

# String theories as the adiabatic limit of Yang-Mills theory

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(Received 4 June 2015; published 4 August 2015)

We consider Yang-Mills theory with a matrix gauge group  $G$  on a direct product manifold  $M = \Sigma_2 \times H^2$ , where  $\Sigma_2$  is a two-dimensional Lorentzian manifold and  $H^2$  is a two-dimensional open disc with the boundary  $S^1 = \partial H^2$ . The Euler-Lagrange equations for the metric on  $\Sigma_2$  yield constraint equations for the Yang-Mills energy-momentum tensor. We show that in the adiabatic limit, when the metric on  $H^2$  is scaled down, the Yang-Mills equations plus constraints on the energy-momentum tensor become the equations describing strings with a world sheet  $\Sigma_2$  moving in the based loop group  $\Omega G = C^\infty(S^1, G)/G$ , where  $S^1$  is the boundary of  $H^2$ . By choosing  $G = \mathbb{R}^{d-1,1}$  and putting to zero all parameters in  $\Omega\mathbb{R}^{d-1,1}$  besides  $\mathbb{R}^{d-1,1}$ , we get a string moving in  $\mathbb{R}^{d-1,1}$ . In another paper of the author, it was described how one can obtain the Green-Schwarz superstring action from Yang-Mills theory on  $\Sigma_2 \times H^2$  while  $H^2$  shrinks to a point. Here we also consider Yang-Mills theory on a three-dimensional manifold  $\Sigma_2 \times S^1$  and show that in the limit when the radius of  $S^1$  tends to zero, the Yang-Mills action functional supplemented by a Wess-Zumino-type term becomes the Green-Schwarz superstring action.

DOI: 10.1103/PhysRevD.92.045003

PACS numbers: 11.15.-q, 11.15.Kc, 11.25.-w

## I. INTRODUCTION

Superstring theory has a long history [1–3] and pretends on description of all four known forces in nature. In the low-energy limit superstring theories describe supergravity in ten dimensions or supergravity interacting with supersymmetric Yang-Mills (SYM) theory. On the other hand, Yang-Mills and SYM theories in four dimensions give descriptions of three main forces in nature not including gravity [4–7]. The aim of this short paper is to show that bosonic strings (both open and closed) as well as type I, IIA, and IIB superstrings can be obtained as a subsector of pure Yang-Mills theory with some constraints on the Yang-Mills energy-momentum tensor. Put differently, knowing the action for superstrings with a world sheet  $\Sigma_2$ , we introduce a Yang-Mills action functional on  $\Sigma_2 \times H^2$  or on  $\Sigma_2 \times S^1$  such that the Yang-Mills action becomes the Green-Schwarz superstring action while  $H^2$  or  $S^1$  shrink to a point. We will work in Lorentzian signature, but all calculations can be repeated for the Euclidean signature of spacetime.

## II. YANG-MILLS EQUATIONS

Consider Yang-Mills theory with a matrix gauge group  $G$  on a direct product manifold  $M = \Sigma_2 \times H^2$ , where  $\Sigma_2$  is a two-dimensional Lorentzian manifold (flat case is included) with local coordinates  $x^a, a, b, \dots = 1, 2$ , and a metric tensor  $g_{\Sigma_2} = (g_{ab})$ ,  $H^2$  is the disc with coordinates  $x^i, i, j, \dots = 3, 4$ , satisfying the inequality  $(x^3)^2 + (x^4)^2 < 1$ , and the metric  $g_{H^2} = (g_{ij})$ . Then  $(x^\mu) = (x^a, x^i)$  are local coordinates on  $M$  with  $\mu = 1, \dots, 4$ .

We start with the gauge potential  $\mathcal{A} = \mathcal{A}_\mu dx^\mu$  with values in the Lie algebra  $\mathfrak{g} = \text{Lie } G$  having scalar product  $(\cdot, \cdot)$  defined either via trace  $\text{Tr}$  or, for Abelian groups like  $\mathbb{R}^{p,q}$ ,  $T^{p,q}$  etc., via a metric on vector spaces. The gauge field  $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$  is the  $\mathfrak{g}$ -valued 2-form

$$\mathcal{F} = \frac{1}{2} \mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu \quad \text{with} \quad \mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. \quad (1)$$

The Yang-Mills equations on  $M$  with the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{ab} dx^a dx^b + g_{ij} dx^i dx^j \quad (2)$$

have the form

$$D_\mu \mathcal{F}^{\mu\nu} := \frac{1}{\sqrt{|\det g|}} \partial_\mu (\sqrt{|\det g|} \mathcal{F}^{\mu\nu}) + [\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}] = 0, \quad (3)$$

where  $g = (g_{\mu\nu})$  and  $\partial_\mu = \partial/\partial x^\mu$ .

Equations (3) follow from the standard Yang-Mills action on  $M$ ,

$$S = \frac{1}{4} \int_M d^4x \sqrt{|\det g|} (\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}), \quad (4)$$

where  $(\cdot, \cdot)$  is the scalar product on the Lie algebra  $\mathfrak{g}$ . Note that the metric  $g_{\Sigma_2}$  on  $\Sigma_2$  is not fixed and the Euler-Lagrange equations for  $g_{\Sigma_2}$  yield the constraint equations

$$T_{ab} = g^{\lambda\sigma} (\mathcal{F}_{a\lambda}, \mathcal{F}_{b\sigma}) - \frac{1}{4} g_{ab} (\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}) = 0 \quad (5)$$

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for the Yang-Mills energy-momentum tensor  $T_{\mu\nu}$ ; i.e., its components along  $\Sigma_2$  are vanishing. Note that these constraints can be satisfied for many gauge configurations; e.g., for self-dual gauge fields, not only does  $T_{ab} = 0$  but even  $T_{\mu\nu} = 0$ .

