Bekenstein bounds and inequalities between size, charge, angular momentum, and energy for bodies

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Bekenstein bounds for the entropy of a body imply a universal inequality between size, energy, angular momentum, and charge. We prove this inequality in electromagnetism. We also prove it, for the particular case of zero angular momentum, in general relativity. We further discuss the relation of these inequalities with inequalities between size, angular momentum, and charge recently studied in the literature.

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I. INTRODUCTION

A universal bound on the entropy of a macroscopic body has been proposed by Bekenstein [1],

$$\frac{\hbar c}{2\pi k_B} S \leq \mathcal{E}\mathcal{R}, \tag{1}$$

where S is the entropy, k_B is Boltzmann's constant, $\mathcal{R}$ is the radius of the smallest sphere that can enclose the body, $\mathcal{E}$ is the total energy, $\hbar$ is the reduced Planck constant, and c is the speed of light. Using similar heuristic arguments, a generalization of Eq. (1) including the electric charge Q and the angular momentum J of the body has also been proposed [2–4],

$$\frac{\hbar c}{2\pi k_B} S \leq \sqrt{(\mathcal{E}\mathcal{R})^2 - c^2 J^2} - \frac{Q^2}{2}. \tag{2}$$

The original physical arguments used to present these inequalities involve black holes. However, a remarkable feature of these inequalities is that the gravitational constant G does not appear in them.

The bound (1) has been extensively studied; see, for example, the review articles [5–7] and references therein. However, the generalization (2) appears to have received much less attention. In particular, since the entropy S is always non-negative, the bound (2) implies the following inequality in which the entropy S and the constant $\hbar$ are not involved:

$$\frac{Q^4}{4\mathcal{R}^2} + \frac{c^2 J^2}{\mathcal{R}^2} \leq \mathcal{E}^2. \tag{3}$$

The equality in Eq. (3) implies, by Eq. (2), that the entropy of the body is zero and hence the system should be in a very particular state. Then, we expect some kind of rigidity statement for the equality in Eq. (3).

The main purpose of this article is to study the inequality (3). The only fundamental constant that appears in Eq. (3) is c. Hence, the obvious theory to test Eq. (3) is electromagnetism. To the best of our knowledge, such a basic study, in full generality, has not been done before. In Sec. II we prove that Eq. (3) holds as a consequence of Maxwell's equations. This theorem provides an indirect but highly nontrivial evidence in favor of the bound (2).

In Sec. III we first discuss the relation of the inequality (3) with inequalities between size, angular momentum, and charge recently studied in general relativity [8]. Then, we point out that a result of Reiris [9] proves Eq. (3) in spherical symmetry in general relativity. Finally, we generalize this result and prove Eq. (3), with $J = 0$, for time-symmetric initial data.

II. ELECTROMAGNETISM

To fix the notation, let us write Maxwell's equations in Gaussian units,

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}, \qquad \nabla \cdot \mathbf{E} = 4\pi\rho, \tag{4}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \tag{5}$$

where $\mathbf{E}$, $\mathbf{B}$ are the electric and magnetic field, and ρ, $\mathbf{j}$ are the charge and current density. These equations are written in terms of inertial coordinates $(t, \mathbf{x})$ where t is the time coordinate and $\mathbf{x}$ are spatial coordinates centered at an arbitrary point x_0.

Let U be an arbitrary region in space. The electric charge contained in U is given by

$$Q(U) = \int_U \rho, \tag{6}$$

and the energy of the electromagnetic field in U is

$$\mathcal{E}(U) = \frac{1}{8\pi}\int_U |\mathbf{E}|^2 + |\mathbf{B}|^2. \tag{7}$$

The angular momentum in the region U in the direction of the unit vector $\mathbf{k}$ with respect to the point x_0 is given by

$$\mathbf{J}\cdot\mathbf{k} = \frac{1}{4\pi c}\int_U (\mathbf{x}\times(\mathbf{E}\times\mathbf{B}))\cdot\mathbf{k}. \tag{8}$$

Finally, in order to study Eq. (3) we need to provide a definition of the radius $\mathcal{R}$ for an arbitrary region U.

Definition 2.1. We define the radius $\mathcal{R}$ of the region U as the radius of the smallest sphere that encloses U.

Given a domain U, we denote by $B_{\mathcal{R}}$ the smallest ball that encloses U and x_0 denotes the center of this ball. Note that, in general, x_0 is not in U; see Fig. 1. We denote by $\partial B_{\mathcal{R}}$ the boundary of $B_{\mathcal{R}}$, that is, the sphere of radius $\mathcal{R}$ centered at x_0.

Before dealing with the general case, it is useful to begin with electrostatics, which in particular implies $J = 0$. We will see that the proof for the dynamical case is based on the proof for the electrostatics case. Also, in electrostatics it is simpler to discuss the scope of Eq. (3).

The equations of electrostatics are given by

$$\nabla\cdot\mathbf{E} = 4\pi\rho, \qquad \nabla\times\mathbf{E} = 0. \tag{9}$$

The potential Φ is defined by $\mathbf{E} = -\nabla\Phi$ and it satisfies the Poisson equation

$$\Delta\Phi = -4\pi\rho. \tag{10}$$

Using Eq. (10) and Gauss's theorem, we obtain that the charge can be written as a boundary integral

$$Q(U) = -\frac{1}{4\pi}\oint_{\partial U}\partial_n\Phi, \tag{11}$$

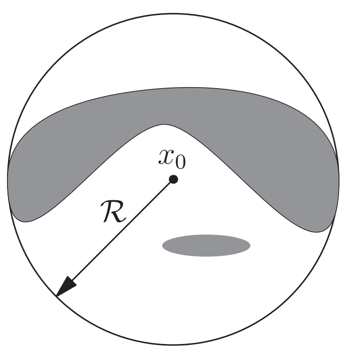

FIG. 1. The domain U is colored with gray. The radius $\mathcal{R}$ is defined as the radius of the smallest sphere that encloses U. For this particular domain U the center x_0 of that sphere is not in U.

