First order post-Newtonian gravitational waveforms of binaries on eccentric orbits with Hansen coefficients

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The inspiral and merger of supermassive black hole binary systems with high orbital eccentricity are among the promising sources of the advanced gravitational wave observatories. In this paper we compute gravitational waveforms in the frequency domain to the first post-Newtonian order, emitted by compact binary systems with arbitrary eccentricity. Our results are fully analytic, ready-to-use expressions of the waveforms in terms of a suitable generalization of Hansen coefficients known from celestial mechanics. Secular terms induced by the eccentricity are eliminated by introducing a suitable phase shift. The obtained waveforms have a rather simple structure, greatly facilitating their use in applications.

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I. INTRODUCTION

Compact binaries (i.e., black holes, neutron stars, and white dwarfs) with nonvanishing eccentricity are promising gravitational wave (GW) sources. Stellar mass compact binaries that are driven by GW emission may be detected within the sensitivity band of the forthcoming gravitational wave observatories [1] advanced LIGO [2] and Virgo [3]. The source signals will be visible by these detectors for a longer time period because of their increased sensitivity, hence the need for an accurate description of both the orbital evolution and the emitted GWs of these systems. In the adiabatic inspiral regime of binary systems when the inspiral time scale is much larger than the time scale of the orbital evolution the perturbative post-Newtonian (PN) description can be applied to high accuracy [4]. For isolated binaries, radiation reaction drives the system toward the circularization of the orbit leading to the disappearance of any initial eccentricity [5]. The evolution of circular sources is extensively described in the literature, and by now the theoretical predictions of compact binaries with negligible eccentricity have even reached the level of fourth order in the PN approximation; see, e.g., [6].

In spite of the general circularization of binary orbits due to GW emission, when the interaction of binaries with their environment is relevant they can remain on orbits with non-negligible eccentricity even towards the end of their evolution. For example, there are indications that binaries in dense galactic nuclei [7,8], embedded in a gaseous disk [9,10] can remain eccentric until the end of their inspiral. Moreover, the interaction of supermassive black hole binaries with star populations [11,12] and the Kozai mechanism and relativistic orbital resonances in hierarchial triples [13–16] can also increase orbital eccentricity.

A standard reference to the description of the 1PN corrected Kepler motion is the work of Damour and Deruelle in Ref. [17], where *three* eccentricities, (*radial*, *time* and *angular eccentricity*) have been introduced. With the help of the Damour-Deruelle parametrization the evolution of the *semi-major axis* and the radial eccentricity due to radiation reaction has been computed by Junker and Schäfer in Ref. [18]. Recently, the explicit time evolution of the semi-major axis and the radial eccentricity has been given in Ref. [19], while the explicit phase of eccentric binaries can be found in Ref. [20]. These results have been generalized for the time and phase functions in Ref. [21].

In matched filtering the measured signal output of the detectors is cross-correlated with theoretical waveform templates. The presence of orbital eccentricity influences significantly the properties of the gravitational waveforms, resulting in the decrease of their detectability when using circular templates. Leading-order PN gravitational waveforms for binaries with eccentricity were presented in Ref. [22]. A rather complete, explicit description of the time-dependent waveform to leading PN order was given in Ref. [23], making use of the Fourier-Bessel expansion for the unperturbed motion. Frequency domain waveforms for arbitrary eccentricities with the inclusion of the relativistic pericenter precession effect were first presented in [21]. In Ref. [21] the parameter estimation accuracy for leading PN-order waveforms with different initial eccentricities using the Fisher matrix method has been analyzed, taking into account the evolution of the orbital frequency and eccentricity due to the radiation reaction. It has been shown in [21] that the precision of source localization improves significantly for supermassive black hole binaries when the eccentricity is properly taken into account. Recent works have shown that even if Fisher matrix analysis remains quite robust for high SNR signals, sources that may be detected by ground based detectors may require a different approach, e.g., Bayesian analysis, [24–26].

The first post-Newtonian eccentric waveforms with a Newtonian type parametrization for bound orbits has been computed quite some time ago by Wagoner and Will [27]. In the parametrization used in Ref. [27], secular terms have appeared, however, in the gravitational waveforms, which had to be eliminated in the 1PN-order terms. Using the Damour-Deruelle parametrization, secular terms can be eliminated with the help of the eccentric anomaly parameter as implemented in the work of Ref. [18]. Gravitational waveforms with eccentricity in the Fourier domain have been given up to 1PN order using the Damour-Deruelle parametrization [28] and more recently to 2PN order in Ref. [29]. 3PN instantaneous contributions to the spherical harmonic modes of gravitational waveforms for binary systems in general orbits were given in [30]. Moreover, for sources entering the sensitivity band of the advanced detectors with small eccentricities ($e < 0.4$) the postcircular or small-eccentricity approximation can also provide ready-to-use Fourier-domain waveforms for data analysis of eccentric inspirals [31,32].

In data analysis small residual eccentricities $e \leq 0.05$ have negligible effects and a circular search is satisfactory for the detection of such binaries [33,34]. In the case of higher eccentricities, searches that specifically target eccentric sources will be necessary. For low-mass ($\leq 10 M_\odot$) spinning and eccentric ($e < 0.6$) compact binary coalescences, however, seedless clustering provides a robust and computationally efficient method for their detection [35,36]. Another important sources of GWs are highly eccentric binaries formed, e.g., through various n-body interactions. Such binary systems emit a sequence of largely isolated gravitational-wave bursts prior to merger. As existing GW searches are not well suited to detect these signals, searches for excess power over an ensemble of time-frequency tiles have been developed [37,38]. This method achieves substantially better sensitivity to eccentric binary signals than existing localized burst searches and allows for model-independent tests of Einsteins theory in the high-velocity, strong-field regime.

The present work builds on our previous results in [21]; in this paper we compute ready-to-use eccentric 1PN waveforms in time and frequency domain using the stationary phase approximation (thereafter, SPA). We use the generalized true anomaly parametrization, which has the advantage that the solution of the equations of motion can be expressed with two eccentricities, instead of three. Secular terms appearing in the waveforms are eliminated by the use of the Poincaré-Lindstedt method in [39] and the introduction of the drift true anomaly parameter, similar to that in Ref. [40]. This new type of parametrization can also be useful in higher orders of the PN expansion. The resulting waveforms have a remarkably simple, compact analytic structure, making them quite suitable for the application in gravitational-wave parameter estimation studies. It is clearly important to extend the leading-order Fisher-type parameter estimation of the binary system to the full 1PN order, where it is essential to maximally simplify the gravitational waveforms. Our results can be used as a starting point for such a full 1PN-order Fisher-type analysis.

We present fully analytical expressions for the evolution of the orbital frequency and of the radial eccentricity up to 1PN accuracy. More specifically, we derive explicit 1PN formulas for arbitrary orbital eccentricities, including the radiation reaction terms. The solution up to 1PN order is given in terms of Appell functions, generalizing slightly some results in Ref. [21]. The time and phase functions appear explicitly in the 1PN-order frequency domain gravitational waveforms.

In order to express the gravitational waveforms with arbitrary eccentricity as simply as possible, we have made extensive use of the Hansen expansion applied in celestial mechanics. In the present work we have also given a (slight) generalization of the venerable Hansen coefficients up to 1PN order. The use of a Hansen-type expansion was natural because of the appearance of different eccentricities in the radial parametrization and the time-evolution equation (i.e., the Kepler equation). In the course of the present work we also had to extend the standard Hansen expansion to allow for cases, when the phase is not an integer multiple of the drift true anomaly. Our work focuses on the explicit, nonsecular, ready-to-use, 1PN-accuracy eccentric waveforms, which can be applied in Fisher method for gravitational waves data analysis.

The paper is organized as follows. We introduce the generalized true anomaly parametrization and the original 1PN waveform in Sec. II. Section III contains the extension of the Hansen coefficients to 1PN order and the Fourier domain SPA waveforms. The radiation reaction problem and the evolution of the time and phase functions to 1PN order are given in Sec. IV. Some of the technical details are presented in Appendixes A–E, i.e., tensor spherical harmonics (A), Damour-Deruelle parametrization (B), Hansen coefficients (C), and parametrization of the waveforms (D–E).

II. PARAMETRIZATION OF THE 1PN DYNAMICS

In this section we describe the 1PN orbital dynamics of compact binaries. Moreover, we compute the full eccentric 1PN waveform with the use of the generalized true anomaly parametrization ϕ without the appearance of secular terms in the expressions. The time-domain waveforms are given with the application of the generalized Hansen expansion. Our aim is to express the full analytic eccentric frequency-domain waveform up to 1PN order.

The equations of motion of the Newtonian and 1PN dynamics is given by [17]. The radial and angular motion can be separated in the leading order, so the Euler-Lagrange equations are

$$\left(\frac{dr}{dt}\right)^2 = D_1 + \frac{2D_2}{r} + \frac{D_3}{r^2} + \frac{D_4}{r^3}, \tag{1}$$

$$\frac{d\theta}{dt} = \frac{D_5}{r^2} + \frac{D_6}{r^3}, \tag{2}$$

where r is the relative distance, θ is the azimuthal polar angle in the orbital plane and the constants D_{1-6} depend on the conserved quantities of the perturbed motion such as the energy and the magnitude of the orbital angular momentum, see Appendix B. The constants D_{1-3}, D_5 contain Newtonian and 1PN terms while D_4 and D_6 are purely 1PN corrections.

