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Plane wave holonomies in quantum gravity. II. A sine wave solution

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This paper constructs an approximate sinusoidal wave packet solution to the equations of canonical gravity. The theory uses holonomy-flux variables with support on a lattice (LHF = lattice-holonomy flux). There is an SU(2) holonomy on each edge of the LHF simplex, and the goal is to study the behavior of these holonomies under the influence of a passing gravitational wave. The equations are solved in a small sine approximation: holonomies are expanded in powers of sines and terms beyond $\sin^2$ are dropped; also, fields vary slowly from vertex to vertex. The wave is unidirectional and linearly polarized. The Hilbert space is spanned by a set of coherent states tailored to the symmetry of the plane wave case. Fixing the spatial diffeomorphisms is equivalent to fixing the spatial interval between vertices of the loop quantum gravity lattice. This spacing can be chosen such that the eigenvalues of the triad operators are large, as required in the small sine limit, even though the holonomies are not large. Appendices compute the energy of the wave, estimate the lifetime of the coherent state packet, discuss circular polarization and coarse-graining, and determine the behavior of the spinors used in the U(N) SHO realization of LQG.

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I. INTRODUCTION

This is the second of two papers with the goal of developing intuition on the behavior of holonomies and fluxes in the presence of a gravitational wave. The previous paper (paper I) constructs a lattice-holonomy flux (LHF) theory having planar symmetry [1]. The gravitational excitation is assumed to be unidirectional and singly polarized. Constraints are evaluated in a small sine (SS), slow variation (SV) limit. Small sine: holonomies are expanded in powers of sine,

$$\begin{aligned}\mathbf{h}(\theta,\hat{n})^{1/2} &= \cos(\theta/2)\mathbf{1} + i\sin(\theta/2)\hat{n}\cdot\boldsymbol{\sigma}\\ &= \mathbf{1} + i\sin(\theta/2)\hat{n}\cdot\boldsymbol{\sigma} + \mathrm{O}\,\sin^2,\end{aligned}$$

where $\mathbf{h}$ is a spin $1/2$ holonomy, a rotation around axis $\hat{n}$ through angle θ. Terms of order $\sin^3$ and higher in the constraints are dropped. Slow variation: dynamical functions f are assumed to vary slowly from vertex to vertex: $\delta f/f \ll 1$. The two assumptions, small sine and slow variation, are closely connected, and for brevity sometimes we will refer to them simply as the SS approximation.

Paper I imposed all gauges at the classical level, except the spatial diffeomorphism gauge. A diffeomorphism gauge is chosen in Sec. II of the present paper. Some discussion is required; the gauge fixing constant C scales with peak angular momentum of the coherent state.

Section III quantizes the theory. As emphasized in that section (and in paper I), any classical solution to the constraints yields a corresponding solution to the quantum constraints, because coherent states are used as a basis for the Hilbert space. Section IV constructs such an approximate classical solution, an undamped sine wave. Section V adds the damping.

Section VI sketches the construction of the coherent states. Section VI A compares the SU(2) coherent states to the familiar coherent states for the free particle. This analogy is used to justify the form of the SU(2) states, in a manner which is qualitative, but should be intuitively convincing. Full details of the construction are given in Ref. [2]; see also [3]. Section VI B summarizes the most important matrix elements of the SU(2) coherent states.

The coherent states depend upon a number of angle and angular momentum parameters. Section VII determines parameter values such that the expectation values of the triads reproduce the sinusoidal solution constructed in Sec. V.

Appendix A computes the ADM energy of the wave. In LHF the energy of the packet, and therefore the lifetime, depends on the quantized frontal area of the packet.

Experimentally, it is clear that SU(2) holonomies (which are just rotation matrices) can be superimposed to form a coherent state, because the earth (for example) presumably is described by a superposition of Legendre polynomials (rotation matrices again); yet both its angular momentum and conjugate angle are sharp.

Theoretically, however, matters are less clear. Coherent states eventually spread, unless the system has the equally spaced energies characteristic of the SHO. It is necessary to show under what conditions the spreading is limited. Appendix B estimates the lifetime of coherent state wave packets.

Appendix D discusses the circularly polarized case. Appendix E discusses coarse-graining. The present solution is an especially simple example which illustrates the

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method of coarse-graining proposed in Refs. [4–7]. Appendix F discusses the SHO/U(N) formulation of LQG.

