the initial condition and solve for the heat kernel. The modified initial condition will be

$$\varphi(x,y;0) = \frac{1}{(4\pi a^2)^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4a^2}}. \tag{48}$$

Using this initial condition, we solve Eq. (37) and obtain the zeroth-order term as

$$\varphi_0(x,y;\sigma) = \frac{1}{[4\pi(\sigma+a^2)]^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4(\sigma+a^2)}}. \tag{49}$$

Since we are interested only up to second-order terms in a, we expand this as

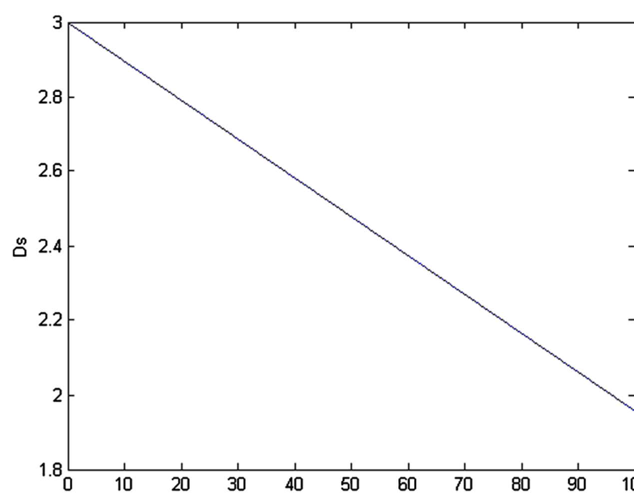

FIG. 2 (color online). Spectral dimension as a function of σ for $n = 4$, $a = 1$, $c = k = 1$.

$$\varphi_0(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}} \times \left(1 + \frac{a^2}{4\sigma^2}\Sigma_{i=1}^{n-1}(x_i-y_i)^2 - (n-1)\frac{a^2}{2\sigma}\right). \tag{50}$$

Similarly we obtain φ_1 from Eq. (40), valid up to first order in a as

$$\varphi_1(x,y;\sigma) = \frac{\alpha}{(4\pi\sigma)^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}. \tag{51}$$

The equation for φ_2 will be the same as Eq. (43) since we are interested only in terms of the order a^2. The resulting solution will be same as Eqs. (44) and (45). Using this we obtain the spectral dimension as

$$D_s = (n-1) - \frac{a^2c^4}{96k^4}\sigma - (n-1)\frac{a^2}{\sigma}. \tag{52}$$

By comparing with Eq. (47), here we have an extra term $-\frac{a^2}{\sigma}(n-1)$ due to the extended nature of the probe. Thus, we see that the extended nature of the probe introduce a correction to the spectral dimension which depends on the inverse power of the diffusion scale σ.

III. MODIFIED κ-DIFFUSION EQUATION AND SPECTRAL DIMENSION

In this section, we study alternative diffusion equations to the ones analyzed in the previous section. Here, we generalize the approach where one starts from the diffusion equation and replaces the Laplacian [$\mathcal{L}$ in Eq. (4)] with the Beltrami-Laplace operator. Thus, we start with the κ-deformed diffusion equation derived in Eq. (14), but use the κ-deformed Beltrami-Laplace operator in place of the Laplacian ∇_{n-1}^2, keeping all other terms of Eq. (14) unchanged. Thus, in this approach we include the possible modification of the diffusion equation in κ spacetime coming from two sources: first, due to the additional terms in the diffusion equation involving the derivative with respect to the diffusion time σ and, second, due to the nonlocal and higher-derivative terms appearing through the deformed Beltrami-Laplace operator. As earlier, here too we analyze the spectral dimension using two different choices of κ-deformed Beltrami-Laplace operators.