where ∂_n denotes a partial derivative along the exterior unit normal vector of the boundary ∂U. The total electrostatics energy is given by

$$\mathcal{E} = \frac{1}{8\pi}\int_{\mathbb{R}^3}|\mathbf{E}|^2. \tag{12}$$

Theorem 2.2. Assume that the charge density ρ has compact support contained in the region U. In electrostatics [i.e., we assume Eq. (9)], the following inequality holds:

$$Q^2 \le 2\mathcal{E}\mathcal{R}, \tag{13}$$

where Q is the charge contained in U, $\mathcal{R}$ is the radius of U defined above, and $\mathcal{E}$ is the total electromagnetic energy given by Eq. (12). The equality in Eq. (13) holds if and only if the electric field is equal to the electric field produced by a spherical thin shell of constant surface charge density and radius $\mathcal{R}$. In particular, this implies that the electric field vanishes inside U.

Proof.—The system has electric field $\mathbf{E}$ (with potential Φ), charge density ρ with support in U, and total charge Q. Let $\mathcal{R}$ be the radius of the domain U defined in definition 2.1 and $B_{\mathcal{R}}$ its corresponding ball centered at x_0.

Consider the auxiliary potential defined by

$$\Phi_0 = \begin{cases} \frac{Q}{r} & \text{if } r \ge \mathcal{R}, \\ \frac{Q}{\mathcal{R}} & \text{if } r \le \mathcal{R}, \end{cases} \tag{14}$$

where r is the radial distance to x_0. The potential Φ_0 corresponds to the potential of a spherical thin shell of radius $\mathcal{R}$, constant surface charge density, and total charge Q.

We define Φ_1 by the difference

$$\Phi_1 = \Phi - \Phi_0. \tag{15}$$

By construction Φ_1 satisfies

$$\Delta\Phi_1 = \begin{cases} 0 & \text{if } r > \mathcal{R}, \\ -4\pi\rho & \text{if } r < \mathcal{R}, \end{cases} \tag{16}$$

and

$$\oint_{\partial B_{\mathcal{R}}}\partial_r\Phi_1 = 0. \tag{17}$$

Equation (17) follows since in the definition of Φ_0 we have used the total charge Q of the potential Φ.

The total energy of the system is given by

$$\mathcal{E} = \frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\Phi|^2 \tag{18}$$

$$= \frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla\Phi_0|^2 + |\nabla\Phi_1|^2 + 2\nabla\Phi_0\cdot\nabla\Phi_1, \tag{19}$$

where in Eq. (19) we have used the definition (15). To calculate the last term in Eq. (19) we decompose the domain of the integral in $\mathbb{R}^3\backslash B_{\mathcal{R}}$ and $B_{\mathcal{R}}$. We have

$$\int_{B_{\mathcal{R}}} \nabla\Phi_0 \cdot \nabla\Phi_1 = 0, \tag{20}$$

since Φ_0 is constant in B_R. For the other integral we have

$$\int_{\mathbb{R}^3\backslash B_{\mathcal{R}}} \nabla\Phi_0 \cdot \nabla\Phi_1 = \int_{\mathbb{R}^3\backslash B_{\mathcal{R}}} \nabla\cdot(\Phi_0\nabla\Phi_1) - \Phi_0\Delta\Phi_1. \tag{21}$$

Since $\Delta\Phi_1 = 0$ on $\mathbb{R}^3\backslash B_R$ the second term on the right-hand side of Eq. (21) vanishes. The first term can be converted into a boundary integral

$$\int_{\mathbb{R}^3\backslash B_{\mathcal{R}}} \nabla(\Phi_0\nabla\Phi_1) = \lim_{r\to\infty}\oint_{\partial B_r} \Phi_0\partial_r\Phi_1 - \oint_{\partial B_{\mathcal{R}}} \Phi_0\partial_r\Phi_1. \tag{22}$$

The first term on the right-hand side of Eq. (22) vanishes by the decay conditions of Φ_0 and Φ_1. For the second term we have

$$\oint_{\partial B_{\mathcal{R}}} \Phi_0\partial_r\Phi_1 = \Phi_0\oint_{\partial B_{\mathcal{R}}} \partial_r\Phi_1 \tag{23}$$

$$= 0, \tag{24}$$

where in Eq. (23) we have used that Φ_0 is constant on spheres and in Eq. (24) we have used Eq. (17). Hence, we have proved that

$$\mathcal{E} = \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla\Phi_0|^2 + |\nabla\Phi_1|^2. \tag{25}$$

The first term in Eq. (25) can be computed explicitly using Eq. (14). It is the binding energy of a spherical shell of radius $\mathcal{R}$ with constant charge surface density and total charge Q. Then, we finally obtain

$$\mathcal{E} = \frac{Q^2}{2\mathcal{R}} + \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla\Phi_1|^2. \tag{26}$$

This equality proves Eq. (13) and also the rigidity statement: if the equality in Eq. (13) holds, then Eq. (26) implies $\nabla\Phi_1 = 0$ and hence $\mathbf{E} = \nabla\Phi_0$. ∎

Note that the equality (26) implies the following estimate for the fields inside the domain U:

$$\mathcal{E} - \frac{Q^2}{2\mathcal{R}} \geq \frac{1}{8\pi}\int_U |\mathbf{E}|^2, \tag{27}$$

where we have used that in U we have $\nabla\Phi_1 = \nabla\Phi = \mathbf{E}$.

