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We consider a class of Lorentz-violating theories of gravity involving a timelike unit vector field (the aether) coupled to a metric, two examples of which are Einstein-aether theory and Hořava gravity. The action always includes the Ricci scalar of the metric and the invariants quadratic in covariant derivatives of the aether, but the theories differ in how the aether is constructed from other fields, and whether those fields are varied in the action. Fields that are not varied define background structures breaking diffeomorphism invariance, including threadings, foliations, and clocks, which generally produce novel degrees of freedom arising from the violation of what would otherwise be initial value constraints. The principal aims of this paper are to survey the nature of the theories that arise and to understand the consequences of breaking diffeomorphism invariance in this setting. In a companion paper [A. J. Speranza, arXiv:1504.03305], we address some of the phenomenology of the “ponderable aether” case in which the presence of a background clock endows the aether with a variable internal energy density that behaves in some respects like dark matter.

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I. INTRODUCTION AND SUMMARY

Longstanding puzzles of cosmology and quantum gravity have led some to question the fundamental assumption of general relativity—that the spacetime manifold has no structure other than that determined by the metric. In particular, the cosmological constant problem, dark energy, dark matter, the trans-Planckian puzzle, the need for a UV completion of general relativity, the problem of time and the interpretation of quantum cosmology have motivated exploration of modified gravity theories with vacuum structure violating local Lorentz boost symmetry. If exact rotational symmetry is preserved, a Lorentz violating vacuum structure selects a preferred timelike direction at each spacetime point. The integral curves of this field of directions may be thought of as the flow of an “aether fluid.” The four-velocity u^a of the aether is the unit timelike vector field tangent to this flow.

In constructing a theory with such an aether, one must decide whether the aether is to be treated as dynamical, i.e. varied in the action principle, or instead as background. If the aether is dynamical, then one must further specify how it is constructed in terms of the fields that are varied in the action. Actually it turns out that the distinction between varied and not varied fields is not so clear-cut: the equations of motion for scalar fields are often a consequence of the equations of motion of the other fields. Such scalars, and the structures they define, can therefore be regarded as “background” structure, even though they might also be

varied in the action. What is important for the physics, however, is not how we refer to them, but how these choices affect the degrees of freedom and behavior of the theories. The purpose of this paper is to examine this question for a variety of related aether theories.

It is natural to assume that the aether-metric dynamics is governed by an (effective) action involving the metric, its curvature, the aether, and covariant derivatives of the aether and the curvature. Before beginning with the detailed analysis, we would like to point out that the theory would generally be dynamically overconstrained, i.e. “inconsistent,” if derivatives of the aether were not included in the action. Suppose, for example, the action $S[g_{ab}, \phi, u^a]$ is a scalar constructed from the metric g_{ab} , a scalar field ϕ , and a unit vector field u^a . Suppose further that the aether enters the Lagrangian density only via the coupling $\frac{1}{2}\sqrt{-g}(u^a\phi_{,a})^2$. The variations of u^a must be orthogonal to u^a to preserve the unit condition, and these impose the equation of motion $(\delta S/\delta u^a)(\delta_b^a - u^a u_b) = \sqrt{-g}(u^m\phi_{,m}) \times [\phi_{,b} - (u^n\phi_{,n})u_b] = 0$. This extremely restrictive condition requires that either ϕ is constant along the flow lines of u^a , or the flow of u^a is hypersurface orthogonal and ϕ is constant on the orthogonal hypersurfaces. This eliminates virtually all of the solutions to the scalar equation of motion. Moreover, even if we choose to not impose the aether equation of motion, the scalar field is still overconstrained since, as shown below, the other equations of motion imply that the Lie derivative $\mathcal{L}_u[(\delta S/\delta u^a) \times (\delta_b^a - u^a u_b)]$ vanishes. Although a weaker condition, this still eliminates almost all scalar field solutions. The situation is quite different, however, if the action includes

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terms quadratic in aether derivatives. Then $\delta S/\delta u^a$ includes second derivatives of the aether, so instead of overconstraining the scalar field the extra conditions can be propagation equations for the aether.

At lowest order in a derivative expansion, the most general action for the metric and aether is given (up to the integral of a total divergence) by¹

$$S[g_{ab}, u^a(\zeta, g_{ab})] = \frac{-1}{16\pi G_0} \int d^4x \sqrt{-g} \times \left(R + \frac{c_\theta}{3} \theta^2 + c_\sigma \sigma^2 + c_\omega \omega^2 + c_a a^2 \right), \quad (1)$$

where ζ denotes collectively independently varied fields used in the construction of the aether $u^a(\zeta, g_{ab})$. The terms in the integrand are the Ricci scalar R of the metric, and the expansion $\theta = \nabla_a u^a$, shear $\sigma_{ab} = \nabla_{(a} u_{b)} + \frac{1}{3}(\nabla_c u^c)h_{ab} - u_{(a} u_{b)}$, twist $\omega_{ab} = \nabla_{[a} u_{b]} - u_{[a} a_{b]}$ and acceleration $a_b = u^a \nabla_a u_b$ of the aether flow (here $h_{ab} = u_a u_b - g_{ab}$ is the spatial metric). In this paper we examine how variations on the construction of $u^a(\zeta, g_{ab})$ affect the resulting theory.

An important distinction is whether or not the aether vector is necessarily hypersurface-orthogonal, and hence nontwisting. The minimal structure required to determine a twist-free aether is a *foliation* by spacelike hypersurfaces, whereas the minimal structure required to determine a twisting aether is a timelike congruence of curves, i.e. a *threading* [1,2]. A well-known example of a foliation theory results if u^a is constructed from a scalar field T as

$$u^a(T, g_{ab}) = g^{ab} T_{,b} / |dT|, \quad (2)$$

where $|dT| = (g^{ab} T_{,a} T_{,b})^{1/2}$ is the norm of the gradient. By construction, the aether (2) is orthogonal to the constant T surfaces. This is khronometric theory [3–5], a.k.a. the infrared limit of the nonprojectable version of Hořava-Lifshitz gravity [6]. Among threading theories, a well-known example results if u^a is constructed from an independent vector field A^a as

$$u^a(A^m, g_{mn}) = A^a / |A|, \quad (3)$$

where $|A| = (g_{mn} A^m A^n)^{1/2}$ is the norm of A^a . This is Einstein-aether theory [7,8], written in the form given in Ref. [9]. In (3) the unit constraint on u^a holds by

¹We use the metric signature $(+ - - -)$. Abstract indices are denoted by Latin letters, spacetime coordinate indices by Greek letters, and comma and semicolon before an index denote partial and covariant derivative respectively. Quantities with density weight 1 are written in caligraphic font, $\mathcal{E}, \mathcal{F}, \mathcal{C}$, or carry a tilde, unless they involve the metric determinant or are written explicitly as a variational derivative.

construction, rather than being imposed via a Lagrange multiplier term as is more commonly done.

In the constructions we study, the foliation or threading is in most cases described covariantly using scalar fields which enter the construction of u^a through their gradients. Since the action (1) involves first derivatives of u^a , it involves second derivatives of the scalars, raising the concern that the resulting theory might suffer from an Ostrogradski instability [10]. However, diffeomorphism invariance implies that the Lagrangian is degenerate, so that there is a possibility that the instability is absent. In fact, as we explain in Sec. II, for all theories we consider, the scalar field equations are redundant with the other field equations. Therefore the scalars need not be varied in the action principle, so we may fix their values by a choice of coordinates at the level of the action. The gauge-fixed action is no longer invariant under the full group of diffeomorphisms, yet it gives a theory that is equivalent to the one defined by the original diffeomorphism invariant action.² There exists a coordinate gauge choice for which the scalars' gradients have constant components, and in such a gauge the action is only first order in derivatives of the remaining dynamical variables. (For example, in the khronometric theory with one dynamical scalar T , u^a in (2) is first order in derivatives, but with the gauge choice $x^0 = T$ its components become $u^a = g^{a0} / \sqrt{g^{00}}$, which contain no derivatives.) We conclude from this that there is no Ostrogradski instability in any gauge.

In Einstein-aether theory, the threading is determined by a *line field*, i.e. a vector field modulo local scaling. We call this the *dynamical aether theory*, since it arises from a dynamical field A^a that appears with only first derivatives in the action. In subsection III A, we consider a related theory where the threading is determined by three scalar fields, that can be fixed as background structures as explained above. We show that this *fixed threading theory* is equivalent to Einstein-aether theory except that it admits violation of the spatial initial value constraints. The constraint violation is characterized by a spatial covector density that is preserved along the aether flow, and does not affect the energy-momentum tensor.