A. Diffusion equation with Beltrami-Laplace operator and corresponding spectral dimension

In this subsection, we rewrite the diffusion equation Eq. (14) using the Casimir (general form of Laplacian) of the kappa-Euclidean space. The Casimir of the d-dimensional κ-deformed Euclidean space is given by [17–19]

$$D_\mu D_\mu = \Box\left(1 - \frac{a^2}{4}\Box\right) \tag{53}$$

$$\Box = \nabla_{d-1}^2 \frac{e^{-A}}{\varphi^2} - \partial_d^2 \frac{2(1-\cosh A)}{A^2}, \tag{54}$$

where $\nabla_{d-1}^2 = \Sigma_{i=1}^{d-1}\frac{\partial^2}{\partial x_i^2}$ and $\partial_d^2 = \frac{\partial^2}{\partial x_d^2}$. Here x_d is the Euclidean time coordinate and x_i, $i = 1, 2, \ldots, d-1$ are the space coordinates.

Equation (14) for a generic n-dimensional Euclidean space reads as

$$\frac{\partial\varphi}{\partial\sigma} = \nabla_n^2\varphi + \frac{a^2c^2}{8k^2}\nabla_n^2\varphi + \frac{a^2}{4}\nabla_n^4\varphi - \frac{a^2}{2}\frac{\partial}{\partial\sigma}\nabla_n^2\varphi - \frac{a^2c^2}{6k^2}\frac{\partial\varphi}{\partial\sigma} + \frac{a^2c^4}{48k^2}\varphi. \tag{55}$$

Since the above equation is valid for any dimensions, we use $D_\mu D_\mu$ for ∇_n^2, which is the general form of the Beltrami-Laplace operator in the κ-deformed Euclidean space.

We expand Eq. (53) up to first nonvanishing terms in a,

$$D_\mu D_\mu = \nabla_{d-1}^2 + \partial_d^2 - \frac{a^2}{3}\partial_d^4 - \frac{a^2}{2}\nabla_{d-1}^2\partial_d^2 - \frac{a^2}{4}\nabla_{d-1}^4, \tag{56}$$

and use this in Eq. (55) and keep terms up to second order in a

$$\begin{aligned}\frac{\partial\varphi}{\partial\sigma} &= \nabla_{n-1}^2\varphi + \partial_n^2\varphi - \frac{a^2}{3}\partial_n^4\varphi - \frac{a^2}{2}\nabla_{n-1}^2\partial_n^2\varphi - \frac{a^2}{4}\nabla_{n-1}^4\varphi \\ &\quad + \frac{a^2c^2}{8k^2}[\nabla_{n-1}^2\varphi + \partial_n^2\varphi] \\ &\quad + \frac{a^2}{4}[\nabla_{n-1}^4\varphi + \partial_n^4\varphi + 2\nabla_{n-1}^2\partial_n^2\varphi] \\ &\quad - \frac{a^2}{2}\frac{\partial}{\partial\sigma}[\nabla_{n-1}^2\varphi + \partial_n^2\varphi] - \frac{a^2c^2}{6k^2}\frac{\partial\varphi}{\partial\sigma} + \frac{a^2c^4}{48k^2}\varphi.\end{aligned} \tag{57}$$

By comparing with Eq. (55), we see that there are three extra terms in the above equation and they modify the spectral dimension (obtained in Sec. II B). Note that the extra terms are of higher derivatives in space and Euclidean time coordinates. We have a term which is quartic derivative in Euclidean time, terms involving product of derivatives in space and Euclidean time and a term having quartic derivatives in space coordinate. We solve the above diffusion equation perturbatively using Eq. (15) for φ. By equating the zeroth-order terms in a, we obtain

$$\frac{\partial}{\partial\sigma}\varphi_0(x,y;\sigma) = \nabla_{n-1}^2\varphi_0(x,y;\sigma) + \partial_n^2\varphi_0(x,y;\sigma). \tag{58}$$

This is the usual diffusion equation in n-dimension whose solution is

$$\varphi_0(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}}. \tag{59}$$

The first-order term in a will give

$$\frac{\partial}{\partial\sigma}\varphi_1(x,y;\sigma) = \nabla^2_{n-1}\varphi_1(x,y;\sigma) + \partial^2_n\varphi_1(x,y;\sigma). \tag{60}$$

It is clear that $\varphi_1(x,y;\sigma)$ also satisfy the usual heat equation and, thus,

$$\varphi_1(x,y;\sigma) = \frac{\alpha}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}}. \tag{61}$$