A. Plane waves in classical general relativity

The classical literature uses primarily two gauges: the one used in this paper, in which $-g_{tt} = g_{zz} = 1$, $g_{\mu\nu} = g_{\mu\nu}(t, z)$; and a gauge

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + f(u, x, y)du^2,$$

$u = (z - t)/\sqrt{2}$. The first gauge was used by Baldwin and Jefferys in their pioneering paper [8]. Peres derived an exact solution for an undamped sinusoidal plane wave using the second gauge [9]. The Peres solution, when converted to the gauge used in this paper, becomes the undamped solution of Sec. IV. Griffiths [10] shows how to convert between the two gauges and describes additional exact nonsinusoidal solutions.

II. FIXING THE DIFFEOMORPHISM GAUGE

Coherent states work best when eigenvalues are large; yet fields must be weak. This is an apparent contradiction. How can fields be small, if eigenvalues must be large?

The LHF $\tilde{\mathrm{E}}$ operators contain area factors not present in their field theory (FT) analogs:

$$(2/\kappa\gamma)\mathrm{E}^i_I(\mathrm{LHF}) = (2/\kappa\gamma)\Delta x^j \wedge \Delta x^k \mathrm{E}^i_I(\mathrm{FT}). \quad (1)$$

(In this paper "Field Theory" refers to a classical theory based on fluxes and connections which have support on the continuum; LHF refers to a theory based on fluxes and holonomies which have support on a lattice. The LHF theory may be classical or quantum depending on context.) The area is in Planck lengths squared, because of the $\kappa\gamma$ factor. Suppose the Δx^i are taken to be 10^2 Planck lengths (an extremely tiny length, by classical standards). The classical triad may be order unity; yet the quantum eigenvalue will be order 10^4. Therefore, *typical angular momenta in the wave function can be order* 10^4, far from order unity, *even though classical values are order unity.* This fact resolves the apparent contradiction discussed in the previous paragraph.

As in the literature for classical general relativity, the diffeomorphism gauge is chosen such that $g_{zz} = 1$. In the notation of paper I, this gauge has parameter $p = 1/2$.

$$\begin{aligned} {}^{(2)}\tilde{\mathrm{E}}(\mathrm{FT}) &= (\mathrm{C_{FT}} \mathrm{E}^z_Z)^{p+1/2} \\ &= \mathrm{C_{FT}} \mathrm{E}^z_Z; \\ \text{equivalently,} \quad (e^Z_z)^2 &= \mathrm{sgn}(e)\mathrm{C_{FT}}. \end{aligned} \quad (2)$$

$\mathrm{C_{FT}}$ is a constant. ${}^{(2)}\tilde{\mathrm{E}}$ is the determinant of the 2×2 transverse (x, y) triads. On the last line of Eq. (2) we have expanded the $\tilde{\mathrm{E}}$ in triads and used $e^z_A = 0, A = X, Y$; $\mathrm{sgn}(e)$ is the sign of the 3×3 triad determinant.

$$e = \mathrm{sgn}(e)|e|.$$

Since e^Z_z must match to flat space at the front of the packet,

$$\begin{aligned} \mathrm{C_{FT}} &= \mathrm{sgn}(e); \\ e^Z_z &= \pm 1 \coloneqq \mathrm{sgn}(z). \end{aligned} \quad (3)$$

We now take this over to LHF. The spatial diffeomorphism gauge must be chosen such that, when factors of Δx^i are stripped out, one recovers the classical gauge fixing.

$$\begin{aligned} {}^{(2)}\tilde{\mathrm{E}}(\mathrm{LHF}) &= \mathrm{C_{LHF}} \mathrm{E}^z_Z(\mathrm{HF}); \\ \mathrm{C_{LHF}} &= (\Delta z)^2 \mathrm{C_{FT}}. \end{aligned} \quad (4)$$

Each E^x in ${}^{(2)}\tilde{\mathrm{E}}$ (LHF) will have an area factor $\Delta y \Delta z$; each E^y in ${}^{(2)}\tilde{\mathrm{E}}$ will have a factor $\Delta x \Delta z$. The factors of Δx, Δy are also present in E^z, but not the Δz. Therefore the missing Δz factors turn up in $\mathrm{C_{LHF}}$. $\mathrm{C_{FT}}$ is still the classical value, $\mathrm{sgn}(e)$. When we pick $\mathrm{C_{LHF}}$, we are picking a value for Δz [in Planck units, after both sides of Eq. (4) are divided by $(\kappa\gamma/2)^2$; compare Eq. (1).]

