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Colliding impulsive gravitational waves and a cosmological constant

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We present a space-time model of the collision of two homogeneous, plane impulsive gravitational waves (each having a delta function profile) propagating in a vacuum before collision and for which the postcollision space-time has constant curvature. The profiles of the incoming waves are $k\delta(u)$ and $l\delta(v)$ where k, l are real constants and $u = 0, v = 0$ are intersecting null hypersurfaces. The cosmological constant Λ in the postcollision region of the space-time is given by $\Lambda = -6kl$.

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I. INTRODUCTION

Finding the space-time structure after the collision of gravitational and/or electromagnetic waves is a difficult problem in general relativity due to the nonlinearity of the field equations. The problem is simplified by specializing to impulsive and/or shock waves which are plane and homogeneous and then exact solutions can be found, with the Khan-Penrose [1] and Bell-Szekeres [2] solutions among the most famous. Up to now no solution where a cosmological constant appears after the collision of two homogeneous, plane, impulsive gravitational waves has yet been found. This paper provides such an exact solution.

Impulsive lightlike signals, i.e. signals traveling with the speed of light and having a delta function profile, are idealized models of more realistic lightlike signals having a profile with a finite width such as, for instance, a burst of lightlike matter or of radiation. In some situations they may provide solvable models to describe their interactions and are sometimes used as classical models of quantum phenomena. In black hole physics one has the examples of mass inflation [3], the limiting curvature principle [4], Hawking radiation and quantum fluctuations [5] and internal structure of a Schwarzschild black hole [6,7]. In general relativity an impulsive lightlike signal exists whenever the Riemann curvature tensor of the space-time manifold exhibits a delta function term with support on a null hypersurface, with the latter representing the space-time history of the signal and across which the first derivatives of the metric tensor are discontinuous. This signal can be a thin shell of lightlike matter, an impulsive gravitational wave or a mixture of both [8]. Recently the cosmological constant has received much attention in connection with a possible description of dark energy. Such exotic matter is described by a perfect fluid with an equation of state for which the sum of the energy density and isotropic pressure vanishes. In this paper we examine the possibility that the collision of two impulsive gravitational waves will produce such exotic matter by adopting a mathematical point of view, i.e. by solving Einstein's field equations with appropriate boundary conditions. No attempt is made to propose a physical mechanism.

For most of the known wave collision models in general relativity the same field equations apply before and after the collision. However the choice of field equations before collision does not determine the choice of field equations after collision. This freedom offers an opportunity to explore new and potentially interesting models. The well-known space-time model of a head-on collision of two homogeneous, plane impulsive gravitational waves, traveling in a vacuum, involves the assumption that *the postcollision region of space-time is a vacuum space-time*. With this assumption the postcollision region is described by the Khan-Penrose [1] solution of Einstein's vacuum field equations (for a derivation see [9]). We demonstrate here that if the postcollision region of space-time is assumed to be a solution of *Einstein's field equations with a cosmological constant* then an exact solution of these field equations can be found satisfying the same conditions on the null hypersurface boundaries of the postcollision region as the Khan-Penrose solution. In addition the cosmological constant can be expressed simply in terms of two parameters which label each of the incoming waves. The postcollision region is a space-time of constant curvature and is thus curvature-singularity free, in contrast to the Khan-Penrose model. The solution derived here is not an extension of the Khan-Penrose solution since it has the property that if the cosmological constant vanishes then at least one of the incoming waves vanishes. The postcollision model presented here can be explained in terms of a redistribution of the energy in the incoming waves and this is described in some detail.

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In Sec. II the incoming plane, impulsive gravitational waves propagating through a vacuum are introduced, the collision problem is specified (as a lightlike boundary value problem) and the solution of Einstein's field equations with a cosmological constant in the postcollision region is given. This is followed in Sec. III by a detailed study of the physical properties of the products of the collision which, in addition to a cosmological constant, include impulsive gravitational waves (as in the Khan-Penrose collision) and lightlike shells of matter. When reasonable physical restrictions are invoked the postcollision region of space-time is anti–de Sitter space-time in this case.