### III. ADIABATIC LIMIT

On  $M = \Sigma_2 \times H^2$  we have the obvious splitting

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu = \mathcal{A}_a dx^a + \mathcal{A}_i dx^i, \quad (6)$$

$$\begin{aligned} \mathcal{F} &= \frac{1}{2} \mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} \mathcal{F}_{ab} dx^a \wedge dx^b + \mathcal{F}_{ai} dx^a \wedge dx^i \\ &\quad + \frac{1}{2} \mathcal{F}_{ij} dx^i \wedge dx^j, \end{aligned} \quad (7)$$

$$T = T_{\mu\nu} dx^\mu dx^\nu = T_{ab} dx^a dx^b + 2T_{ai} dx^a dx^i + T_{ij} dx^i dx^j. \quad (8)$$

By using the adiabatic approach in the form presented in [8,9], we deform the metric (2) and introduce the metric

$$ds_\varepsilon^2 = g_{ab} dx^a dx^b + \varepsilon^2 g_{ij} dx^i dx^j, \quad (9)$$

where  $\varepsilon \in [0, 1]$  is a real parameter. It is assumed that the fields  $\mathcal{A}_\mu$  and  $\mathcal{F}_{\mu\nu}$  smoothly depend in  $\varepsilon^2$ , i.e.,  $\mathcal{A}_\mu = \mathcal{A}_\mu^{(0)} + \varepsilon^2 \mathcal{A}_\mu^{(1)} + \dots$  and  $\mathcal{F}_{\mu\nu} = \mathcal{F}_{\mu\nu}^{(0)} + \varepsilon^2 \mathcal{F}_{\mu\nu}^{(1)} + \dots$ . Furthermore, we have  $\det g_\varepsilon = \varepsilon^4 \det(g_{ab}) \det(g_{ij})$  and

$$\begin{aligned} \mathcal{F}_\varepsilon^{ab} &= g_\varepsilon^{ac} g_\varepsilon^{bd} \mathcal{F}_{cd} = \mathcal{F}^{ab}, \\ \mathcal{F}_\varepsilon^{ai} &= g_\varepsilon^{ac} g_\varepsilon^{ij} \mathcal{F}_{cj} = \varepsilon^{-2} \mathcal{F}^{ai} \quad \text{and} \\ \mathcal{F}_\varepsilon^{ij} &= g_\varepsilon^{ik} g_\varepsilon^{jl} \mathcal{F}_{kl} = \varepsilon^{-4} \mathcal{F}^{ij}, \end{aligned} \quad (10)$$

where indices in  $\mathcal{F}^{\mu\nu}$  are raised by the nondeformed metric tensor  $g^{\mu\nu}$ .

For the deformed metric (9) the action functional (4) is changed to

$$\begin{aligned} S_\varepsilon &= \frac{1}{4} \int_M d^4x \sqrt{|\det g_{\Sigma_2}|} \sqrt{\det g_{H^2}} \{ \varepsilon^2 (\mathcal{F}_{ab}, \mathcal{F}^{ab}) \\ &\quad + 2(\mathcal{F}_{ai}, \mathcal{F}^{ai}) + \varepsilon^{-2} (\mathcal{F}_{ij}, \mathcal{F}^{ij}) \}. \end{aligned} \quad (11)$$

The term  $\varepsilon^{-2} (\mathcal{F}_{ij}, \mathcal{F}^{ij})$  in the Yang-Mills Lagrangian (11) diverges when  $\varepsilon \rightarrow 0$ . To avoid this we impose the flatness condition

$$\mathcal{F}_{ij}^{(0)} = 0 \Rightarrow \lim_{\varepsilon \rightarrow 0} (\varepsilon^{-1} \mathcal{F}_{ij}) = 0 \quad (12)$$

on the components of the field tensor along  $H^2$ . Here  $\mathcal{F}_{ij}^{(0)} = 0$ , but  $\mathcal{F}_{ij}^{(1)}$  etc. in the  $\varepsilon^2$  expansion must not be zero. For the deformed metric (9), the Yang-Mills equations have the form

$$\varepsilon^2 D_a \mathcal{F}^{ab} + D_i \mathcal{F}^{ib} = 0, \quad (13)$$

$$\varepsilon D_a \mathcal{F}^{aj} + \varepsilon^{-1} D_i \mathcal{F}^{ij} = 0. \quad (14)$$

In the deformed metric (9) the constraint equations (5) become

$$\begin{aligned} T_{ab}^\varepsilon &= \varepsilon^2 \left\{ g^{cd} (\mathcal{F}_{ac}, \mathcal{F}_{bd}) - \frac{1}{4} g_{ab} (\mathcal{F}_{cd}, \mathcal{F}^{cd}) \right\} \\ &\quad + g^{ij} (\mathcal{F}_{ai}, \mathcal{F}_{bj}) - \frac{1}{2} g_{ab} (\mathcal{F}_{ci}, \mathcal{F}^{ci}) \\ &\quad - \frac{1}{4} \varepsilon^{-2} g_{ab} (\mathcal{F}_{ij}, \mathcal{F}^{ij}) = 0. \end{aligned} \quad (15)$$