Subsection III B considers a different theory, in which the aether threading is determined by a line field as in Einstein-aether theory, and there is an additional scalar field which determines a preferred clock constrained to measure proper time along the threads. This *fixed clock theory* is equivalent to Einstein-aether theory except that it admits violation of a single initial value constraint per spatial point. The constraint violation is characterized by a scalar density that is preserved along the aether flow, and appears in the energy-momentum tensor like a rest mass density of

²It is notoriously challenging to define what it means for a *theory itself*—as opposed to its formulation—to be diffeomorphism invariant [11,12].

the aether fluid. We call this a *ponderable aether*, invoking the 19th-century adjective that was used to distinguish ordinary matter from aether. Finally in subsection III C we consider the *fixed aether theory*, containing both a fixed threading and a fixed clock. The four scalars that describe the background structure completely determine the aether vector, and hence this theory is equivalent to Einstein-aether theory with the vector u^a taken to be nondynamical.

In Sec. IV, we consider foliation-type theories. After reviewing the khronometric theory, we consider in Sec. IV B introducing an independent clock field. As before this leads to a violated constraint as well as an additional component in the stress tensor. Finally, we show that when the clock field is constrained to coincide with the preferred foliation, the resulting theory is projectable Hořava gravity, and we again find a violated constraint. This constraint violation was already studied in Ref. [13], which referred to it as “dark matter as an integration constant.” Unlike before, however, the violation in this case is not preserved under flows of u^a for a generic form of the action (1).

II. VARIATIONAL PRELIMINARIES

We begin by establishing the notation and several key results that are used throughout this paper. We define the following tensor densities, resulting from variations of the Einstein-aether action (1),

$$\mathcal{E}_{ab} \equiv \left. \frac{\delta S}{\delta g^{ab}} \right|_{u^c}, \quad \mathcal{F}_{ab} \equiv \left. \frac{\delta S}{\delta g^{ab}} \right|_{u_c}, \quad (4)$$

$$\mathcal{E}_c \equiv \left. \frac{\delta S}{\delta u^c} \right|_{g^{ab}} = g_{cd} \left. \frac{\delta S}{\delta u_d} \right|_{g^{ab}}. \quad (5)$$

The object to the right of the vertical line in these expressions indicates which tensors are held fixed when computing the variation, i.e. \mathcal{E}_{ab} and \mathcal{F}_{ab} differ in that contravariant u^c is held fixed in the former, while covariant u_c is fixed in the latter. They are related by

$$\mathcal{F}_{ab} = \mathcal{E}_{ab} + \mathcal{E}_{(a} u_{b)}. \quad (6)$$

As discussed in the Introduction, we are interested in cases where u^a is constructed from g_{ab} and other fields collectively denoted ζ . The Einstein equation results from varying the action holding ζ fixed, and hence receives a contribution from the explicit metrics appearing in the action, as well as from the metric dependence of u^a . In all cases considered, u^a depends algebraically on the metric, and the Einstein equation takes the form

$$\mathcal{E}_{ab} + \mathcal{E}_c \frac{\delta u^c}{\delta g^{ab}} = 0. \quad (7)$$

For the foliation-type theories of Sec. IV, it is more convenient to work with covariant u_c , in which case the expression for the Einstein equation is

$$\mathcal{F}_{ab} + \mathcal{E}_c \frac{\delta u_c}{\delta g^{ab}} = 0. \quad (8)$$

In some cases the Einstein-aether action is supplemented with a Lagrange multiplier term enforcing the unit constraint,

$$S_\lambda = - \int d^4x \tilde{\lambda} (g_{ab} u^a u^b - 1). \quad (9)$$

This will contribute a $\tilde{\lambda}$ -dependent term to the Einstein equation as well as the aether variations. We will write such terms explicitly when they appear; the quantities \mathcal{E}_{ab} , \mathcal{F}_{ab} and \mathcal{E}_a are always defined by (4) and (5), with the action S given by (1).

Most of the theories considered in this paper involve scalar fields that determine the background structures on which the theory is based. We will often make use of the fact that the scalar equations of motion are implied by the other field equations. The proof of this is straightforward: consider an action $S[g_{ab}, \chi, \Phi^i]$ that is a diffeomorphism invariant functional of the metric, other tensor fields χ , and scalar fields Φ^i . Under a diffeomorphism generated by ξ^a , the action varies by

$$\delta S = \int \left(\frac{\delta S}{\delta g^{ab}} \xi_\xi g^{ab} + \frac{\delta S}{\delta \chi} \xi_\xi \chi + \frac{\delta S}{\delta \Phi^i} \xi_\xi \Phi^i \right) \quad (10)$$

This variation must vanish for all vectors ξ^a , and since the first two terms are zero when the metric and χ field equations hold, we find that

$$\frac{\delta S}{\delta \Phi^i} \nabla_a \Phi^i = 0. \quad (11)$$

As long as the gradients $d\Phi^i$ are (nonvanishing and) linearly independent, which can hold for up to four scalars, this implies that the scalar field equations hold, $\delta S / \delta \Phi^i = 0$. If their equations of motion are automatic in this way, we can fix the scalars at the level of the action without losing any dynamical information. The gauge-fixed action is no longer invariant under the full diffeomorphism group. In our application, where a vector field u^a is constructed using scalars, the gradients of the scalars must be linearly independent in order for u^a to be nonsingular and nonvanishing. Thus, for the physically relevant configurations, those scalar field equations are automatic.³ A scalar Lagrange multiplier, on the other hand, need not have nonvanishing gradient, so its field equation

³It can happen that the gradients fail to be independent on a set of measure zero, e.g. on a codimension one surface, but with u^a remaining well defined in the limit as that surface is approached. In that case presumably continuity implies that the scalar field equations also hold directly on that surface, at least provided the fields are all nonsingular.

(i.e. the corresponding constraint) should be imposed directly.

Finally, we recall that in any diffeomorphism invariant theory, some of the field equations are constraints on initial data, rather than evolution equations. For Einstein-aether theory, which contains a dynamical, contravariant vector field u^a , the quantities

$$\mathcal{C}_b^{(t)} = \nabla_a t (2\mathcal{E}^a_b + u^a \mathcal{E}_b). \quad (12)$$

contain no more than first partial derivatives with respect to t , for any choice of t and the remaining three coordinates [14,15]. When the field equations are satisfied, we have $\mathcal{E}^a_b = -\tilde{\lambda} u^a u_b$ and $\mathcal{E}_b = 2\tilde{\lambda} u_b$, where $\tilde{\lambda}$ terms arise from the Lagrange multiplier term (9). The $\tilde{\lambda}$ terms, thus, cancel, so in Einstein-aether theory $\mathcal{C}_b^{(t)}$ vanishes on shell. When t is a time evolution coordinate, these constraints, thus, restrict the allowed initial data. For a covariant aether vector u_a , the expression for the constraint has a different appearance,

$$\mathcal{C}_b^{(t)} = \nabla_a t (2\mathcal{F}^a_b - \mathcal{E}^a u_b), \quad (13)$$

but is in fact the same as a consequence of (6).

More generally, in the various theories we consider here, although the quantities (12) [or (13)] will have no higher than first t derivatives (in appropriately adapted gauges), some or all of them may not vanish when the field equations hold, because the u^a (or u_a) field equation per se is not imposed. For this reason, we refer to them generally as “constraint quantities,” rather than as “constraints.” This failure of constraint equations to hold corresponds to the lack of diffeomorphism invariance of the gauge-fixed action. We shall analyze the form of the constraint violation in each case as a means of characterizing the extra freedom available in solutions to these theories.

III. TWISTING AETHER: THREADING THEORIES

A twisting aether flow does not determine a preferred foliation of spacetime by spacelike hypersurfaces, but it does define a preferred threading of spacetime. In Einstein-aether theory, this threading is specified by an independently varied vector field modulo local scale, from which the aether four-velocity u^a (3) is constructed with the use of the metric (alternatively, one can use a Lagrange multiplier term to enforce the unit constraint on u^a). In this section we consider three other ways of constructing u^a . In the first subsection, the threading is determined by three scalar fields which are Lagrangian (comoving) coordinates for the aether. In the following two subsections, an additional scalar field ψ is introduced into both the line field and the Lagrangian coordinate constructions of u^a . The field ψ is an independent “clock” that marks time along the threads, and is constrained to agree with proper time by a Lagrange

multiplier term. These constructions are all very closely related to each other, but they yield theories that differ insofar as different integration constants are required to determine a solution, corresponding to different initial value constraints that are violated.