II. COLLIDING WAVES

A plane, homogeneous gravitational impulse wave propagating in a vacuum is described in general relativity by a space-time with line element

$$ds^2 = -(1+ku_+)^2dx^2 - (1-ku_+)^2dy^2 + 2du dv, \quad (2.1)$$

where k is a constant (introduced for convenience) and $u_+ = u\vartheta(u)$ where $\vartheta(u) = 1$ for $u > 0$ and $\vartheta(u) = 0$ for $u < 0$ is the Heaviside step function. The metric given via this line element satisfies Einstein's vacuum field equations everywhere (in particular on $u = 0$). The only nonvanishing Newman-Penrose component of the Riemann curvature tensor on the tetrad given via the 1-forms $\vartheta^1 = (1+ku_+)dx$, $\vartheta^2 = (1-ku_+)dy$, $\vartheta^3 = dv$, $\vartheta^4 = du$ is

$$\Psi_4 = -k\delta(u). \quad (2.2)$$

Thus the curvature tensor is type N (the radiative type) in the Petrov classification with the vector field $\partial/\partial v$ the degenerate principal null direction and therefore the propagation direction of the history of the wave (the null hypersurface $u = 0$) in space-time. The wave profile is the delta function, singular on $u = 0$, and thus the wave is an impulsive wave. There are two families of intersecting null hypersurfaces $u = \text{constant}$ and $v = \text{constant}$ in the space-time with line element (2.1). A homogeneous, plane impulsive gravitational wave propagating in a vacuum in the opposite direction to that with history $u = 0$ has history $v = 0$ and this is described by a space-time with line element

$$ds^2 = -(1+lv_+)^2dx^2 - (1-lv_+)^2dy^2 + 2du dv, \quad (2.3)$$

where l is a convenient constant and $v_+ = v\vartheta(v)$. The Ricci tensor vanishes everywhere when calculated with the metric tensor given by this line element. The only nonvanishing Newman-Penrose component of the Riemann curvature tensor on the tetrad given via the 1-forms $\vartheta^1 = (1+lv_+)dx$, $\vartheta^2 = (1-lv_+)dy$, $\vartheta^3 = dv$, $\vartheta^4 = du$ is

$$\Psi_0 = -l\delta(v), \quad (2.4)$$

indicating a Petrov type N curvature tensor with degenerate principal null direction $\partial/\partial u$.

For the collision problem we envisage a precollision vacuum region of space-time $v < 0$ with line element (2.1) and a precollision vacuum region of space-time $u < 0$ with line element (2.3) (with both line elements coinciding when $v < 0$ *and* $u < 0$). The waves collide at $u = v = 0$ and the postcollision region of the space-time corresponds to $u > 0$ *and* $v > 0$. In this region the line element has the form [1,10,11]

$$ds^2 = -e^{-U}(e^V dx^2 + e^{-V} dy^2) + 2e^{-M} du dv, \quad (2.5)$$

where U, V, M are each functions of u, v. These functions must satisfy the following conditions on the null hypersurface boundaries of the region $u > 0, v > 0$:

$$v = 0, \qquad u \geq 0 \Rightarrow e^{-U} = 1 - k^2u^2,$$
$$e^V = \frac{1+ku}{1-ku}, \qquad M = 0, \quad (2.6)$$

and

$$u = 0, \qquad v \geq 0 \Rightarrow e^{-U} = 1 - l^2v^2,$$
$$e^V = \frac{1+lv}{1-lv}, \qquad M = 0. \quad (2.7)$$

Einstein's field equations with a cosmological constant Λ in the region $u > 0, v > 0$ calculated with the metric tensor given by the line element (2.5) read

$$U_{uv} = U_u U_v - \Lambda e^{-M}, \quad (2.8)$$

$$2V_{uv} = U_u V_v + U_v V_u, \quad (2.9)$$

$$2U_{uu} = U_u^2 + V_u^2 - 2M_u U_u, \quad (2.10)$$

$$2U_{vv} = U_v^2 + V_v^2 - 2M_v U_v, \quad (2.11)$$

$$2M_{uv} = V_u V_v - U_u U_v, \quad (2.12)$$

where the subscripts denote partial derivatives. To implement our strategy below for solving (2.8)–(2.12) subject to the boundary conditions (2.6) and (2.7) we will need to know V_v at $v = 0$, which we denote by $(V_v)_{v=0}$, and V_u at $u = 0$, which we denote by $(V_u)_{u=0}$. We already have from (2.6) and (2.7)

$$(V_u)_{v=0} = \frac{2k}{1-k^2u^2} \quad \text{and} \quad (V_v)_{u=0} = \frac{2l}{1-l^2v^2}, \quad (2.13)$$

and also

$$(U_u)_{v=0}=\frac{2k^2u}{1-k^2u^2}\quad\text{and}\quad(U_v)_{u=0}=\frac{2l^2v}{1-l^2v^2}.\tag{2.14}$$