In the adiabatic limit  $\varepsilon \rightarrow 0$ , the Yang-Mills equations (13) and (14) become

$$D_i \mathcal{F}^{ib} = 0, \quad (16)$$

$$D_a \mathcal{F}^{aj} = 0, \quad (17)$$

since the  $\varepsilon^{-1}$  term vanishes due to (12). We also keep (17) since it follows from the action (11) after taking the limit  $\varepsilon \rightarrow 0$ . One can see that the constraint equations (15) are nonsingular in the limit  $\varepsilon \rightarrow 0$  also due to (12):

$$T_{ab}^0 = g^{ij} (\mathcal{F}_{ai}, \mathcal{F}_{bj}) - \frac{1}{2} g_{ab} (\mathcal{F}_{ci}, \mathcal{F}^{ci}) = 0. \quad (18)$$

Note that for the adiabatic limit of instanton equations [8,9] the constraints (15) disappear since the energy-momentum tensor for self-dual and anti-self-dual gauge fields vanishes on any four-manifold  $M$ .

### IV. FLAT CONNECTIONS

Now we start to consider the flatness equation (12), the equations (16), (17), and the constraint equations (18). From now on we will consider only zero modes in  $\varepsilon^2$  expansions and equations on them. For simplicity of notation we will omit the index “(0)” from all  $\mathcal{A}^{(0)}$  and  $\mathcal{F}^{(0)}$  tensor components. In the adiabatic approach it is assumed that all fields depend on coordinates  $x^a \in \Sigma_2$  only via moduli parameters  $\phi^\alpha(x^a)$ ,  $\alpha, \beta = 1, 2, \dots$ , appearing in the solutions of the flatness equation (12).

Flat connection  $\mathcal{A}_{H^2} := \mathcal{A}_i dx^i$  on  $H^2$  has the form

$$\mathcal{A}_{H^2} = g^{-1} \hat{d}g \quad \text{with} \quad \hat{d} = dx^i \partial_i \quad \text{for} \quad \partial_i = \frac{\partial}{\partial x^i}, \quad (19)$$

where  $g = g(\phi^\alpha(x^a), x^i)$  is a smooth map from  $H^2$  into the gauge group  $G$  for any fixed  $x^a \in \Sigma_2$ .

Let us introduce on  $H^2$  spherical coordinates:  $x^3 = \rho \cos \varphi$  and  $x^4 = \rho \sin \varphi$ . Using these coordinates, we impose on  $g$  the condition  $g(\varphi = 0, \rho^2 \rightarrow 1) = \text{Id}$  (framing) and denote by  $C_0^\infty(H^2, G)$  the space of framed

flat connections on  $H^2$  given by (19). On  $H^2$ , as on a manifold with a boundary, the group of gauge transformations for any fixed  $x^a \in \Sigma_2$  is defined as (see e.g., [9–11])

$$\mathcal{G}_{H^2} = \{g: H^2 \rightarrow G | g \rightarrow \text{Id for } \rho^2 \rightarrow 1\}. \quad (20)$$

Hence the solution space of the equation (12) is the infinite-dimensional group  $\mathcal{N} = C_0^\infty(H^2, G)$ , and the moduli space of solutions is the based loop group [9,10,12]

$$\mathcal{M} = C_0^\infty(H^2, G)/\mathcal{G}_{H^2} = \Omega G. \quad (21)$$

This space can also be represented as  $\Omega G = LG/G$ , where  $LG = C^\infty(S^1, G)$  is the loop group with the circle  $S^1 = \partial H^2$  parametrized by  $e^{i\varphi}$ .

### V. MODULI SPACE

On the group manifold (21) we introduce local coordinates  $\phi^\alpha$  with  $\alpha = 1, 2, \dots$  and recall that  $\mathcal{A}_\mu$ 's depend on  $x^a \in \Sigma_2$  only via moduli parameters  $\phi^\alpha = \phi^\alpha(x^a)$ . Then moduli of gauge fields define a map

$$\phi: \Sigma_2 \rightarrow \mathcal{M} \quad \text{with} \quad \phi(x^a) = \{\phi^\alpha(x^a)\}. \quad (22)$$

These maps are constrained by Eqs. (16), (17), and (18). Since  $\mathcal{A}_{H^2}$  is a flat connection for any  $x^a \in \Sigma_2$ , the derivatives  $\partial_a \mathcal{A}_i$  have to satisfy the linearized (around  $\mathcal{A}_{H^2}$ ) flatness condition; i.e.,  $\partial_a \mathcal{A}_i$  belong to the tangent space  $T_{\mathcal{A}}\mathcal{N}$  of the space  $\mathcal{N} = C_0^\infty(H^2, G)$  of framed flat connections on  $H^2$ . Using the projection  $\pi: \mathcal{N} \rightarrow \mathcal{M}$  from  $\mathcal{N}$  to the moduli space  $\mathcal{M}$ , one can decompose  $\partial_a \mathcal{A}_i$  into the two parts

$$T_{\mathcal{A}}\mathcal{N} = \pi^* T_{\mathcal{A}}\mathcal{M} \oplus T_{\mathcal{A}}\mathcal{G} \Leftrightarrow \partial_a \mathcal{A}_i = (\partial_a \phi^\beta) \xi_{\beta i} + D_i \epsilon_a, \quad (23)$$

where  $\mathcal{G}$  is the gauge group  $\mathcal{G}_{H^2}$  for any fixed  $x^a \in \Sigma_2$ ,  $\{\xi_\alpha = \xi_{ai} dx^i\}$  is a local basis of tangent vectors at  $T_{\mathcal{A}}\mathcal{M}$  (they form the loop Lie algebra  $\Omega\mathfrak{g}$ ) and  $\epsilon_a$  are  $\mathfrak{g}$ -valued gauge parameters ( $D_i \epsilon_a \in T_{\mathcal{A}}\mathcal{G}$ ) which are determined by the gauge-fixing conditions

$$g^{ij} D_i \xi_{aj} = 0 \Leftrightarrow g^{ij} D_i D_j \epsilon_a = g^{ij} D_i \partial_a \mathcal{A}_j. \quad (24)$$

