Stationary cylindrically symmetric spacetimes with a massless scalar field and a nonpositive cosmological constant

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The general stationary cylindrically symmetric solution of Einstein-massless scalar field system with a nonpositive cosmological constant is presented. It is shown that the general solution is characterized by four integration constants. Two of these essential parameters have a local meaning and characterize the gravitational field strength. The other two have a topological origin, as they define an improper coordinate transformation that provides the stationary solution from the static one. The Petrov scheme is considered to explore the effects of the scalar field on the algebraic classification of the solutions. In general, these spacetimes are of type I. However, the presence of the scalar field allows us to find a nonvacuum type O solution and a wider family of type D spacetimes, in comparison with the vacuum case. The mass and angular momentum of the solution are computed using the Regge-Teitelboim method in the case of a negative cosmological constant. In absence of a cosmological constant, the curvature singularities in the vacuum solutions can be removed by including a phantom scalar field, yielding nontrivial locally homogeneous spacetimes. These spacetimes are of particular interest, as they have all their curvature invariants constant.

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I. INTRODUCTION

In vacuum, the static cylindrically symmetric spacetimes, in the absence of a cosmological constant, was found by Levi-Civita [1] just a few years after the emerging of general relativity. However, the inclusion of a nonzero cosmological constant was only achieved almost 70 years later by Linet [2] and Tian [3]. More recently, some geometrical properties of these spacetimes, such as the presence of conical singularities, were reviewed in [4,5]. The stationary cylindrically symmetric vacuum solution was discovered independently by Lanczos [6] and Lewis [7]. The general solution contains a number of integration constant, whose physical interpretation has been studied in [8,9]. In vacuum, the cylindrical stationary spacetime with a nonvanishing cosmological constant was derived in [10] and [11]. The interpretation of the integration constants was clarified in [12], where it was proved that three of them are indeed essential parameters. Two integration constants have a topological origin [13], and a third one characterizes the local gravitational field.

Despite the static cylindrically symmetric spacetimes are widely known in vacuum, exact solutions containing a massless scalar field as matter source in presence of a cosmological constant have received almost null attention until now. Previously, solutions with plane symmetry, which are a particular case of the cylindrical ones, have been reported [14,15] and other particular solutions in [16,17].[1] The main efforts on this subject can be found in [18,19] and [20] for the static and stationary cases, respectively. In these articles the existence of soliton and wormhole solutions in the presence of an arbitrary self-interaction potential for the scalar field was analyzed, providing also a useful method for obtaining general cylindrically symmetric solutions.

In this article, the general stationary cylindrically symmetric solution of Einstein-massless scalar field system with a nonpositive cosmological constant Λ is found, and its geometrical properties are studied. The aim of this work is to determine the implications of a massless scalar field in a cylindrically symmetric system. Due to the high interest in exact solutions whose asymptotic behavior approaches the anti-de Sitter spacetime, we include in the analysis a negative cosmological constant. In fact, the solutions presented here, for $\Lambda < 0$, have that asymptotic behavior. Moreover, we study the effect of a massless scalar field in the case of a vanishing cosmological constant, i.e., we explore the backreaction generated by the scalar field in the well-known Lanczos-Lewis and Levi-Civita spacetimes.

As is expected, in the absence of suitable potentials and nonminimal couplings for the scalar field, the no-hair theorem rules out solutions having event horizons, and this is precisely our case. We are just considering a massless scalar field with a constant potential (zero or negative). Thus, in general, the solutions presented here contain naked

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[1]Unfortunately, along these two articles there are inconsistencies in the signs of the cosmological constant and the kinetic term of the scalar field.

singularities, which however could have some physical interest [21].

We find that the general stationary cylindrically symmetric solution contains two different classes. In the first of them, the stationary spacetimes become static by adjusting smoothly the integration constants related with the rotation. For the second class this process is not possible, and in consequence, such a class of solutions does not have a static limit. In this sense, this class has an unclear physical relevance and therefore we focus on the analysis of the solutions belonging to the first class. For this reason, hereafter we will refer to the first class as the general solution.

The article is organized as follows. In the next section, the action and the ansatz are established and the general solution is presented as a linear combination of three functions, according to the cosmological constant. Then, for a negative cosmological constant, the local properties of the solutions are studied using the Newman-Penrose (NP) formalism, where the Weyl-NP scalars allow to obtain the Petrov classification of these spacetimes. It is shown that a parameter included through the scalar field enlarges the family of spacetimes with respect to the vacuum ones. Afterwards, following [13], the stationary spacetime is obtained from the static one by means of a topological construction. These formalisms allow us to identify the four essential parameters of the general solution. One of them is the amplitude of the scalar field, which in conjunction with a second one describes the strength of the gravitational field. The remaining parameters have a topological origin and are just globally defined, because they cannot be removed by a proper coordinate transformation. Moreover, the mass and angular momentum are computed by using the Regge-Teitelboim method [22]. These conserved charges illustrate the physical meaning of the essential parameters. The case of a vanishing cosmological constant is considered in Sec. IV. We note that it is necessary to integrate the field equations from scratch, because a special class of solutions is not available by just taking the limit $\Lambda \to 0$ in the solutions presented in Sec. II. We found that these spacetimes have all their scalar invariants constant, and are supported by a phantom scalar field. After of some concluding remarks, an appendix is included. The appendix offers a very detailed derivation of the general solution. The key point is to reduce the field equations to a very simple uncoupled system of differential equations, which allow us to find (i) all the solutions and (ii) figure out how they split in two classes: the physical one, which contains the static solution, and the one lacking a static limit.

II. GENERAL STATIONARY CYLINDRICALLY SYMMETRIC SOLUTIONS

We consider the Einstein-Hilbert action with a massless scalar field and a cosmological constant Λ,

$$I = \int d^4x\sqrt{-g}\left[\frac{R-2\Lambda}{2\kappa} - \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\right], \qquad (1)$$

where $\kappa = 8\pi G$ is the gravitational constant. The stress-energy tensor turns out to be

$$T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi, \qquad (2)$$

and the field equations are given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \qquad \Box\Phi = 0. \qquad (3)$$

The general stationary, cylindrically symmetric[2] configuration can be described by the line element

$$ds^2 = g_{tt}(r)dt^2 + g_{\phi\phi}(r)d\phi^2 + g_{zz}(r)dz^2 + 2g_{t\phi}(r)dtd\phi + dr^2, \qquad (4)$$

where the coordinates range as $t \in (-\infty,\infty)$, $r \in [0,\infty)$, $z \in (-\infty,\infty)$ and $\phi \in [0, 2\pi)$, and a scalar field depending just on the radial coordinate, $\Phi = \Phi(r)$.