Let us discuss the scope of Eq. (13). The first important observation is that in Eq. (13) the energy $\mathcal{E}$ is the total energy of the system, which in electrostatics is equivalent to the binding energy. That is, $\mathcal{E}$ represents the work needed to assemble the charge configuration from infinity. Inequality is clearly false if instead of the total energy we use the integral of the energy density on the domain U given by Eq. (7). For example, take the spherical shell of radius $\mathcal{R}$ and constant surface charge density. Then, the domain U is given by the ball $B_{\mathcal{R}}$, but the integral of the energy density over $B_{\mathcal{R}}$ is zero since the electric field vanishes in $B_{\mathcal{R}}$.

Equation (13) is not valid if we consider many disconnected regions and take Q and $\mathcal{R}$ to be the corresponding radius and charge of only one region and $\mathcal{E}$ the total energy of the system. The counterexample is the following. Consider two spherical thin shells of constant surface density with radii R_1 and R_2 and total charges Q and $-Q$. The separation between the centers is L, and we assume that they do not overlap, i.e., $L \geq R_1 + R_2$. The total energy of this system is given by

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 - \frac{Q^2}{L}, \tag{28}$$

where the self-energy of each shell is given by

$$\mathcal{E}_1 = \frac{Q^2}{2R_1}, \qquad \mathcal{E}_2 = \frac{Q^2}{2R_2}. \tag{29}$$

For a simple way to compute the third term in Eq. (28) (namely, the interaction energy) see, for example, page 75 of Ref. [10]. At the contact point $L = R_1 + R_2$ we have

$$\mathcal{E} - \mathcal{E}_1 = \frac{Q^2(R_1 - R_2)}{2R_2(R_1 + R_2)}. \tag{30}$$

Take $R_2 > R_1$; then, if the shells are close enough to the contact point, from Eq. (30) we deduce that

$$\mathcal{E} - \mathcal{E}_1 < 0. \tag{31}$$

But then

$$\mathcal{E} < \mathcal{E}_1 = \frac{Q^2}{2R_1}, \tag{32}$$

and hence Eq. (13) is not valid for the shell R_1 if we take $\mathcal{E}$ as the total energy and Q and $\mathcal{R}$ as the charge and radius of the shell in Eq. (13).

An alternative and useful way to prove Eq. (13) in electrostatics is the following. By Thomson's theorem the electrostatic energy of a body of fixed shape, size, and charge is minimized when its charge Q distributes itself to make the electrostatic potential constant throughout the body (see, for example, page 128 of Ref. [10]). That is, the original configuration is replaced by a conductor with the same total charge and size which has less or equal energy. For conductors, Eq. (13) is related with the capacity of the conductor, defined as follows. Consider a conductor U and define the potential Φ_1 by

$$\Delta\Phi_1 = 0 \text{ in } \mathbb{R}^3\backslash U, \tag{33}$$

$$\Phi_1 = 1 \text{ at } \partial U, \tag{34}$$

$$\lim_{r\to\infty}\Phi_1 = 0. \tag{35}$$

The capacity of U is given by

$$C = -\frac{1}{4\pi}\oint_{\partial U}\partial_n\Phi_1. \tag{36}$$

The capacity C satisfies the well-known relation

$$\mathcal{E} = \frac{Q^2}{2C}, \tag{37}$$

where $\mathcal{E}$ is the total electrostatic energy of the conductor. Then, for a conductor, Eq. (13) is equivalent to

$$C \le \mathcal{R}. \tag{38}$$

Since (by Thomson's theorem) conductors minimize the energy, we have proved that Eq. (13) for general configurations reduces to the inequality (38) for conductors.

To prove Eq. (38) we use the variational characterization of C,

$$C = \frac{1}{4\pi}\inf_{\Phi\in K}\int_{\mathbb{R}^3\backslash U}|\nabla\Phi|^2, \tag{39}$$

where K is the set of all functions Φ that decay at infinity and are equal to 1 at ∂U. Consider the following test function:

$$\Phi_R = \begin{cases} \frac{\mathcal{R}}{r} & \text{if } r \ge \mathcal{R}, \\ 1 & \text{if } r \le \mathcal{R}. \end{cases} \tag{40}$$

We have that $\Phi_R \in K$ and hence we can use Eq. (22) to obtain

$$C \le \frac{1}{4\pi}\int_{\mathbb{R}^3\backslash B_{\mathcal{R}}}|\nabla\Phi_R|^2 = \mathcal{R}. \tag{41}$$

This characterization in terms of the capacity is useful to find interesting examples and estimates. In particular, it allows one to prove the following relevant statement: Eq. (13) is not valid if we replace the definition of $\mathcal{R}$ by the area radius, namely

$$\mathcal{R}_A = \sqrt{\frac{A}{4\pi}}, \tag{42}$$

where A is the area of the boundary ∂U. Note that the area radius $\mathcal{R}_A$ represents perhaps the simplest definition of radius that can be directly translated into curved spaces. The following counterexample shows that even in flat space $\mathcal{R}_A$ is not an appropriate measure of size in our context.

Consider a prolate conducting ellipsoid with radii a and b with $a > b$. The capacity of this conductor is given by (see page 22 of Ref. [11])

$$C = \frac{\sqrt{a^2-b^2}}{\cosh^{-1}a/b}, \tag{43}$$

and the surface area is given by

$$A = 2\pi b^2\left(1 + \frac{a}{b}\frac{\sin^{-1}e}{e}\right), \qquad e^2 = 1 - \frac{b^2}{a^2}. \tag{44}$$

We calculate the dimensionless quotient

$$\frac{C}{\mathcal{R}_A} = \frac{\sqrt{2}\sqrt{\frac{a^2}{b^2}-1}}{(\cosh^{-1}\frac{a}{b})\sqrt{1+\frac{a}{b}\frac{\sin^{-1}e}{e}}}. \tag{45}$$

Note that $C/\mathcal{R}_A$ depends only on the dimensionless parameter a/b. We take the limit $a/b \to \infty$,

$$\lim_{a/b\to\infty}\frac{C}{\mathcal{R}_A} \approx \frac{2}{\sqrt{\pi}}\frac{\sqrt{a/b}}{\log(a/b)} \to \infty, \tag{46}$$

and hence Eq. (38) is not satisfied for $\mathcal{R}_A$.