Note also that since  $\mathcal{A}_i(\phi^\alpha, x^j)$  depends on  $x^a$  only via  $\phi^\alpha$ , we have

$$\partial_a \mathcal{A}_i = \frac{\partial \mathcal{A}_i}{\partial \phi^\beta} \partial_a \phi^\beta \stackrel{(24)}{\Rightarrow} \epsilon_a = (\partial_a \phi^\beta) \epsilon_\beta, \quad (25)$$

where the gauge parameters  $\epsilon_\beta$  are found by solving the equations

$$g^{ij} D_i D_j \epsilon_\beta = g^{ij} D_i \frac{\partial \mathcal{A}_j}{\partial \phi^\beta}. \quad (26)$$

Recall that  $\mathcal{A}_i$  are given explicitly by (19) and  $\mathcal{A}_a$  are yet free. It is natural to choose  $\mathcal{A}_a = \epsilon_a$  [5,6] and obtain

$$\mathcal{F}_{ai} = \partial_a \mathcal{A}_i - D_i \mathcal{A}_a = (\partial_a \phi^\beta) \xi_{\beta i} = \pi_* \partial_a \mathcal{A}_i \in T_{\mathcal{A}}\mathcal{M}. \quad (27)$$

Thus, if we know the dependence of  $\phi^\alpha$  on  $x^a$ , then we can construct

$$(\mathcal{A}_\mu) = (\mathcal{A}_a, \mathcal{A}_i) = ((\partial_a \phi^\beta) \epsilon_\beta, g^{-1}(\phi^\alpha, x^j) \partial_i g(\phi^\beta, x^k)), \quad (28)$$

which are in fact the components  $\mathcal{A}_\mu^{(0)} = \mathcal{A}_\mu(\epsilon = 0)$ .

### VI. EFFECTIVE ACTION

For finding equations for  $\phi^\alpha(x^a)$ , we substitute (27) into (16) and see that (16) are resolved due to (24). Substituting (27) into (17), we obtain the equations

$$\frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left( \sqrt{|\det g_{\Sigma_2}|} g^{ab} \partial_b \phi^\beta \right) g^{ij} \xi_{\beta j} + g^{ab} g^{ij} (D_a \xi_{\beta j}) \partial_b \phi^\beta = 0. \quad (29)$$

We should project (29) on the moduli space  $\mathcal{M} = \Omega G$ , metric  $\mathbb{G} = (G_{\alpha\beta})$ , which is defined as

$$G_{\alpha\beta} = \langle \xi_\alpha, \xi_\beta \rangle = \int_{H^2} d\text{vol} g^{ij} (\xi_{ai}, \xi_{bj}). \quad (30)$$

The projection is provided by multiplying (29) by  $\langle \xi_\alpha, \cdot \rangle$  (cf., e.g., [13,14]). We obtain

$$\begin{aligned} & \frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left( \sqrt{|\det g_{\Sigma_2}|} g^{ab} \partial_b \phi^\beta \right) \langle \xi_\alpha, \xi_\beta \rangle + g^{ab} \langle \xi_\alpha, D_a \xi_\beta \rangle \partial_b \phi^\beta \\ &= \frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left( \sqrt{|\det g_{\Sigma_2}|} g^{ab} \partial_b \phi^\beta \right) G_{\alpha\beta} + \langle \xi_\alpha, \nabla_\gamma \xi_\beta \rangle g^{ab} \partial_a \phi^\gamma \partial_b \phi^\beta \\ &= G_{\alpha\sigma} \left\{ \frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left( \sqrt{|\det g_{\Sigma_2}|} g^{ab} \partial_b \phi^\sigma \right) + \Gamma_{\beta\gamma}^\sigma g^{ab} \partial_a \phi^\beta \partial_b \phi^\gamma \right\} = 0, \end{aligned} \quad (31)$$

where

$$\Gamma_{\beta\gamma}^{\sigma} = \frac{1}{2} G^{\sigma\lambda} (\partial_{\gamma} G_{\beta\lambda} + \partial_{\beta} G_{\gamma\lambda} - \partial_{\lambda} G_{\beta\gamma}) \quad \text{with} \quad \partial_{\gamma} := \frac{\partial}{\partial \phi^{\gamma}} \quad (32)$$

are the Christoffel symbols and  $\nabla_{\gamma}$  are the corresponding covariant derivatives on the moduli space  $\mathcal{M}$  of flat connections on  $H^2$ .

The equations

$$\frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left( \sqrt{|\det g_{\Sigma_2}|} g^{ab} \partial_b \phi^{\alpha} \right) + \Gamma_{\beta\gamma}^{\alpha} g^{ab} \partial_a \phi^{\beta} \partial_b \phi^{\gamma} = 0 \quad (33)$$

are the Euler-Lagrange equations for the effective action

$$S_{\text{eff}} = \int_{\Sigma_2} dx^1 dx^2 \sqrt{-\det(g_{ab})} g^{cd} G_{\alpha\beta} \partial_c \phi^{\alpha} \partial_d \phi^{\beta} \quad (34)$$

obtained from the action functional (11) in the adiabatic limit  $\varepsilon \rightarrow 0$ ; it appears from the term  $(\mathcal{F}_{ai}, \mathcal{F}^{ai})$  in (11) (other terms vanish). Equations (33) are the standard sigma-model equations defining maps from  $\Sigma_2$  into the based loop group  $\Omega G$ .