As shown in the Appendix, the general solution (4) of the field equations (3) can be written as a linear combination of three functions

$$\begin{aligned} g_{tt}(r) &= a_1 g_1(r) - a_0 g_0(r),\\ g_{\phi\phi}(r) &= b_1 g_1(r) - b_0 g_0(r),\\ g_{t\phi}(r) &= \sqrt{a_0 b_0}\, g_0(r) - \sqrt{a_1 b_1}\, g_1(r),\\ g_{zz}(r) &= c_0 g_2(r), \end{aligned} \qquad (5)$$

where for a negative cosmological constant $\Lambda = -3l^{-2}$,

$$g_i(r) = \left(\frac{e^{3r/l}-b}{e^{3r/l}+b}\right)^{K_i}(e^{3r/l} - b^2 e^{-3r/l})^{2/3}, \quad i = \{0,1,2\}, \qquad (6)$$

and the scalar field is given by

$$\Phi(r) = \Phi_0 + \frac{1}{2}\sqrt{\frac{\alpha}{2\kappa}}\log\left(\frac{e^{3r/l}-b}{e^{3r/l}+b}\right)^2. \qquad (7)$$

For $\Lambda = 0$, the functions are

$$g_i(r) = r^{2/3+K_i}, \qquad i = \{0,1,2\}, \qquad (8)$$

and the scalar field is

[2] In order to include spacetimes lacking a regular axis, we are adopting the less restrictive definition of cylindrical symmetry given in [12].

$$\Phi = \Phi_0 + \sqrt{\frac{\alpha}{2\kappa}}\log(r), \tag{9}$$

where the origin has been chosen at $r = 0$.

Here K_i, a_0, a_1, b, b_0, b_1, c_0, α and Φ_0 are integration constants. The constants K_i are not independent, since they verify the algebraic relations

$$K_0 + K_1 + K_2 = 0, \tag{10}$$

$$K_0K_1 + K_1K_2 + K_2K_0 = -\frac{4}{3} + \alpha. \tag{11}$$

In order to ensure a real metric and scalar field, the previous algebraic relations fix bounds for the constants. The constant α runs in the interval $0 \leq \alpha \leq 4/3$, and the constants $|K_i|$ are bounded from above by $\frac{2}{3}\sqrt{4-3\alpha}$, $\frac{1}{3}\sqrt{4-3\alpha}$, and $\frac{1}{3}\sqrt{4-3\alpha}$ in any order.

Note that the presence of the scalar field is encoded in the additional integration constant α in (11). In absence of the scalar field, the stationary solutions presented in [11], and the static ones in [2,3], are recovered.

The constant c_0 can be absorbed by rescaling the noncompact coordinate z, and only one of the constants a_0, a_1, b_0, b_1 is essential, as it will become clear in the next section.

III. ANALYSIS OF THE SOLUTIONS WITH $\Lambda < 0$

In order to get insight about the parameter b, it is convenient to start with static metric

$$ds^2 = -g_0(r)dt^2 + g_1(r)l^2d\phi^2 + g_2(r)dz^2 + dr^2. \tag{12}$$

The constant b determines the location of the axis of symmetry at $r_0 = l/3\log|b|$, and it can be removed from the scalar field by a shift of the radial coordinate $r \to r + r_0$. With this shift, b just appears as a multiplicative factor $b^{2/3}$ in g_i, and consequently, the invariants do not depend on b. In other words, b could be removed from the solution by rescaling the coordinates t, z, ϕ. However, ϕ is a compact coordinate and global properties will be modified with this rescaling. In fact, the metric with the shifted radial coordinate reduces in the absence of the scalar field to that shown in [5], where a conicity parameter equivalent to $b^{-1/3}$ is explicitly exhibited. In summary, b has no relevance for the local properties, but it is a topological parameter that contributes to the mass of the solution (see subsection III D). On the contrary, note that for $\Lambda = 0$ a shift of the radial coordinate does not have any local or global implication.

The general solution previously considered for the vacuum case does not contain a locally anti-de Sitter (AdS) spacetime [4]. Indeed, the locally AdS solution appears as a special branch disconnected from the general one [5]. The advantage of our general static solution is that it is smoothly connected to a locally AdS spacetime, and in fact, this is achieved just doing $b = 0$ in (12). Explicitly, we obtain

$$ds^2 = dr^2 + e^{2r/l}(-dt^2 + l^2d\phi^2 + dz^2), \tag{13}$$

which becomes the background required for computing the conserved charges in subsection III D.

A. Local properties

In order to obtain a deeper insight into the geometrical properties of the solution, we make use of an invariant characterization of the spacetimes. Spacetimes are usually classified according to the Petrov classification of their Weyl invariants. Note that for analyzing the local properties it is enough to consider the static solutions because, as it will be shown in the next subsection, the stationary solutions can be obtained from a topological construction, and therefore they are locally equivalent. The general solution presented above, (12), is of type I (named normally algebraically general). However, as Linet pointed out in [2], a particular choice of the constants K_0, K_1, and K_2, makes the solution an algebraically special spacetime of type D. We find that, with the inclusion of the scalar field, i.e., by means of the constant α, the Petrov type D spacetimes are no longer determined only by those particular values of K_i, but by a range of values driven by α. Namely, Petrov type D spacetimes are found for values of K_i taken as any ordering of $\pm\frac{2}{3}\sqrt{4-3\alpha}$, $\mp\frac{1}{3}\sqrt{4-3\alpha}$, and $\mp\frac{1}{3}\sqrt{4-3\alpha}$, provided $0 \leq \alpha < 4/3$. These type D spacetimes have a planar section (two K_i are equal), which allows an additional symmetry. This fourth Killing vector corresponds to a rotation or a boost in this plane depending on its signature.

A novel feature introduced by the scalar field is a nontrivial Petrov type O subfamily. In fact, for $\alpha = \frac{4}{3}$, $b \neq 0$ and vanishing K_i, a conformally flat spacetime arises, and it is given by

$$ds^2 = dr^2 + (e^{3r/l} - b^2e^{-3r/l})^{2/3}(-dt^2 + dz^2 + l^2d\phi^2). \tag{14}$$

In other words, the scalar field gives rise to a wider family of spacetimes. This Petrov type O is a new subfamily parametrized by b, which strictly emerges due to the scalar field. In this case the number of isometries is enlarged to six since we are dealing with a conformally flat spacetime. It is remarkable to have such a number of symmetries in a space endowed with a matter source, in particular since for the vacuum (nontrivial) case there are at most four Killing vectors [4]. For $b = 0$ the scalar field is trivial—it is a constant—and (14) reduces to the locally AdS spacetime (13).

Studying the Weyl and Ricci scalars of the Newman-Penrose formalism it is shown that they are singular at the axis for the whole family of solutions, except in two cases.