Caps ΔX^I denote local Lorentz coordinates; lower case Δx^i denote coordinates on the global manifold. The two sets of coordinates are related.

$$\begin{aligned} \tilde{\mathrm{E}}^i_I(\mathrm{LHF}) &= \mathrm{sgn}(e)(e e^i_I)(\mathrm{FT})\Delta x^j \Delta x^k \\ &= \mathrm{sgn}(e) e^J_j e^K_k \Delta x^j \Delta x^k \\ &= \mathrm{sgn}(e) \Delta X^J \Delta X^K. \end{aligned} \quad (5)$$

Equation (5) may also be written as

$$\tilde{\mathrm{E}}^i(\mathrm{FT})\Delta x^j \Delta x^k = \mathrm{sgn}(e)\Delta X^J \Delta X^K. \quad (6)$$

The weak-classical-field-but-large-eigenvalue connection emerges if one multiplies the last equation by $(2/\kappa\gamma)$ and equates the result to a spin eigenvalue j.

$$(2/\kappa\gamma)\tilde{\mathrm{E}}^i(\mathrm{FT})\Delta x^j \Delta x^k = \mathrm{O}\ j.$$

j can be large, even though $\tilde{\mathrm{E}}$ (FT) is small, because of the large area.

The $\tilde{\mathrm{E}}$ (FT) will have z dependence, therefore from Eq. (6) the $\Delta x^j \Delta x^k$ vary with z, or the $\Delta X^J \Delta X^K$, or both. We assume the global coordinates x^i are fixed; the variation is in the Lorentz lengths X^I. Equivalently, $\tilde{\mathrm{E}}$ (FT) and $\tilde{\mathrm{E}}$ (LHF) have the same variation with z, since $\tilde{\mathrm{E}}$ (FT) and $\tilde{\mathrm{E}}$ (LHF) differ only by factors of Δx^j, which are held fixed.

Support for this assumption comes from a later result in the sections on coherent state parameters. The coherent states are approximate eigenstates of the $\tilde{\mathrm{E}}$ (LHF) in Eq. (5), with eigenvalues equal to an angular momentum or Z coordinate of angular momentum.

$$\tilde{\mathrm{E}}^a_A(LQG)|\mathrm{coh}\rangle = (\kappa\gamma/2)L^a_A|\mathrm{coh}\rangle;, \qquad a = x, y;$$
$$\tilde{\mathrm{E}}^z_Z(\mathrm{LHF})|\mathrm{coh}\rangle = (\kappa\gamma/2)m|\mathrm{coh}\rangle. \tag{7}$$

The last line of Eq. (5) gives

$$(\kappa\gamma/2)(L^a_A \quad \text{or} \quad m) = \mathrm{sgn}(e)\Delta X^J\Delta X^K. \tag{8}$$

If the Lorentz lengths ΔX^I are taken as fixed, then the canonical momenta cannot vary in the presence of a gravitational wave, a reduction to the absurd.

When quantizing plane waves in FT, using ADM variables, one can renormalize constraints by dividing out a factor $\Delta x\Delta y$. The FT expressions then contain only integrals over z. Such a renormalization is not possible in LHF, because not every term contains an overall factor of $\Delta x\Delta y$. Some $(\Delta x, \Delta y)$ are hidden in holonomies and do not cancel out. In FT the integrals over transverse directions are infinite, and renormalization is mandatory. In LHF the transverse integrals range over the circumferences of the x and y circles and are finite. Renormalization is not necessary.

A. N can be fixed at unity

Paper I introduced a modified lapse $\underline{\mathrm{N}}$ which obeys simpler boundary conditions than the usual lapse N. However, it is desirable to arrange $\mathrm{N} > 0$, so that dt and dT "run" in the same direction.

$$\mathrm{dT} = e^T_t\mathrm{d}t = \mathrm{N}\mathrm{d}t;$$
$$e^T_i = 0; \qquad i = x, y, z. \tag{9}$$

The second line is the usual gauge choice which reduces full Lorentz symmetry to SU(2).

Fortunately, the diffeomorphism gauge just chosen leads to a simple relation between $\underline{\mathrm{N}}$ and N.

$$\underline{\mathrm{N}}(\mathrm{FT}) \coloneqq \mathrm{N}(\mathrm{E}^z_Z/|e|)_{\mathrm{FT}} = \mathrm{N}e^z_Z = \mathrm{N}\mathrm{sgn}(z). \tag{10}$$

$\mathrm{sgn}(z)$ is the sign of e^Z_z and E^z_Z. Also, the constraints of paper I require $\underline{\mathrm{N}}$ to be a constant: $\delta_{(c)}\underline{\mathrm{N}} = 0$. If the constant is chosen appropriately, N becomes unity:

$$\underline{\mathrm{N}}(\mathrm{FT}) = \mathrm{sgn}(z). \tag{11}$$

A corollary: with this choice, neither $\underline{\mathrm{N}}$ nor N can vanish. □

The LHF $\underline{\mathrm{N}}$ follows from the field theory $\underline{\mathrm{N}}$, Eq. (11), except one replaces field theory $\tilde{\mathrm{E}}$ by LHF $\tilde{\mathrm{E}}$.

$$\underline{\mathrm{N}}(\mathrm{LHF}) = \underline{\mathrm{N}}(\mathrm{FT})/\Delta z = 1/\Delta Z. \tag{12}$$

$\underline{\mathrm{N}}$ is a contravariant rank one tensor, and therefore needs a $1/\Delta z$ to make it diffeomorphism invariant.