A. Fixed threading theory

A threading can be specified as the curves along which three scalar fields φ^I , $I = 1, 2, 3$ are all constant. If the theory is to depend only on these curves as one dimensional submanifolds, and not on any parametrization, the action must be invariant under all smooth invertible field redefinitions of the scalars,

$$\varphi^I \mapsto \bar{\varphi}^I(\varphi^I). \quad (14)$$

This can be achieved by restricting the action to depend on φ^I only via the unit aether four-velocity

$$u^a(\varphi^I, g_{ab}) = \tilde{J}^a / |\tilde{J}|, \quad (15)$$

with \tilde{J}^a the metric-independent vector density

$$\tilde{J}^a = \tilde{\epsilon}^{abcd} \varphi^1_{,b} \varphi^2_{,c} \varphi^3_{,d}, \quad (16)$$

where $\tilde{\epsilon}^{abcd}$ is the alternating symbol, i.e. the Levi-Civita tensor density of weight 1. The vector field defined in (15) is invariant under the “ φ -diffeos” (14), since both the numerator and denominator are rescaled by the Jacobian determinant $\det(\partial\bar{\varphi}^I/\partial\varphi^I)$. The corresponding action (1) is then a functional of the metric and the three scalar fields.

Note that the action is quadratic in second derivatives of the scalars φ^I . This implies that the field equations will be fourth order in derivatives of the scalars, and third order in derivatives of the metric (arising from the Christoffel connection terms). However, as explained in Sec. II, we may treat the scalars as fixed, not varied in the action, without changing the dynamical content of the theory. Since the φ^I define a threading, we call this the fixed threading theory. In the co-moving gauge, where φ^I are equal to the spatial coordinates, u^a contains no derivatives, and the field equations arising from metric variations are of second order.

1. Relation to zero temperature perfect fluid

The dynamics of perfect fluids was formulated long ago in terms of three Euler potentials φ^I [16,17], a formulation that has recently been fruitfully exploited with the application of ideas from effective field theory (see e.g. [18]). In that setting, the vector density \tilde{J}^a represents the conserved entropy current and $|\tilde{J}|$ is the entropy density in the fluid rest frame. The entropy current is invariant under φ -diffeomorphisms with unit Jacobian determinant. Unlike for our aetherial application, full

φ -diffeo invariance is not imposed, because the entropy density is physically meaningful. The presence of conserved particle number necessitates an additional scalar field with a shift symmetry in the action. Our “clock field” ψ introduced below [see e.g. (46)] is directly analogous to this, although the corresponding chemical potential $u^a \nabla_a \psi$ is required by the unit norm constraint to be everywhere equal to unity.

For fluids without conserved particle number, the action at first order in derivatives is the integral of minus the energy density expressed as a function $\rho(b)$ of the entropy density scalar $b = |\tilde{J}|/\sqrt{-g}$. The function $\rho(b)$ determines the equation of state of the fluid. The aether fluid has the property that at first derivative order its energy density is independent of the entropy density, as required by the full φ -diffeo symmetry. This happens for a thermal fluid only at zero temperature, hence the aether can be considered a “zero temperature fluid.” The action for such a fluid is just proportional to the spacetime volume, so the stress-energy tensor at this derivative order is nothing but a (possibly vanishing) cosmological constant, motivating the name *vacuum fluid* for the aether. The dynamics of the vacuum fluid is governed at lowest derivative order by the action (1) involving the “strain” of the fluid.

2. Comparison with Einstein-aether theory

Under variations of the fields, in $u^a(A^m, g_{mn})$ (3) and $u^a(\varphi^I, g_{mn})$ (15), the variation of u^a has both a parallel and a perpendicular part,

$$\delta u^a = \delta u_{\parallel}^a + \delta u_{\perp}^a. \quad (17)$$

The metric variation generates only δu_{\parallel}^a , while the A^m and φ^I variations generate only δu_{\perp}^a . The metric-induced variation in both cases is just what is needed to keep u^a a unit vector: $0 = \delta(g_{mn}u^m u^n) = (\delta g_{mn})u^m u^n + 2g_{ma}u^m \delta u^a$ implies that $\delta u_{\parallel}^a = -\frac{1}{2}u^a u^m u^n \delta g_{mn}$; hence,

$$\delta u_{\parallel}^a = \frac{1}{2}u^a u_m u_n \delta g^{mn}. \quad (18)$$

The perpendicular part of the variation is given in Einstein-aether theory by

$$\delta u_{\perp}^a = (\delta_m^a - u^a u_m) \delta A^m, \quad (19)$$

and in the fixed threading theory by

$$\delta u_{\perp}^a = \frac{1}{2|\tilde{J}|} (\delta_m^a - u^a u_m) \tilde{c}^{abcd} \epsilon_{IJK} \varphi_b^I \varphi_c^J \delta \varphi_{,d}^K. \quad (20)$$

Thus, the metric equation of motion (7) is the same in terms of g_{mn} and u^m in the background threading theory as it is in Einstein-aether theory,

$$\mathcal{E}_{ab} + \frac{1}{2}u^c \mathcal{E}_c u_a u_b = 0. \quad (21)$$

The remaining equations of motion arise in both theories from the variation δu_{\perp}^a , and here a discrepancy arises.

In Einstein-aether theory the equation of motion arising from perpendicular aether variation (19) is

$$\mathcal{E}_m^{\perp} \equiv (\delta_m^a - u^a u_m) \mathcal{E}_a = 0, \quad (22)$$

while in background threading theory it is

$$\mathcal{E}_K \equiv \frac{\delta S}{\delta \varphi^K} = 0. \quad (23)$$

Diffeomorphism invariance of the action implies that the scalar equations (23) hold as a consequence of the Einstein equation [cf. discussion around Eq. (11)], so they add no new information. On the other hand, the perpendicular aether equation (22) adds restrictions in Einstein-aether theory.

To discover the precise relation between the equations of motion (22) and (23), note that since $u^a(\varphi^I, g_{mn})$ is constructed covariantly, diffeo variations of its arguments induce its diffeo variation $\delta u^a = \mathcal{L}_{\xi} u^a$ as a vector field. In particular, the perpendicular component of $\mathcal{L}_{\xi} u^a$ is equal to the variation induced via $\mathcal{L}_{\xi} \varphi^K$, so the corresponding contributions to the variation of the action $S[g_{ab}, u^a(\varphi^I, g_{mn})]$ are also equal,

$$\int \mathcal{E}_m^{\perp} \mathcal{L}_{\xi} u^m = \int \mathcal{E}_K \mathcal{L}_{\xi} \varphi^K. \quad (24)$$

Now using $\mathcal{L}_{\xi} u^a = -\mathcal{L}_u \xi^a$ and integrating by parts, we obtain

$$\int (\mathcal{L}_u \mathcal{E}_m^{\perp} - \mathcal{E}_K \varphi_{,m}^K) \xi^m = 0 \quad (25)$$

for all vector fields ξ^m , which yields the identity

$$\mathcal{L}_u \mathcal{E}_m^{\perp} = \mathcal{E}_K \varphi_{,m}^K. \quad (26)$$

Thus, (22) implies (23) (provided again that the gradients $\varphi_{,m}^K$ are linearly independent), but (23) implies only that the Lie derivative of (22) holds.

We, thus, see that in the fixed threading theory, the Einstein equation implies

$$\mathcal{L}_u \mathcal{E}_m^{\perp} = 0. \quad (27)$$

Put differently, instead of the aether equation (22), one has

$$\mathcal{E}_m^{\perp} = -\tilde{\mu}_m^{\perp}, \quad (28)$$

where the “source term” $\tilde{\mu}_m^{\perp}$ is a covector density that satisfies $u^m \tilde{\mu}_m^{\perp} = 0$ and is conserved along the aether flow,

$$\mathcal{L}_u \tilde{\mu}_m^\perp = 0. \tag{29}$$

A transparent way to express the conservation law (29) is to use adapted coordinates, $x^I = \varphi^I$ and to choose $x^0 = \tau$ with $u^\alpha \tau_{,\alpha} = 1$, so that the components of the aether four-velocity are all constant, $u^\alpha = \delta_\tau^\alpha$. Then the components of the Lie derivative are just the partial derivatives with respect to τ , and (29) takes the simple form

$$\partial_\tau \tilde{\mu}_\alpha^\perp = 0. \tag{30}$$

The three components $\tilde{\mu}_I^\perp$ are then just constants of integration on each thread, while $\tilde{\mu}_\tau^\perp$ vanishes identically. The freedom to choose these integration constants different from zero is what distinguishes the fixed threading theory from Einstein-aether theory. For lack of a better name, we shall call $\tilde{\mu}_m^\perp$ the *vector source density* (VSD).