The first one, corresponds to the CSI spacetimes, which will be discussed in Sec. IV B. The second case appears for a constant scalar field ($\alpha = 0$) provided the constants K_i take the values $\{\pm\frac{4}{3}, \mp\frac{2}{3}, \mp\frac{2}{3}\}$, or any permutation of them [2]. Since this special solution is regular at the axis, a change of the radial coordinate r can be performed to prove that this type D solution is a black string. In fact, for $K_0 = 4/3$, and $K_1 = K_2 = -2/3$ the transformation reads

$$r = \frac{2l}{3}\log\left[\frac{\rho^{3/2} + \sqrt{\rho^3 - 4bl^3}}{2l^3}\right], \tag{15}$$

yielding the black string

$$ds^2 = -\left(\frac{\rho^2}{l^2} - \frac{4lb}{\rho}\right)dt^2 + \frac{d\rho^2}{\frac{\rho^2}{l^2} - \frac{4lb}{\rho}} + \frac{\rho^2}{l^2}dz^2 + \rho^2 d\phi^2. \tag{16}$$

Note that the original axis of symmetry at $r_0 = l/3 \log|b|$ is mapped to the horizon $\rho_+ = 2^{2/3} l b^{1/3}$, and the new axis of symmetry is located at $\rho = 0$. This black string was previously found by solving the Einstein field equations in [23], and by using an adequate coordinate transformation in [5].

B. Topological construction of the rotating solution from a static one

As explained in [13], a diagonal static metric with dependence on the spacelike coordinates r and z, and with the "angular" coordinate stretched to infinity, can be locally equivalent but globally different to a stationary axisymmetric metric obtained from a topological identification in the static spacetime. This identification is defined by two essential parameters. This kind of essential parameters cannot be removed by a permissible change of coordinates since they encode topological information. In this section we are going to build the stationary solution (4) with the metric coefficients (5), using the procedure presented in [13] in the particular case of cylindrical symmetry.

Let us consider the static solution with scalar field

$$ds^2 = -g_0(r)d\hat{t}^2 + g_1(r)l^2 d\hat{\phi}^2 + g_2(r)dz^2 + dr^2, \tag{17}$$

where g_i is given by (6) in a coordinate system $(\hat{t}, r, z, \hat{\phi})$ with $\hat{t} \in (-\infty, \infty)$, $r \in [0, \infty)$, $z \in (-\infty, \infty)$ and $\hat{\phi} \in (-\infty, \infty)$. Note that $\hat{\phi}$ is not a compact coordinate. We perform a coordinate transformation on the $(\hat{t}, \hat{\phi})$ plane given by

$$\hat{t} = \beta_0\phi + \beta_1 t, \qquad \hat{\phi} = \alpha_0\phi + \alpha_1 t, \tag{18}$$

where α_0, α_1, β_0 and β_1 are parameters. This transforms (17) into (5) by defining these parameters as follows

$$\begin{aligned}
\alpha_0 &= \frac{\sqrt{b_1}}{l}, \\
\alpha_1 &= -\frac{\sqrt{a_1}}{l}, \\
\beta_0 &= -\sqrt{b_0}, \\
\beta_1 &= \sqrt{a_0}.
\end{aligned} \tag{19}$$

As shown in [13], α_1 and β_1 are not essential parameters, and they can be set as $\alpha_1 = 0$ and $\beta_1 = 1$. On the contrary, α_0 and β_0 are essential. However, after a topological identification, which transforms the $(\hat{t}, \hat{\phi})$ plane into a cylinder, one can fix the period of the angular coordinate ϕ to 2π by choosing $\alpha_0 = 1$. Since in (17) all the coordinates are not compact, b can be absorbed by rescaling the coordinates. After identification, $\hat{\phi}$ becomes periodic and b has a topological meaning. The parameter α_0 plays the same topological role, and in fact it redefines b. Therefore, without loss of generality α_0 can be fixed, but not simultaneously with b. In other words, since from the beginning the static solution contains an arbitrary conicity parameter b, the constant α_0 can be fixed. Going back to relations (19) we find that $a_0 = 1$, $a_1 = 0$, and $b_1 = l^2$ reproduce the set of values chosen for α_0, α_1, and β_1. Then, after fixing the period as 2π there is just one essential parameter β_0 in the transformation, which will be named $-a$ hereafter. Then, the transformation (18) reduces to

$$\begin{aligned}
\hat{t} &= t - a\phi, \\
\hat{\phi} &= \phi.
\end{aligned} \tag{20}$$

In summary, a topological construction can bring the solution (17) into a locally equivalent, but globally different, solution by doing the transformation (20) to get

$$ds^2 = -g_0(r)(dt - ad\phi)^2 + g_1(r)l^2 d\phi^2 + g_2(r)dz^2 + dr^2. \tag{21}$$

Transformation (20) is not a proper coordinate transformation, since it converts an exact 1-form into a closed but not exact 1-form, as was discussed in detail in [24]. Hence, (20) only preserves the local geometry, but not the global one. Therefore, the resulting manifold is globally stationary but locally static. Hereafter, we will consider (21) instead of (5) as the general solution, because it already contains all the local and global essential information.

C. Asymptotic behavior

In order to display the asymptotic behavior of the fields, it is convenient to use the coordinate $\rho = le^{r/l}$. In this way, the behavior at large ρ is given by

$$g_{tt}(\rho) = -\frac{\rho^2}{l^2} + \frac{2blK_0}{\rho} + O(\rho^{-4}),$$
$$g_{\phi\phi}(\rho) = \rho^2\left(1 - \frac{a^2}{l^2}\right) + \frac{2lb(-l^2K_1 + a^2K_0)}{\rho} + O(\rho^{-4}),$$
$$g_{t\phi}(\rho) = \frac{\rho^2 a}{l^2} - \frac{2blaK_0}{\rho} + O(\rho^{-4}),$$
$$g_{zz}(\rho) = \frac{\rho^2}{l^2} - \frac{2blK_2}{\rho} + O(\rho^{-4}), \qquad g_{\rho\rho}(\rho) = \frac{l^2}{\rho^2},$$
$$\Phi(\rho) = \Phi_0 + \sqrt{\frac{2\alpha}{\kappa}}\frac{bl^3}{\rho^3} + O(\rho^{-9}). \tag{22}$$

One can note that the metric asymptotically approaches a locally AdS spacetime, as the scalar field becomes constant. The background is fixed by setting $a = b = \alpha = \Phi_0 = 0$, which corresponds to a locally AdS spacetime.

D. Mass and angular momentum

The mass and angular momentum of the solutions are determined using the Regge-Teitelboim method [22]. In the canonical formalism, the generator of an asymptotic symmetry associated to the vector $\xi = (\xi^\perp, \xi^i)$ is built as a linear combination of the constraints $\mathcal{H}_\perp, \mathcal{H}_i$, with an additional surface term $Q[\xi]$

$$H[\xi] = \int d^3x(\xi^\perp\mathcal{H}_\perp + \xi^i\mathcal{H}_i) + Q[\xi]. \tag{23}$$

A suitable choice of this surface term attains the generator has well-defined functional derivatives with respect to the canonical variables [22]. The surface term $Q[\xi]$ is the conserved charge under deformations ξ provided the constraints vanish. For the action (1), the variation of $Q[\xi]$ is given by

$$\delta Q[\xi] = \oint d^2S_l\left[\frac{G^{ijkl}}{2\kappa}(\xi^\perp\delta g_{ij;k} - \xi^\perp{}_{,k}\delta g_{ij}) + 2\xi_k\delta\pi^{kl} + (2\xi^k\pi^{jl} - \xi^l\pi^{jk})\delta g_{jk} - (\sqrt{g}\xi^\perp g^{lj}\Phi_{,j} + \xi^l\pi_\Phi)\delta\Phi\right], \tag{24}$$

where $G^{ijkl} \equiv \sqrt{g}(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl})/2$ and the semicolon stands for the covariant differentiation in the spacelike hypersurface. The canonical variables are the spatial metric g_{ij} and the scalar field Φ together with their respective conjugate momenta π^{ij} and π_Φ.