With this example we conclude the study of Eq. (13) in electrostatics. From now on, we will deal with the full Maxwell's equations (4)–(5). As a preliminary step, we prove Eq. (3) with $Q = 0$ and $J \neq 0$. This particular case will also be used in the general proof of Eq. (3).

Theorem 2.3. Consider a solution of Maxwell's equations (4)–(5) in the domain U. Let $\mathcal{R}$ be the radius of U defined in definition 2.1 and let x_0 be the center of the corresponding sphere. Then the following inequality holds:

$$c|J(U)| \le \mathcal{R}\mathcal{E}(U), \tag{47}$$

where $J(U)$ is the angular momentum of the electromagnetic field given by Eq. (8) with respect to the point x_0. Moreover, the equality in Eq. (47) holds if and only if the electromagnetic field vanishes in U.

Note that Eq. (47) is purely quasilocal, in contrast with the previous inequality (13): in Eq. (47) there appear only quantities defined on the domain U and not global quantities like the total energy $\mathcal{E}$. Of course, since $\mathcal{E} \ge \mathcal{E}(U)$, Eq. (47) implies the global inequality

$$c|J(U)| \leq \mathcal{R}\mathcal{E}. \tag{48}$$

Moreover, theorem 2.3 also implies a rigidity statement for the inequality (48): equality holds if and only if the electromagnetic field vanishes everywhere.

Proof.—We estimate the difference

$$\mathcal{E}(U) - \frac{c}{\mathcal{R}}|J(U)| = \frac{1}{8\pi}\int_U |\mathbf{E}|^2 + |\mathbf{B}|^2 - \frac{1}{4\pi\mathcal{R}}\left|\int_U (\mathbf{x}\times(\mathbf{E}\times\mathbf{B}))\cdot\mathbf{k}\right| \tag{49}$$

$$\geq \frac{1}{8\pi}\int_U |\mathbf{E}|^2 + |\mathbf{B}|^2 - \frac{2}{\mathcal{R}}|(\mathbf{x}\times(\mathbf{E}\times\mathbf{B}))\cdot\mathbf{k}|. \tag{50}$$

The integrand of the angular momentum (i.e., the angular momentum density) satisfies the elementary inequality

$$|(\mathbf{x}\times(\mathbf{E}\times\mathbf{B}))\cdot\mathbf{k}| \leq |(\mathbf{x}\times(\mathbf{E}\times\mathbf{B}))||\mathbf{k}| \tag{51}$$

$$= |(\mathbf{x}\times(\mathbf{E}\times\mathbf{B}))| \tag{52}$$

$$\leq |\mathbf{x}||\mathbf{E}||\mathbf{B}|, \tag{53}$$

where in Eq. (51) we used the inequality $|\mathbf{a}\cdot\mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$, in Eq. (52) we used that $\mathbf{k}$ is a unit vector, and in Eq. (53) we used the inequality $|\mathbf{a}\times\mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$. Using this inequality, we obtain

$$|\mathbf{E}|^2 + |\mathbf{B}|^2 - \frac{2}{\mathcal{R}}|(\mathbf{x}\times(\mathbf{E}\times\mathbf{B}))\cdot\mathbf{k}| \geq |\mathbf{E}|^2 + |\mathbf{B}|^2 - 2\frac{|\mathbf{x}|}{\mathcal{R}}|\mathbf{E}||\mathbf{B}|. \tag{54}$$

We write the right-hand side of the inequality as follows:

$$|\mathbf{E}|^2 + |\mathbf{B}|^2 - 2\frac{|\mathbf{x}|}{\mathcal{R}}|\mathbf{E}||\mathbf{B}| \tag{55}$$

$$= |\mathbf{E}|^2 + |\mathbf{B}|^2 - \frac{|\mathbf{x}|}{\mathcal{R}}(|\mathbf{E}|^2 + |\mathbf{B}|^2) + \frac{|\mathbf{x}|}{\mathcal{R}}(|\mathbf{E}|^2 + |\mathbf{B}|^2) - 2\frac{|\mathbf{x}|}{\mathcal{R}}|\mathbf{E}||\mathbf{B}| \tag{56}$$

$$= \left(1 - \frac{|\mathbf{x}|}{\mathcal{R}}\right)(|\mathbf{E}|^2 + |\mathbf{B}|^2) + \frac{|\mathbf{x}|}{\mathcal{R}}(|\mathbf{E}| - |\mathbf{B}|)^2 \tag{57}$$

$$\geq \left(1 - \frac{|\mathbf{x}|}{\mathcal{R}}\right)(|\mathbf{E}|^2 + |\mathbf{B}|^2). \tag{58}$$

Collecting these inequalities, we arrive to our final result,

$$\mathcal{E}(U) - \frac{c}{\mathcal{R}}|J(U)| \geq \frac{1}{8\pi}\int_U \left(1 - \frac{|\mathbf{x}|}{\mathcal{R}}\right)(|\mathbf{E}|^2 + |\mathbf{B}|^2). \tag{59}$$

By the definition of $\mathcal{R}$ we have $|\mathbf{x}| \leq \mathcal{R}$ on U, and hence the integrand on the right-hand side of Eq. (59) is nonnegative. This proves Eq. (47). Moreover, Eq. (59) also proves the rigidity statement: if equality holds, then the integrand on the right-hand side of Eq. (59) should vanish. Then, for every $\mathbf{x} \in U$ that is not on the sphere $\partial B_{\mathcal{R}}$ we have that both $\mathbf{E}$ and $\mathbf{B}$ are zero. By continuity, the fields are also zero on the points on the sphere $\partial B_{\mathcal{R}}$. ■