The identity (26) shows that the φ^K equation of motion is “weaker” than the perpendicular u^m equation of motion, but this discrepancy remains a bit mysterious, since it would seem that variations of φ^K produce all possible perpendicular variations of u^m . Of course the difference must arise because φ^K occurs in the action with an extra derivative, but why exactly is that important? The answer lies in the boundary conditions. When we drop boundary terms we are holding φ^K fixed at the boundaries, in particular the initial and final boundary. This entails an integral constraint on the u^m variations (the endpoints of each thread are fixed), which translates into the fact that the φ^K variations imply only the time derivative of the u^m equation of motion.

A simple example serves to illustrate this point. Suppose we have a mechanical system in one dimension with Lagrangian $L(x, \dot{x})$, and we make the replacement $x = \dot{y}$, and treat $y(t)$ as the basic dynamical variable. Then the action variation is $\delta S = \int (\delta S / \delta x) \delta \dot{y} = - \int (d/dt) (\delta S / \delta x) \delta y$, so the y equation of motion is the time derivative of the x equation of motion. It is weaker than the x equation of motion because not all x variations are included in the y version of Hamilton’s principle. Since the initial and final values $y_{1,2}$ are fixed in the y variations, there is an implicit constraint on the integral of x due to the fact that $\int \dot{y} dt = \int y dt = y_2 - y_1$. We could include this constraint directly in the x version of Hamilton’s principle with the addition of a Lagrange multiplier term $\lambda (\int x dt - \Delta y)$. The result would be the equation of motion $\delta S / \delta x = \lambda$, where the Lagrange multiplier λ is an undetermined constant corresponding to a constant external force. In the y equation, λ corresponds to the extra integration constant needed to specify a solution.

How does a nonzero VSD for the aether field equation change the aether theory? In another paper, we find that it does not alter the Newtonian limit or static, spherical stars (assuming no radial aether component) [19], and by symmetry homogeneous, isotropic cosmology is unaltered.

However, it acts as an external force for wave modes, shifting the equilibrium amplitude away from zero. In the next subsection we show that, more generally, the source density integration constants characterize a violation of the initial value constraints of Einstein-aether theory. In the following subsection we show that magnitude of the source density is diluted as the aether expands with the universe, which suppresses its observable consequences.

3. Initial value constraint violation

The VSD $\tilde{\mu}_m^\perp$ in (28) suggests that the fixed threading theory requires more initial data than Einstein aether theory, since $\tilde{\mu}_m^\perp$ is a freely specifiable initial source for the aether equation. This new freedom can be characterized in terms of violated Einstein-aether initial value constraints.

For an arbitrary fourth coordinate x^0 , the constraint quantities (12) take the form

$$C_\alpha^{(0)} = 2\mathcal{E}^0_\alpha + u^0 \mathcal{E}_\alpha. \tag{31}$$

When the metric field equation (21) holds, (28) implies that these quantities are nonvanishing and instead satisfy

$$C_\alpha^{(0)} = -u^0 \tilde{\mu}_\alpha^\perp. \tag{32}$$

The u^α component constraint $u^\alpha \tilde{C}_\alpha^{(0)} = 0$ holds, since $u^\alpha \tilde{\mu}_\alpha^\perp = 0$, but the three “perpendicular constraints” are violated.

In adapted coordinates, $x^I = \varphi^I$, the constraint violation is preserved in time. This is easiest to see with the choice $x^0 = \tau$, with τ the proper time along the threads. Then we have $u^0 = 1$, and (30) shows that the components of the constraint $\tilde{C}_\alpha^{(\tau)}$ are preserved in τ . In fact the same result holds for any choice of the fourth coordinate x^0 : the condition $u^\alpha \tilde{\mu}_\alpha^\perp = 0$ implies that under a change from τ to x^0 , the components of the covector density $\tilde{\mu}_\alpha^\perp$ change only by the Jacobian factor $\partial\tau/\partial x^0$, while $u^0 = (\partial x^0 / \partial \tau) u^\tau$. Therefore the components of the x^0 -constraint (32) in (x^0, x^I) coordinates are the same as those of the τ -constraint in (τ, x^I) coordinates. Since $u^0 \partial_0 = \partial_\tau$, the previous result implies that also⁴

$$\partial_0 C_\alpha^{(0)} = 0. \tag{33}$$

This equation shows that the new freedom takes the form of an infinite collection of conserved quantities. The constraint violation may be freely specified at an initial time, but remains constant at all subsequent times.

⁴A covariant version of this argument uses (29) and the identity $\mathcal{L}_{u/(u \cdot dx^0)}[(u \cdot dx^0) \tilde{\mu}_m^\perp] = \mathcal{L}_u \tilde{\mu}_m^\perp$, which holds in view of the unit density weight and $u^m \tilde{\mu}_m^\perp = 0$.

The vanishing of the constraint quantities in Einstein-aether theory is a consequence of full spacetime diffeomorphism symmetry. The fixed threading theory respects only the thread preserving diffeomorphisms, which in adapted coordinates take the form $t \mapsto \bar{t}(t, x^j)$, $x^j \mapsto \bar{x}^j(x^j)$. Intuitively, since we cannot perform arbitrary gauge transformations of the spatial coordinates as we evolve in time, there should be no constraints associated with those diffeomorphisms imposed on the dynamics. This is why we find that $C_I^{(0)}$, the spatial constraint quantities for evolution along the threads, are nonvanishing.

By contrast, for evolution with respect to a parameter that is constant on the threads, say x^3 , all constraint quantities vanish, since $u^a \nabla_a x^3 = 0$, so the number of initial value constraints remains equal to four. This might be expected since as we evolve in x^3 , we can perform both time and spatial diffeomorphisms. (That these are required to preserve the fibers evidently does not cause the constraints to be lost.) This gauge symmetry means the dynamics cannot be fully deterministic, so that some field equations must be constraints.

4. Cosmological evolution of source density

In homogeneous isotropic symmetry, the VSD necessarily vanishes, and the background threading theory is identical to Einstein-aether theory. It is natural to imagine some kind of fluctuations around the symmetric configuration however. Since the VSD arises as integration constants, its power spectrum cannot be derived from the properties of quantum vacuum fluctuations. At this point we have identified no principle to select a primordial spectrum of VSD. What we can say however is that the amplitude will decrease as the universe expands.

To characterize the amplitude of the VSD, we use the scalar quantity

$$\kappa \equiv [g^{ab} \tilde{\mu}_a^\perp \tilde{\mu}_b^\perp / (-g)]^{1/2}. \quad (34)$$

An approximate redshift law for κ can be easily obtained by using for g_{ab} the homogeneous isotropic metric $ds^2 = dt^2 - a(t)^2 dx^i dx^i$, and neglecting the anisotropic corrections to the conservation law (29). Then the coordinates (x^i, t) are adapted to u^a , and the conservation law takes the form $\partial_i \tilde{\mu}_i^\perp = 0$, so (34) yields $\kappa \propto a^{-4}$. The physical effects of the VSD therefore decrease like those of radiation as the universe expands.

B. Fixed clock theory: A ponderable aether

In Sec. III A, we introduced three scalar fields that defined a threading. In the comoving gauge, these scalars have the effect of breaking spatial diffeomorphism symmetry when fixed at the level of the action. In this section we consider a different theory, in which temporal rather than spatial diffeomorphism symmetry is broken. This

involves introducing a ‘‘clock’’ field ψ that defines a preferred notion of time along the aether flow.

Since the clock field ψ is a scalar, we may again fix ψ to a background value at the level of the action. In analogy to the fixed threading we expect this fixed clock to lead to a violation of an initial value constraint and therefore to produce, in effect, an additional degree of freedom. The constraint violation in this case is quite analogous to the dark matter as an integration constant [13] in projectable Hořava gravity. The latter is due to the absence of the local Hamiltonian constraint in that theory. That constraint normally arises from the variation of the lapse function $N = (g^{tt})^{-1/2}$, but in projectable Hořava gravity $N = N(t)$ depends only on t . It is therefore not varied independently at each point on a constant t surface, so the associated local constraint is not imposed. The covariant construction of an aether with a fixed clock given here yields a similar effect. Unlike in the projectable Hořava case, however, the ‘‘dark matter mass current’’ is conserved in the fixed clock aether theory.