Now equate the second-order terms in a in Eq. (57) to get

$$\begin{aligned}\frac{\partial\varphi_2}{\partial\sigma} &= \nabla^2_{n-1}\varphi_2 + \partial^2_n\varphi_2 - \frac{1}{3}\partial^4_n\varphi_0 - \frac{1}{2}\nabla^2_{n-1}\partial^2_n\varphi_0 \\ &\quad - \frac{1}{4}\nabla^4_{n-1}\varphi_0 + \frac{c^2}{8k^2}(\nabla^2_{n-1}\varphi_0 + \partial^2_n\varphi_0) \\ &\quad + \frac{1}{4}(\nabla^4_{n-1}\varphi_0 + \partial^4_n\varphi_0 + 2\nabla^2_{n-1}\partial^2_n\varphi_0) \\ &\quad - \frac{1}{2}\frac{\partial}{\partial\sigma}(\nabla^2_{n-1}\varphi_0 + \partial^2_n\varphi_0) - \frac{c^2}{6k^2}\frac{\partial\varphi_0}{\partial\sigma} + \frac{c^4}{48k^2}\varphi_0.\end{aligned} \tag{62}$$

Substitute for φ_0 and using Eq. (23) we calculate φ_2. Using this heat kernel, we obtain the return probability as

$$\begin{aligned}P(\sigma) &= \frac{1}{(4\pi\sigma)^{\frac{n}{2}}}\left[1 + a\alpha + a^2\beta - (1+n)^2\frac{a^2}{16\sigma} + a^2\frac{c^4}{48k^4}\sigma \right.\\ &\quad \left. - a^2\frac{n(n+2)}{16\sigma} + \frac{a^2c^2}{48k^2}n\right].\end{aligned} \tag{63}$$

The logarithmic derivative of the above expression gives the spectral dimension as

$$D_s = n - \frac{a^2}{8\sigma}(1 + 4n + 2n^2) - \frac{a^2c^4}{24k^4}\sigma. \tag{64}$$

In the commutative limit we find $D_s = n$. The spectral dimension as a function of σ with $n = 4$ and $a = k = c = 1$ is shown in Fig. 3. We see that in the limit $\sigma \to 0$, the spectral dimension $D_s \to -\infty$. As σ increases, D_s reaches a value close to 3 and thereafter decreases with increase in σ. We note that one of the corrections depends on the diffusion scale linearly while the other changes as the inverse of σ. This feature is the same as the spectral dimension obtained in Eq. (28). The requirement of the positivity of the spectral dimension gives a bound on the deformation parameter as $a^2 < \frac{96\sigma}{147+\sigma^2}$.

The use of an extended probe would result in the spectral dimension

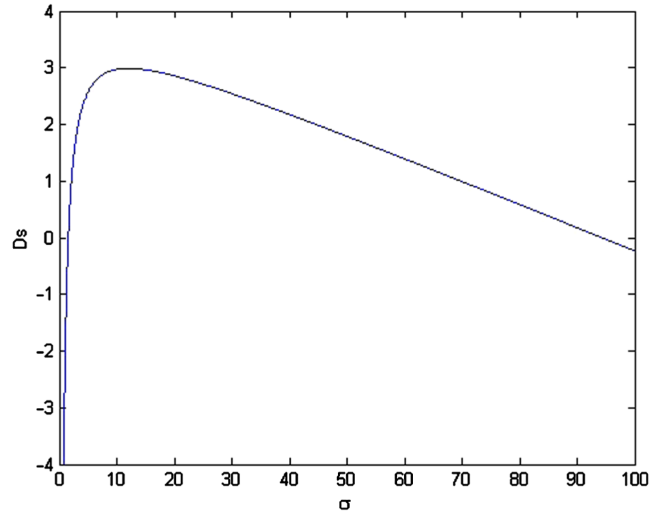

FIG. 3 (color online). Spectral dimension as a function of σ for $n = 4$, $a = 1$, $c = k = 1$.

$$D_s = n - \frac{a^2}{8\sigma}(1 + 4n + 2n^2) - \frac{a^2c^4}{24k^4}\sigma - \frac{a^2n}{\sigma}. \tag{65}$$

By comparing with Eq. (64), we find an additional term $-\frac{a^2n}{\sigma}$ due to the finite width of the probe. This new term is proportional to the initial dimension we start with and inversely proportional to σ. Note that the extended probe does not change the generic behavior of the dimensional flow.