In the following we consider the Euler-Lagrange equations and give their solution with the generalized true anomaly similarly to the Damour-Deruelle parametrization. We introduce the generalized true anomaly parametrization ϕ (denoted by χ in [41]) as

$$r = \frac{a_r(1-e_r^2)}{1+e_r\cos\phi}, \tag{3}$$

where a_r is the semi-major axis and e_r is the radial eccentricity. This parametrization has the same form as the standard, Keplerian one, with the orbital parameters a_r, e_r, containing only the leading-order (Newtonian) terms. The radial motion can be computed using the true anomaly parametrization, so we obtain a Keplerian equation for the 1PN dynamics [17]:

$$n(t-t_0) = u - e_t\sin u \equiv \mathcal{M}, \tag{4}$$

where n, u and $\mathcal{M}$ are, respectively, the mean motion, eccentric and mean anomaly parameters. We introduce yet another parametrization, where only two eccentricities appear, and the evolution of the azimuthal angle θ is still governed by a simple equation. The relations between u and ϕ is given by

$$\tan\frac{\phi}{2} = \sqrt{\frac{1+e_r}{1-e_r}}\tan\frac{u}{2}. \tag{5}$$

The angular evolution of this Keplerian motion can also be expressed with the help of the generalized true anomaly parameter. From Eqs. (3) and (4) the time evolution of the true anomaly up to 1PN order is found to be given as

$$\dot{\phi} = \frac{na_r^2(1-e_r^2)^{3/2}}{r^2(1-e_te_r+(e_r-e_t)\cos\phi)}. \tag{6}$$

The integration of Eq. (6) with the help of (3) leads to the relation

$$\theta - \theta_0 = (1+\kappa_1)\phi + \kappa_2\sin\phi, \tag{7}$$

where we have introduced the 1PN-order quantities,

$$\kappa_1 = \frac{3GM}{c^2a_r(1-e_r^2)}, \qquad \kappa_2 = \frac{G\mu e_r}{2c^2a_r(1-e_r^2)}. \tag{8}$$

In our formulas G and c denote, respectively, the gravitational constant and the speed of light, m_1, m_2 are masses of the compact binary, $M = m_1 + m_2$ is the total mass, and $\mu = m_1m_2/M$ is the reduced one. In this parametrization only radial and time eccentricities (e_r, e_t) appear while angle eccentricity does not. In Eqs. (4) and (7) t_0 and θ_0 are integration constants, and in our calculations we set $t_0 = \theta_0 = 0$.

Based on Ref. [42] the gravitational radiation field up to 1PN order can be written as

$$h_{ij}^{TT} = \frac{G}{c^4D_L}\left[\sum_{m=-2}^{2}\overset{(2)}{I}_{2m}T_{ij}^{E2,2m} + \frac{1}{c}\left(\sum_{m=-2}^{2}\overset{(2)}{S}_{2m}T_{ij}^{B2,2m} + \sum_{m=-3}^{3}\overset{(3)}{I}_{3m}T_{ij}^{E2,3m}\right) + \frac{1}{c^2}\left(\sum_{m=-3}^{3}\overset{(3)}{S}_{3m}T_{ij}^{B2,3m} + \sum_{m=-4}^{4}\overset{(4)}{I}_{4m}T_{ij}^{E2,4m}\right)\right]. \tag{9}$$

In Eq. (9) D_L denotes the luminosity distance, $T_{ij}^{E2,km}$ and $T_{ij}^{B2,km}$ are the tensorial *electric* and *magnetic* scalar harmonics which are given by Eq. (2.30d) in [42]; see Appendix A. The quantities $\overset{(k)}{I}_{km}$, $\overset{(k)}{S}_{km}$ are the kth time derivatives of the mass and current multipole moments. The explicit form of these multipoles was given by Junker and Schäfer in 1PN order with eccentric anomaly u in Refs. [18] and [28]. Later, in [29] the authors have computed the explicit time-dependent multipoles $\overset{(k)}{I}_{km}$ and $\overset{(k)}{S}_{km}$ up to 2PN.

The polarization states up to 1PN order are expressed as

$$h_{+,\times}(\phi) = h_{+,\times}^N(\phi) + h_{+,\times}^H(\phi) + h_{+,\times}^{\rm PN}(\phi), \tag{10}$$

with the Newtonian $h_{+,\times}^N(\phi)$, the half PN order $h_{+,\times}^H(\phi)$, and 1PN $h_{+,\times}^{\rm PN}(\phi)$ contributions (see Appendixes D and E). Thereafter in this section, we omit the $+/\times$ notation of the polarization states. We parametrize the 1PN gravitational waveforms using the generalized true anomaly, ϕ, and the drift anomaly, ϕ', as

$$h^{\rm PN}(\phi) = \sum_{m=0}^{4}\sum_{l=0,2,4}[(c_m^{cl}\cos m\phi + s_m^{cl}\sin m\phi)\cos l\phi' + (c_m^{sl}\cos m\phi + s_m^{sl}\sin m\phi)\sin l\phi'], \tag{11}$$

where the coefficients c_m^{cl}, s_m^{cl}, c_m^{sl}, s_m^{sl}, $c_m \equiv c_m^{s0}$ and $s_m \equiv s_m^{s0}$ depend on the radial eccentricity e_r, the mass parameters M, μ, η and the two polar angles Θ and Φ of the line of sight (see Appendix E). The drift anomaly parameter [40], ϕ', is defined as

$$\phi' = (1+\kappa_1)\phi. \tag{12}$$

The reason to introduce the drift anomaly parameter is to avoid the known secular terms appearing in the eccentric waveforms [27], when simply expanding up to 1PN order the functions of the angle θ in terms of ϕ, and instead we use the following expansion:

$$\cos\theta \approx \cos\phi' - \kappa_2 \sin\phi \sin\phi', \tag{13}$$

$$\sin\theta \approx \sin\phi' + \kappa_2 \sin\phi \cos\phi'. \tag{14}$$

The above relations will be used to eliminate secular terms in the waveforms. The dependence on the drift anomaly, ϕ', of the waveforms $h^{\mathrm{PN}}(\phi)$ can be easily eliminated by Eq. (12); however, this leads to the appearance of noninteger harmonics in the arguments.

To obtain the time dependent waveform we express the trigonometric functions of the generalized true anomaly as

$$\cos\lambda\phi = \sum_{k=0}^{\infty} C_k^{\lambda} \cos k\mathcal{M}, \qquad \sin\lambda\phi = \sum_{k=0}^{\infty} S_k^{\lambda} \sin k\mathcal{M}, \tag{15}$$

where $\lambda \in \mathbb{R}$ and C_k^{λ}, S_k^{λ} stand for the (generalized) Fourier-Bessel coefficients. We note that in the unperturbed Keplerian case, $e = e_r = e_t = e_\theta$, $\cos\lambda\phi$ and $\sin\lambda\phi$ can be expressed in terms of $\cos\phi$ and $\sin\phi$. Let us recall the classical result for Keplerian motion; see, e.g., [43]:

$$\cos\phi = -e + \frac{2(1-e^2)}{e} \sum_{k=1}^{\infty} J_k(ke) \cos k\mathcal{M}, \tag{16}$$

$$\sin\phi = 2\sqrt{1-e^2} \sum_{k=1}^{\infty} \frac{J_k'(ke)}{k} \sin k\mathcal{M}, \tag{17}$$

where the prime denotes the derivative with respect to the eccentricity e. This classical expansion gets more and more complicated for increasing values of λ.

In the following, we introduce Hansen coefficients and their extension to 1PN order in the next chapter.

III. GENERALIZATION OF HANSEN COEFFICIENTS

In celestial mechanics Hansen expansion has been well known since the 19th century (see Appendix C). In our description of time-dependent waveforms there appear Hansen coefficients and it is important to extend the Hansen expansion up to 1PN order. Hansen coefficients $X_k^{n,m}$ are introduced in the expansion

$$\left(\frac{r}{a}\right)^n \exp(im\phi) = \sum_{k=-\infty}^{\infty} X_k^{n,m} \exp(ik\mathcal{M}). \tag{18}$$

In this section n and k denote indices, not to be confused with the mean motion and pericenter drift in Sec. II and Appendix B. The definition of Hansen coefficients is

$$X_k^{n,m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{r}{a}\right)^n \exp(im\phi - ik\mathcal{M}) d\mathcal{M}. \tag{19}$$

In the waveforms there appear trigonometric functions of $m\phi$, where m is not an integer parameter. So we have to generalize the formula of the Keplerian Hansen coefficients [44],

$$X_k^{n,m} = (1+\beta^2)^{-n-1} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \binom{n-m+1}{s} \times \binom{n+m+1}{t} (-\beta)^{s+t} I_p(ke), \tag{20}$$

where $p = k - m - s + t$, $\beta = (1-\sqrt{1-e^2})/e$, and the function $I_p(z)$ is given by the contour integral

$$I_p(z) = \frac{1}{2\pi i} \oint u^{-1-p} \exp\frac{z(u-u^{-1})}{2} du. \tag{21}$$

For an integer p (i.e., m is integer) $I_p(z) = J_p(z)$, where $J_p(z)$ is the Bessel function (e.g., for the Newtonian waveform see [21]). When p is noninteger, $I_p(z) = J_p(z) + g_p(z)$, where $g_p(z)$ is the correction integral [45]

$$g_p(z) = -\frac{\sin p\pi}{\pi} \int_0^{\infty} \exp(-pu - z\sinh u) du, \tag{22}$$

for $R(z) > 0$.