We start as in Einstein-aether theory with a dynamical vector field A^a but, rather than defining the aether four-velocity dividing by $|A|$, we define it by

$$u^a(A^m, \psi) = \frac{A^a}{A^m \psi_{,m}}, \quad (35)$$

where ψ is the clock field. By construction we have $u^a \psi_{,a} = 1$, so ψ is a parameter on the aether flow compatible with u^a . Note that (35) is unchanged under a thread-dependent shift $\psi \mapsto \psi + v$, with v constant along each thread, $A^a \nabla_a v = 0$ (note this symmetry was called a ‘‘chemical shift’’ in the works on effective field theory for fluids [18]). The requirement of this symmetry precludes standard kinetic or potential terms for ψ and, since $u^a \psi_{,a} = 1$, a term like $(u^a \psi_{,a})^2$ only adds a constant to the action.

Unlike its Einstein-aether cousin $u^a(A^m, g_{mn})$ (3), $u^a(A^m, \psi)$ is not a unit vector by construction, so we impose the unit constraint by adding a Lagrange multiplier term (9) to the action, enforcing the relation

$$(A^m \psi_{,m})^2 = g_{mn} A^m A^n. \quad (36)$$

It seems at first that this could be satisfied either by solving a first order ODE for ψ on each thread, or by restricting g_{mn} (the condition is independent of the scale of A^m so it can not be satisfied by restricting that scale). However, solving (36) for ψ by integrating along each thread would be inconsistent with fixing ψ at both endpoints in Hamilton’s principle unless further constraints on variations of A^a and g_{ab} are imposed. Instead, it is simplest to view the unit constraint as fixing a component of the metric in terms of A^m and $\psi_{,m}$. Since ψ is a scalar field, its equation of motion is satisfied by virtue of the other equations of motion

(provided $\psi_{,m} \neq 0$). It can therefore be considered fixed. We call this the fixed clock theory since, when the unit constraint is satisfied, ψ marks proper time on each thread.

The variation of (35) is given by

$$\delta u^a = -u^a u^m \delta \psi_{,m} + \frac{1}{A \cdot d\psi} (\delta_m^a - u^a \psi_{,m}) \delta A^m. \quad (37)$$

The ψ equation of motion, thus, takes the form of a current conservation law,

$$(\tilde{\mu} u^m)_{,m} = 0, \quad \tilde{\mu} \equiv \mu \sqrt{-g} \equiv u^a (2\tilde{\lambda} u_a - \mathcal{E}_a). \quad (38)$$

The A^a equation of motion is

$$\frac{1}{A \cdot d\psi} (\delta_m^a - u^a \psi_{,m}) (\mathcal{E}_a - 2\tilde{\lambda} u_a) = 0, \quad (39)$$

which is equivalent to

$$\mathcal{E}_a - 2\tilde{\lambda} u_a = -\tilde{\mu} \psi_{,a}. \quad (40)$$

In this theory, u^a has no metric dependence, so the Einstein equation is

$$\mathcal{E}_{ab} + \tilde{\lambda} u_a u_b = 0, \quad (41)$$

which in light of (38) becomes

$$\mathcal{E}_{ab} + \frac{1}{2} u^c \mathcal{E}_c u_a u_b + \frac{1}{2} \tilde{\mu} u_a u_b = 0. \quad (42)$$

The aether stress tensor, thus, picks up the extra dustlike contribution, $\mu u_a u_b$, which is not present in Einstein-aether theory. Together with the conservation equation (38) this suggests the interpretation of $\tilde{\mu}$ as the internal energy density of the aether, and motivates the descriptive term ‘‘ponderable aether.’’ Notice that even in the absence of the aether terms in the action (1), the Lagrange multiplier term (9) alone suffices to introduce the aethereal dust stress tensor $\mu u_a u_b$.

The sign of μ is not fixed, so the aethereal dust can contribute negative energy density to the source in the metric field equation. Normally the presence of negative energy could produce an instability, since the system could evolve to a highly excited state of compensating positive and negative energy components. However, as a consequence of the conservation equation (38), the spatial integral of μ is conserved along any bundle of aether worldlines, precluding instabilities of that sort.

The initial value formulation of the fixed clock theory differs from that of Einstein-aether theory by a violated constraint equation. As explained previously for the fixed threading theory, the quantity $\mathcal{C}_\alpha^{(t)}$ defined in (12) has only first t -derivatives of the metric and the aether four-velocity. For the background clock theory it has second derivatives,

since the aether four-velocity (35) involves the derivative of ψ . In the adapted coordinate ‘‘clock gauge’’ $x^0 = \psi$, however, u^α is algebraic,

$$u^\alpha(A^\mu, \psi) = A^\alpha/A^0, \quad (43)$$

so $\tilde{\mathcal{C}}_\alpha^{(t)}$ has only first t -derivatives for any choice of t . When the A^a and metric equations (40) and (41) hold, we have

$$\mathcal{C}_b^{(t)} = -(u^a t_{,a}) \tilde{\mu} \psi_{,b}. \quad (44)$$

The right hand side vanishes when contracted with any vector tangent to the constant ψ surface, so the presence of nonzero $\tilde{\mu}$ in the aether equation (40) leads to a single constraint violation. The additional freedom in the theory is parametrized by $\tilde{\mu}$. If we choose $t = \psi$ as the evolution parameter and use the clock gauge, (44) takes the form

$$\mathcal{C}_\alpha^{(\psi)} = -\tilde{\mu} \delta_\alpha^0. \quad (45)$$

If we further choose spatially adapted coordinates, so that $u^\alpha = \delta_0^\alpha$, the right hand side of equation (45) is constant in x^0 as a consequence of (38). As in the fixed threading theory, the new degrees of freedom are, in this sense, ‘‘totally integrable.’’

We have found that for evolution with respect to any coordinate t such that $u \cdot dt \neq 0$, the ψ -component of the constraint quantities does not vanish. This is because the fixed clock breaks time diffeomorphism symmetry. The clock shift symmetry remains, but it allows for only a single, time independent shift, so the ψ surfaces cannot be deformed as we evolve along the threads. The components of the constraints in the directions tangent to the ψ surfaces are preserved, since the threads are determined by a dynamical vector field, rather than by a background structure.

The contribution $\mu u_a u_b$ has the form of a pressureless fluid source in the Einstein equation, but its divergence is not zero when the aether is not geodesic. In homogeneous isotropic cosmology, however, it does behave exactly as pressureless dust, with $\mu \propto a^{-3}$. During an inflationary period μ would be exponentially suppressed, so in the standard inflationary cosmological model it would presumably be too small today to have any observable effect. If there were some way to transcend the classical conservation law for μ and generate a nonzero value around the time of matter radiation equality, it could play the role of the homogeneous dark matter in a Λ CDM model.⁵ This leads

⁵If it were generated earlier, e.g. at reheating, it would quickly dominate unless fine-tuned to an extremely small value relative to the radiation energy density. Mechanisms for generating dark matter after inflation have been proposed for the related projectable Hořava gravity [13] and mimetic dark matter [20,21] theories.

to the question of how it would behave as structure forms. Its nongeodesic character suggests that it would not form structure in the manner of geodesic dark matter. In another paper [19] we have examined the growth of linearized perturbations, and found that if μ were to comprise the homogeneous dark matter density at early times it would lead to an unacceptably high growth rate on superhorizon scales and no growth on subhorizon scales.

C. Fixed aether theory

In the previous two sections, we considered a theory with broken spatial diffeomorphisms, the fixed threading, and one with broken temporal diffeomorphisms, the fixed clock. In this section we combine these features and consider an aether theory with broken spatial and temporal diffeomorphisms.

We now define the aether four-velocity by

$$u^a(\varphi^I, \psi) = \frac{\tilde{J}^a}{\tilde{J}^m \psi_{,m}}, \quad (46)$$

where \tilde{J}^a is the vector density constructed from φ^I defined in equation (16), and ψ is a scalar clock field. Like the aether four-velocity of the background threading theory (15), $u^a(\varphi^I, \psi)$ is unchanged under all φ -diffeos (14) and, as in the fixed clock theory, it has the clock shift symmetry, $\psi \mapsto \psi + v(\varphi^I)$ (again, this corresponds to the ‘‘chemical shift’’ in the effective theory of fluid dynamics [18]). Also, as in the latter theory, it is not normalized by construction, so the Lagrange multiplier term (9) is again used to impose the unit constraint.