To evaluate $\delta Q[\xi]$ we consider as asymptotic conditions just the asymptotic behavior of the solutions with a negative cosmological constant (22), where the integration constants K_i, a, b, α are allowed to be varied. The additive constant of the scalar field Φ_0 is considered as a fixed constant without variation, in order to save the asymptotic scale invariance.[3]

Since the solution is in the comoving frame along z, the corresponding momentum $Q[\partial_z]$ vanishes. Then, the only nonvanishing charges are those associated to symmetry under time translations and the rotational invariance, the mass and angular momentum, respectively. Defining $q[\xi]$ as the charge by unit length $Q[\xi] = \int q[\xi]dz$, we can obtain from (22) and (24), the explicit form of $\delta q[\xi]$

$$\delta q[\xi] = \frac{6\pi}{\kappa}[-\xi^t\delta(b(K_1 + K_2)) + \xi^\phi\delta(ab(K_1 - K_0))]. \tag{25}$$

Thus, using $\kappa = 8\pi G$, the mass $M = q[\partial_t]$ and angular momentum $J = q[\partial_\phi]$ per unit length are

$$M = \frac{3b}{4G}K_0, \qquad J = \frac{3ab}{4G}(K_1 - K_0). \tag{26}$$

These global charges are defined up to an additive constant without variation. In order to set the locally AdS spacetime (13) as a background, these additive constants must be chosen to be null.

As we can see from the expression for the angular momentum, there are two manners of turning off the angular momentum. The first one is by doing $a = 0$, which cancels the off-diagonal term $g_{t\phi}$ in the metric. The second way is less obvious, since it is achieved by considering $K_0 = K_1$. Indeed, this particular choice of the parameters yields a static solution of type D. This can be shown from the coordinate transformation

$$d\phi \to d\phi + \frac{a}{(a^2 - l^2)}dt, \qquad dt \to dt. \tag{27}$$

As analyzed in [13], this transformation contains an inessential parameter $\alpha_1 = a/(a^2 - l^2)$, which does not change the topology. Therefore, the solution with $K_0 = K_1$ is no just locally equivalent to the static solution, but also globally.

IV. ANALYSIS OF SOLUTIONS WITH $\Lambda = 0$

The limit $\Lambda \to 0$, or equivalently $l \to \infty$, in the configurations given by Eqs. (6) and (7) in Sec. II, does not provide all the solutions coming from a direct integration of the field equations. In fact, as shown in the Appendix, two classes of solutions are obtained. The first type corresponds to solutions that match the limit $\Lambda \to 0$ in the configurations introduced in Sec. III, and they are dubbed as Levi-Civita type spacetimes. The second type is formed by spacetimes having all their invariants constant. These two types will be analyzed in detail below. The discussion in this section is focused on static solutions. The topological construction explained in Sec. III does not depend on the value of the cosmological constant, and in consequence, the stationary solutions for $\Lambda = 0$ can be obtained from the

[3]For $\delta\Phi_0 \neq 0$, $\delta Q[\xi]$ contains a term proportional to $\oint d^2S\xi^t\sqrt{\alpha}b\delta\Phi_0$. The integration of this term requires a boundary condition relating Φ_0 with α and b.

improper transformation (20). Since (20) is a local transformation, the static configuration and its stationary counterpart share the same local properties.

A. Levi-Civita type spacetimes

In this subsection, we analyze a Levi-Civita type spacetime in presence of a massless scalar field. The algebraic relations (10) and (11) determine two essential constants related to the gravitational and scalar field strengths. Since ϕ is an angular coordinate with a given period, the constant b_1 in (5) cannot be absorbed by a rescaling of this coordinate keeping the same period. Then, b_1 is a third essential parameter and plays a topological role in the same way as b in Sec. III. The transformation (20) provides the fourth essential parameter for the stationary solution.

As in Sec. III, we study the local properties through the Petrov classification. Normally the solution is algebraically general as occurs in vacuum [25], but algebraically special spacetimes are also possible to be found. The scalar field parametrizes three families of type D spacetimes, which will be described in Table 1. Two of these families (S_1 and S_2) are allowed only for a nonvanishing scalar field, while the third one (S_3) reduces to the three known vacuum type D Levi-Civita spacetimes by switching off the scalar field and by circular permutations of K_i. A nontrivial type O spacetime emerges strictly from the scalar field. In this case $K_0 = K_1 = K_2 = 0$ and $\alpha = 4/3$ yielding the conformally flat metric

$$ds^2 = dr^2 + r^{2/3}(-dt^2 + dz^2 + b_1 d\phi^2). \qquad (28)$$

This is the counterpart with $\Lambda = 0$ of the conformally flat spacetime described in (14).

It is found that the nonvanishing components of the Riemann tensor $R^{\mu\nu}{}_{\lambda\rho}$ and Kretschmann scalar are proportional to r^{-2} and r^{-4}, respectively. Then, the spacetime is asymptotically locally flat.

Until now, we have considered a nonvanishing constant u_1 [defined by (A38) in the Appendix]. However, when we consider $u_1 = 0$, the functional form of $g_i(r)$ is drastically modified. This new branch of solutions, which is not directly provided by the limit $\Lambda \to 0$ in Sec. III, are analyzed in next subsection.

TABLE I. Petrov D spacetimes for $\Lambda = 0$. The constants K_i are classified in three sets, and depend on the amplitude of the scalar field α. Within each set K_0, K_1, and K_2 can be taken in any order. The last column shows the range of α allowed for each set. The first two sets are exclusive for a nonconstant scalar field ($\alpha \neq 0$), and the third one also includes a trivial scalar field.