In order to compute $(V_v)_{v=0}$ and $(V_u)_{u=0}$ we must first calculate $(U_v)_{v=0}$ and $(U_u)_{u=0}$. We obtain these latter quantities by evaluating (2.8) at $u=0$ and at $v=0$ and solving the resulting first order ordinary differential equations. The constants of integration which arise are determined from the fact that U_v and U_u both vanish when $u=0$ *and* $v=0$, which follows from (2.14). We then find that

$$(U_v)_{v=0}=-\frac{\Lambda u(1-\frac{1}{3}k^2u^2)}{1-k^2u^2}\quad\text{and}$$
$$(U_u)_{u=0}=-\frac{\Lambda v(1-\frac{1}{3}l^2v^2)}{1-l^2v^2}.\tag{2.15}$$

Now evaluating (2.9) at $v=0$ and at $u=0$ provides us with a pair of first order ordinary differential equations for $(V_v)_{v=0}$ and $(V_u)_{u=0}$. These equations are straightforward to solve and the resulting constants of integration are determined from the fact that $V_u=2k$ and $V_v=2l$ when $u=0$ *and* $v=0$, which follows from (2.13). The final results are

$$(V_v)_{v=0}=\left(2l+\frac{\Lambda}{3k}\right)(1-k^2u^2)^{-1/2}-\frac{\Lambda}{3k}\left(\frac{1+k^2u^2}{1-k^2u^2}\right),\tag{2.16}$$

$$(V_u)_{u=0}=\left(2k+\frac{\Lambda}{3l}\right)(1-l^2v^2)^{-1/2}-\frac{\Lambda}{3l}\left(\frac{1+l^2v^2}{1-l^2v^2}\right).\tag{2.17}$$

Dividing (2.9) successively by V_u and by V_v and then differentiating the resulting equations and combining them we obtain

$$2\frac{\partial^2}{\partial u\partial v}\log\frac{V_u}{V_v}=\left(U_u\frac{V_v}{V_u}\right)_u-\left(U_v\frac{V_u}{V_v}\right)_v.\tag{2.18}$$

This suggests that we examine the possibility of a separation of variables,

$$\frac{V_u}{V_v}=\frac{A(u)}{B(v)},\tag{2.19}$$

for some functions $A(u)$ and $B(v)$. The resulting mathematical simplification is that (2.19) becomes a first order wave equation for V (see below) and that (2.18) becomes a second order wave equation for U. From a physical point of view we have shown [8] that if, as is the case in general, two systems of backscattered gravitational waves exist in the postcollision region (one with propagation direction $\partial/\partial u$ in space-time and one with propagation direction $\partial/\partial v$) then (2.19) implies that there exists a frame of reference in which the energy densities of the two systems of waves are equal. Using (2.13), (2.16) and (2.17) determines the right-hand side of (2.19) and the result is

$$\frac{V_u}{V_v}=\frac{k[(1+\frac{\Lambda}{6kl})\sqrt{1-l^2v^2}-\frac{\Lambda}{6kl}(1+l^2v^2)]}{l[(1+\frac{\Lambda}{6kl})\sqrt{1-k^2u^2}-\frac{\Lambda}{6kl}(1+k^2u^2)]}.\tag{2.20}$$

Hence this equation can be written as a first order wave equation

$$V_{\bar u}=V_{\bar v},\tag{2.21}$$

with $\bar u(u)$ and $\bar v(v)$ given by the differential equations

$$\frac{d\bar u}{du}=k\left[\left(1+\frac{\Lambda}{6kl}\right)\sqrt{1-k^2u^2}-\frac{\Lambda}{6kl}(1+k^2u^2)\right]^{-1},\tag{2.22}$$

$$\frac{d\bar v}{dv}=l\left[\left(1+\frac{\Lambda}{6kl}\right)\sqrt{1-l^2v^2}-\frac{\Lambda}{6kl}(1+l^2v^2)\right]^{-1}.\tag{2.23}$$

These two equations are interesting in general but we shall concentrate in this paper on two standout special cases: $\Lambda=0$ and $\Lambda=-6kl$. The case $\Lambda=0$ is shown in the Appendix to correspond to the Khan-Penrose [1] space-time.