Since $u^a(\varphi^I, \psi)$ is constructed entirely from four scalar fields, its variations arise solely from variations of the four scalars. As explained above, provided the four gradients $\varphi^I_{,a}$ and $\psi_{,a}$ are linearly independent, which is required for them to define a threading with parameter ψ , the equations of motion for the four scalars will follow from the Einstein equation. They can therefore be held fixed in the action. This fixed aether theory is, thus, equivalent to Einstein-aether theory with an aether vector u^a that is not varied in the action; that is, the aether field equation \mathcal{E}_a of Einstein-aether theory is not imposed.

When the g_{ab} and $\tilde{\lambda}$ equations of motion hold, however, the (vanishing) variation of the action $S[g_{ab}, u^a, \tilde{\lambda}]$ with respect to a diffeomorphism generated by ξ^a is given by $\int (\mathcal{E}_a - 2\tilde{\lambda}u_a)\mathcal{L}_\xi u^a = \int \xi^a \mathcal{L}_u (\mathcal{E}_a - 2\tilde{\lambda}u_a)$. Since this vanishes for all ξ^a we infer that

$$\mathcal{L}_u (\mathcal{E}_a - 2\tilde{\lambda}u_a) = 0. \quad (47)$$

This is similar to (27) in the fixed threading theory, but includes the component of \mathcal{E}_a along u_a . Thus, although the aether field equation is not imposed, it holds with the addition of an undetermined, ‘‘constant’’ source term,

$$\mathcal{E}_a - 2\tilde{\lambda}u_a = -\tilde{\mu}_a, \quad \mathcal{L}_u \tilde{\mu}_a = 0. \quad (48)$$

Equation (48) implies

$$\tilde{\lambda} = \frac{1}{2}(\tilde{\mu} + u^a \mathcal{E}_a), \quad (49)$$

with

$$\tilde{\mu} \equiv \mu \sqrt{-g} \equiv u^a \tilde{\mu}_a. \quad (50)$$

Therefore, as in the fixed clock theory, the aether stress tensor contribution $(\tilde{\lambda}/\sqrt{-g})u_a u_b$ picks up the extra term $\mu u_a u_b$ not present in Einstein-aether theory. Also, (48) and $\mathcal{L}_u u^a = 0$ imply $\mathcal{L}_u \tilde{\mu} = (\tilde{\mu} u^a)_{,a} = 0$, so that μ acts like a ‘‘dark matter’’ source of gravity that can be interpreted as the internal energy density of a ponderable aether.

When we work in co-moving, clock gauge ($x^I = \varphi^I$, $x^0 = \psi$), the diffeomorphism symmetry is broken down to time independent transformations $x^I \rightarrow f^I(x^I)$ and $x^0 \rightarrow x^0 + f(x^I)$, so we should expect all four constraints to be violated. When the metric and $\tilde{\lambda}$ equations are satisfied, the x^0 -constraint quantity (12) for the fixed aether theory in these coordinates takes the form

$$\mathcal{C}_a^{(0)} = 2\mathcal{E}^0_a + u^0 \mathcal{E}_a = -\tilde{\mu}_a. \quad (51)$$

This indeed confirms that all four initial value constraints of Einstein-aether theory are violated. This is as expected, since for evolution with respect to any parameter that advances along the threads, there remains no diffeomorphism freedom that would make the dynamics underdetermined. For evolution with respect to a parameter s that is constant along the threads, we again find that all constraints vanish. In addition to the s -dependent thread preserving diffeomorphisms, the clock field’s shift symmetry allows for s -dependent changes in ψ . Thus, we find four additional initial value freedoms per spatial point, which again by (48) are ‘‘totally integrable.’’

IV. FOLIATION THEORIES

We now turn to theories involving a foliation of spacetime by spacelike hypersurfaces. These theories are distinct from threading theories because the aether vector, constructed as the unit normal to the foliation, is necessarily twist-free. The simplest foliation-type theory is khronometric theory, the low energy limit of non-projectable Hořava gravity. After reviewing its construction in Sec. IV A, we proceed in Sec. IV B to add a fixed clock to the theory as was done for the threading theories in Sec. III B. As in that case, the resulting theory exhibits constraint violation, and similarly contains a ‘‘dark matter’’ component in the Einstein equation. Finally, in

Sec. IV C we consider the case where the foliation and the clock field coincide. This results in the projectable version of Hořava gravity, and we discuss the relation between the constraint violation and the dark matter as an integration constant of that theory [13].

A. Khronometric theory

A twist-free aether can be described by a scalar field T , dubbed the “khronon,” whose level sets define the hypersurfaces orthogonal to the aether four-velocity [3–5]. In order that the theory depend only on the foliation by hypersurfaces, and not the values of T , the action must be invariant under monotonic reparametrizations

$$T \rightarrow \tilde{T}(T). \quad (52)$$

The gradient of T transforms as $\tilde{T}_{,a} = (d\tilde{T}/dT)T_{,a}$, so the numerator and denominator of the aether four-velocity (2) both acquire a factor $d\tilde{T}/dT$, and these factors cancel. Therefore the action $S[g_{ab}, u^a(T, g_{mn})]$ (1) is invariant under T reparametrizations.

Just as in the threading theory, this action is quadratic in second derivatives of T , so when T is varied it yields equations of motion that are fourth order in derivatives of T and third order in derivatives of the metric. Again, as explained in Sec. II, we may fix T at the level of the action without changing the dynamics. In the adapted gauge where T is identified with one of the spacetime coordinates, we have from (2)

$$u^\alpha(T, g_{ab}) = g^{\alpha T} / \sqrt{g^{TT}}. \quad (53)$$

Since the aether four-velocity is an algebraic function of the metric components in this gauge, and the action (1) produces a second order field equation for the metric (terms with more than two *spatial* derivatives occur in the full Hořava-Lifshitz theory). In this formulation, which is equivalent to Hořava’s original one [6], the action is invariant only under T -foliation preserving diffeomorphisms, together with T -reparametrizations (52).

We can now examine the constraints for this theory. The metric dependence of u_c induces a variation $\delta u_c = -\frac{1}{2}u_c u_a u_b \delta g^{ab}$, so the metric field equation (8) reads

$$\mathcal{F}_{ab} - \frac{1}{2}\mathcal{E}^c u_c u_a u_b = 0. \quad (54)$$

When this equation is satisfied, the constraint (13) is equal to

$$\mathcal{C}_b^{(t)} = -(\nabla_a t)\mathcal{E}_\perp^a u_b. \quad (55)$$

If we choose $t = T$ then, since $dT \propto u$, the right hand side vanishes, so all the constraints $\mathcal{C}_b^{(T)}$ vanish in the adapted gauge.

This is complementary to the situation described in Sec. III A 3 for the threading theory. There we found that for evolution with respect to a parameter constant on the threads, the constraint quantities vanish. Here we find a similar result: for evolution with respect to a parameter s that is constant on the foliation, the constraint quantities vanish. We can still make s -dependent time reparametrizations and spatial diffeomorphisms under this evolution, so we expect constraints associated with these gauge transformations. If instead we were to consider evolution with respect to a different parameter s' that is *not* constant on the foliation, we would find the T -component of the constraint violated. This is because the foliation cannot be deformed in an s' -dependent fashion, so the theory loses that gauge symmetry and the associated constraint.

B. Fixed clock foliation theory

We can add a fixed clock to the foliation theory by following the method introduced above for the twisting aether. We introduce the clock field ψ and define the aether four-velocity covector as

$$u_a(T, g_{mn}, \psi) = \frac{T_{,a}}{g^{mn} T_{,m} \psi_{,n}}, \quad (56)$$

which is constrained by a Lagrange multiplier term to have unit norm. The unit constraint requires that the lapse $N = (g^{ab} T_{,a} T_{,b})^{-1/2}$ be equal to $(g^{ab} T_{,a} \psi_{,b})^{-1}$, which freezes one metric degree of freedom (in the adapted gauge, it fixes $(g^{T\psi})^2 = g^{TT}$). The field equations for both of the scalars T and ψ again follow from the Einstein equation, provided $T_{,a}$ and $\psi_{,a}$ are linearly independent. Although the gradients dT and $d\psi$ are both timelike, they will generically be independent whenever the aether is accelerated, since

$$\mathcal{L}_u(u_{[a}\psi_{,b]}) = a_{[a}\psi_{,b]}. \quad (57)$$

Thus, the gauge in which both T and ψ are set equal to coordinates will generically be nonsingular.