B. $\underline{\mathrm{N}}$ may be chosen unity

It is convenient to make the light cone variable du equal to the inertial frame dU,

$$\mathrm{d}u = (e^z_Z\mathrm{d}Z - e^t_T\mathrm{d}T)/\sqrt{2} = (\mathrm{sgn}(z)\mathrm{d}Z - \mathrm{d}T)/\sqrt{2}, \tag{13}$$

This requires

$$\mathrm{sgn}(z) = +1 = \underline{\mathrm{N}}(\mathrm{FT}). \tag{14}$$

C. Some triads can vanish

The gauge choice Eq. (3) forbids zeros of e^Z_z, but not zeros of ${}^{(2)}\mathrm{e}$, the determinant of the transverse e^A_a. ${}^{(2)}\tilde{\mathrm{E}}$ and E^z_Z each contain one power of ${}^{(2)}\mathrm{e}$ and could conceivably pass through zero simultaneously, when away from the small sine limit.

D. A comment on signs

As indicated in the previous sections, there is a natural choice for $\mathrm{sgn}(z)$, and similarly for the other signs. Most formulas of this paper are worked out for both signs of the $\mathrm{sgn}(i)$, although this is overkill. One may always choose $\mathrm{sgn}(i) = +1$, and the sign does not change in the small sine limit. Working out the results for both signs does help in checking the algebra. Section IX and the conclusion summarize results for the choice $\mathrm{sgn}(i) = +1$.

III. THE QUANTUM SCALAR CONSTRAINT

Our final formula for the scalar constraint $\tilde{\mathrm{H}}$ in paper I was

$$\tilde{\mathrm{H}} = \sum_n(1/\kappa)\{(1/2)(\delta_{(c)}\mathrm{E}^y_Y/\mathrm{E}^y_Y - \delta_{(c)}\mathrm{E}^x_X/\mathrm{E}^x_X)^2\mathrm{E}^z_Z$$
$$+ \delta_{(c)}\mathrm{E}^z_Z[-(\delta_{(c)}{}^{(2)}\tilde{\mathrm{E}})/2{}^{(2)}\tilde{\mathrm{E}}] + \delta_{(c)}(\delta_{(c)}\mathrm{E}^z_Z)\}$$
$$= 0. \tag{15}$$

The gauge choice Eq. (4) implies

$$\delta_{(c)}{}^{(2)}\tilde{\mathrm{E}}/{}^{(2)}\tilde{\mathrm{E}} = \delta_{(c)}\mathrm{E}^z_Z/\mathrm{E}^z_Z$$
$$= \delta_{(c)}\mathrm{E}^x_X/\mathrm{E}^x_X + \delta_{(c)}\mathrm{E}^y_Y/\mathrm{E}^y_Y. \tag{16}$$

This and the next equation use a distributive law for the difference which is valid given the slow variation (SV) assumption.

$$\delta_{(c)}(AB) = (\delta_{(c)}A)B + A(\delta_{(c)}B) \quad (\mathrm{SV}).$$

In Eq. (15), one can divide through by E^z_Z and use Eq. (16) to eliminate $\delta_{(c)}\mathrm{E}^z_Z$. The double difference may be rewritten using

$$\begin{aligned}\delta_{(c)}\{\delta_{(c)}\mathrm{E}^z_Z\}/\mathrm{E}^z_Z &= \delta_{(c)}\{[\delta_{(c)}\mathrm{E}^x_X/\mathrm{E}^x_X + \delta_{(c)}\mathrm{E}^y_Y/\mathrm{E}^y_Y]\mathrm{E}^z_Z\}/\mathrm{E}^z_Z \\ &= [\delta_{(c)}(\delta_{(c)}\mathrm{E}^x_X)/\mathrm{E}^x_X + \delta_{(c)}(\delta_{(c)}\mathrm{E}^y_Y)/\mathrm{E}^y_Y] \\ &\quad - [\delta_{(c)}\mathrm{E}^x_X/\mathrm{E}^x_X]^2 - [\delta_{(c)}\mathrm{E}^y_Y/\mathrm{E}^y_Y]^2 \\ &\quad + [\delta_{(c)}\mathrm{E}^x_X/\mathrm{E}^x_X + \delta_{(c)}\mathrm{E}^y_Y/\mathrm{E}^y_Y]^2. \qquad (17)\end{aligned}$$