Due to the appearance of different eccentricities, we have to generalize Hansen coefficients in a different way for the 1PN order. The expressions for r/a and the evolution of the mean anomaly, Eq. (C3), contain PN corrections,

$$\frac{r}{a} = (1+\beta_r^2)^{-1}(1-\beta_r y)(1-\beta_r y^{-1}), \tag{23}$$

$$\frac{d\mathcal{M}}{du} = 1 - \frac{e_t}{2}(y + y^{-1}), \tag{24}$$

where we have introduced the complex quantity $y = \exp iu$ and $\beta_r = \beta(e_r)$; see Appendix C. Then the integrand is

$$(X_k^{n,m})_{\rm PN} = \frac{(1+\beta_r^2)^{-n}}{2\pi}\int_{-\pi}^{\pi} y^{m-k}(1-\beta_r y^{-1})^{n+m} \times (1-\beta_r y)^{n-m}\left(1-\frac{e_t(y+y^{-1})}{2}\right) \times \exp\frac{ke_t(y-y^{-1})}{2}du, \tag{25}$$

which can be extended in the form of an infinite sum. The generalized Hansen coefficients up to 1PN order are

$$(X_k^{n,m})_{\rm PN} = (1+\beta_r^2)^{-n}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\binom{n-m}{s} \times \binom{n+m}{t}(-\beta_r)^{s+t}\tilde{I}_p(ke_t), \tag{26}$$

with the notation

$$\tilde{I}_p(ke_t) = I_p(ke_t) - \frac{e_t}{2}[I_{p-1}(ke_t) + I_{p+1}(ke_t)]. \tag{27}$$

We note that when m is an integer, $\tilde{I}_p(ke_t)$ can be written as

$$\tilde{I}_p(ke_t) = \left(1-\frac{p}{k}\right)J_p(ke_t) + \frac{\sin(p\pi)}{k\pi}. \tag{28}$$

In the waveform expressions, we introduce the coefficients (omitting the $+/\times$ notations for the polarizations):

$$C_m^{\pm4} = \frac{c_m^{c4} \mp s_m^{s4}}{2}, \qquad S_m^{\pm4} = \frac{s_m^{c4} \pm c_m^{s4}}{2}, \tag{29}$$

$$C_m^{\pm2} = \frac{c_m^{c2} \mp s_m^{s2}}{2}, \qquad S_m^{\pm2} = \frac{c_m^{c2} \pm s_m^{s2}}{2}. \tag{30}$$

The explicit time-dependent waveforms, Eq. (11), become

$$h^{PN}(t) = \sum_{m=0}^{4}\sum_{k=0}^{\infty}(\mathcal{C}_k^m \cos k\mathcal{M} + \mathcal{S}_k^m \sin k\mathcal{M}), \tag{31}$$

where

$$\mathcal{C}_k^m = C_m^{-4}C_k^{m_{4-}} + C_m^{+4}C_k^{m_{4+}} + C_m^{-2}C_k^{m_{2-}} + C_m^{+2}C_k^{m_{2+}} + c_m C_k^m, \tag{32}$$

$$\mathcal{S}_k^m = S_{m,}^{-4}S_k^{m_{4-}} + S_m^{+4}S_k^{m_{4+}} + S_m^{-2}S_k^{m_{2-}} + S_m^{+2}S_k^{m_{2+}} + s_m S_k^m. \tag{33}$$

Moreover, $C_0^\lambda = X_0^{0,\lambda}$, $C_k^\lambda = X_k^{0,\lambda} + X_{-k}^{0,\lambda}$ and $S_k^\lambda = X_k^{0,\lambda} - X_{-k}^{0,\lambda}$, where $X_k^{n,\lambda}$ are the generalized Hansen coefficients. These waveforms have a simple compact structure compared to the corresponding expressions in [28].

The waveform in Fourier space can be described in the stationary phase approximation of the time-dependent waveform (see Eqs. (B2) and (B3) in the Appendix B of [21]). Taking an arbitrary harmonic function $\mathcal{A}(t)\cos\Phi(t)$, where the conditions $\dot{\mathcal{A}}/\mathcal{A} \ll \dot\Phi$ and $\ddot\Phi \ll \dot\Phi^2$ hold for the amplitude $\mathcal{A}(t)$ and phase $\Phi(t)$, the Fourier transform of the function is written as

$$\mathcal{F}[\mathcal{A}(t)\sin\Phi(t)] = \mathcal{A}[f(\mathcal{T})]\sqrt{\frac{\pi}{2|\ddot\Psi[f(\mathcal{T})]|}}\exp[i(\Psi[f(\mathcal{T})] + \pi/4)], \tag{34}$$

$$\mathcal{F}[\mathcal{A}(t)\cos\Phi(t)] = \mathcal{A}[f(\mathcal{T})]\sqrt{\frac{\pi}{2|\ddot\Psi[f(\mathcal{T})]|}}\exp[i(\Psi[f(\mathcal{T})] - \pi/4)]. \tag{35}$$

Here $\Psi[f(\mathcal{T})] = 2\pi f(\mathcal{T})t[\nu(\mathcal{T})] - \Phi[\nu(\mathcal{T})]$ is the phase function, $\nu = n/(2\pi)$ is the orbital frequency, $\mathcal{T}$ is the saddle point, and the functions $t[\nu(\mathcal{T})]$ and $\Phi[\nu(\mathcal{T})]$ appearing in the above expressions can be derived from the leading-order equations for gravitational radiation by Appell functions (see the Appendix in [21]). It is necessary to add that here the phase and frequency do not split into a triplet due to pericenter precession, as happened when the precession was separately treated as in Ref. [21]), because it is taken into account in the full 1PN equations of motion. Therefore, the 1PN waveform depends on the single frequency f and phase $\Psi_k = 2\pi f(\mathcal{T}_k)t[\mathcal{T}_k] - \Phi[\mathcal{T}_k]$. The saddle points $\mathcal{T}_k$ are computed by the SPA condition $|\dot\Psi[f(\mathcal{T}_k)]| = 0$. The eccentric waveform is a trigonometric function of $k\mathcal{M}$ [see Eq. (15)], and the condition $f = k\nu$ holds. Accordingly, the waveform, Eq. (31), in the Fourier space becomes

$$h_+^{\rm PN}(f) = (4k\dot\nu)^{-1/2}\sum_{m=0}^{4}\sum_{k=0}^{\infty}[\mathcal{C}_k^{+,m}\exp(i\Psi_-) + \mathcal{S}_k^{+,m}\exp(i\Psi_+)], \tag{36}$$

$$h_\times^{\rm PN}(f) = (4k\dot\nu)^{-1/2}\sum_{m=0}^{4}\sum_{k=0}^{\infty}[\mathcal{C}_k^{\times,m}\exp(i\Psi_-) + \mathcal{S}_k^{\times,m}\exp(i\Psi_+)], \tag{37}$$

where the phase functions are $\Psi_\pm = \Psi_k \pm \pi/4$. As a next step we shall compute the phase $\Phi(\mathcal{T})$ and time $t(\mathcal{T})$ functions appearing in the 1PN waveform.

IV. RADIATION REACTION TO 1PN ORDER

To leading order, the averaged radiative change of the Newtonian semimajor axis a and eccentricity e is governed by the quadrupole formula; see Peters [5]. Using Kepler's third law, the semimajor axis a can be replaced by the orbital frequency ν to have the following relations:

$$\dot{\nu}_N = \frac{48(G\mathcal{M}_c)^{5/3}(2\pi\nu)^{11/3}}{5c^5\pi(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right), \tag{38}$$

$$\dot{e}_N = -\frac{304(G\mathcal{M}_c)^{5/3}(2\pi\nu)^{8/3}}{15c^5(1-e^2)^{5/2}}e\left(1+\frac{121}{304}e^2\right). \tag{39}$$

Here $\mathcal{M}_c = M\eta^{3/5}$ is the chirp mass of the binary system and $\eta = \mu/M$, is the symmetric mass ratio. Peters's equations, (38) and (39), can be integrated and the solution for the phase and time functions can be expressed in terms of Appell functions [21].

The averaged losses of the radial orbital parameters due to gravitational radiation reaction up to 1PN order is given by Junker and Schäfer [18]. The relation between the orbital frequency, ν, and the semi-major axis, a_r, to 1PN order can be written as

$$a_r = \frac{(GM)^{1/3}}{(2\pi\nu)^{2/3}}\left[1+(\eta-9)\frac{(2\pi GM\nu)^{2/3}}{3c^2}\right]. \tag{40}$$

The contributions to a_r can be found from D_1, D_2 defined by Eqs. (B7) in Appendix B. The time evolution of ν and e_r up to 1PN order can be written as

$$\dot{\nu} = \dot{\nu}_N + \dot{\nu}_{PN}, \tag{41}$$

$$\dot{e} = \dot{e}_N + \dot{e}_{PN}, \tag{42}$$

with

$$\dot{\nu}_{PN} = \frac{(G\mathcal{M}_c)^{7/3}(2\pi\nu)^{13/3}}{560c^7\pi\eta^{2/5}(1-e^2)^{9/2}}[16(1273-924\eta) - 24(2561+2254\eta)e^2 - 42(3885+158\eta)e^4 - (13147-1036\eta)e^6], \tag{43}$$

$$\dot{e}_{PN} = -\frac{(G\mathcal{M}_c)^{7/3}(2\pi\nu)^{10/3}}{2520c^7\eta^{2/5}(1-e^2)^{7/2}}e[8(26493-22540\eta) - 60(11598+1001\eta)e^2 - (168303-16940\eta)e^4], \tag{44}$$

where Eqs. (35) and (36) of Ref. [18] and Eq. (40) have been used. From now on the subscript r of from e_r and a_r will be omitted. Thereafter, we determine the perturbative solution to the above equations up to 1PN order.