The ψ field equation gives a conservation law, corresponding to its shift symmetry $\psi \rightarrow \psi + \text{const}$,

$$[(\mathcal{E}^a - 2\tilde{\lambda}u^a)u_a u^m]_{,m} = 0. \quad (58)$$

Defining, as usual, the scalar density $\tilde{\mu} = (2\tilde{\lambda}u^a - \mathcal{E}^a)u_a$, we now have

$$\tilde{\lambda} = \frac{1}{2}(\tilde{\mu} + u_a \mathcal{E}^a), \quad (59)$$

as in the fixed aether case (49). The metric equation of motion receives a contribution from the metric variation in

u_a , namely $\delta u_c = -u_c u_{(a} \psi_{.b)} \delta g^{ab}$. The metric field equation is then

$$\mathcal{F}_{ab} - \tilde{\lambda} u_a u_b - (\mathcal{E}^c - 2\tilde{\lambda} u^c) u_c u_{(a} \psi_{.b)} = 0. \quad (60)$$

Rearranging to compare with (54), this becomes

$$\mathcal{F}_{ab} - \frac{1}{2} \mathcal{E}^c u_c u_a u_b + \frac{1}{2} \tilde{\mu} (u_a u_b + 2u_{(a} \psi_{.b)}^\perp) = 0, \quad (61)$$

where $\psi_{.b}^\perp = \psi_{.b} - u_b$ is the projection of $\psi_{.b}$ perpendicular to u_b . The $\tilde{\mu}$ term gives the difference between this theory and the khronometric theory, and it takes the form of nongeodesic ‘‘dark matter’’ with momentum density.

Unlike the previous fixed clock theories, here the clock field itself has an effect on the dynamics via the perpendicular component of $\psi_{.b}$ in the stress tensor (61). Since $u^m \psi_{.m} = 1$, ψ is determined on each thread by its value at one point. Hence the value of ψ on one spacelike hypersurface must be chosen as initial data in order to integrate the equations of motion.

We now examine the constraints. Using T as the time coordinate, enforcing the Einstein equation (61), and using the fact that $u \propto dT$, we find for the constraint quantities

$$C_b^{(T)} = -N^{-1} \tilde{\mu} \psi_{.b}. \quad (62)$$

Thus, the T -surface constraint is violated in the ψ component for essentially the same reasons given in Sec. III B for the fixed clock theory. The conservation law (58) implies $\mathcal{L}_u \tilde{\mu} = 0$, and we have $\mathcal{L}_u d\psi = 0$, so the constraint violation satisfies

$$\mathcal{L}_u (N C_b^{(T)}) = 0. \quad (63)$$

This conservation law is more complicated than the analogous one for the threading theory (33). In the adapted gauge where T and ψ are coordinates (and in which the field equations are second order in derivatives), the vector u^a cannot be chosen to be proportional to ∂_T , since $u \cdot d\psi = 1$. Instead, we can choose a gauge where $u = N^{-1} \partial_T + \partial_\psi$, in which case (63) becomes

$$\partial_T C_\beta + \partial_\psi (N C_\beta) = 0. \quad (64)$$

Hence, we see that the constraint violation evolves according to a first order differential equation that also involves ψ -derivatives of the metric component N .

C. Projectable Hořava gravity

Our final example of a foliation theory is projectable Hořava gravity, which can be obtained from khronometric gravity (Sec. IV A) by imposing in the action the

restriction that the lapse function be constant on each foliation surface, $N = N(T)$. The aether then takes the form $u_a = N(T) T_{.a}$. An aether of this form satisfying the unit constraint is geodesic ($u^a \nabla_a u_b = u^a \nabla_b u_a = \frac{1}{2} \nabla_b (u^a u_a) = 0$), so the acceleration term $c_a a^2$ in the action (1) becomes superfluous.

The projectability restriction can be implemented by adding to the Lagrangian density a Lagrange multiplier term $\tilde{\lambda}^{ab} N_{.a} T_{.b}$. Here $\tilde{\lambda}^{ab}$ is an antisymmetric tensor density, and $N(g, T) = (g^{mn} T_{.m} T_{.n})^{-1/2}$ is the lapse function. In the adapted gauge $x^0 = T$, the new constraint term becomes $\tilde{\lambda}^{i0} N_{.i}$, with $N = (g^{00})^{-1/2}$. The Hamiltonian constraint, which arises from variations of the lapse, now contains an additional piece $-\partial_i \tilde{\lambda}^{i0}$ from the Lagrange multiplier term. This term represents a local violation of the Hamiltonian constraint. Since the violation is a spatial divergence, the integral of the Hamiltonian constraint will still be imposed on a compact space without boundary. This integrated constraint is the generator of global time reparametrizations, and follows from the global variations of the lapse function $N(T)$ in the original 3+1 formulation of projectable Hořava gravity. With asymptotically flat boundary conditions, the integrated Hamiltonian constraint will not vanish, but equal the flux of $\tilde{\lambda}^{i0}$ through the sphere at spatial infinity. This is consistent with the fact that the metric is fixed at infinity, so, in particular, the variational principle does not include global variations of the lapse.

An alternative version of the projectable theory lacks the global time reparametrization gauge symmetry, and correspondingly the global Hamiltonian constraint is not imposed [22–24]. This version arises from a fixed clock foliation theory (Sec. IV B) when the clock is required in the action to be constant on the foliation slices. To implement this we may add to the Lagrangian a Lagrange multiplier term $\tilde{\lambda}^{ab} \psi_{.a} T_{.b}$, with $\tilde{\lambda}^{ab}$ an antisymmetric tensor density. The corresponding constraint implies that $d\psi \propto dT$, so that u^a (56) takes the form $u_a = \psi_{.a} / |d\psi|^2$. Having imposed this form for u_a , the $\tilde{\lambda}^{ab}$ constraint can be omitted. At this point T has disappeared from the action, its role being taken over by ψ .⁶ The unit constraint then implies that $|d\psi| = 1$, so that $u_a = \psi_{.a}$, and the lapse in the adapted gauge $x^0 = \psi$ is fixed equal to 1. There is no global lapse variation, and no global Hamiltonian constraint. This covariant formulation of projectable Hořava gravity has been described before in [3,25].⁷

⁶We could have kept the role of T explicit, using $d\psi \propto dT$ to express (56) as $u_a = T_{.a} / (\psi' |dT|^2)$. The unit constraint then implies $\psi' |dT| = 1$, so that we have $u_a = \psi'(T) T_{.a}$. The lapse function is, thus, given by $N = N(T) = \psi'(T)$.

⁷For some closely related theories, see [20,21,26–28].

The Einstein equation in this formulation of projectable Hořava gravity results from explicit metric variations alone, since u_a is metric-independent, and it reads

$$\mathcal{F}_{ab} - \tilde{\lambda}u_a u_b = 0. \quad (65)$$

Using the definition of $\tilde{\mu}$ in (59) we can rewrite this as

$$\mathcal{F}_{ab} - \frac{1}{2}\mathcal{E}^c u_c u_a u_b - \frac{1}{2}\tilde{\mu}u_a u_b = 0, \quad (66)$$

which looks like the khronometric theory equation (54) with an additional “dark matter” component.⁸ We note a peculiar difference from the foliation + clock theory: $\tilde{\mu}$ appears in (66) with the opposite sign as in (61). Thus, in projectable Hořava theory, positive $\tilde{\mu}$ represents *negative* energy density, whereas it gives positive energy density in the foliation + clock theory.

The on-shell value of the constraint quantity (13) associated with the clock field ψ is

$$\mathcal{C}_b^{(\psi)} = \tilde{\mu}u_b, \quad (67)$$

so again we find a single constraint violated, due to the presence of the “dark matter energy density” $\tilde{\mu}$.