## VII. VIRASORO CONSTRAINTS

The last undiscussed equations are the constraints (18). Substituting (27) into (18), we obtain

$$g^{ij}(\xi_{ai}, \xi_{\beta j}) \partial_a \phi^{\alpha} \partial_b \phi^{\beta} - \frac{1}{2} g_{ab} g^{cd} g^{ij}(\xi_{ai}, \xi_{\beta j}) \partial_c \phi^{\alpha} \partial_d \phi^{\beta} = 0. \quad (35)$$

Integrating (35) over  $H^2$  (projection on  $\mathcal{M}$ ), we get

$$G_{\alpha\beta} \partial_a \phi^{\alpha} \partial_b \phi^{\beta} - \frac{1}{2} g_{ab} g^{cd} G_{\alpha\beta} \partial_c \phi^{\alpha} \partial_d \phi^{\beta} = 0. \quad (36)$$

These are equations which one will obtain from (34) by varying with respect to  $g_{ab}$ . Thus,

$$T_{ab}^V = G_{\alpha\beta} \partial_a \phi^{\alpha} \partial_b \phi^{\beta} - \frac{1}{2} g_{ab} g^{cd} G_{\alpha\beta} \partial_c \phi^{\alpha} \partial_d \phi^{\beta} \quad (37)$$

is the traceless stress-energy tensor and Eqs. (36) are the Virasoro constraints accompanying the Polyakov string action (34).

## VIII. B FIELD

In string theory the action (34) is often extended by adding the  $B$ -field term. This term can be obtained from the topological Yang-Mills term

$$\frac{1}{2} \int_M d^4 x \sqrt{\det g_{H_2}^{\varepsilon}} \varepsilon_{\mu\nu\lambda\sigma} (\mathcal{F}_{\varepsilon}^{\mu\nu}, \mathcal{F}_{\varepsilon}^{\lambda\sigma}), \quad (38)$$

which in the adiabatic limit  $\varepsilon \rightarrow 0$  becomes

$$\int_M d^4 x \sqrt{\det g_{H_2}} \varepsilon^{ab} \varepsilon^{ij} (\mathcal{F}_{ai}, \mathcal{F}_{bj}) = \int_{\Sigma_2} dx^1 dx^2 \varepsilon^{cd} B_{\alpha\beta} \partial_c \phi^{\alpha} \partial_d \phi^{\beta}, \quad (39)$$

where

$$B_{\alpha\beta} = \int_{H^2} d\text{vol} \varepsilon^{ij}(\xi_{ai}, \xi_{\beta j}) \quad (40)$$

are components of the 2-form  $\mathbb{B} = (B_{\alpha\beta})$  on the moduli space  $\mathcal{M} = \Omega G$ .

## IX. REMARKS ON SUPERSTRINGS

The adiabatic limit of supersymmetric Yang-Mills theories with a (partial) topological twisting on Euclidean manifold  $\Sigma \times \bar{\Sigma}$ , where  $\Sigma$  and  $\bar{\Sigma}$  are Riemann surfaces, was considered in [15]. Several sigma models with fermions on  $\Sigma$  (including supersymmetric ones) were obtained. Switching to Lorentzian signature and adding constraints of type (18), which were not considered in [15], one can get stringy sigma-model resembling NSR strings. However, analysis of these sigma models demands more efforts and goes beyond the scope of our paper.

Another possibility is to consider ordinary Yang-Mills theory (11) but with Lie supergroup  $G$  as the structure group. We restrict ourselves to the  $N = 2$  super translation group with ten-dimensional Minkowski space  $\mathbb{R}^{9,1}$  as the bosonic part. This super translation group can be represented as the coset [16,17]

$$G = \text{SUSY}(N = 2)/\text{SO}(9, 1), \quad (41)$$

with coordinates  $(X^{\alpha}, \theta^{Ap})$ , where  $\theta^p = (\theta^{Ap})$  are two Majorana-Weyl spinors in  $d = 10, \alpha = 0, \dots, 9, A = 1, \dots, 32$  and  $p = 1, 2$ . The generators of  $G$  obey the Lie superalgebra  $\mathfrak{g} = \text{Lie}G$ ,

$$\{\xi_{Ap}, \xi_{Bq}\} = (\gamma^{\alpha} C)_{AB} \delta_{pq} \xi_{\alpha}, \quad [\xi_{\alpha}, \xi_{Ap}] = 0, \quad [\xi_{\alpha}, \xi_{\beta}] = 0, \quad (42)$$

where  $\gamma^{\alpha}$  are the  $\gamma$  matrices in  $\mathbb{R}^{9,1}$  and  $C$  is the charge conjugation matrix. On the superalgebra  $\mathfrak{g}$ , we introduce the standard metric

$$\langle \xi_{\alpha} \xi_{\beta} \rangle = \eta_{\alpha\beta}, \quad \langle \xi_{\alpha} \xi_{Ap} \rangle = 0 \quad \text{and} \quad \langle \xi_{Ap} \xi_{Bq} \rangle = 0, \quad (43)$$

where  $(\eta_{\alpha\beta}) = \text{diag}(-1, 1, \dots, 1)$  is the Lorentzian metric on  $\mathbb{R}^{9,1}$ .