The proof of Eq. (47) (but not the rigidity statement) can be directly generalized to any classical field theory. It is a direct consequence of the dominant energy condition.[1] Let $T_{\mu\nu}$ be the electromagnetic energy-momentum tensor of the theory. The indices $\mu, \nu, \ldots$ are four-dimensional and we are using the signature $(-+++)$. For example, for electromagnetism we have

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\lambda\gamma}F^{\lambda\gamma}\right), \tag{60}$$

where $F_{\mu\nu}$ is the (antisymmetric) electromagnetic field tensor that satisfies Maxwell's equations. Consider a spacelike surface U with normal t^μ. The energy is given by

$$\mathcal{E} = \int_U T_{\mu\nu}t^\mu t^\nu. \tag{61}$$

Let η^μ be a Killing vector field that corresponds to space rotations. The angular momentum corresponding to the rotation η^μ is given by

$$J(U) = \frac{1}{c}\int_U T_{\mu\nu}t^\mu \eta^\nu. \tag{62}$$

Choosing coordinates such that x^i are spacelike Cartesian coordinates on the surface U and $t^\mu = (1,0,0,0)$, then the space rotations are characterized by

$$\eta_i = \epsilon_{ijk}k^j x^k, \tag{63}$$

where k is an arbitrary constant spacelike unit vector that represents the axis of rotation and the indices $i, j, k\ldots$ are three-dimensional. For the case of electromagnetism, it is easy to check [using Eq. (60)] that the definition (62) coincides with Eq. (8).

Assume that $T_{\mu\nu}$ satisfies the dominant energy condition, namely

$$T_{\mu\nu}\xi^\mu k^\nu \geq 0, \tag{64}$$

for all future-directed timelike or null vectors k^μ and ξ^μ.

We denote by η the square norm of η^i, that is $\eta = \eta^i\eta_i = \eta^\mu\eta_\mu$, and define the unit vector $\hat{\eta}^\mu = \eta^\mu\eta^{-1/2}$. Then, the vector

[1] I thank G. Dotti for providing me with this argument.

$$k^\mu = t^\mu - \hat{\eta}^\mu \tag{65}$$

is null future directed (since $t^\mu \eta_\mu = 0$). Choosing $\xi^\mu = t^\mu$ and k^μ given by Eq. (65), from Eq. (4) we obtain

$$T_{\mu\nu} t^\mu t^\nu \geq T_{\mu\nu} t^\mu \hat{\eta}^\nu. \tag{66}$$

Since η is the square of the distance to the axis, we have that

$$\eta \leq \mathcal{R}^2, \tag{67}$$

where $\mathcal{R}$ is the radius of a the ball that encloses U. Hence we deduce

$$J(U) = \frac{1}{c}\int_U T_{\mu\nu} t^\mu \eta^\nu = \frac{1}{c}\int_U T_{\mu\nu} t^\mu \eta^{1/2} \hat{\eta}^\nu \tag{68}$$

$$\leq \frac{\mathcal{R}}{c}\int_U T_{\mu\nu} t^\mu \hat{\eta}^\nu \tag{69}$$

$$\leq \frac{\mathcal{R}}{c}\int_U T_{\mu\nu} t^\mu t^\nu \tag{70}$$

$$= \frac{\mathcal{R}\mathcal{E}(U)}{c}. \tag{71}$$

Hence, we have proved Eq. (47) for a general energy-momentum tensor that satisfies the dominant energy condition (64). Note, however, that we have not proved the rigidity statement as in theorem 2.3.

Finally, we prove Eq. (3) for electromagnetism in full generality.

Theorem 2.4. Assume that $\rho(x, t_0)$, for some t_0, has compact support contained in U. Consider a solution of Maxwell's equations (4)–(5) that decay at infinity. Then the following inequality holds at t_0:

$$\frac{c|J(U)|}{\mathcal{R}} + \frac{Q^2}{2\mathcal{R}} \leq \mathcal{E}. \tag{72}$$

In particular, Eq. (72) implies

$$\frac{Q^4}{4\mathcal{R}^2} + \frac{c^2|J(U)|^2}{\mathcal{R}^2} \leq \mathcal{E}^2. \tag{73}$$

Moreover, if the equality in Eq. (72) holds, then the electromagnetic field is that produced by an electrostatic spherical thin shell of radius $\mathcal{R}$ and charge Q. For that case, the magnetic field vanishes everywhere and hence $J = 0$.

Proof.—Consider the Coulomb gauge[2]

$$B = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{74}$$

where the potential $\mathbf{A}$ satisfies the Coulomb gauge condition

$$\nabla \cdot \mathbf{A} = 0. \tag{75}$$

In this gauge, the total energy can be written in the following form:

$$\mathcal{E} = \frac{1}{8\pi}\int_{\mathbb{R}^3} |\mathbf{E}|^2 + |\mathbf{B}|^2 \tag{76}$$

$$= \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla\Phi|^2 + 2\nabla\Phi \cdot \frac{\partial \mathbf{A}}{\partial t} + \left|\frac{\partial \mathbf{A}}{\partial t}\right|^2 + |\mathbf{B}|^2, \tag{77}$$

where in Eq. (77) we have used the expression (74) for the electric field in terms of the potential $\mathbf{A}$. For the second term in the integrand of Eq. (77) we use the identity

$$\nabla\Phi \cdot \frac{\partial \mathbf{A}}{\partial t} = \nabla \cdot \left(\Phi \frac{\partial \mathbf{A}}{\partial t}\right) - \Phi \frac{\partial \nabla \cdot \mathbf{A}}{\partial t} \tag{78}$$

$$= \nabla \cdot \left(\Phi \frac{\partial \mathbf{A}}{\partial t}\right), \tag{79}$$

where in Eq. (79) we have used the Coulomb gauge condition (75). Using the asymptotic falloff conditions for Φ and $\mathbf{A}$ and Gauss's theorem, from Eq. (79) we obtain

$$\int_{\mathbb{R}^3} \nabla\Phi \cdot \frac{\partial \mathbf{A}}{\partial t} = 0. \tag{80}$$