With $\Lambda=-6kl$ we can solve (2.22) and (2.23), requiring $\bar u=0$ when $u=0$ and $\bar v=0$ when $v=0$, with

$$\bar u=\tan^{-1}ku,\qquad\bar v=\tan^{-1}lv.\tag{2.24}$$

By (2.21) we have $V=V(\bar u+\bar v)$ and the boundary condition (2.6) written in terms of $\bar u,\bar v$ reads as follows: when $\bar v=0$, $V=\log(\frac{1+\tan\bar u}{1-\tan\bar u})$. Hence

$$V(\bar u+\bar v)=\log\left(\frac{1+\tan(\bar u+\bar v)}{1-\tan(\bar u+\bar v)}\right),\tag{2.25}$$

and restoring the coordinates u,v we have

$$V(u,v)=\log\left(\frac{1-kluv+ku+lv}{1-kluv-ku-lv}\right),\tag{2.26}$$

for $u\ge0$, $v\ge0$ provided $\Lambda=-6kl$. Next writing (2.9) in terms of the variables $\bar u,\bar v$ and using (2.21) and (2.25) we have

$$U_{\bar u}+U_{\bar v}=\frac{8\tan(\bar u+\bar v)}{1-\tan^2(\bar u+\bar v)},\tag{2.27}$$

which is easily integrated to yield

$$e^{-U} = C(\bar{u} - \bar{v})\left(\frac{1 - \tan^2(\bar{u} + \bar{v})}{1 + \tan^2(\bar{u} + \bar{v})}\right), \qquad (2.28)$$

where $C(\bar{u} - \bar{v})$ is a function of integration. When $\bar{v} = 0$ the boundary condition (2.6) requires $e^{-U} = 1 - \tan^2 \bar{u}$ and so $C(\bar{u}) = 1 + \tan^2 \bar{u}$. Hence restoring the coordinates u, v we have $U(u, v)$ given by

$$e^{-U} = \frac{(1 - kluv)^2 - (ku + lv)^2}{(1 + kluv)^2}, \qquad (2.29)$$

for $u \geq 0, v \geq 0$. In light of (2.28) we see that U is a linear combination of a function of $\bar{u} - \bar{v}$ and a function of $\bar{u} + \bar{v}$ and thus satisfies the second order wave equation $U_{\bar{u}\bar{u}} = U_{\bar{v}\bar{v}}$. This wave equation is the equation that (2.18) reduces to when (2.21) holds and the barred coordinates are used.

With $V(u, v)$ and $U(u, v)$ given by (2.26) and (2.29) we use the field equation (2.8) with $\Lambda = -6kl$ to calculate $M(u, v)$. The result is

$$M(u, v) = 2\log(1 + kluv), \qquad (2.30)$$

and this clearly satisfies the boundary conditions (2.6) and (2.7). Now with V, U and M determined a lengthy calculation verifies that the remaining field equations (2.10)–(2.12) are automatically satisfied. Thus the line element (2.5) of the postcollision region reads

$$ds^2 = \frac{-(1 - kluv + ku + lv)^2 dx^2 - (1 - kluv - ku - lv)^2 dy^2 + 2dudv}{(1 + kluv)^2}. \qquad (2.31)$$

If in the metric tensor components here we replace u, v by $u_+ = u\vartheta(u), v_+ = v\vartheta(v)$ we obtain in a single line element the expressions (2.1) and (2.3) for the precollision regions and (2.31) for the postcollision region. In particular this will enable us to calculate the physical properties of the boundaries $v = 0, u \geq 0$ and $u = 0, v \geq 0$ of the postcollision region.

III. POSTCOLLISION PHYSICAL PROPERTIES

With our sign conventions, choice of units for which $c = G = 1$, and energy-momentum-stress tensor T_{ab}, Einstein's field equations with a cosmological constant Λ read

$$R_{ab} = \Lambda g_{ab} - 8\pi\left(T_{ab} - \frac{1}{2}T^c{}_c g_{ab}\right). \qquad (3.1)$$

Also for a perfect fluid with proper density ρ and isotropic pressure p we have

$$T_{ab} = (\rho + p)u_a u_b - p g_{ab}, \qquad (3.2)$$

where u_a, satisfying $u_a u^a = 1$, is the 4-velocity of a fluid particle. We note that for the exotic matter mentioned in Sec. I, $\rho + p = 0$ and thus $T_{ab} = \rho g_{ab}$ and, with our sign conventions, Einstein's field equations with this energy-momentum-stress tensor are equivalent to the field equations with a cosmological constant $\Lambda = 8\pi\rho$.