It was shown in [18] that the action functional for Yang-Mills theory on  $\Sigma_2 \times H^2$  with the gauge group  $G$ , defined by (42)

$$S_\varepsilon = \frac{1}{2\pi} \int_{\Sigma_2 \times H^2} d^4x \sqrt{|\det g_{\Sigma_2}|} \sqrt{|\det g_{H^2}|} \{ \varepsilon^2 \langle \mathcal{F}_{ab} \mathcal{F}^{ab} \rangle + 2 \langle \mathcal{F}_{ai} \mathcal{F}^{ai} \rangle + \varepsilon^{-2} \langle \mathcal{F}_{ij} \mathcal{F}^{ij} \rangle \} \quad (44)$$

plus the Wess-Zumino-type term

$$S_{WZ} = \frac{1}{\pi} \int_{\Sigma_3 \times H^2} dx^{\hat{a}} \wedge dx^{\hat{b}} \wedge dx^{\hat{c}} \wedge dx^3 \wedge dx^4 f_{\Gamma\Delta\Lambda} \mathcal{F}_{\hat{a}i}^\Gamma \xi^i \mathcal{F}_{\hat{b}j}^\Delta \xi^j \mathcal{F}_{\hat{c}k}^\Lambda \xi^k \quad (45)$$

yield the Green-Schwarz superstring action [17] in the adiabatic limit  $\varepsilon \rightarrow 0$ . Here  $\Sigma_3$  is a Lorentzian manifold with the boundary  $\Sigma_2 = \partial\Sigma_3$  and local coordinates  $x^{\hat{a}}$ ,  $\hat{a} = 0, 1, 2$ ; the structure constants  $f_{\Gamma\Delta\Lambda}$  are given in [16] and  $(\xi_i) = (\sin \varphi, -\cos \varphi)$  is the unit vector on  $H^2$  running the boundary  $S^1 = \partial H^2$ .

## X. SUPERSTRINGS FROM $d = 3$ YANG-MILLS

Here we will show that the Green-Schwarz superstrings with a world sheet  $\Sigma_2$  can also be associated with a Yang-Mills model on  $\Sigma_2 \times S^1$ . When the radius of  $S^1$  tends to zero, the action of this Yang-Mills model becomes the Green-Schwarz superstring action. So, we consider Yang-Mills theory on a direct product manifold  $M^3 = \Sigma_2 \times S^1$ , where  $\Sigma_2$  is a two-dimensional Lorentzian manifold discussed before and  $S^1$  is the unit circle parametrized by  $x^3 \in [0, 2\pi]$  with the metric tensor  $g_{S^1} = (g_{33})$  and  $g_{33} = 1$ . As the structure group  $G$  of Yang-Mills theory, we consider the super translation group in  $d = 10$  auxiliary dimensions (41) with the generators (42) and the metric (43) on the Lie superalgebra  $\mathfrak{g} = \text{Lie}G$ . As in (20), we impose framing over  $S^1$ , i.e., consider the group of gauge transformations equal to the identity over  $S^1$ . Coordinates on  $G$  are  $X^\alpha$  and  $\theta^{A\rho}$  introduced in the previous section. The 1-forms

$$\Pi^\Delta = \{ \Pi^\alpha, \Pi^{A\rho} \} = \{ dX^\alpha - i\delta_{pq} \bar{\theta}^p \gamma^\alpha d\theta^q, d\theta^{A\rho} \} \quad (46)$$

form a basis of 1-forms on  $G$  [16].

By using the adiabatic approach, we deform the metric on  $\Sigma_2 \times S^1$  and introduce

$$ds_\varepsilon^2 = g_\varepsilon^{\mu\nu} dx^\mu dx^\nu = g_{ab} dx^a dx^b + \varepsilon^2 (dx^3)^2, \quad (47)$$

where  $\varepsilon \in [0, 1]$  is a real parameter,  $a, b = 1, 2$ ,  $\mu, \nu = 1, 2, 3$ . This is equivalent to the consideration of the circle  $S_\varepsilon^1$  of radius  $\varepsilon$ . It is assumed that for the fields  $\mathcal{A}_\mu$  and  $\mathcal{F}_{\mu\nu}$ , there exist limits  $\lim_{\varepsilon \rightarrow 0} \mathcal{A}_\mu$  and  $\lim_{\varepsilon \rightarrow 0} \mathcal{F}_{\mu\nu}$ . Indices are raised by  $g_\varepsilon^{\mu\nu}$  and we have

$$\begin{aligned} \mathcal{F}_\varepsilon^{ab} &= g_\varepsilon^{ac} g_\varepsilon^{bd} \mathcal{F}_{cd} = \mathcal{F}^{ab}, \\ \mathcal{F}_\varepsilon^{a3} &= g_\varepsilon^{ac} g_\varepsilon^{33} \mathcal{F}_{c3} = \varepsilon^{-2} \mathcal{F}^{a3}, \end{aligned} \quad (48)$$

where indices in  $\mathcal{F}^{\mu\nu}$  are raised by the nondeformed metric tensor.