Then, we have the following expression for the total energy:

$$\mathcal{E} = \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla\Phi|^2 + \left|\frac{\partial \mathbf{A}}{\partial t}\right|^2 + |\mathbf{B}|^2. \tag{81}$$

The potential $\Phi(x, t)$ satisfies the Poisson equation

$$\Delta\Phi(x, t) = -4\pi\rho(x, t) \tag{82}$$

for all t. At a fixed t, we can perform the same decomposition (15) for the potential $\Phi(x, t)$ used in theorem 2.2. Then, using Eq. (26), we obtain

$$\mathcal{E} = \frac{Q^2}{2\mathcal{R}} + \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla\Phi_1|^2 + \left|\frac{\partial \mathbf{A}}{\partial t}\right|^2 + |\mathbf{B}|^2, \tag{83}$$

where Φ_1 is defined by Eqs. (15) and (14). By the same integration by parts argument used to deduce Eq. (80), we obtain that

$$\int_{\mathbb{R}^3} \nabla\Phi_1 \cdot \frac{\partial \mathbf{A}}{\partial t} = 0. \tag{84}$$

[2] I thank O. Reula for suggesting the idea of using the Coulomb gauge.

Hence, we can write the energy (83) in the following way:

$$\mathcal{E} = \frac{Q^2}{2\mathcal{R}} + \frac{1}{8\pi}\int_{\mathbb{R}^3}\left|\nabla\Phi_1 + \frac{\partial \mathbf{A}}{\partial t}\right|^2 + |\mathbf{B}|^2. \tag{85}$$

We decompose the integral in Eq. (85) over the domains $\mathbb{R}^3\setminus U$ and U and we use the following simple but important identity:

$$\int_{\mathbb{R}^3}\left|\nabla\Phi_1 + \frac{\partial \mathbf{A}}{\partial t}\right|^2 = \int_{\mathbb{R}^3\setminus U}\left|\nabla\Phi_1 + \frac{\partial \mathbf{A}}{\partial t}\right|^2 + \int_{U}\left|\nabla\Phi_1 + \frac{\partial \mathbf{A}}{\partial t}\right|^2 \tag{86}$$

$$= \int_{\mathbb{R}^3\setminus U}\left|\nabla\Phi_1 + \frac{\partial \mathbf{A}}{\partial t}\right|^2 + \int_{U}\left|\nabla\Phi + \frac{\partial \mathbf{A}}{\partial t}\right|^2 \tag{87}$$

$$= \int_{\mathbb{R}^3\setminus U}\left|\nabla\Phi_1 + \frac{\partial \mathbf{A}}{\partial t}\right|^2 + \int_{U}|\mathbf{E}|^2, \tag{88}$$

where in Eq. (87) we have used that $\nabla\Phi_1 = \nabla\Phi$ in U since Φ_0 is constant in U, and in Eq. (88) we have used the expression for the electric field in the Coulomb gauge (74). Then we obtain the following expression for the energy $\mathcal{E}$:

$$\mathcal{E} = \frac{Q^2}{2\mathcal{R}} + \mathcal{E}(U) + \frac{1}{8\pi}\int_{\mathbb{R}^3\setminus U}\left|\nabla\Phi_1 + \frac{\partial \mathbf{A}}{\partial t}\right|^2 + |\mathbf{B}|^2, \tag{89}$$

where $\mathcal{E}(U)$ is the electromagnetic energy density integrated over the domain U, namely

$$\mathcal{E}(U) = \frac{1}{8\pi}\int_U |\mathbf{E}|^2 + |\mathbf{B}|^2. \tag{90}$$

We use theorem 2.3 to bound $\mathcal{E}(U)$ [i.e., the estimate (59)] and we finally have

$$\mathcal{E} - \frac{Q^2}{2\mathcal{R}} - \frac{c|J(U)|}{\mathcal{R}} \geq \frac{1}{8\pi}\left(\int_{\mathbb{R}^3\setminus U}\left|\nabla\Phi_1 + \frac{\partial \mathbf{A}}{\partial t}\right|^2 + |\mathbf{B}|^2 + \int_U\left(1 - \frac{|\mathbf{x}|}{\mathcal{R}}\right)(|\mathbf{E}|^2 + |\mathbf{B}|^2)\right). \tag{91}$$

Since the left-hand side of Eq. (91) is non-negative we have proved the inequality (72). Equation (91) also implies the rigidity statement. We assume the equality in Eq. (72); then, the integrand on the right-hand side of Eq. (91) should vanish. This implies that $\mathbf{B} = 0$ everywhere, and hence the potential $\mathbf{A}$ is a gradient. Using Eq. (75) and the falloff condition for $\mathbf{A}$ we deduce that $\mathbf{A} = 0$. Then, using Eq. (91) again, we obtain that $\nabla\Phi_1 = 0$ and hence the statement is proved.

Taking the square of Eq. (72), we obtain

$$\frac{c|J|Q^2}{\mathcal{R}} + \frac{Q^4}{4\mathcal{R}^2} + \frac{c^2J^2}{\mathcal{R}^2} \leq \mathcal{E}^2, \tag{92}$$

which, in particular, implies Eq. (73). ∎

III. GENERAL RELATIVITY

In this section we study Eq. (3) in general relativity. In Sec. III A we discuss a remarkable relation between this inequality and inequalities between size, charge, and angular momentum. In Sec. III B we present a proof of Eq. (3), with $J = 0$, for time-symmetric initial conditions.