On the half null tetrad $\vartheta^1 = e^{-\frac{1}{2}(U-V)}dx$, $\vartheta^2 = e^{-\frac{1}{2}(U+V)}dy$, $\vartheta^3 = e^{-\frac{1}{2}M}dv$, $\vartheta^4 = e^{-\frac{1}{2}M}du$ with V, U, M given by (2.26), (2.29) and (2.30) with u, v replaced by u_+, v_+ the Ricci tensor components of the space-time are given by

$$R_{ab} = -6kl\vartheta(u)\vartheta(v)g_{ab} + \frac{2klu_+(k^2u_+^2 - 3)}{1 - k^2u_+^2}\delta(v)\delta_a^3\delta_b^3 + \frac{2klv_+(l^2v_+^2 - 3)}{1 - l^2v_+^2}\delta(u)\delta_a^4\delta_b^4. \qquad (3.3)$$

This confirms that the space-time region $u > 0, v > 0$ is a solution of the field equations with a cosmological constant, $R_{ab} = \Lambda g_{ab}$, with $\Lambda = -6kl$, and that there are lightlike shells with the boundaries $v = 0, u \geq 0$ and $u = 0, v \geq 0$ as histories, corresponding to the delta function terms in (3.3). Here g_{ab} are the (constant) metric tensor components on the half null tetrad given via the basis 1-forms $\{\vartheta^a\}$. The lightlike shells have no isotropic surface pressure [12] and the surface energy densities are $\mu_{(1)}$ and $\mu_{(2)}$ given by

$$8\pi\mu_{(1)} = \frac{\Lambda u}{3}\left(\frac{k^2u^2 - 3}{1 - k^2u^2}\right) \quad \text{on} \quad v = 0, u \geq 0, \qquad (3.4)$$

and

$$8\pi\mu_{(2)} = \frac{\Lambda v}{3}\left(\frac{l^2v^2 - 3}{1 - l^2v^2}\right) \quad \text{on} \quad u = 0, v \geq 0. \qquad (3.5)$$

The lightlike shells must have positive surface energy densities. The only way to realize this on $v = 0, u \geq 0$ (respectively on $u = 0, v \geq 0$) is to have $kl > 0$ and $k^2u^2 < 1$ (respectively $kl > 0$ and $l^2v^2 < 1$). Thus the cosmological constant $\Lambda = -6kl$ must be negative. These restrictions on the coordinates are less restrictive than the condition $k^2u^2 + l^2v^2 < 1$ for $u \geq 0$ and $v \geq 0$ required in the Khan-Penrose postcollision space-time on account of the presence of the curvature singularity. These restrictions on the coordinates also avoid infinite surface energy densities in the shells which are arguably as serious as a curvature singularity. Lightlike shells did not appear in the Khan-Penrose model and their presence here is due to the nonzero cosmological constant.

This result implies that the energy density of the exotic matter is negative and that consequently only the so-called strong energy condition [13], namely, $\rho + p \geq 0$ and $\rho + 3p \geq 0$, can be satisfied.

The Newman-Penrose components of the Weyl conformal curvature tensor are given by

$$\Psi_0 = -\frac{l(1+k^2u_+^2)}{1-k^2u_+^2}\delta(v), \qquad \Psi_4 = -\frac{k(1+l^2v_+^2)}{1-l^2v_+^2}\delta(u),$$
$$\Psi_1 = \Psi_2 = \Psi_3 = 0. \tag{3.6}$$

Thus the boundaries $v = 0$, $0 \leq k^2u^2 < 1$ and $u = 0, 0 \leq l^2v^2 < 1$ are the histories of impulsive gravitational waves corresponding to the delta function terms here. The post-collision region $u > 0, v > 0$ is conformally flat and is a space-time of constant curvature with Riemann curvature tensor components given by

$$R_{abcd} = -2kl(g_{ad}g_{bc} - g_{ac}g_{bd}). \tag{3.7}$$

Hence this region of space-time does not possess a curvature singularity, in striking contrast to the postcollision region of the Khan-Penrose space-time.

IV. SUMMARY

We can briefly summarize our results as follows: for this model collision the energy in the incoming impulsive gravitational waves is redistributed after the collision into two lightlike shells of matter and two impulsive gravitational waves moving away from each other followed by a space-time of constant curvature. When the surface energy densities of the postcollision lightlike shells of matter are required to be positive the space-time of constant curvature must be anti–de Sitter space-time.