We consider the Yang-Mills action of the form

$$S_\varepsilon = \int_{M^3} d^3x \sqrt{|\det g_{\Sigma_2}|} \left\{ \frac{\varepsilon^2}{2} \langle \mathcal{F}_{ab} \mathcal{F}^{ab} \rangle + \langle \mathcal{F}_{a3} \mathcal{F}^{a3} \rangle \right\}, \quad (49)$$

which for  $\varepsilon = 1$  coincides with the standard Yang-Mills action. Variations with respect to  $\mathcal{A}_\mu$  and  $g_{ab}$  yield the equations

$$\varepsilon^2 D_a \mathcal{F}^{ab} + D_3 \mathcal{F}^{3b} = 0, \quad D_a \mathcal{F}^{a3} = 0, \quad (50)$$

$$\begin{aligned} T_{ab}^\varepsilon &= \varepsilon^2 \left( g^{cd} \langle \mathcal{F}_{ac} \mathcal{F}^{bd} \rangle - \frac{1}{4} g_{ab} \langle \mathcal{F}_{cd} \mathcal{F}^{cd} \rangle \right) \\ &+ \langle \mathcal{F}_{a3} \mathcal{F}_{b3} \rangle - \frac{1}{2} g_{ab} \langle \mathcal{F}_{c3} \mathcal{F}^{c3} \rangle. \end{aligned} \quad (51)$$

In the adiabatic limit  $\varepsilon \rightarrow 0$ , Eqs. (50) and (51) become

$$D_3 \mathcal{F}^{3b} = 0, \quad D_a \mathcal{F}^{a3} = 0, \quad (52)$$

$$T_{ab}^0 = \langle \mathcal{F}_{a3} \mathcal{F}_{b3} \rangle - \frac{1}{2} g_{ab} \langle \mathcal{F}_{c3} \mathcal{F}^{c3} \rangle. \quad (53)$$

Notice that as a function of  $x^3 \in S^1$ , the field  $\mathcal{A}_3$  belongs to the loop algebra  $L\mathfrak{g} = \mathfrak{g} \oplus \Omega\mathfrak{g}$ , where  $\Omega\mathfrak{g}$  is the Lie superalgebra of the based loop group  $\Omega G$ . Let us denote by  $\mathcal{A}_3^0$  the zero mode in the expansion of  $\mathcal{A}_3$  in  $\exp(ix^3) \in S^1$  (Wilson line). The generic  $\mathcal{A}_3$  can be represented in the form

$$\mathcal{A}_3 = h^{-1} \mathcal{A}_3^0 h + h^{-1} \partial_3 h, \quad (54)$$

where the  $G$ -valued function  $h$  depends on  $x^a$  and  $x^3$ . For fixed  $x^a \in \Sigma_2$ , one can choose  $h \in \Omega G = \text{Map}(S^1, G)/G$ . We denote by  $\mathcal{N}$  the space of all  $\mathcal{A}_3$  given by (54) and define the projection  $\pi: \mathcal{N} \rightarrow G$  on the space  $G$  parametrizing  $\mathcal{A}_3^0$  since we want to keep only  $\mathcal{A}_3^0$  in the limit  $\varepsilon \rightarrow 0$ . We denote by  $Q$  the fibers of the projection  $\pi$ .

In the adiabatic approach, it is assumed that  $\mathcal{A}_3^0$  depends on  $x^a \in \Sigma_2$  only via the moduli parameters  $(X^\alpha, \theta^{A\rho}) \in G$ . Therefore, the moduli define the maps

$$(X, \theta^p): \Sigma_3 \rightarrow G \quad (55)$$

which are not arbitrary; they are constrained by Eqs. (52) and (53). The derivatives  $\partial_a \mathcal{A}_3$  of  $\mathcal{A}_3 \in \mathcal{N}$  belong to the tangent space  $T_{\mathcal{A}_3} \mathcal{N}$  of the space  $\mathcal{N}$ . Using the projection  $\pi: \mathcal{N} \rightarrow G$ , one can decompose  $\partial_a \mathcal{A}_3$  into two parts,

$$T_{\mathcal{A}_3} \mathcal{N} = \pi^* T_{\mathcal{A}_3^0} G \oplus T_{\mathcal{A}_3} \mathcal{Q} \Leftrightarrow \partial_a \mathcal{A}_3 = \Pi_a^\Delta \xi_{\Delta 3} + D_3 \epsilon_a, \quad (56)$$

where  $\Delta = (\alpha, Ap)$  and

$$\Pi_a^\alpha := \partial_a X^\alpha - i \delta_{pq} \bar{\theta}^p \gamma^\alpha \partial_a \theta^q, \quad \Pi_a^{Ap} := \partial_a \theta^{Ap}. \quad (57)$$

In (56),  $\epsilon_a$  are  $\mathfrak{g}$ -valued parameters ( $D_3 \epsilon_a \in T_{\mathcal{A}_3} \mathcal{Q}$ ) and the vector fields  $\xi_{\Delta 3}$  on  $G$  can be identified with the generators  $\xi_\Delta = (\xi_\alpha, \xi_{Ap})$  of  $G$ .

On  $\xi_{\Delta 3}$  we impose the gauge-fixing condition

$$D_3 \xi_{\Delta 3} = 0 \stackrel{(56)}{\Rightarrow} D_3 D_3 \epsilon_a = D_3 \partial_a \mathcal{A}_3. \quad (58)$$

Recall that  $\mathcal{A}_3$  is fixed by (54) and  $\mathcal{A}_a$  are yet free. In the adiabatic approach, one chooses  $\mathcal{A}_a = \epsilon_a$  (cf., [5,6]) and obtains

$$\mathcal{F}_{a3} = \partial_a \mathcal{A}_3 - D_3 \mathcal{A}_a = \Pi_a^\Delta \xi_{\Delta 3} \in T_{\mathcal{A}_3^0} G. \quad (59)$$