A. Inequalities between size, charge, and angular momentum

For a black hole the entropy is given by the horizon area A,

$$S_{\rm bh} = \frac{k_B c^3}{4G\hbar}A. \tag{93}$$

Equation (2) is constructed in such a way that for a Kerr-Newman black hole, using the formula (93), we get an equality. Moreover, Szabados [12] observed that for dynamical black holes this inequality is also expected to hold. It is the generalization of the Penrose inequality including charge and angular momentum (see the review article [13] and Refs. [14,15] and the discussion therein).

For ordinary bodies, Eq. (3) is closely related to inequalities between size, angular momentum, and charge, which was recently studied in Ref. [8]. To show this relation we argue as follows. The hoop conjecture essentially says that if matter is enclosed in a sufficiently small region, then the system should collapse to a black hole [16]. Then, if the body is not a black hole we expect an inequality of the form

$$\frac{G}{c^4}\mathcal{E} \leq k\mathcal{R}, \tag{94}$$

where k is a universal dimensionless constant of order one. The exact value of k will depend on the precise formulation of the hoop conjecture and this is not important in what follows.

Using Eq. (94) to bound $\mathcal{E}$ in Eq. (3), we obtain

$$\frac{Q^4}{4} + c^2J^2 \leq k^2\frac{c^8}{G^2}\mathcal{R}^4. \tag{95}$$

Note that the constant G appears in Eq. (95). That is, Eq. (95) involves two fundamental constants (c and G), in contrast to Eq. (3) which involves only one (c). On the other hand, Eq. (95) involves fewer physical quantities (charge, angular momentum, and size) than Eq. (3) (charge, angular momentum, size, and energy).

The bound (95) implies

$$\frac{G}{c^3}|J| \leq k\mathcal{R}^2. \tag{96}$$

Equation (96) was conjectured in Ref. [8] using different kinds of arguments as those leading to Eq. (3). With an appropriate definition of size, a version of this inequality was proved for constant-density bodies in Ref. [8]. Recently, Khuri [17] has proved it in the general case, using the same measure of size as in Ref. [8]. However, these inequalities are not expected to be sharp. We will came back to this point below.

Also, from Eq. (95) we get the inequality

$$|Q| \leq (2k)^{1/2}\frac{c^2}{G^{1/2}}\mathcal{R}. \tag{97}$$

This inequality can also be deduced using similar arguments as in Ref. [8] and it was studied for some particular examples in Ref. [18]. Recently, Khuri [19] has proved a general version of Eq. (97) using a similar (but not identical) measure of size as the one used in Eq. (96). As in the case of angular momentum, this result is not expected to be sharp.

The relation between the Bekenstein bounds and inequalities (96) and (97) provides two important new insights. The first one is the following. We pointed out that Eq. (96) was conjectured in ref. [8] using heuristic physical arguments and also that Eq. (97) can be deduced using similar kinds of arguments. However, with these arguments Eqs. (96) and (97) are deduced individually. These kinds of arguments do not seem to provide a way of deducing the complete inequality (95), which is obtained here for the first time using the Bekenstein bounds. Moreover, the arguments presented above suggest that there is only one universal constant k to be fixed. This constant can be fixed by analyzing a simple limit case, for example spherical symmetry, with $J = 0$. We are currently working on this problem [20].

The second, and perhaps most important point concerns the rigidity of the inequality (95). The arguments presented in Ref. [8] do not give any insight about what happens when equality is reached in Eq. (95). The Bekenstein bounds provide such a statement. Let us assume that the equality is reached in Eq. (95). Since we have assumed that it is not a black hole we can use the hoop conjecture inequality (94) to obtain

$$\frac{Q^4}{4} + c^2J^2 = k^2\frac{c^8}{G^2}\mathcal{R}^4 \geq \mathcal{E}^2. \tag{98}$$

But then we can use Eq. (3) to conclude that if the equality is reached in Eq. (95) then the equality should also hold in Eq. (3). By the Bekenstein bound (2), this implies that the entropy of the body is zero. Hence we have the following rigidity statement for Eq. (95): the equality is achieved if and only the entropy of the body is zero. In general relativity, this statement appears to imply that in fact the equality is achieved if and only if the spacetime is flat. We will further discuss this point in the next section.

B. Proof of the inequality between charge, energy, and size for time-symmetric initial data

In general relativity, the inequality (95) was proved in spherical symmetry (in a different context) by Reiris [9]. In the following we generalize this result to time-symmetric initial data.

The most important difficulty in studying these kind of inequalities in curved spaces is how to define the measure of size $\mathcal{R}$. We propose a new measure of size which is tailored to the proof of theorem 3.2. This measure of size represents a natural generalization to curved spaces of definition 2.1 used in Sec. II.

The definition of size and the proof of the theorem is based on the *inverse mean curvature flow* (IMCF). A family of 2-surfaces on a Riemannian manifold evolves under the IMCF if the outward normal speed at which a point on the surface moves is given by the reciprocal of the mean curvature of the surface. For the precise definition and properties of the IMCF we refer to Ref. [21]. The IMCF has played a key role in the proof of the Riemannian Penrose inequality [21].

Using the IMCF we define the following radius $\mathcal{R}$ of a region U in a Riemannian manifold.

Definition 3.1. Consider a region U on a complete, asymptotically flat, Riemannian manifold. Take a point x_0 on the manifold and consider the inverse mean curvature flow starting at this point. Consider the area of the first 2-surface on the flow that encloses the region U, and define $\mathcal{R}_{x_0}$ to be the area radius of this surface. The radius $\mathcal{R}$ of the region U is defined as the infimum of $\mathcal{R}_{x_0}$ over all points x_0 on the manifold.