	K_0	K_1	K_2	α
S_1	$\frac{2}{3}\sqrt{4-3\alpha}$	$-\frac{1}{3}\sqrt{4-3\alpha}$	$-\frac{1}{3}\sqrt{4-3\alpha}$	$(0,\frac{4}{3})$
S_2	$-\frac{2}{3}$	$\frac{1}{3}\pm\sqrt{1-\alpha}$	$\frac{1}{3}\mp\sqrt{1-\alpha}$	$(0,1]$
S_3	$-\frac{2}{3}\sqrt{4-3\alpha}$	$\frac{1}{3}\sqrt{4-3\alpha}$	$\frac{1}{3}\sqrt{4-3\alpha}$	$[0,\frac{4}{3})$

B. CSI spacetimes

In general, the Levi-Civita type spacetimes discussed above possess curvature invariants which are singular at $r = 0$. However, it is possible to find regular spacetimes, i.e., spacetimes free of any curvature singularity, where in addition, all polynomial scalar invariants constructed from the Riemann tensor and its covariant derivatives are constant. These spacetimes are known as constant scalar invariant (CSI) spacetimes. In this subsection, a nontrivial CSI spacetime due to the presence of the scalar field is presented. It is found that it is required to switch off the cosmological constant in order to get this class of spacetimes. This case is of particular interest since it provides a nonvacuum solution with constant curvature scalars. For simplicity, only the static cases will be considered, since the stationary CSI spacetimes containing a static limit can be obtained by performing the coordinate transformation (20).

From the field equations (3) one can obtain the Ricci scalar, which reads

$$R = 4\Lambda + \kappa\Phi'^2 = 4\Lambda + \frac{P_0^2}{2U^2}, \qquad (29)$$

where the last equality comes from (A28). Assuming $P_0 \neq 0$, i.e., a nontrivial scalar field, $U = u_0 =$ constant is a necessary condition for a CSI spacetime. Since $U' = 0$ is not a solution for a nonvanishing Λ, there are no CSI spacetimes in this case. However, for $\Lambda = 0$ the function U becomes a constant by setting $u_1 = 0$ in Eq. (A38), and consequently, a must be negative. Thus, the candidates to CSI spacetimes are the ones lacking a static limit. Nevertheless, if a phantom scalar field is considered, i.e., if we replace P_0^2 by $-P_0^2$, there are no restrictions on the sign of a for $u_1 = 0$. In this way, it is possible to find the static CSI solution with $a \geq 0$.

The solution in this case are given by exponentials,

$$g_i(r) = e^{K_i r}, \qquad i = \{0, 1, 2\}, \qquad (30)$$

and the scalar field is a linear function, $\Phi(r) = \Phi_0 + \sqrt{\alpha/(2\kappa)}r$, where $\alpha = -P_0^2/u_0^2$. The integration constants satisfy the algebraic relations,

$$K_0 + K_1 + K_2 = 0,$$
$$K_0K_1 + K_1K_2 + K_2K_0 = \alpha. \qquad (31)$$

Thus, from (31) we obtain,

$$K_0 = \frac{1}{2}\left(-K_2 \pm \sqrt{-3(K_2)^2 - 4\alpha}\right), \qquad (32)$$

$$K_1 = \frac{1}{2}\left(-K_2 \mp \sqrt{-3(K_2)^2 - 4\alpha}\right). \qquad (33)$$

Note that the reality condition of the line element demands $\alpha < -\frac{3}{4}(K_2)^2$ in accordance with the fact of considering a phantom scalar field. This means that the presence of a phantom scalar field makes it possible to remove curvature singularities present in the vacuum solutions. The Petrov classification indicates that these spacetimes are type D.

In order to verify that these spacetimes are indeed CSI spacetimes, we make use of a theorem proved in [26,27]. The theorem states that any four dimensional locally homogeneous spacetime is a CSI spacetime. The static line element with the metric coefficients (30) has three trivial Killing vectors ∂_t, ∂_z, and ∂_ϕ. However, it is possible to find a fourth Killing vector given by

$$\xi^{(4)} = \left(-\frac{1}{2}K_0 t, -\frac{1}{2}K_1\phi, -\frac{1}{2}K_2 z, 1\right), \tag{34}$$

in the coordinate system (t, ϕ, z, r), which in addition to the trivial ones, form a transitive group of isometries. Therefore, this spacetime is locally homogeneous.

V. CONCLUDING REMARKS

In this paper, the general stationary cylindrically symmetric solution of Einstein-massless scalar field system with a nonpositive cosmological constant has been found, and its local and global properties has been studied. As shown in the Appendix, there is an additional class of solutions, which fail in having a static limit. Due to their still unclear physical relevance, we have exclude them from the discussion along the main text.

Four integration constants are essential parameters for the general solution. This means that these parameters encode all the relevant physical information. One is the amplitude of the scalar field, which beside a second one present in the metric, characterize the gravitational field strength. The other two parameters have a topological origin, since they appear in the improper gauge transformation that allows us to obtain the stationary solution from the static one. The meaning of these parameters can be also analyzed from the expressions for the mass and angular momentum of the solutions with a negative cosmological constant.

The Petrov classification was performed to explore the effects of the scalar field on the vacuum solutions for a negative and a vanishing cosmological constant. The inclusion of the scalar field enlarges the family of solutions in comparison with the vacuum case. Thus, type D solutions are now parametrized by the amplitude of the scalar field and nontrivial type O solutions have been found in presence of a nonvanishing scalar field. These conformally flat solutions endowed with a matter field have six Killing vectors. Note that in the vacuum case, there are not type O solutions apart from the trivial ones, the locally Minkowski (for $\Lambda = 0$) and the locally AdS spacetime (for $\Lambda < 0$).

Another interesting case occurs for $\Lambda = 0$. There are special type D solutions which are possible only if the scalar field is present. We have shown that these spacetimes have a fourth Killing vector, which completes a transitive group of isometries, and consequently they are locally homogeneous. Thus, these solutions become CSI spacetimes dressed by a phantom scalar field.

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APPENDIX: SOLVING THE FIELD EQUATIONS

In this Appendix we present a complete and detailed derivation of the general solution for a massless stationary cylindrically symmetric scalar field in the presence of a nonpositive cosmological constant in four spacetime dimensions. The methodology used in this derivation is based on that proposed in [20].[4]

Let us consider the general stationary cylindrically symmetric metric

$$ds^2 = e^{2\alpha}dr^2 + e^{2\mu}dz^2 + e^{2\beta}d\phi^2 - e^{2\gamma}(dt - Se^{-2\gamma}d\phi)^2, \tag{A1}$$

where $\alpha, \beta, \gamma, \mu, S$ are functions of the radial coordinate r. The nonvanishing components of the Ricci tensor are

$$R^r{}_r = \bar{R}^r{}_r + 2\omega^2, \tag{A2}$$

$$R^z{}_z = \bar{R}^z{}_z, \tag{A3}$$

$$R^\phi{}_\phi = \bar{R}^\phi{}_\phi + 2\omega^2 + WSe^{-2\gamma}, \tag{A4}$$

$$R^t{}_t = \bar{R}^t{}_t - 2\omega^2 - WSe^{-2\gamma}, \tag{A5}$$

$$R^\phi{}_t = -W, \tag{A6}$$

$$R^t{}_\phi = e^{-2\gamma}[S(\bar{R}^\phi{}_\phi - \bar{R}^t{}_t + 4\omega^2) + W(e^{2\beta} + S^2e^{-2\gamma})]. \tag{A7}$$