The relation between ν and e from Eqs. (41) and (42) up to 1PN order is

$$\frac{d\nu}{de} = \frac{\dot{\nu}_N}{\dot{e}_N} + \frac{\dot{\nu}_{PN}}{\dot{e}_N} - \frac{\dot{\nu}_N\dot{e}_{PN}}{\dot{e}_N^2}. \tag{45}$$

The general solution in the Newtonian order, [i.e., without the two last terms in the right-hand side of Eq. (45)], is given as

$$\nu_N = \frac{Ce^{-18/19}(1-e^2)^{3/2}}{(1+\frac{121}{304}e^2)^{1305/2299}}, \tag{46}$$

where C is an integration constant. For later use we rewrite ν_N as

$$\nu_N = \nu_0\sigma(e)/\sigma(e_0), \quad \text{where}$$
$$\sigma(e) = e^{-18/19}(1-e^2)^{3/2}\left(1+\frac{121}{304}e^2\right)^{-1305/2299} \tag{47}$$

where ν_0 and e_0 are the initial values for $\nu_N(e_0) = \nu_0$. The general solution of Eq. (45) can be written as

$$\nu = (b_N + b_{PN})^{-3/2}, \tag{48}$$

where b_N and b_{PN} are given as

$$b_N = \frac{Ce^{12/19}(1+\frac{121}{304}e^2)^{870/2299}}{1-e^2}, \tag{49}$$

$$b_{PN} = \frac{(2\pi G\mathcal{M}_c)^{2/3}}{c^2\eta^{2/5}(1-e^2)}\left(1+\frac{121}{304}e^2\right)^{870/2299} \times\left[\frac{B_1+B_2e^2+B_3e^4}{(1+\frac{121}{304}e^2)^{3169/2299}} + {}_2F_1\left(\frac{870}{2299},\frac{13}{19},\frac{32}{19};-\frac{121}{304}e^2\right)B_4e^2\right], \tag{50}$$

together with the coefficients

$$B_1 = \frac{1153}{3192} - \frac{89}{114}\eta,$$
$$B_2 = -\frac{2293125927}{558758080} + \frac{60619}{6984476}\eta,$$
$$B_3 = -\frac{86928802699}{93871357440} + \frac{129501097}{670509696}\eta,$$
$$B_4 = \frac{703785517}{4014235680} - \frac{49913}{735208}\eta. \tag{51}$$

Here ${}_2F_1(\alpha,\beta,\gamma;z)$ is the hypergeometric function [47]. These general solutions for $\nu(e)$ and $a(e)$ are consistent with the 1PN Kepler equation (40). The evolution equations due to radiation reaction for $\dot{a}$ and $\dot{e}$ up to 1PN order imply that the solution for the semimajor axis, $a(e)$, is proportional to $b_N + b_{PN}$.

Let us identify the Newtonian expression $b_N^{-3/2} \equiv \nu_N = C_0\sigma(e)$, where $C_0 = \nu_0/\sigma(e_0)$. The integration constant C has leading-order corrections at 1PN order; therefore, we have to require the equation $\nu(e_0) = \nu_0$ to hold, in order to get the correct perturbative solution to 1PN accuracy for the orbital frequency ν, Eq. (48), which can be written as

$$\nu = \nu_0 \frac{\sigma(e)}{\sigma(e_0)} \left[1 + \frac{3}{2}\nu_0^{2/3}\left(b_{PN}(e_0) - \left(\frac{\sigma(e)}{\sigma(e_0)}\right)^{2/3} b_{PN}(e)\right)\right]. \tag{52}$$

Then our aim is to compute the time and phase functions,

$$t - t_c = \int_0^e \frac{de'}{\dot{e}(e')}, \tag{53}$$

$$\Phi - \Phi_c = 2\pi \int_0^e \frac{\nu(e')}{\dot{e}(e')} de', \tag{54}$$

up to 1PN order. The integrals in the Newtonian case are given in Appendix A of [21]. Such type of integrals can be given by extended hypergeometric functions, i.e., Appell functions [47] and similar integrands appear in 1PN order. Then we can compute the integrand of time function to 1PN order as

$$(t - t_c)_{PN} = -\frac{5c^3\Lambda_0^2}{76G\mathcal{M}_c\eta^{2/5}} \int_0^e \frac{e'^{17/19}(1-e'^2)^{-3/2}}{(1+\frac{121}{304}e'^2)^{-1181/2299}} \times \left[\frac{\tilde{B}_1 + \tilde{B}_2 e'^2 + \tilde{B}_3 e'^2}{(1+\frac{121}{304}e'^2)^{3169/2299}} + {}_2F_1\left(\frac{870}{2299}, \frac{13}{19}, \frac{32}{19}; -\frac{121}{304}e'^2\right)\tilde{B}_4 e'^2\right] de' - 4\nu_0^{2/3} b_{PN}(e_0)(t-t_c)_N, \tag{55}$$

where we have introduced the notation $\Lambda_0 = \sigma(e_0)/(2\pi\nu_0)$ which depends on the initial eccentricity and orbital frequency and $(t - t_c)_N$ is the integrated Newtonian-order time function, and the parameters $\tilde{B}_i$ are given as

$$\tilde{B}_1 = -\frac{2467}{1216} + \frac{93}{304}\eta,$$
$$\tilde{B}_2 = -\frac{4092801021}{1955653280} + \frac{202947069}{223503232}\eta,$$
$$\tilde{B}_3 = -\frac{2398183171}{7822613120} + \frac{1154703}{3492238}\eta,$$
$$\tilde{B}_4 = \frac{703785517}{1338078560} - \frac{149739}{735208}\eta. \tag{56}$$

The numerical values of the parameters, $\tilde{B}_i$, for equal-mass binaries ($\eta = 1/4$) are: $\tilde{B}_1 = -1.95$, $\tilde{B}_2 = -1.87, \tilde{B}_3 = -0.22$ and $\tilde{B}_4 = 0.48$. Note that in Eq. (55), the last term is originating from the solution of the orbital frequency in Eq. (52). As one can easily see (e.g., from a suitable integral representation [47]), the hypergeometric function, ${}_2F_1(a,b,c;-z)$, is monotonously decreasing for $a \in \mathbb{R}, \{b, c-b, z\} > 0$. Therefore ${}_2F_1(\frac{870}{2299}, \frac{13}{19}, \frac{32}{19}; -\frac{121}{304}e^2)$ decreases monotonously from 1 to ~0.9474 as e varies between [0, 1], and the trivial approximation ${}_2F_1(\frac{870}{2299}, \frac{13}{19}, \frac{32}{19}; -\frac{121}{304}e^2) \simeq 1$ to evaluate the integral in Eq. (55) is sufficient for our purposes. For example, the contribution coming from the integral proportional to ${}_2F_1$ is approximated to better than 2% accuracy for $e \in [0, 0.6]$. For $\eta = 1/4$ and $e = 0.6$ the integral of the first three terms in the square bracket of Eq. (55) is −0.6014, while the integral of the last term containing ${}_2F_1$ is much smaller, 0.02653. Then the time function will be approximated by:

$$(t - t_c)_{PN} \simeq -\frac{5c^3\Lambda_0^2}{76G\mathcal{M}_c\eta^{2/5}} \times \left[\int_0^e \frac{e'^{17/19}(1-e'^2)^{-3/2}(\tilde{B}_1 + \tilde{B}_2 e'^2 + \tilde{B}_3 e'^4)}{(1+\frac{121}{304}e'^2)^{1988/2299}} de' + \tilde{B}_4 \int_0^e \frac{e'^{55/19}(1-e'^2)^{-3/2}}{(1+\frac{121}{304}e'^2)^{-1181/2299}} de'\right] - 4\nu_0^{2/3} b_{PN}(e_0)(t-t_c)_N. \tag{57}$$

The phase function $\Phi - \Phi_c$ can be computed similarly. The integrand of the phase function $\Phi - \Phi_c$ up to linear order is

$$(\Phi - \Phi_c)_{PN} = -\frac{5c^3\Lambda_0}{76G\mathcal{M}_c\eta^{2/5}} \int_0^e \times \frac{e'^{-1/19}}{(1+\frac{121}{304}e'^2)^{124/2299}} \left[\frac{\hat{B}_1 + \hat{B}_2 e'^2 + \hat{B}_3 e'^4}{(1+\frac{121}{304}e'^2)^{3169/2299}} + {}_2F_1\left(\frac{870}{2299}, \frac{13}{19}, \frac{32}{19}; -\frac{121}{304}e'^2\right)\hat{B}_4 e'^2\right] de' - \frac{15}{2}\Gamma_0(\Phi - \Phi_c)_N, \tag{58}$$

where $(\Phi - \Phi_c)_N$ is the integrated Newtonian-order time function and we have introduced the quantities $\hat{B}_i = \tilde{B}_i - 9B_i/8$.

In summary, the integrated time and phase functions up to 1PN are expressed as $t - t_c = t_N + t_{PN}$ and $\Phi - \Phi_c = \Phi_N + \Phi_{PN}$ where

TABLE I. Constants of the time and phase functions.