Unlike previous cases considered in this paper, $\tilde{\mu}$ is generically not conserved along the aether flow in projectable Hořava gravity. The conservation equation comes from the clock field equation of motion, which reads

$$\nabla_a[\mathcal{E}^a - 2\tilde{\lambda}u^a] = 0. \quad (68)$$

Decomposing this equation into parallel and perpendicular components, we find evolution equation for $\tilde{\mu}$,

$$\mathcal{L}_u \tilde{\mu} = \nabla_a \mathcal{E}_\perp^a. \quad (69)$$

Since the aether equation of motion is not imposed, this means the “dark matter” may be generated or destroyed along the flow of u^a . Nonconservation of the dark matter as an integration constant was pointed out in [13], where it was suggested that this could provide a mechanism for the generation of dark matter during the early universe. In that paper, it was assumed that the theory agrees with general relativity in the IR, so that the coupling parameters c_i are zero. (Recall that the Lagrange multiplier term results in nonzero $\tilde{\mu}$ even when the aether couplings are zero.) In this limit, $\mathcal{E}^a = 0$, so we would recover the conservation equation $\mathcal{L}_u \tilde{\mu} = 0$ were it not for the higher-derivative terms included in the full Hořava-Lifshitz theory [13]. We note that the nonconservation of $\tilde{\mu}$ is potentially

⁸In the formulation with the integrated constraint, we would find the same Einstein equation, with the identification of $\tilde{\mu} = u_c \nabla_d \tilde{\lambda}^{dc}$

problematic. Apparently nothing enforces that $-\tilde{\mu}$ remain positive, so instabilities might arise.⁹

V. DISCUSSION

In this paper we studied a variety of aether theories including and modifying Einstein-aether theory and the IR limit of Hořava-Lifshitz gravity (khronometric gravity), which differ only in how the aether is constructed from the independently varied fields in the action. When those fields are scalars, their equations of motion are implied by the other equations of motion, so they may be regarded as defining fixed background structures. We found that it can be consistent to include such background structures, and that they often induce extra degrees of freedom owing to the loss of diffeomorphism constraints.

The specific structures we considered were the fixed threading and fixed foliation, as well as a fixed clock field that could be included in either the threading or foliation theories. We also considered a nonfixed threading, described by a vector field rather than a triple of scalar fields. For the fixed threading theory, the Einstein equation was unaltered relative to Einstein-aether theory, but the perpendicular aether equation of motion and the corresponding constraint equations were modified by a constant source term. The foliation theory without additional structure is equivalent to nonprojectable Hořava gravity. For this theory, the initial value constraints hold in the adapted gauge, but the Einstein equation differs from the threading-type theories since the aether appears naturally as a covector u_a .

The addition of the (fixed) clock field ψ modifies the Einstein equation by an additional term that has the form of a pressureless dust stress tensor, which can be thought of as due to an internal energy density of the aether. We therefore called such aethers “ponderable.” The fixed clock also leads to a violation of the ψ component of the initial value constraint. When a fixed clock is added to the fixed threading theory, we obtain a fixed aether theory, which is equivalent to describing the aether by a vector field that is not varied in the action. When a fixed clock is added to the foliation theory and constrained to be constant on the preferred foliation, the projectable version of Hořava gravity results.

The appearance of a dark-matter-like component in the Einstein equation is also a feature of the recently proposed mimetic dark matter theory [20]. In this theory, the physical metric g_{ab} is constructed from another metric \bar{g}_{ab} and a scalar ϕ such that the gradient $\nabla_a \phi$ is unit by construction, $g_{ab} = (\bar{g}^{cd} \nabla_c \phi \nabla_d \phi) \bar{g}_{ab}$. It was shown in [30] that this theory is equivalent to ordinary Einstein gravity, supplemented with a scalar field ϕ that appears in the action only via the constraint term imposing that $\nabla_c \phi$ is unit. The

⁹Only when $\tilde{\mu}$ is conserved due to the aether parameters being zero has it been shown that fluctuations around positive energy backgrounds have positive energy [3,29].

discussion of Sec. IV C, therefore, demonstrates that this theory is equivalent to projectable Hořava gravity with vanishing aether action, that is, with the parameters c_i set to zero.¹⁰ Thus, the mimetic dark matter theory can be viewed as a special case of the aetheral theories described here.

The threading theory formalism discussed here resembles the Lagrangian description of perfect fluids. The aether, constructed from the comoving potentials φ^I , acts as a zero temperature vacuum fluid, according to the thermodynamic relations developed in [18]. The vanishing of the temperature is closely tied with the enhanced symmetry of the aether fluid, which includes all φ^I -diffeomorphisms rather than only the volume preserving ones. It was mentioned in [31] that such an enhanced symmetry is not possible without adding more fields, but this is true only for an action that is first order in derivatives.¹¹ The lowest order terms in the aether action involve second derivatives of φ^I . A derivative expansion for a fluid action was discussed in [18,32], and the aetheral terms invariant under all φ^I -diffeomorphisms appear in the latter reference.

The clock field ψ in the aether theories is analogous to the phase field introduced in [18] for fluids carrying a conserved particle number. In particular, it possesses the same chemical shift symmetry, $\psi \mapsto \psi + v(\varphi^I)$, which, in the aetheral case, corresponds to a freedom to shift the

¹⁰More generally, all the background clock theories discussed here can be formulated using the \bar{g}_{ab} metric construction of [20] (see also [29] for the case of a vector field), instead of imposing the unit constraint with a Lagrange multiplier term. If we define the physical metric as $g_{ab} = (\bar{g}^{cd}u_c u_d)\bar{g}_{ab}$ for a covariant aether, or as $g_{ab} = (\bar{g}_{cd}u^c u^d)^{-1}\bar{g}_{ab}$ for a contravariant aether, the aether vector automatically has unit norm with respect to the physical metric. The action depends only on the conformal class of \bar{g}_{ab} , so the \bar{g}_{ab} variation gives a trace free equation, $(G^{ab} - T^{ab}) - (G - T)u^a u^b = 0$, where G^{ab} and T^{ab} are the variational derivatives of the action with respect to g_{ab} , and the traces G and T are their contractions with g_{ab} . For the fixed aether theory, there is no u^a variation equation, and instead of the usual Einstein equation we have only this Einstein equation with an additional source with pressureless dust energy-momentum tensor and energy density $G - T$. Thus, we have recovered the ponderable aether discussed in the text. The equivalence to a mimetic dark matter theory arises in the case when there is no aether action. Then T^{ab} is just the matter stress tensor and is conserved when the matter satisfies its equation of motion, so the Bianchi identity implies the conservation law $\nabla_a[(G - T)\hat{u}^a \hat{u}^b] = 0$. The extra term, thus, behaves in this case as geodesic dark matter dust.

¹¹With the addition of another scalar field it is possible. An example is provided by a Lagrangian $F(y)$ that is a function only of the chemical potential $y = u^a \nabla_a \psi$, where u^a is the fluid velocity (15). This would not contain higher-derivative terms, and it is invariant under full φ^I diffeomorphisms (not just volume preserving ones). It also has the chemical shift symmetry, so its symmetries are the same as those of the fixed aether theory. It can be shown that an $F(y)$ Lagrangian possesses the same dynamics as an uncharged perfect fluid. It cannot produce an equation of state $p = 0$, whereas the formulation using only φ^I cannot produce $\rho = 0$.

initial value of the clock along each thread. The scalar $y = u^a \nabla_a \psi$ has a thermodynamic interpretation as the chemical potential for particles charged under the shift symmetry. In our aetheral setting, however, y is fixed everywhere equal to unity by construction of u^a [see, e.g. Eq. (46)], and ψ measures proper time along the flow.

Having elucidated the basic structure of these theories, it is interesting to consider the phenomenological consequences of the presence of the background structures. In a companion paper [19], we considered some of the astrophysical and cosmological implications of the source densities for the threading-type theories. In particular, we examined whether the new component in the Einstein equation for ponderable aethers with fixed clocks could play the role of dark matter. The two main results of that analysis are that (i) the ‘‘aetheral dark matter fluid’’ has pressure, hence does not seed structure formation on subhorizon scales, so another dark matter component must be present, and (ii) during matter domination, the presence of a homogeneous ponderable aether energy density causes problematic growth of the isocurvature modes on superhorizon scales. In particular, for isocurvature amplitudes of order 10^{-5} at radiation-matter equality (which would be the value expected from inflation [33]), the growth at large scales becomes inconsistent with CMB and large scale structure observations when the ponderable aether contributes more than 1% of the homogeneous energy density. On the other hand, these results do not apply to the dark matter as an integration constant [13] in projectable Hořava gravity. That theory results from taking the limit $c_a \rightarrow \infty, c_\omega \rightarrow \infty$ of Einstein-aether theory [4,9]. In that limit the dark matter fluid is pressureless, and the large-scale isocurvature modes are decaying.

Finally, it should perhaps be emphasized that, in a theory with conserved ‘‘aetheral dark matter’’ current, the primordial value of the internal energy density of the aether would be driven very nearly to zero if there is an early period of inflation. This leads to the curious conclusion that the nondynamical and dynamical aether theories could appear to be essentially equivalent in their phenomenological predictions.

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