The auxiliary functions ω and W appearing above are defined as

[4]Our results contain the static solution with $\Lambda < 0$ in [19], and the stationary solution with $\Lambda = 0$ in [20], in the case of a normal scalar field.

$$\omega \equiv \frac{1}{2} e^{\gamma-\beta-\alpha} (S e^{-2\gamma})', \tag{A8}$$

$$W \equiv e^{-\alpha-\beta-\gamma-\mu} (\omega e^{2\gamma+\mu})', \tag{A9}$$

and the barred symbols denote the Ricci tensor components for the static metric obtained from (A1) by setting $S = 0$, which are

$$-e^{2\alpha} \bar{R}^r{}_r = \beta'' + \gamma'' + \mu'' + \beta'^2 + \gamma'^2 + \mu'^2 - \alpha'(\beta' + \gamma' + \mu'), \tag{A10}$$

$$-e^{2\alpha} \bar{R}^z{}_z = \mu'' + \mu'(\beta' + \gamma' + \mu' - \alpha'), \tag{A11}$$

$$-e^{2\alpha} \bar{R}^\phi{}_\phi = \beta'' + \beta'(\beta' + \gamma' + \mu' - \alpha'), \tag{A12}$$

$$-e^{2\alpha} \bar{R}^t{}_t = \gamma'' + \gamma'(\beta' + \gamma' + \mu' - \alpha'). \tag{A13}$$

The field equations can be expressed as

$$R^\mu{}_\nu = \Lambda \delta^\mu{}_\nu + \kappa \partial^\mu \Phi \partial_\nu \Phi \equiv \tau^\mu{}_\nu, \tag{A14}$$

where $\tau^\mu{}_\nu$ is reduced to $\mathrm{diag}(\Lambda + \kappa e^{-2\alpha} \Phi'^2, \Lambda, \Lambda, \Lambda)$ for the metric (A1) and a scalar field depending only on r.

Since $\tau^\phi{}_t = 0$, Eq. (A6) implies $W = 0$. Then, from (A9) we obtain

$$\omega = \omega_0 e^{-2\gamma-\mu}, \tag{A15}$$

where ω_0 is an integration constant. The definition (A8) yields $(S e^{-2\gamma})' = 2\omega_0 e^{\beta+\alpha-3\gamma-\mu}$, so that

$$S = e^{2\gamma} \left(S_0 + 2\omega_0 \int e^{\alpha+\beta-3\gamma-\mu} dr \right), \tag{A16}$$

where S_0 is a second integration constant (hereafter, all the quantities with subscripts 0, 1, or 2 denote integration constants). Then, the field equations are reduced to

$$\bar{R}^r{}_r = \Lambda + \kappa e^{-2\alpha} \Phi'^2 - 2\omega^2, \tag{A17}$$

$$\bar{R}^z{}_z = \Lambda, \tag{A18}$$

$$\bar{R}^\phi{}_\phi = \Lambda - 2\omega^2, \tag{A19}$$

$$\bar{R}^t{}_t = \Lambda + 2\omega^2. \tag{A20}$$

The components ϕ_t and t_ϕ of the field equations are satisfied by virtue of $W = 0$ and Eqs. (A19), (A20). The equation for the scalar field,

$$\Phi'' + (\beta' + \gamma' + \mu' - \alpha')\Phi' = 0, \tag{A21}$$

admits a first integral given by

$$\Phi' = \frac{P_0}{\sqrt{2\kappa}} e^{\alpha-\beta-\gamma-\mu}. \tag{A22}$$

We choose the gauge $\alpha = 0$. Introducing the functions U, V, σ as follows,

$$\mu = \log U - \sigma, \qquad \beta = \frac{1}{2}(\sigma - \log V), \qquad \gamma = \frac{1}{2}(\sigma + \log V), \tag{A23}$$

an equivalent system of equations is obtained,

$$\bar{R}^z{}_z + \bar{R}^\phi{}_\phi + \bar{R}^t{}_t = -\frac{U''}{U} = 3\Lambda, \tag{A24}$$

$$\bar{R}^\phi{}_\phi + \bar{R}^t{}_t = -\sigma'' - \frac{U'}{U}\sigma' = 2\Lambda, \tag{A25}$$

$$\bar{R}^\phi{}_\phi - \bar{R}^t{}_t = \frac{V''}{V} - \frac{V'^2}{V^2} + \frac{U'V'}{UV} = -\frac{4\omega_0^2}{U^2V^2}, \tag{A26}$$

$$\bar{R}^r{}_r = -\frac{3}{2}\sigma'^2 - \frac{U''}{U} + \frac{2\sigma' U'}{U} - \frac{V'^2}{2V^2} = \Lambda + \frac{P_0^2}{2U^2} - \frac{2\omega_0^2}{U^2V^2}. \tag{A27}$$

In terms of these functions, the scalar field and the metric reads

$$\Phi = \Phi_0 + \frac{P_0}{\sqrt{2\kappa}} \int \frac{dr}{U}, \tag{A28}$$

and

$$ds^2 = dr^2 + U^2 e^{-2\sigma} dz^2 + \frac{e^\sigma}{V} d\phi^2 - V e^\sigma \left(dt - \frac{S}{V e^\sigma} d\phi \right)^2, \tag{A29}$$

respectively, with

$$S = V e^\sigma \left(S_0 + 2\omega_0 \int \frac{dr}{UV^2} \right). \tag{A30}$$

Note that $U^2 V' V \times \mathrm{Eq.(A26)} = -(U^2 \times \mathrm{Eq.(A27)})'$, so that in the general case $V' \neq 0$ it is enough to consider just Eq. (A27) because it implies (A26). In the special case $V' = 0$, Eq. (A26) yields $\omega_0 = 0$ and (A27) becomes

$$-\frac{3}{2}\sigma'^2 + \frac{2\sigma' U'}{U} = -2\Lambda + \frac{P_0^2}{2U^2}. \tag{A31}$$

Equations (A25) and (A24) yield

$$\sigma = \begin{cases} \sigma_0 + \sigma_1 \int \frac{dr}{U} & : \Lambda = 0, \\ \sigma_0 + \log U^{2/3} + \sigma_1 \int \frac{dr}{U} & : \Lambda \neq 0. \end{cases} \tag{A32}$$

Replacing (A32) and (A24) in (A27) we get

$$U^2V'^2 - aV^2 - 4\omega_0^2 = 0, \tag{A33}$$

where $a = 4\sigma_1 U' - 3\sigma_1^2 - P_0^2$ for $\Lambda = 0$, and $a = 4U'^2/3 + 4\Lambda U^2 - P_0^2 - 3\sigma_1^2$ otherwise. Equation (A24) implies that U' and $4U'^2/3 + 4\Lambda U^2$ are constants for $\Lambda = 0$ and $\Lambda \neq 0$, respectively. Therefore, in both cases a is a constant.