$\delta=-\frac{121}{304}$	N	PN
$t-t_c$	$\gamma=\frac{3}{2}$	$\gamma=\frac{3}{2}$
	$\hat{\beta}=-\frac{1181}{2299}$	$\beta=\frac{1988}{2299}$
	$\alpha=-\frac{10}{19}$	$\alpha_0=\frac{17}{19}$, $\alpha_1=\frac{55}{19}$, $\alpha_2=\frac{93}{19}$
$\Phi-\Phi_c$	$\gamma=0$	$\gamma=0$
	$\breve{\beta}=\frac{124}{2299}$	$\tilde{\beta}=\frac{3293}{2299}$
	$\tilde{\alpha}=\frac{8}{19}$	$\tilde{\alpha}_0=\frac{1}{19}$, $\tilde{\alpha}_1=\frac{37}{19}$, $\tilde{\alpha}_2=\frac{75}{19}$

$$
\begin{aligned}
t_N &= -\frac{15c^5\Lambda_0^{8/3}}{304(G\mathcal{M}_c)^{5/3}}F(e,\alpha,\hat{\beta},\gamma,\delta),\\
t_{PN} &= -\frac{5c^3\Lambda_0^2}{76G\mathcal{M}_c\eta^{2/5}}\left[\sum_{i=1}^{3}F(e,\alpha_i,\beta,\gamma,\delta)\tilde{B}_i\right.\\
&\quad\left.+F(e,\alpha_2,\hat{\beta},\gamma,\delta)\tilde{B}_4\right]-4\Gamma_0 t_N,\\
\Phi_N &= -\frac{15c^5\Lambda_0^{5/3}}{304(G\mathcal{M}_c)^{5/3}}F(e,\tilde{\alpha},\breve{\beta},0,\delta),\\
\Phi_{PN} &= -\frac{5c^3\Lambda_0}{76G\mathcal{M}_c\eta^{2/5}}\left[\sum_{i=1}^{3}F(e,\tilde{\alpha}_i,\hat{\beta},0,\delta)\hat{B}_i\right.\\
&\quad\left.+F(e,\tilde{\alpha}_2,\tilde{\beta},0,\delta)\hat{B}_4\right]-\frac{15}{2}\Gamma_0\Phi_N, \qquad (59)
\end{aligned}
$$

where $\Gamma_0=\nu_0^{2/3}b_{PN}(e_0)$, $F(e,\alpha,\beta,\gamma,\delta)\doteq F_1(\frac{\alpha}{2},\beta,\gamma,\frac{2+\alpha}{2};\delta e^2,e^2)e^{\alpha}/\alpha$ with $F_1(\alpha,\beta,\beta',\gamma;x,y)$ denoting an Appell function; see, e.g., [47]. The numerical values of the constants appearing in Eq. (59) are summarized in Table I. We note that $F(e,\alpha,\beta,0,\delta)={}_2F_1(e,\alpha,\beta,\delta)$.

The qualitative behavior of the orbital evolution is depicted on Figs. 1–5. On Figs. 1–3 the initial conditions have been chosen so that the inspiral time takes one year to reach the last stable orbit (LSO), as defined in Ref. [48] for the eccentric case. From this the frequency $\nu_{\rm LSO}=[(1-e_{\rm LSO}^2)/(6+2e_{\rm LSO})]^{3/2}(2\pi M)^{-1}$ where the $e_{\rm LSO}$ is the final eccentricity at LSO. We note that the frequency, $\nu_{\rm LSO}$ is used in the present case for illustrative purposes only. The orbital evolutions are plotted in Figs. 2–5 until the circular limit is reached. Near the LSO, the PN expansion is not expected to be convergent; therefore, the part of the curves on Figs. 1–5 near the LSO can be taken at best as illustrative.

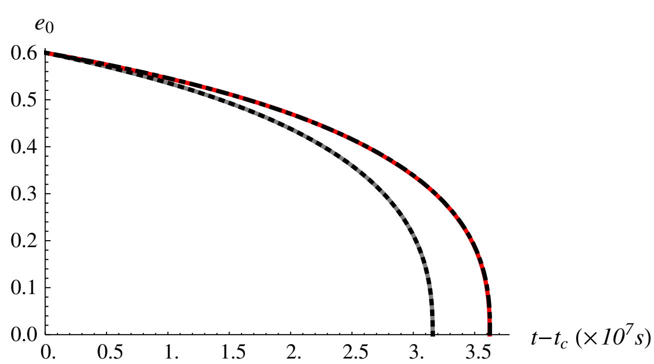

FIG. 1 (color online). Comparison of the analytic and numerical results for the evolution of the eccentricity. The initial eccentricity is $e_0=0.6$ and the masses of binary are $m_i=10^6(1+z)M_\odot$ with redshift $z=1$. The initial frequency is $\nu_0=8.09$ μHz. The dotted black line denotes the analytic, while the gray line the numerical solution in the Newtonian case. The analytic and numerical solutions for the 1PN orbital evolution are depicted by the dot-dashed and solid (red) lines, respectively. It can be seen that the analytic solution is in perfect agreement with the numerical one.

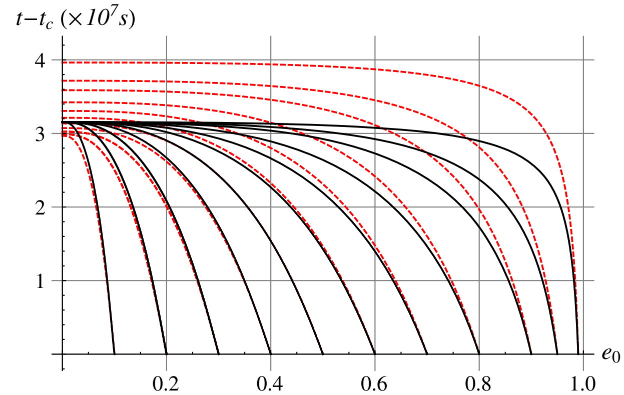

FIG. 2 (color online). The time function for supermassive black hole binaries with various initial eccentricities. The Newtonian and 1PN expressions are denoted by solid (black) and dashed (red) lines, respectively. The masses of the components are $m_i=10^6(1+z)M_\odot$ with redshift $z=1$.

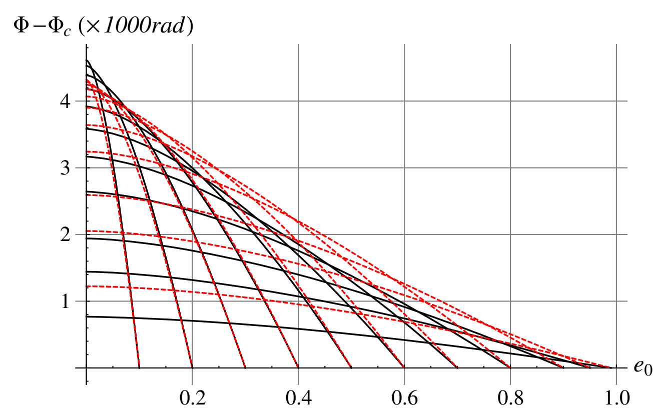

FIG. 3 (color online). The phase function for supermassive black hole binaries with various initial eccentricities. The Newtonian and 1PN expressions are denoted by solid (black) and dashed (red) lines, respectively. The masses of the components are $m_i=10^6(1+z)M_\odot$ with redshift $z=1$.

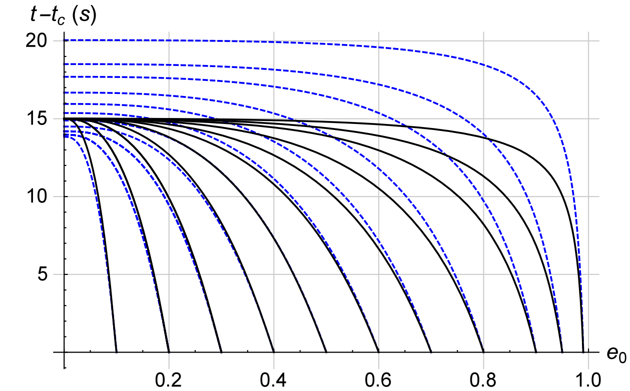

FIG. 4 (color online). The time function for neutron star binaries with various initial eccentricities. The Newtonian and 1PN expressions are denoted by solid (black) and dashed (blue) lines, respectively. The masses of the neutron stars are $m_i = 1.4(1+z)M_\odot$ with redshift $z = 1$. The initial conditions are such that the starting time is set to 15 s before reaching the LSO. The initial orbital frequencies for $e_0 = 0.1$, $e_0 = 0.5$, and $e_0 = 0.8$ are $\nu_0 = 15.46, 10.67$, and 4.15 Hz.

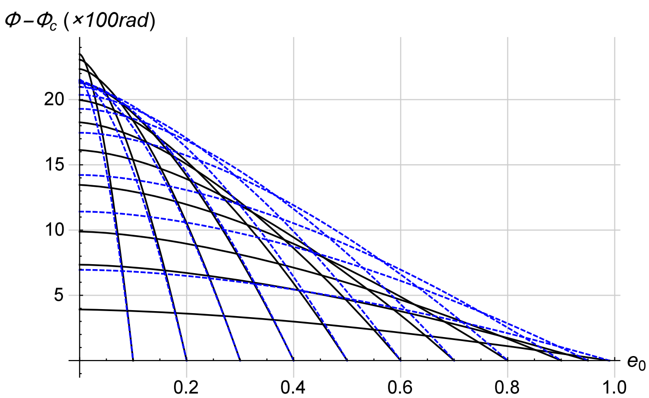

FIG. 5 (color online). The phase function for neutron star binaries with various initial eccentricities. The Newtonian and 1PN expressions are denoted by solid (black) and dashed (blue) lines, respectively. The masses of the neutron stars are $m_i = 1.4(1+z)M_\odot$ with redshift $z = 1$. The initial conditions are such that the starting time is set to 15 s before reaching the LSO.

V. SUMMARY

In our work we have investigated the orbital evolution and the emitted radiation of eccentric binary systems up to 1PN order. Extending our previous results in [21] we have presented fully analytical expressions for the orbital evolution of eccentric binaries and the resulting ready-to-use 1PN waveforms.

For the description of the orbital dynamics, the generalized true anomaly parametrization was introduced resulting in the appearance of only two eccentricities in the solution of the equations of motion. It is usual to expand the azimuthal angle of the separation vector in terms of the generalized true anomaly parameter; however, secular terms appear during this process. These secular terms were eliminated by the introduction of the drift true anomaly parameter, which may also be useful in higher-order PN expansion. The evolution of the orbital frequency and the radial eccentricity is derived up to 1PN order including radiation reaction contributions. The solution is expressed in terms of Appell functions. One important result is the explicit 1PN expressions for the time and phase functions appearing explicitly in the 1PN order frequency domain waveforms.