B. Spectral dimension with $\Box$ as the Beltrami-Laplace operator

It is easy to see from Eq. (53) and Eq. (54) that the $\Box$ operator has the same commutative limit as $D_\mu D_\mu$. Equation (37) in generic n-dimension spacetime is of the form

$$\frac{\partial\varphi}{\partial\sigma} = \nabla^2_n\varphi - \frac{a^2c^2}{24k^2}\frac{\partial\varphi}{\partial\sigma} + a^2\frac{c^4}{192k^4}\varphi. \tag{66}$$

Now we use $\Box$ as the general form of Beltrami-Laplace operator in the above equation, in place of ∇^2_n. We expand the $\Box$ operator and keep terms up to the first nonvanishing terms in a,

$$\Box = \nabla^2_{d-1} + \partial^2_d - \frac{a^2}{12}\partial^4_d. \tag{67}$$

Now substituting Eq. (67) in Eq. (66) and keeping terms up to second order in a, we get

$$\frac{\partial\varphi}{\partial\sigma} = \nabla^2_{n-1}\varphi + \partial^2_n\varphi - \frac{a^2}{12}\partial^4_n\varphi - \frac{a^2c^2}{24k^2}\frac{\partial\varphi}{\partial\sigma} + a^2\frac{c^4}{192k^4}\varphi. \tag{68}$$

We note that Eq. (68) has one extra term compared to Eq. (66) which is quartic derivative in the Euclidean time.

We solve the above diffusion equation perturbatively using Eq. (15) for φ, as earlier. By equating the zeroth-order terms in a, we obtain

$$\frac{\partial}{\partial\sigma}\varphi_0(x,y;\sigma) = \nabla^2_{n-1}\varphi_0(x,y;\sigma) + \partial^2_n\varphi_0(x,y;\sigma) \quad (69)$$

and the corresponding solution is

$$\varphi_0(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}}. \quad (70)$$

The first-order terms in a give

$$\frac{\partial}{\partial\sigma}\varphi_1(x,y;\sigma) = \nabla^2_{n-1}\varphi_1(x,y;\sigma) + \partial^2_n\varphi_1(x,y;\sigma), \quad (71)$$

whose solution is given by

$$\varphi_1(x,y;\sigma) = \frac{\alpha}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}}. \quad (72)$$

Second-order terms in a will result in

$$\frac{\partial\varphi_2}{\partial\sigma} = \nabla^2_{n-1}\varphi_2 + \partial^2_n\varphi_2 - \frac{1}{12}\partial^4_n\varphi_0 - \frac{c^2}{24k^2}\frac{\partial\varphi_0}{\partial\sigma} + \frac{c^4}{192k^4}\varphi_0. \quad (73)$$

We solve this equation by substituting for φ_0 and using Eq. (23). The solutions of Eq. (69), (71) and (73) are used to obtain the return probability,

$$P(\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}}\left[1 + a\alpha + a^2\beta - \frac{a^2}{16\sigma} + \frac{a^2c^4}{192k^4}\sigma + \frac{a^2c^2}{48k^2}n\right], \quad (74)$$

and using Eq. (6), we find the spectral dimension as

$$D_s = n - \frac{a^2}{8\sigma} - \frac{a^2c^4}{96k^4}\sigma. \quad (75)$$

Note that the spectral dimension has one term which is linear in σ and another which is proportional to σ^{-1}. For $n = 4$ and $k = c = 1$, it is easy to see from Fig. 4 that the spectral dimension increases with σ initially and then decreases as σ increases. It is clear that the spectral dimension is positive for $\frac{192}{a^2} - \sqrt{\frac{36864}{a^4} - 12} < \sigma < \frac{192}{a^2} + \sqrt{\frac{36864}{a^4} - 12}$.