The change of variable

$$x = \int \frac{dr}{U}, \tag{A34}$$

transforms Eq. (A33) into

$$\left(\frac{dV}{dx}\right)^2 - aV^2 - 4\omega_0^2 = 0, \tag{A35}$$

which can be integrated by quadrature yielding

$$V = e^{\sqrt{a}(x-x_0)} - \frac{\omega_0^2}{a} e^{-\sqrt{a}(x-x_0)}, \quad \text{if } a > 0, \tag{A36a}$$

$$V = 2\omega_0 x + V_0, \quad \text{if } a = 0, \tag{A36b}$$

$$V = \frac{2\omega_0}{\sqrt{-a}} \sin[\sqrt{-a}(x - x_0)], \quad \text{if } a < 0. \tag{A36c}$$

The integral appearing in Eq. (A30) is equivalent to $\int dx V^{-2}$. Then, from (A36) we obtain

$$\int \frac{dx}{V^2} = \begin{cases} -\frac{\sqrt{a}}{2} \frac{e^{-\sqrt{a}(x-x_0)}}{a e^{\sqrt{a}(x-x_0)} - \omega_0^2 e^{-\sqrt{a}(x-x_0)}} & : a > 0, \\ -\frac{1}{2\omega_0(2\omega_0 x + V_0)} & : a = 0, \omega_0 \neq 0, \\ \frac{x}{V_0^2} & : a = 0, \omega_0 = 0, \\ -\frac{\sqrt{-a}}{4\omega_0^2 \tan[\sqrt{-a}(x-x_0)]} & : a < 0. \end{cases} \tag{A37}$$

An important consequence can be derived from Eq. (A33) [or equivalently from (A35)]. For $a < 0$ there are no real nonvanishing solutions for this equation if $\omega_0 = 0$. This means that all the real solutions are stationary, but they do not contain a static limit. This class of solutions has a "windmill" form, as explained in [13]. On the contrary, the solutions in the case $a > 0$ can be reduced to static ones. The case $a = 0$ has two different branches. The first one, defined by the conditions $V' \neq 0, \omega_0 \neq 0$, provides stationary solutions that fail in containing a static limit. The second branch, $V' = 0, \omega_0 = 0$, corresponds to the special case mentioned before. In fact, Eq. (A31) implies $a = 0$ regardless of the value of the cosmological constant. This special branch contains solutions with static limit.

In what follows, we explicitly show all the possible solutions, which will classify according the value of a.

1. General solution with $\Lambda = 0$

From (A24) we obtain

$$U = u_0 + u_1 r, \tag{A38}$$

and from (A32),

$$\sigma = \begin{cases} \sigma_0 + \frac{\sigma_1}{u_1}\log(u_0 + u_1 r) & \text{if } u_1 \neq 0, \\ \sigma_0 + \frac{\sigma_1}{u_0} r & \text{if } u_1 = 0. \end{cases} \tag{A39}$$

Moreover, from (A34) we get

$$x = \frac{\log(u_0 + u_1 r)}{u_1} \quad \text{if } u_1 \neq 0, \tag{A40a}$$

$$x = \frac{r}{u_0} \quad \text{if } u_1 = 0, \tag{A40b}$$

and the constant a is given by

$$a = 4\sigma_1 u_1 - 3\sigma_1^2 - P_0^2. \tag{A41}$$

a. Type A solutions: $a > 0$

A necessary condition for $a > 0$ is $\sigma_1 u_1 > 0$. Then, from (A36a) and (A40a) we obtain

$$V = V_0 (u_0 + u_1 r)^{\frac{\sqrt{a}}{u_1}} - \frac{\omega_0^2}{aV_0}(u_0 + u_1 r)^{-\frac{\sqrt{a}}{u_1}}, \tag{A42}$$

where the constant V_0 is a redefinition of $e^{-\sqrt{a}x_0}$.

After some algebraic manipulations we can express the general solution in the manner indicated in the main text. The functions g_0, g_1, and g_2 are given by

$$g_i = (r + \bar{u}_0)^{K_i + \frac{2}{3}}, \tag{A43}$$

where

$$K_0 = \frac{\sigma_1 + \sqrt{a}}{u_1} - \frac{2}{3},$$
$$K_1 = \frac{\sigma_1 - \sqrt{a}}{u_1} - \frac{2}{3},$$
$$K_2 = \frac{4}{3} - \frac{2\sigma_1}{u_1}, \tag{A44}$$

and $\bar{u}_0 = u_0/u_1$. The constants a_0, a_1, b_0, b_1, c_0, and α are given by

$$a_0 = e^{\sigma_0} V_0 u_1^{\frac{\sigma_1+\sqrt{a}}{u_1}}, \qquad b_0 = S_0^2 a_0, \tag{A45}$$

$$a_1 = \frac{e^{\sigma_0}\omega_0^2}{aV_0} u_1^{\frac{\sigma_1-\sqrt{a}}{u_1}},$$
$$b_1 = \frac{e^{\sigma_0}}{aV_0}\left(1+\frac{\omega_0 S_0}{\sqrt{a}}\right) u_1^{\frac{\sigma_1-\sqrt{a}}{u_1}}, \tag{A46}$$

$$c_0 = e^{-2\sigma_0} u_1^{2-\frac{2\sigma_1}{u_1}}, \qquad \alpha = \frac{P_0^2}{u_1^2}. \tag{A47}$$

It is possible to map the condition $a > 0$ to an equivalent one in terms of K_2 and α,

$$(K_2)^2 < \frac{4}{3}\left(\frac{4}{3} - \alpha\right). \tag{A48}$$

b. Type B solutions: $a = 0$

As before, this case requires $\sigma_1 u_1 > 0$. From (A36b) and (A40a) we get

$$V = V_0 + \frac{2\omega_0}{u_1}\log(u_0 + u_1 r). \tag{A49}$$

In the general case $\omega_0 \neq 0$, the function V is not a constant and the metric has no static limit. The special case $V = V_0$ appears provided $\omega_0 = 0$ and the corresponding metric can be obtained from an improper gauge transformation in the $t - \phi$ plane. In this case, the functions g_0, g_1, and g_2 are given by (A43), where

$$K_0 = K_1 = \frac{\sigma_1}{u_1} - \frac{2}{3},$$
$$K_2 = \frac{4}{3} - \frac{2\sigma_1}{u_1}. \tag{A50}$$

The constants a_0, a_1, b_0, b_1, c_0, and α are given by

$$a_0 = e^{\sigma_0} V_0 u_1^{\frac{\sigma_1}{u_1}}, \qquad b_0 = S_0^2 a_0, \tag{A51}$$

$$a_1 = 0, \qquad b_1 = \frac{e^{\sigma_0}}{V_0} u_1^{\frac{\sigma_1}{u_1}}, \tag{A52}$$

$$c_0 = e^{-2\sigma_0} u_1^{2-\frac{2\sigma_1}{u_1}},$$
$$\alpha = \frac{P_0^2}{u_1^2}. \tag{A53}$$

In terms of K_2 and α, the condition $a = 0$ becomes

$$(K_2)^2 = \frac{4}{3}\left(\frac{4}{3} - \alpha\right). \tag{A54}$$

c. Type C solutions: $a < 0$

From (A36c) and (A40a)–(A40b), we get

$$V = \begin{cases} \frac{2\omega_0}{\sqrt{-a}} \sin\left[\sqrt{-a}\left(\frac{\log(u_0+u_1 r)}{u_1} - x_0\right)\right] & : u_1 \neq 0 \\ \frac{2\omega_0}{\sqrt{-a}} \sin\left[\sqrt{-a}\left(\frac{r}{u_0} - x_0\right)\right] & : u_1 = 0. \end{cases} \tag{A55}$$

Since that V has no definitive sign, the norm of the Killing vectors ∂_t and ∂_ϕ does not maintain a fixed sign. This type of solution has no static limit.