APPENDIX: THE CASE $\Lambda = 0$

With $\Lambda = 0$ we can solve (2.22) and (2.23), requiring $\bar{u} = 0$ when $u = 0$ and $\bar{v} = 0$ when $v = 0$, with

$$\bar{u} = \sin^{-1}ku, \qquad \bar{v} = \sin^{-1}lv. \tag{A1}$$

By (2.21) we have $V = V(\bar{u}+\bar{v})$ and the boundary condition (2.6) written in terms of $\bar{u}, \bar{v}$ reads as follows: when $\bar{v} = 0, V = \log(\frac{1+\sin\bar{u}}{1-\sin\bar{u}})$. Hence

$$V(\bar{u}+\bar{v}) = \log\left(\frac{1+\sin(\bar{u}+\bar{v})}{1-\sin(\bar{u}+\bar{v})}\right) = \log\left[\left(\frac{\cos\bar{u}+\sin\bar{v}}{\cos\bar{u}-\sin\bar{v}}\right)\left(\frac{\cos\bar{v}+\sin\bar{u}}{\cos\bar{v}-\sin\bar{u}}\right)\right]. \tag{A2}$$

Restoring the coordinates u, v we have

$$V(u,v) = \log\left[\left(\frac{\sqrt{1-k^2u^2}+lv}{\sqrt{1-k^2u^2}-lv}\right)\left(\frac{\sqrt{1-l^2v^2}+ku}{\sqrt{1-l^2v^2}-ku}\right)\right], \tag{A3}$$

for $u \geq 0, v \geq 0$ provided $\Lambda = 0$. Writing (2.9) in terms of the variables $\bar{u}, \bar{v}$ and using (2.21) and (A2) we have

$$U_{\bar{u}} + U_{\bar{v}} = 2\tan(\bar{u}+\bar{v}). \tag{A4}$$

Integrating this results in

$$e^{-U} = D(\bar{u}-\bar{v})\cos(\bar{u}+\bar{v}), \tag{A5}$$

where $D(\bar{u}-\bar{v})$ is a function of integration. With $\bar{v} = 0$ the boundary condition (2.6) requires $e^{-U} = \cos^2\bar{u}$ and so $D(\bar{u}) = \cos\bar{u}$. Restoring the coordinates u, v we have $U(u,v)$ given by

$$e^{-U} = \cos(\bar{u}-\bar{v})\cos(\bar{u}+\bar{v}) = 1 - k^2u^2 - l^2v^2. \tag{A6}$$

Now (2.8) with $\Lambda = 0$ is automatically satisfied. Combining (2.8) with $\Lambda = 0$ and (2.12) we have

$$(2M + U)_{uv} = V_uV_v. \tag{A7}$$

Changing the independent variables u, v to $\bar{u}, \bar{v}$ using (A1), and using (A2), this reads

$$(2M + U)_{\bar{u}\,\bar{v}} = 4\sec^2(\bar{u}+\bar{v}). \tag{A8}$$

Integrating and using (A6) we arrive at

$$e^{-2M} = \frac{\cos^3(\bar{u}+\bar{v})}{\cos(\bar{u}-\bar{v})}f(\bar{u})g(\bar{v}), \tag{A9}$$

where $f(\bar{u})$ and $g(\bar{v})$ are functions of integration. From the boundary conditions (2.6) and (2.7) we see that $M = 0$ when $\bar{v} = 0$ and $M = 0$ when $\bar{u} = 0$ and so it follows that $f(\bar{u})g(\bar{v}) = \sec^2\bar{u}\sec^2\bar{v}$. Hence

$$e^{-M} = \sqrt{\frac{\cos^3(\bar{u}+\bar{v})}{\cos(\bar{u}-\bar{v})}}\sec\bar{u}\sec\bar{v}. \tag{A10}$$

Restoring the coordinates u, v and simplifying this becomes

$$e^{-M} = \frac{(1-k^2u^2-l^2v^2)^{3/2}}{(\sqrt{1-k^2u^2}\sqrt{1-l^2v^2}+kluv)^2\sqrt{1-k^2u^2}\sqrt{1-l^2v^2}}, \tag{A11}$$

for $u \geq 0, u \geq 0$ when $\Lambda = 0$. Substituting (A3), (A6) and (A11) into the field equations (2.10) and (2.11) verifies that these latter equations are automatically satisfied. The line element (2.5) with V, U and M given by (A3), (A6) and (A11) is the Khan-Penrose postcollision line element. No derivation is given in [1].

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