We have presented both time and frequency domain waveforms in a simple analytic form. The simplicity of the waveforms relies on the application of the Hansen expansion. This important method of celestial mechanics proved to be useful in expressing time-domain gravitational waveforms. According to the required accuracy the Hansen coefficients were generalized to 1PN order. With the introduction of the drift true anomaly parameter the orbital phase becomes a noninteger multiple of the drift true anomaly parameter after expansion. To cover these cases the Hansen coefficients were extended to noninteger values in their parameters. As a result, we have presented explicit ready-to-use 1PN eccentric waveforms with no secular dependence in their expressions. The compact analytic structure makes these waveforms a good candidate for parameter estimation studies up to 1PN order based on the Fisher analysis.

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APPENDIX A: TENSOR SPHERICAL HARMONICS

Following the notation of [49] the traceless, symmetric and unit basis tensors can be written as

$$\begin{aligned} t^{\pm 2} &= \frac{1}{2}(e_x \otimes e_x - e_y \otimes e_y) \pm \frac{i}{2}(e_x \otimes e_y + e_y \otimes e_x), \\ t^{\pm 1} &= \mp \tfrac{1}{2}(e_x \otimes e_z + e_z \otimes e_x) - \tfrac{i}{2}(e_y \otimes e_z + e_z \otimes e_y), \\ t^0 &= \frac{1}{\sqrt{6}}(-e_x \otimes e_x - e_y \otimes e_y + 2e_z \otimes e_z). \end{aligned} \tag{A1}$$

The scalar harmonic tensors on this basis are given by

$$T^{2l',lm} = \sum_{m'=-l'}^{l'} \sum_{m''=-2}^{2} (l'2m'm'', lm) Y^{l'm'} t^{m''}, \tag{A2}$$

where $(l'2m'm'', lm)$ denotes the Clebsch-Gordan coefficients and Y^{lm} is the conventional spherical harmonic. Then the electric and magnetic tensor harmonics can be expressed as

$$T^{E2,lm} = \sqrt{\frac{l(l+1)}{2(2l+1)(2l+3)}}T^{2l+2,lm} + \sqrt{\frac{3(l-1)(l+2)}{(2l+1)(2l+3)}}T^{2l,lm} + \sqrt{\frac{(l+1)(l+2)}{2(2l-1)(2l+1)}}T^{2l-2,lm}, \tag{A3}$$

$$T^{B2,lm} = -i\sqrt{\frac{l-1}{2l+1}}T^{2l+1,lm} - i\sqrt{\frac{l+2}{2l+1}}T^{2l-1,lm}. \tag{A4}$$

As an example we consider the tensor harmonics $T^{E2,22}$ and $T^{B2,22}$ appearing in the Newtonian waveform. Using the relationship between the Descartes and spherical polar coordinates,

$$\begin{aligned} e_x &= e_r \sin\theta\cos\varphi + e_\theta\cos\theta\cos\varphi - e_\varphi\sin\varphi, \\ e_y &= e_r\sin\theta\sin\varphi + e_\theta\cos\theta\sin\varphi + e_\varphi\cos\varphi, \\ e_z &= e_r\cos\theta - e_\theta\sin\theta, \end{aligned} \tag{A5}$$

the tensor harmonics have the form

$$T^{E2,22} = \frac{1}{8}\sqrt{\frac{5}{2\pi}}[(1+\cos^2\theta)h_+ + 2i\cos\theta h_\times]e^{2i\varphi}, \tag{A6}$$

$$T^{B2,22} = -\frac{1}{16}\sqrt{\frac{5}{2\pi}}[(3+\cos 2\theta)h_\times - 4i\cos\theta h_+]e^{2i\varphi}, \tag{A7}$$

where $h_+ = e_\theta\otimes e_\theta - e_\varphi\otimes e_\varphi$ and $h_\times = e_\theta\otimes e_\varphi + e_\varphi\otimes e_\theta$ are the two independent polarizations. The tensor spherical harmonics up to 2PN are given in [29].

APPENDIX B: DAMOUR-DERUELLE PARAMETRIZATION

In the following, we summarize the first post-Newtonian parametrization of the orbital motion introduced by Damour and Deruelle [17] for the description of compact binaries. The equations of motion, Eqs. (1) and (2), can be solved by the eccentric anomaly quasiparametrization u, that is,

$$r = a_r(1 - e_r\cos u), \tag{B1}$$

where the orbital parameters are the semi-major axis a_r and the radial eccentricity e_r. These orbital parameters are characterized by the turning points ($r_{\max}$ and $r_{\min}$ in [50]) of the radial motion. The Kepler equation and angular evolution can be given as

$$n(t-t_0) = u - e_t\sin u, \tag{B2}$$

$$\theta - \theta_0 = (1+k)v_\theta, \tag{B3}$$

$$v_\theta = 2\arctan\sqrt{\frac{1+e_\theta}{1-e_\theta}}\tan\frac{u}{2}, \tag{B4}$$

in terms of the orbital elements of the 1PN orbital dynamics such as the mean motion n, the time eccentricity e_t, the angle eccentricity e_θ, and the pericenter drift k (which is in relationship with the pericenter precession $\langle\dot{\gamma}\rangle$ averaged over one radial period; see [21]).

The relationship between ϕ and v_θ is given by Eqs. (B3) and (7) up to 1PN order as

$$\phi = v_\theta - \frac{G\mu e_r}{2a_r(1-e_r^2)c^2}\sin v_\theta. \tag{B5}$$

The orbital parameters up to 1PN order are given by [17]

$$\begin{aligned} n &= \frac{(-D_1)^{3/2}}{D_2}, \quad a_r = -\frac{D_2}{D_1} + \frac{D_4}{2D_3}, \\ e_t &= \left[1 - \frac{D_1}{D_2^2}\left(D_3 - \frac{D_2D_4}{D_3}\right)\right]^{1/2}, \quad e_r = \left(1 - \frac{D_1D_3}{2D_2D_3}\right)e_t, \\ e_\theta &= \left(1 + \frac{D_1D_4}{D_2D_3} - \frac{D_1D_6}{D_2D_5}\right)e_t, \quad k = \frac{3Gm}{a_r(1-e_R^2)c^2}, \end{aligned} \tag{B6}$$

with the quantities D_{1-6}

$$\begin{aligned} D_1 &= \frac{2E}{\mu}\left(1 + \frac{3}{2}(3\eta - 1)\frac{E}{\mu c^2}\right), \\ D_2 &= GM\left(1 + (7\eta - 6)\frac{E}{\mu c^2}\right), \\ D_3 &= -\frac{L^2}{\mu^2}\left(1 + 2(3\eta-1)\frac{E}{\mu c^2}\right) + (5\eta - 10)\frac{G^2M^2}{c^2}, \\ D_4 &= (-3\eta + 8)\frac{GML^2}{\mu^2c^2}, \\ D_5 &= \frac{L}{\mu}\left(1 + (3\eta - 1)\frac{E}{\mu c^2}\right), \\ D_6 &= (2\eta - 4)\frac{GML}{\mu c^2}. \end{aligned} \tag{B7}$$

Here E and L are the conserved energy and the magnitude of the orbital angular momentum of the perturbed binary system, respectively.

APPENDIX C: HANSEN COEFFICIENTS

The Hansen coefficients are important functions of celestial mechanics which are known for more than 100 years. Using the standard notation the Hansen expansion is written as [51]

$$\left(\frac{r}{a}\right)^n \exp(im\phi) = \sum_{k=-\infty}^{\infty} X_k^{n,m} \exp(ik\mathcal{M}), \tag{C1}$$

where r is the relative distance, a is the semimajor axis, ϕ is the true anomaly and $\mathcal{M}$ is the mean anomaly. The coefficients $X_k^{n,m}$ are called the Hansen coefficients. Here the constants n and m are integers. The Fourier series representation of the Hansen coefficients is

$$X_k^{n,m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{r}{a}\right)^n \exp(im\phi - ik\mathcal{M}) d\mathcal{M}. \tag{C2}$$

The integration variable can be changed to the eccentric or true anomalies with the use of the leading-order Kepler equation,

$$\frac{d\mathcal{M}}{du} = 1 - e\cos u, \tag{C3}$$

$$\frac{d\mathcal{M}}{d\phi} = \frac{(1-e^2)^{3/2}}{(1+e\cos\phi)^2}. \tag{C4}$$

Introducing the complex variables $x = \exp i\phi$ and $y = \exp iu$ ($z = \exp i\mathcal{M}$ for the contour integral) the relationship between the eccentric and true anomalies [52] with x and y is expressed as

$$\frac{x-1}{x+1} = \frac{1+\beta}{1-\beta}\frac{y-1}{y+1}, \tag{C5}$$

where $\beta = (1-\sqrt{1-e^2})/e$ and one gets the relations for the variable x

$$\exp i\phi = y(1-\beta y^{-1})(1-\beta y)^{-1}, \tag{C6}$$

and the mean anomaly $\mathcal{M}$,

$$\frac{d\mathcal{M}}{du} = \frac{r}{a} = (1+\beta^2)^{-1}(1-\beta y)(1-\beta y^{-1}), \tag{C7}$$

$$\exp i\mathcal{M} = y\exp\left[-\frac{e}{2}(y-y^{-1})\right]. \tag{C8}$$

The integrand with eccentric anomaly u is given as

$$X_k^{n,m} = \frac{(1+\beta^2)^{-n-1}}{2\pi}\int_{-\pi}^{\pi} y^{m-k}(1-\beta y^{-1})^{n+m+1} \times (1-\beta y)^{n-m+1}\exp\left[\frac{ke}{2}(y-y^{-1})\right] du. \tag{C9}$$