Substituting (59) into the first equation in (52), we see that they are resolved due to (58). Substituting (59) into the action  $S_0 = \lim_{\epsilon \rightarrow 0} S_\epsilon$  given by (49) and integrating over  $x^3$ , we obtain the effective action

$$S_0 = 2\pi \int_{\Sigma_2} d^2 x \sqrt{|\det g_{\Sigma_2}|} g^{ab} \Pi_a^\alpha \Pi_b^\beta \eta_{\alpha\beta}, \quad (60)$$

which coincides with the kinetic part of the Green-Schwarz superstring action [17]. One can show (cf., [14]) that the second equations in (52) are equivalent to the Euler-Lagrange equations for  $(X^\alpha, \theta^{Ap})$  following from (60). Finally, substituting (59) into (53), we obtain the equations

$$\Pi_a^\alpha \Pi_b^\beta \eta_{\alpha\beta} - \frac{1}{2} g_{ab} g^{cd} \Pi_c^\alpha \Pi_d^\beta \eta_{\alpha\beta} = 0, \quad (61)$$

which can also be obtained from (60) by variation of  $g^{ab}$ .

For getting the full Green-Schwarz superstring action one should add to (60) a Wess-Zumino-type term which is described as follows [16,17]. One should consider a Lorentzian 3-manifold  $\Sigma_3$  with the boundary  $\Sigma_2 = \partial \Sigma_3$  and coordinates  $x^{\hat{a}}$ ,  $\hat{a} = 0, 1, 2$ . On  $\Sigma_3$ , one introduces the 3-form [16]

$$\Omega_3 = i d x^{\hat{a}} \Pi_{\hat{a}}^\alpha \wedge (\check{d}\bar{\theta}^1 \gamma^\beta \wedge \check{d}\theta^1 - \check{d}\bar{\theta}^2 \gamma^\beta \wedge \check{d}\theta^2) \eta_{\alpha\beta} = \check{d}\Omega_2, \quad (62)$$

where

$$\Omega_2 = -i \check{d} X^\alpha \wedge (\bar{\theta}^1 \gamma^\beta \check{d}\theta^1 - \bar{\theta}^2 \gamma^\beta \check{d}\theta^2) \quad \text{with} \\ \check{d} = d x^{\hat{a}} \frac{\partial}{\partial x^{\hat{a}}}.$$

Then the term

$$S_{\text{WZ}} = \int_{\Sigma_3} \Omega_3 = \int_{\Sigma_2} \Omega_2 \quad (63)$$

is added to (60) with a proper coefficient  $\kappa$  and  $S_{\text{GS}} = S_0 + \kappa S_{\text{WZ}}$  is the Green-Schwarz action for the superstrings of type I, IIA, and IIB.

To get (63) from Yang-Mills theory, we consider the manifold  $\Sigma_3 \times S^1$  and notice that in addition to (59) we now have the components

$$\begin{aligned} \mathcal{F}_{03} &= \Pi_0^\Delta \xi_{\Delta 3} \\ &= (\partial_0 X^\alpha - i \delta_{pq} \bar{\theta}^p \gamma^\alpha \partial_0 \theta^q) \xi_{\alpha 3} + (\partial_0 \theta^{Ap}) \xi_{Ap3}. \end{aligned} \quad (64)$$

We introduce 1-forms  $F_3 := \mathcal{F}_{\hat{a}3} d x^{\hat{a}}$  on  $\Sigma_3$ , where  $\mathcal{F}_{\hat{a}3}(\epsilon)$  are general Yang-Mills fields on  $\Sigma_3 \times S^1$  which take the form (59), (64) only in the limit  $\epsilon \rightarrow 0$ , and consider the functional

$$S_{\text{WZ}}^{\text{YM}} = \int_{\Sigma_3 \times S^1} f_{\Delta\Lambda\Gamma} F_3^\Delta \wedge F_3^\Lambda \wedge F_3^\Gamma \wedge d x^3, \quad (65)$$

where the explicit form of the constant  $f_{\Delta\Lambda\Gamma}$  can be found in [16]. Therefore, the Yang-Mills action (49) plus (65) in the adiabatic limit  $\epsilon \rightarrow 0$  becomes the Green-Schwarz action. This result can be considered as a generalization of the Green result [19] who derived the superstring theory in a fixed gauge from Chern-Simons theory on  $\Sigma_2 \times \mathbb{R}$ .

## XI. CONCLUDING REMARKS

We have shown that bosonic strings and Green-Schwarz superstrings can be obtained via the adiabatic limit of Yang-Mills theory on manifolds  $\Sigma_2 \times H^2$  with a Wess-Zumino-type term. Notice that the constraint equations (15) on the Yang-Mills energy momentum tensor with  $\epsilon > 0$  are important for restoring the unitarity of Yang-Mills theory on  $\Sigma_2 \times H^2$ . More interestingly, the same result is also obtained by considering Yang-Mills theory on three-dimensional manifolds  $\Sigma_2 \times S_\epsilon^1$  with the radius of the circle  $S_\epsilon^1$  given by  $\epsilon \in [0, 1]$ . For  $\epsilon \neq 0$  we have well-defined quantum Yang-Mills theory on  $\Sigma_2 \times S_\epsilon^1$ . For  $\epsilon \rightarrow 0$  we get superstring theories. This raises hopes that various results for superstring theories can be obtained from results of the associated Yang-Mills theory on  $\Sigma_2 \times S_\epsilon^1$ .

## ACKNOWLEDGMENTS

This work was partially supported by the Deutsche Forschungsgemeinschaft Grant No. LE 838/13.

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