In Fig. 2 we draw a schematic picture of the flow starting at a typical point x_0. In flat space, the IMCF starting at a point develop spheres, and hence definition 3.1 coincides with definition 2.1 presented in the previous section. However, we emphasize that this definition is very different than the one used in Refs. [8,17,19].

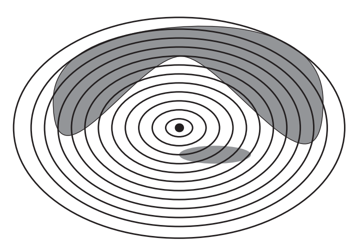

FIG. 2. Schematic drawing of the inverse mean curvature flow from a typical point. The last surface is defined as the first one that enclosed the domain U.

The radius $\mathcal{R}$ defined above certainly involves sophisticated mathematics; however, it is important to recall that it can be explicitly estimated numerically for arbitrary curved backgrounds.

It is important to recall that, in general, the flow will develop singularities. This singular behavior can be treated using the weak formulation discovered in Ref. [21]. In what follows, for simplicity of the presentation, we will assume that the flow is smooth; however, all the arguments are also valid in the weak formulation.

We have the following result.

Theorem 3.2. Consider an asymptotically flat, complete, time-symmetric initial data for Einstein's equations that satisfy the dominant energy condition and with no minimal surfaces. Assume that there is a region U outside of which the initial data are electrovacuum. Then we have

$$Q^2 \leq 2\mathcal{E}\mathcal{R}, \tag{99}$$

where $\mathcal{E}$ is the Arnowitt-Deser-Misner mass, Q is the charge contained in U, and $\mathcal{R}$ is the radius of U defined above. Moreover, if the equality in Eq. (99) holds, then the data is flat inside the region U.

Proof.—The proof is inspired by Reiris's proof [9] and it is a simple consequence of the results presented in Refs. [22] and [21].

The crucial property of the IMCF is the Geroch monotonicity of the Hawking energy. The Hawking energy of a closed 2-surface $\mathcal{S}$ is given by

$$\mathcal{E}_H(\mathcal{S}) = \sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\mathcal{S}} H^2\right), \tag{100}$$

where H is the mean curvature of the surface and A is its area. The Geroch monotonicity can be written in the following form. Assume that the flow runs between a surface $\mathcal{S}_r$ and a surface $\mathcal{S}_s$, with $r < s$; then, we have

$$\mathcal{E}_H(\mathcal{S}_s) \geq \mathcal{E}_H(\mathcal{S}_r) + \frac{1}{(16\pi)^{3/2}}\int_r^s (A_t)^{1/2}\int_{\mathcal{S}_t} R dt, \tag{101}$$

where R is the scalar curvature. Note that the dominant energy condition for time-symmetric data implies that $R \geq 0$. We will use Eq. (101) in two steps.

First, consider an arbitrary point x_0 and run the IMCF from x_0. Since the data satisfy the dominant energy condition, a small sphere around x_0 has non-negative Hawking mass. Moreover, the assumption that there are no minimal surfaces on the data guarantees that the flow runs up to infinity (even in the presence of singularities; see Ref. [21]). Then, using Eq. (101), we conclude that any level set of the flow has non-negative Hawking energy, in particular the surface $\mathcal{S}_{x_0}$ that encloses the region U used in definition 3.1; that is,

$$\mathcal{E}_H(\mathcal{S}_{x_0}) \geq 0. \tag{102}$$

We denote by A_{x_0} the area of $\mathcal{S}_{x_0}$ and the area radius is given by $\mathcal{R}_{x_0} = \sqrt{A_{x_0}/4\pi}$.

In the second step, we continue the flow from the surface $\mathcal{S}_{x_0}$ to infinity. Following Ref. [22], we bound the integral of the scalar curvature in terms of the charge

$$\frac{1}{(16\pi)^{3/2}}\int_{x_0}^{\infty} (A_t)^{1/2}\int_{\mathcal{S}_t} R dt \geq \frac{Q^2}{2\mathcal{R}_{x_0}}, \tag{103}$$

where we have used the fact that the charge is conserved outside $\mathcal{S}_{x_0}$, since by construction $\mathcal{S}_{x_0}$ encloses the region U and by assumption the support of the charge density is contained in U. Using Eqs. (103) and (101), we obtain

$$\mathcal{E} - \frac{Q^2}{2\mathcal{R}_{x_0}} \geq \mathcal{E}_H(\mathcal{S}_{x_0}). \tag{104}$$

Using Eq. (102), we finally get

$$\mathcal{E} - \frac{Q^2}{2\mathcal{R}_{x_0}} \geq 0. \tag{105}$$

In particular, this inequality applies to the radius $\mathcal{R}$ and hence Eq. (99) follows.

If the equality holds in Eq. (104), then we have $\mathcal{E}_H(\mathcal{S}_{x_0}) = 0$ and hence we can use the same rigidity argument as in Ref. [21] to conclude that inside $\mathcal{S}_{x_0}$ the data are flat. ■

We have obtained a similar kind of estimate as in the electromagnetic case (27) in which $\mathcal{E}_H(\mathcal{S}_0)$ is interpreted as the quasilocal energy inside $\mathcal{S}_0$.

Comparing theorem 3.2 with theorem 2.4 in electromagnetism, we see that there is no rigidity statement outside the region U in theorem 3.2. The natural question is whether a similar statement as that in theorem 2.4 holds, namely, that the equality implies that the field is produced by a charged thin shell. However, it is likely that the charged thin shell in general relativity never saturates the inequality (in contrast with electromagnetism). The reason is that the rest energy of the shell is now taken into account. Hence, a stronger rigidity statement is expected for theorem 3.2: the equality holds if and only if the complete data are flat. We are currently working on this problem [20].

It would be interesting to include angular momentum in theorem 3.2. However, this appears to be a difficult problem. In particular, it is not clear how to include angular momentum in the inequality using the IMCF.

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