The integral can be extended to infinity as a series of the Bessel functions:

$$X_k^{n,m} = (1+\beta^2)^{-n-1}\sum_{p=-\infty}^{\infty} E_{k-p}^{n,m} J_p(ke). \tag{C10}$$

The coefficients $E_l^{n,m}$ for $l \geqq m$ and $E_{-l}^{n,-m}$ for $l < m$ can be expressed by the hypergeometric function ${}_2F_1(a,b,c;z)$ as

$$E_l^{n,m} = (-\beta)^{l-m}\binom{n-m+1}{l-m} \times {}_2F_1(l-n-1, -m-n-1, l-m+1; \beta^2). \tag{C11}$$

The first description of this formula was given by Hill [53]. Another representation of the Hansen coefficients can be found in the 1889 work of Tisserand on celestial mechanics [54],

$$X_k^{n,m} = \frac{(-\beta)^{|k-m|}}{(1+\beta^2)^{n+1}}\sum_{s=0}^{\infty}\mathcal{P}_s\mathcal{Q}_s\beta^{2s}, \tag{C12}$$

where

$$\mathcal{P}_s = \left\{\begin{matrix} P_{s+k-m} & k \geqq m \\ P_s & k < m \end{matrix}\right\}, \tag{C13}$$

$$\mathcal{Q}_s = \left\{\begin{matrix} Q_s & k \geqq m \\ Q_{s+m-k} & k < m \end{matrix}\right\}, \tag{C14}$$

and

$$P_s = \sum_{r=0}^{s}\binom{n-m+1}{s-r}\frac{1}{r!}\left(\frac{kr}{2\beta}\right)^r, \tag{C15}$$

$$Q_s = \sum_{r=0}^{s}\binom{n+m+1}{s-r}\frac{1}{r!}\left(-\frac{kr}{2\beta}\right)^r. \tag{C16}$$

APPENDIX D: LEADING- AND HALF-ORDER WAVEFORMS

The leading-order waveform in terms of the true anomaly ϕ is given by the quadrupole formula (the notation of Ref. [21] for the azimuthal angle $\Phi = \gamma$),

$$h_+^N(\phi) = \frac{G^2\mu^2}{4c^4\eta a(1-e^2)D_L}\sum_{m=0}^{3}(c_m^{N+}\cos m\phi + s_m^{N+}\sin m\phi), \tag{D1}$$

$$h_\times^N(\phi) = \frac{G^2\mu^2}{c^4\eta a(1-e^2)D_L}\sum_{m=0}^{3}(c_m^{N\times}\cos m\phi + s_m^{N\times}\sin m\phi), \tag{D2}$$

where

$$
\begin{aligned}
c_0^{N+} &= -2e^2[1-\cos 2\Theta + N_\Theta \cos 2\Phi], & c_0^{N\times} &= 2e^2\cos\Theta\sin 2\Phi,\\
c_1^{N+} &= -e(2-2\cos 2\Theta + 5N_\Theta\cos 2\Phi), & c_1^{N\times} &= 5e\cos\Theta\sin 2\Phi,\\
c_2^{N+} &= -4N_\Theta\cos 2\Phi, & c_2^{N\times} &= 4\cos\Theta\sin 2\Phi,\\
c_3^{N+} &= -eN_\Theta\cos 2\Phi, & c_3^{N\times} &= e\cos\Theta\sin 2\Phi,\\
s_1^{N+} &= -5eN_\Theta\sin 2\Phi, & s_1^{N\times} &= -5\cos\Theta\cos 2\Phi, \qquad \text{(D3)}
\end{aligned}
$$

and $s_{2-3}^{N+} = c_{2-3}^{N+}$ and $s_{2-3}^{N\times} = -c_{2-3}^{N\times}$ after the interchange of $\cos 2\Phi$ and $\sin 2\Phi$. Here $N_\Theta = 3 + \cos 2\Theta$ and we have introduced the shorthand notations $a \equiv a_r$ and $e \equiv e_r$. We note that the first step in the derivation of the frequency domain waveforms is the replacement of the semimajor axis a with the orbital frequency ν; see Eq. (40).

The half-order waveforms (denoted by the superscript H) are

$$
h_+^H(\phi) = \frac{\delta m G^{1/2}\mu^{3/2}}{64c^5[\eta a(1-e^2)]^{3/2}D_L} \times \sum_{m=0}^{5}(c_m^+\cos m\phi + s_m^+\sin m\phi), \qquad \text{(D4)}
$$

$$
h_\times^H(\phi) = \frac{\delta m G^{1/2}\mu^{3/2}}{32c^5[\eta a(1-e^2)]^{3/2}D_L} \times \sum_{m=0}^{5}(c_m^\times\cos m\phi + s_m^\times\sin m\phi), \qquad \text{(D5)}
$$

where

$$
\begin{aligned}
c_0^{H+} &= 8e\{[11-2e^2+(1-6e^2)\cos 2\Theta]\\
&\quad\times\sin\Theta\sin\Phi + e^2\rho_\Theta\sin 3\Phi\},\\
c_1^{H+} &= [(84+77e^2)\sin\Theta + (4-39e^2)\sin 3\Theta]\\
&\quad\times\sin\Phi + 35e^2\rho_\Theta\sin 3\Phi,\\
c_2^{H+} &= 4e(\varrho_\Theta\sin\Phi + 15\rho_\Theta\sin 3\Phi),\\
c_3^{H+} &= e^2\varrho_\Theta\sin\Phi + 2(18+7e^2)\rho_\Theta\sin 3\Phi,\\
c_4^{H+} &= 20e\rho_\Theta\sin 3\Phi,\\
c_5^{H+} &= 3e^2\rho_\Theta\sin 3\Phi,\\
s_1^{H+} &= -[(84+31e^2)\sin\Theta + (4-29e^2)\\
&\quad\times\sin 3\Theta]\cos\Phi - 35e^2\rho_\Theta\cos 3\Phi,
\end{aligned}
$$

and $s_{2-5}^{H+} = -c_{2-5}^{H+}$ after the interchange of $\sin m\Phi$ and $\cos m\Phi$. Here $\delta m = m_1 - m_2$, $\rho_\Theta = 5\sin\Theta + \sin 3\Theta$ and $\varrho_\Theta = 23\sin\Theta - 5\sin 3\Theta$. The coefficients for the cross-polarization are

$$
\begin{aligned}
c_0^{H\times} &= 8e\sin 2\Theta[(3-e^2)\cos\Phi + 2e^2\cos 3\Phi],\\
c_1^{H\times} &= 2\sin 2\Theta[(12-5e^2)\cos\Phi + 35e^2\cos 3\Phi],\\
c_2^{H\times} &= 8e\sin 2\Theta(\cos\Phi + 15\cos 3\Phi),\\
c_3^{H\times} &= 2\sin 2\Theta[e^2\cos\Phi + 2(18+7e^2)\cos 3\Phi],\\
c_4^{H\times} &= 40e\sin 2\Theta\cos 3\Phi,\\
c_5^{H\times} &= 6e^2\sin 2\Theta\cos 3\Phi,\\
s_1^{H\times} &= 2\sin 2\Theta[(12-7e^2)\sin\Phi + 35e^2\sin 3\Phi], \qquad \text{(D6)}
\end{aligned}
$$

and $s_{2-5}^{H\times} = c_{2-5}^{H\times}$ after the interchange of $\cos m\Phi$ and $\sin m\Phi$.

APPENDIX E: 1PN WAVEFORM

The 1PN waveforms are given by Eq. (11) which can be written in the following form:

$$
h_+^{PN}(\phi) = \frac{G^3\mu^3}{768c^7[\eta a(1-e^2)]^2D_L} \times \sum_{m=0}^{4}[h_{m+}^{4\phi'}(\phi) + h_{m+}^{2\phi'}(\phi) + h_{m+}(\phi)], \qquad \text{(E1)}
$$

$$
h_\times^{PN}(\phi) = \frac{G^3\mu^3}{384c^7[\eta a(1-e^2)]^2D_L} \times \sum_{m=0}^{4}[h_{m\times}^{4\phi'}(\phi) + h_{m\times}^{2\phi'}(\phi) + h_{m\times}(\phi)], \qquad \text{(E2)}
$$

with

$$
h_{m+,\times}^{4\phi'}(\phi) = (c_m^{c4+,\times}\cos m\phi + s_m^{c4+,\times}\sin m\phi)\cos 4\phi' + (c_m^{s4+,\times}\cos m\phi + s_m^{s4+,\times}\sin m\phi)\sin 4\phi', \qquad \text{(E3)}
$$

$$
h_{m+,\times}^{2\phi'}(\phi) = 4[(c_m^{c2+,\times}\cos m\phi + s_m^{c2+,\times}\sin m\phi)\cos 2\phi' + (c_m^{s2+,\times}\cos m\phi + s_m^{s2+,\times}\sin m\phi)\sin 2\phi'], \qquad \text{(E4)}
$$

$$
h_{m+,\times}(\phi) = 3(c_m^{+,\times}\cos m\phi + s_m^{+,\times}\sin m\phi). \qquad \text{(E5)}
$$

Here $\phi' = (1+\kappa_1)\phi$ is the drift true anomaly, $\kappa_1 = 3GM/[a(1-e^2)c^2]$ and the coefficients $c_m^{c4+,\times}$, $s_m^{c4+,\times}$, $c_m^{c2+,\times}$, $s_m^{c2+,\times}$ $c_m^{+,\times}$ and $s_m^{+,\times}$ are explicitly given in the next subsections.