2. General solution with $\Lambda = -3l^{-2} < 0$

Let us consider a negative cosmological constant $\Lambda = -3l^{-2}$. Equation (A24) is easily solved. It gives

$$U = c_1 e^{3r/l} - c_2 e^{-3r/l}, \tag{A56}$$

and Eq. (A32) provides σ as

$$\sigma = \sigma_0 + \log U^{2/3} + \sigma_1 \int \frac{dr}{U}, \tag{A57}$$

where

$$x = \int \frac{dr}{U}$$
$$= \begin{cases} \frac{l\,\mathrm{sgn}(c_2)}{6\sqrt{c_1 c_2}} \log\left(\frac{e^{3r/l} - \sqrt{c_2/c_1}}{e^{3r/l} + \sqrt{c_2/c_1}}\right) & : c_1 c_2 > 0, \\ -\frac{l e^{-3r/l}}{3c_1} & : c_2 = 0, \\ -\frac{l e^{3r/l}}{3c_2} & : c_1 = 0, \\ -\frac{l\,\mathrm{sgn}(c_2)}{3\sqrt{-c_1 c_2}} \arctan\left(\sqrt{-c_1/c_2}\, e^{3r/l}\right) & : c_1 c_2 < 0. \end{cases} \tag{A58}$$

In this case, the constant a becomes

$$a = 48 c_1 c_2 l^{-2} - 3\sigma_1^2 - P_0^2. \tag{A59}$$

a. Type A solutions: $a > 0$

The case $a > 0$ requires the necessary condition $c_1 c_2 > 0$. From (A36a) and the first line in (A58)

$$V = V_0 \left(\frac{e^{3r/l} - \sqrt{c_2/c_1}}{e^{3r/l} + \sqrt{c_2/c_1}}\right)^{\frac{l\sqrt{a}}{6\sqrt{c_1 c_2}}} - \frac{\omega_0^2}{aV_0}\left(\frac{e^{3r/l} - \sqrt{c_2/c_1}}{e^{3r/l} + \sqrt{c_2/c_1}}\right)^{-\frac{l\sqrt{a}}{6\sqrt{c_1 c_2}}}. \tag{A60}$$

In the same way as for the case $\Lambda = 0$, algebraic manipulations allow us to express the general solution of this type in the form indicated in the main text. The functions g_0, g_1, and g_2 are given by

$$g_i = (e^{3r/l} - b^2 e^{-3r/l})^{2/3} \left(\frac{e^{3r/l} - b}{e^{3r/l} + b} \right)^{K_i}, \qquad \text{(A61)}$$

where

$$K_0 = \frac{(\sigma_1 + \sqrt{a})l}{6\sqrt{c_1 c_2}}, \qquad K_1 = \frac{(\sigma_1 - \sqrt{a})l}{6\sqrt{c_1 c_2}},$$
$$K_2 = \frac{-2\sigma_1 l}{6\sqrt{c_1 c_2}}, \qquad \text{(A62)}$$

and $b = \sqrt{c_2/c_1}$. The constants a_0, a_1, b_0, b_1, c_0, and α are given by

$$a_0 = e^{\sigma_0} V_0 c_1^{2/3}, \qquad b_0 = S_0^2 a_0, \qquad \text{(A63)}$$

$$a_1 = \frac{e^{\sigma_0}\omega_0^2}{aV_0} c_1^{2/3}, \qquad b_1 = \frac{e^{\sigma_0}}{aV_0}\left(1 + \frac{\omega_0 S_0}{\sqrt{a}}\right) c_1^{2/3}, \qquad \text{(A64)}$$

$$c_0 = e^{-2\sigma_0} c_1^{2/3}, \qquad \alpha = \frac{P_0^2 l^2}{36 c_1 c_2}. \qquad \text{(A65)}$$

In terms of K_2 and α the condition $a > 0$ reads

$$(K_2)^2 < \frac{4}{3}\left(\frac{4}{3} - \alpha\right). \qquad \text{(A66)}$$

b. Type B solutions: $a = 0$

For $a = 0$, the condition $c_1 c_2 > 0$ is also necessary. From (A36b) and the first line in (A58) we get

$$V = V_0 + \frac{\omega_0 l}{3\sqrt{c_1 c_2}} \log\left(\frac{e^{3r/l} - \sqrt{c_2/c_1}}{e^{3r/l} + \sqrt{c_2/c_1}} \right). \qquad \text{(A67)}$$

The special case $V = V_0$ appears provided $\omega_0 = 0$ and a corresponding metric can be obtained from an improper gauge transformation in the $t - \phi$ plane. In this case, the functions g_0, g_1, and g_2 are given by (A61), where

$$K_0 = K_1 = \frac{\sigma_1 l}{6\sqrt{c_1 c_2}}, \qquad K_2 = \frac{-2\sigma_1 l}{6\sqrt{c_1 c_2}}. \qquad \text{(A68)}$$

The constants a_0, a_1, b_0, b_1, c_0, and α are given by

$$a_0 = e^{\sigma_0} V_0 c_1^{2/3}, \qquad b_0 = S_0^2 a_0, \qquad \text{(A69)}$$

$$a_1 = 0, \qquad b_1 = \frac{e^{\sigma_0}}{V_0} c_1^{2/3}, \qquad \text{(A70)}$$

$$c_0 = e^{-2\sigma_0} c_1^{2/3}, \qquad \alpha = \frac{P_0^2 l^2}{36 c_1 c_2}. \qquad \text{(A71)}$$

In terms of K_2 and α, the condition $a = 0$ becomes

$$(K_2)^2 = \frac{4}{3}\left(\frac{4}{3} - \alpha\right). \qquad \text{(A72)}$$

c. Type C solutions: $a < 0$

In this case V is given by (A36c), where x is provided by (A58) according to the constants c_1 and c_2 appearing in the definition of U in Eq. (A56). Analogously to the case of a vanishing cosmological constant, the Killing vectors ∂_t and ∂_ϕ do not have a norm with definite sign, and the solutions do not contain a static limit.

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