The spectral dimension with an extended probe is given by

$$D_s = n - \frac{a^2}{8\sigma} - \frac{a^2c^4}{96k^4}\sigma - \frac{a^2n}{\sigma}, \quad (76)$$

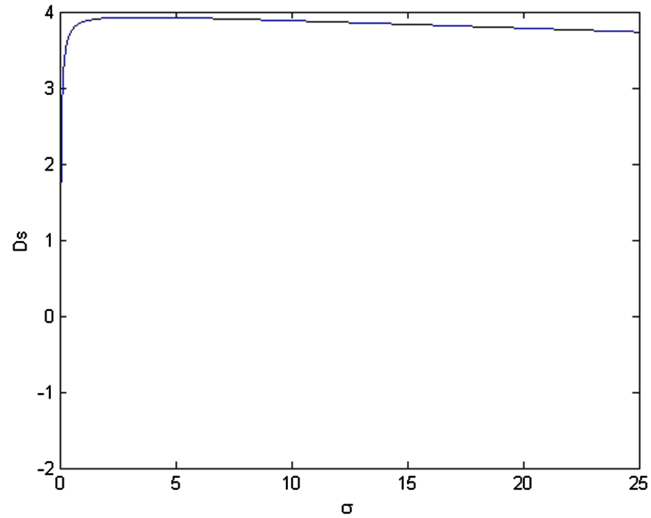

FIG. 4 (color online). Spectral dimension as a function of σ for $n = 4$, $a = 1$, $c = k = 1$.

which has an extra term, $-\frac{a^2n}{\sigma}$, compared to Eq. (75). Note that this additional term depends on the topological dimension n and it is proportional to σ^{-1}. This will not change the general feature of the dimensional flow.

IV. CONCLUSION

In this paper, we have constructed four different modified diffusion equations in the κ spacetime and, using their solutions, analyzed the dimensional flow in the κ spacetime. In these studies, we have used probes which are pointlike as well as probes with finite extension. For all these cases, we get the correct commutative limit, where the spectral dimension matches with the topological dimension. In all four cases studied, the spectral dimension changes with the probe scale. We note that for the three cases studied [see Eqs. (28), (64), and (75)] in the high-energy limit where $\sigma \to 0$, the spectral dimension becomes infinitely negative ($-\infty$). This feature was also observed in [16]. Thus, for these three cases the spectral dimension loses its meaning at high energies. By demanding that the spectral dimension should be positive definite, we obtain bounds on the deformation parameter in terms of diffusion time in these three cases. In the case of the spectral dimension obtained in Eq. (47), we note that as $\sigma \to 0$ the spectral dimension becomes equal to topological dimension. In all four cases, we see a novel feature of spectral dimension of noncommutative spacetime in comparison with the result obtained in [16] as well as in [12–15]. The new fact revealed here is that the spectral dimension goes to $-\infty$ at low energies (i.e., $\sigma \to \infty$). We want to emphasize that this feature is absent in the commutative limit, and in the commutative limit we get back the equality between the spectral dimension and the topological dimension at low energies. From Eqs. (28), (64), and (75), we see that the spectral dimension increases from $-\infty$ as σ rises from zero, reaches a maximum value, and then decreases to $-\infty$. The maximum value of the spectral dimension in all three cases is less than the topological dimension. A similar behavior of the spectral dimension, but in a completely different context, was reported in [35]. Here the spectral dimension of commutative spacetime has been calculated using the relativistic Schrödinger equation analytically continued (RSEAC) and the result is compared with the one derived using the telegraph equation (TE). The analysis of [35] shows that only TE produces the spectral dimension that agrees with the topological dimension in the low energies while both these approaches show a reduction of spectral dimension to two at high energies.