1. The coefficients proportional to cos $4\phi'$ and sin $4\phi'$

There are coefficients proportional to $\cos 4\phi'$ and $\sin 4\phi'$ in the 1PN waveform, Eq. (E3). The coefficients proportional to $\cos m\phi$ are

$$\begin{aligned} c_4^{c4+} &= 24 c_{4\Phi} G_\Theta \lambda_\eta e^4, \\ c_3^{c4+} &= 165 c_{4\Phi} G_\Theta \lambda_\eta e^3, \\ c_2^{c4+} &= 500 c_{4\Phi} G_\Theta \lambda_\eta e^2, \\ c_1^{c4+} &= c_{4\Phi} G_\Theta \lambda_\eta (764+135e^2)e, \\ c_0^{c4+} &= 4 c_{4\Phi} G_\Theta \lambda_\eta (64+71e^2), \end{aligned} \tag{E6}$$

where we have introduced the shorthand notations $c_{m\Phi} = \cos m\Phi$, $s_{m\Phi} = \sin m\Phi$, $c_{m\Theta} = \cos m\Theta$, $s_{m\Theta} = \sin m\Theta$, $G_\Theta = -5 + 4c_{2\Theta} + c_{4\Theta}$ and $\lambda_\eta = 3\eta - 1$. The equality $c_{0-4}^{s4+} = c_{0-4}^{c4+}$ holds after the replacement of $c_{4\Phi}$ with $s_{4\Phi}$. The coefficients proportional to $\sin m\phi$ are

$$\begin{aligned} s_4^{c4+} &= -24 s_{4\Phi} G_\Theta \lambda_\eta e^4, \\ s_3^{c4+} &= -100 s_{4\Phi} G_\Theta \lambda_\eta e^3, \\ s_2^{c4+} &= -360 s_{4\Phi} G_\Theta \lambda_\eta e^2, \\ s_1^{c4+} &= -6 G_\Theta s_{4\Phi} \lambda_\eta (52+9e^2)e, \end{aligned} \tag{E7}$$

and $s_{1-4}^{s4+} = -s_{1-4}^{c4+}$ after the replacement of $s_{4\Phi}$ with $c_{4\Phi}$. The coefficients for cross-polarization are

$$\begin{aligned} c_4^{c4\times} &= 48 s_{4\Phi} H_\Theta \lambda_\eta e^4, \\ c_3^{c4\times} &= 336 s_{4\Phi} H_\Theta \lambda_\eta e^3, \\ c_2^{c4\times} &= 1000 s_{4\Phi} H_\Theta \lambda_\eta e^2, \\ c_1^{c4\times} &= 2 s_{4\Phi} H_\Theta \lambda_\eta (764+135e^2)e, \\ c_0^{c4\times} &= 6 s_{4\Phi} H_\Theta \lambda_\eta (64+71e^2), \end{aligned} \tag{E8}$$

and $c_{0-4}^{s4\times} = -c_{0-4}^{c4\times}$ after the replacement of $s_{4\Phi}$ with $c_{4\Phi}$ and we have introduced the notation $H_\Theta = c_\Theta - c_{3\Theta}$.

$$\begin{aligned} s_4^{c4\times} &= 48 c_{4\Phi} H_\Theta \lambda_\eta e^4, \\ s_3^{c4\times} &= 300 c_{4\Phi} H_\Theta \lambda_\eta e^3, \\ s_2^{c4\times} &= 720 c_{4\Phi} H_\Theta \lambda_\eta e^2, \\ s_1^{c4\times} &= 12 c_{4\Phi} H_\Theta \lambda_\eta (52+9e^2)e, \end{aligned} \tag{E9}$$

and $s_{1-4}^{s4\times} = s_{1-4}^{c4\times}$ after the replacement of $c_{4\Phi}$ with $s_{4\Phi}$.

2. The coefficients proportional to cos $2\phi'$ and sin $2\phi'$

The coefficients proportional to $\cos m\phi$ are

$$\begin{aligned} c_3^{c2+} &= 3c_{2\Phi}[291 - 81\eta + 4(19+9\eta)c_{2\Theta} - \lambda_\eta c_{4\Theta}]e^3, \\ c_2^{c2+} &= 4c_{2\Phi}\{983 - 375\eta - 6(19+3\eta)e^2 + 2[116 + 81\eta - 6e^2(3+\eta)]c_{2\Theta} - (1+6e^2)\lambda_\eta c_{4\Theta}\}e^2, \\ c_1^{c2+} &= c_{2\Phi}\{5948 - 3012\eta + (99 - 1089\eta)e^2 + 4[4(76+81\eta) - 33e^2(1-\eta)]c_{2\Theta} - (4+81e^2)\lambda_\eta c_{4\Theta}\}e, \\ c_0^{c2+} &= 4c_{2\Phi}\{508 - 228\eta + (353 - 465\eta)e^2 + 2[52 + 60\eta + e^2(14+57\eta)]c_{2\Theta} + (4-19e^2)\lambda_\eta c_{4\Theta}\}, \end{aligned} \tag{E10}$$

and $c_{0-3}^{s2+} = s_{0-3}^{c2+}$ after the replacement of $c_{2\Phi}$ with $s_{2\Phi}$. The coefficients proportional to $\sin m\phi$ are

$$\begin{aligned} s_3^{c2+} &= 3s_{2\Phi}[87\eta - 293 - 4(23-3\eta)c_{2\Theta} - \lambda_\eta c_{4\Theta}]e^3, \\ s_2^{c2+} &= 12 s_{2\Phi}\{81\eta - 259 + 2(19+3\eta)e^2 - 4[2(10-\eta) - (3+\eta)e^2]c_{2\Theta} - (3-2e^2)\lambda_\eta c_{4\Theta}\}e^2, \\ s_1^{c2+} &= 3s_{2\Phi}\{204\eta - 900 + 11e^2(1+21\eta) - 4[4(17+\eta) - (1+19\eta)e^2]c_{2\Theta} - 5(4-3e^2)\lambda_\eta c_{4\Theta}\}e, \end{aligned} \tag{E11}$$

and $s_{1-3}^{s2+} = -s_{1-3}^{c2+}$ after the replacement of $s_{2\Phi}$ with $c_{2\Phi}$. The coefficients for the cross-polarization are

$$\begin{aligned} c_3^{c2\times} &= 3s_{2\Phi}[-11(17-3\eta)c_\Theta - 3\lambda_\eta c_{3\Theta}]e^3, \\ c_2^{c2\times} &= 4s_{2\Phi}\{[159\eta - 625 + 6e^2(13+\eta)]c_\Theta - (17-6e^2)\lambda_\eta c_{3\Theta}\}e^2, \\ c_1^{c2\times} &= s_{2\Phi}\{[3(7+155\eta)e^2 - 4(931-321\eta)]c_\Theta + 5(-28+9e^2)\lambda_\eta c_{3\Theta}\}e, \\ c_0^{c2\times} &= 4s_{2\Phi}\{[(201\eta - 199)e^2 - 32(10-3\eta)]c_\Theta - (16-e^2)\lambda_\eta c_{3\Theta}\} \end{aligned} \tag{E12}$$

and $c_{0-3}^{s2\times} = -c_{0-3}^{c2\times}$ after the replacement of $s_{2\Phi}$ with $c_{2\Phi}$. The coefficients proportional to $\cos 2\phi'$ and $\sin 2\phi'$ are

$$\begin{aligned} s_3^{c2\times} &= 6c_{2\Phi}[(27\eta - 97)c_\Theta - \lambda_\eta c_{3\Theta}]e^3, \\ s_2^{c2\times} &= 24c_{2\Phi}\{[(26\eta - 86 + (13+\eta)e^2)]c_\Theta - (2-e^2)\lambda_\eta c_{3\Theta}\}e^2, \\ s_1^{c2\times} &= 6c_{2\Phi}\{[68\eta - 300 + (7+67\eta)e^2]c_\Theta - (12-7e^2)\lambda_\eta c_{3\Theta}\}e, \end{aligned} \tag{E13}$$

and $s_{1-3}^{s2\times} = s_{1-3}^{c2\times}$ after the replacement of $c_{2\Phi}$ with $s_{2\Phi}$.

3. The coefficients without ϕ' dependence

The coefficients without ϕ' dependence are

$$\begin{aligned} c_3^+ &= [11(11-\eta) - 4(29+\eta)c_{2\Theta} + 5\lambda_\eta c_{4\Theta}]e^3, \\ c_2^+ &= 20[37 - 7\eta - 4(9-\eta)c_{2\Theta} + \lambda_\eta c_{4\Theta}]e^2, \\ c_1^+ &= \{4(299-49\eta) + 5(135-53\eta)e^2 - 4[300 - 52\eta + (159-37\eta)e^2]c_{2\Theta} - (4-39e^2)\lambda_\eta c_{4\Theta}\}e, \\ c_0^+ &= 4\{363 - 73\eta - 10(5+\eta)e^2 - 4[91 - 19\eta - 2e^2(7-\eta)]c_{2\Theta} - (1-6e^2)\lambda_\eta c_{4\Theta}\}. \end{aligned} \tag{E14}$$

and the coefficients s_{1-3}^+ are zero. For the cross-polarization,

$$\begin{aligned} s_3^\times &= 4H_\Theta \lambda_\eta e^3, \\ s_2^\times &= 16H_\Theta \lambda_\eta e^2, \\ s_1^\times &= 4H_\Theta \lambda_\eta (4+e^2)e, \end{aligned} \tag{E15}$$

and the coefficients $c_{0-3}^\times$ are zero.

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