We note that the major difference in the present analysis from the earlier ones is the use of a modified diffusion equation(s). In our case, we have not just used the Beltrami-Laplace operator in the usual diffusion equation given in Eq. (4) but derived the modified diffusion equation in the κ-deformed spacetime. This is done by applying the Wick's rotation to the κ-deformed Schrödinger equation, obtained by taking the nonrelativistic limit of the well-studied κ-deformed Klein-Gordon equation. This approach explicitly introduces finite mass for the particle undertaking diffusion on the deformed spacetime. We see from the spectral dimension obtained in Eqs. (28), (47), (64), and (75) that in the limit of a massless probe, the spectral dimension and topological dimension coincide at low energies. We note here that the probes used in earlier studies [12–16] were massless ones. The spectral dimension obtained in Eq. (47) shows the interesting property that in the limit of probe mass set to zero, there are no corrections to the spectral dimension due to the noncommutativity. This feature is unique, as the spectral dimension calculated for the other three cases does have an a-dependent term, even in the limit of vanishing probe mass.

The diffusion equation constructed and analyzed in Sec. II C [see Eq. (37)] do not have any higher-derivative terms unlike the other three cases studied here [see Eqs. (14), (57), and (68)]. The deformed diffusion equation given in Eq. (37) is obtained from a specific choice of the Laplacian (equivalently Klein-Gordon operator in the κ-deformed space-time). The fact that in the massless limit of the probe, the spectral dimension is exactly the same as the topological dimension for all probe scales shows that the noncommutativity between the time and space coordinate does not affect the spectral dimension of the space part of κ spacetime at all.

Equations (14) and (37) are derived from the Wick's rotated nonrelativistic limit of two different choices of the κ-deformed Klein-Gordon equation. In the nonrelativistic limit, one neglects higher-time derivative terms and, thus, keeps only higher-space derivative terms if any appear in the deformed diffusion equation. Thus, we do not have any higher-time derivatives (equivalently, higher derivatives with respect to σ) in these two equations. Further, for the specific choice of the deformed Klein-Gordon equation used in Sec. II C, there are no higher-order spatial derivatives (up to second order in a). This is why the spectral dimension obtained in Eq. (47) has a completely different behavior at high energies. For both the choice of Beltrami-Laplace operator considered in Sec. III, higher derivatives with respect to spatial as well as Euclidean time coordinates are present, and they do appear in the corresponding diffusion equations [see Eqs. (57) and (68)].

It is interesting to note that the three diffusion equations leading to negative spectral dimension of high energies all have the higher-derivative terms. It has been known that such equations result in negative return probabilities [36]. In our formulation, κ-deformed diffusion equations are written down in the commutative spacetime. All the effects of nonlocality inherent in the noncommutative spacetime are contained in the a-dependent terms of the deformed diffusion equation. As it is clear, these terms are all higher-order derivatives and, thus, nonlocal (except for the case studied in Sec. II C). As discussed above, the higher-time derivative drops out in the nonrelativistic limit and this explains why noncommutativity does not play any role in the limit of vanishing mass of the probe for the spectral dimension obtained in Eq. (47).). The κ-deformed Laplacian we used does have higher-derivative terms. These terms summarize the nonlocal effects of the noncommutativity of the spacetime. In the momentum space representation of the Laplacian, this nonlocality appears as higher-power terms of momentum [13–15]. Laplacians with higher derivatives were also analyzed in [36–39].

The negative value of the spectral dimension we see in our analysis might be a reflection of the higher-derivative terms (and, thus, related to the built-in nonlocality of noncommutative space-time). But the higher-derivative terms in the Laplacian (equivalently, Beltrami-Laplace operator) are a characteristic feature of κ-deformed spacetime. Here we have taken a perturbative approach in the analysis of the spectral dimension. A detailed analysis of the issue of higher derivatives requires a field theoretic reinterpretation going beyond the usual diffusion equation [36,37]. The issues related to higher-derivative terms and that of the negative return probability have been analyzed in [36], and the field theoretical reinterpretation of the spectral dimension as a possible way to avoid the negative return probability was also introduced. The spectral dimension calculated in Eqs. (28), (64), and (75), by imposing the requirement that the spectral dimension should be positive definite at high energies, translates into the conditions $a^2 < \frac{72\sigma}{45+\sigma^2}$, $a^2 < \frac{96\sigma}{147+\sigma^2}$, $a^2 < \frac{384\sigma}{12+\sigma^2}$, respectively. This feature suggests the possibility of the multiscale structure of the spacetime at high energies, which has been pointed out earlier [37]. These issues are being investigated now.

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