The constraint simplifies to

$$0 = \delta_{(c)}(\delta_{(c)}\mathrm{E}^x_X)(1/\mathrm{E}^x_X) + \delta_{(c)}(\delta_{(c)}\mathrm{E}^y_Y)(1/\mathrm{E}^y_Y). \qquad (18)$$

At this point one can make the transition from classical to quantum. Classical functions are replaced by quantum operators; brackets become quantum commutators (Dirac rather than Poisson brackets, because the unidirectional constraints are second class). There is a factor ordering question, because Dirac brackets imply the $\tilde{\mathrm{E}}$ no longer commute with themselves. In a typical LQG quantization involving Poisson brackets, the $\tilde{\mathrm{E}}$ are ordered to the right of the K's. Here, the $\delta_{(c)}\tilde{\mathrm{E}}$ are equivalent to K's because of the unidirectional constraints. Therefore triads have been moved to the right of the $\delta_{(c)}\tilde{\mathrm{E}}$.

A. Comparison to classical results

The classical calculation yields the following results for the nonzero components of the Einstein and Weyl tensors:

$$\begin{aligned}\mathrm{G}_{uu} &= \ddot{\mathrm{E}}^x_X/\mathrm{E}^x_X + \ddot{\mathrm{E}}^y_Y/\mathrm{E}^y_Y = 0; \\ \mathrm{C}^x_{uxu} &= \ddot{\mathrm{E}}^x_X/\mathrm{E}^x_X - \ddot{\mathrm{E}}^y_Y/\mathrm{E}^y_Y = -\mathrm{C}^y_{uyu}. \qquad (19)\end{aligned}$$

Variables are x, y, u, v. Fields are single polarization and unidirectional (dependent on u only); dots denote derivatives with respect to u. Gauge is $e^Z_z = \pm 1$. The LHF constraint Eq. (18) is just the classical constraint, with u derivatives replaced by z differences.

From Eq. (19), the classical Weyl tensor is the scalar constraint, with one minus sign change. This same relation (between scalar constraint and Weyl) holds in the quantum case. Therefore

$$\begin{aligned}\delta_{(c)}(\delta_{(c)}\mathrm{E}^x_X)/\mathrm{E}^x_X &= -\delta_{(c)}(\delta_{(c)}\mathrm{E}^y_Y)/\mathrm{E}^y_Y; \\ \mathrm{Weyl} &= 2\delta_{(c)}(\delta_{(c)}\mathrm{E}^x_X)/\mathrm{E}^x_X. \qquad (20)\end{aligned}$$

One can pick a desired curvature, choose an E^x_X which produces this curvature, and immediately have a solution to the scalar constraint.

B. Coherent states, Dirac brackets, and the scalar constraint

In leading order, coherent states do not preserve quantum commutators. Let O_1, O_2 be two quantum operators peaked at values $O_i(\mathrm{cl})$. Then

$$\begin{aligned}\langle\mathrm{coh}|O_1O_2|\mathrm{coh}\rangle &= \langle\mathrm{coh}|O_1|\mathrm{coh}\rangle\langle\mathrm{coh}|O_2|\mathrm{coh}\rangle \\ &\quad + \sum_{SC}\langle\mathrm{coh}|O_1|\mathrm{SC}\rangle\langle\mathrm{SC}|O_2|\mathrm{coh}\rangle \\ &= O_1(\mathrm{cl})O_2(\mathrm{cl}) + \mathrm{order} 1/\sqrt{L}. \qquad (21)\end{aligned}$$

The O_i acting on a coherent state typically give back the coherent state, plus small correction (SC) states which are down by order $1/\sqrt{\mathrm{L}}$ [2]. If we neglect the SC states, then the commutator $\langle[O_1, O_2]\rangle$ is zero. Thiemann and Winkler, without constructing the SC states, have shown that Poisson brackets are preserved in the semiclassical limit [11] in the sense that the quantum commutator is given by $i\hbar$ times the classical Poisson bracket. Since Dirac brackets are functions of Poisson brackets, it is likely that Dirac brackets are preserved also.

The tendency of coherent states to turn quantum operators into classical expressions is helpful in another context. If one has a classical solution to the scalar constraint, one immediately has a quantum solution, because the classical function is the corresponding quantum operator, evaluated at the peak values specified by the coherent state.

$$\begin{aligned}&(H = O_1O_2\ldots)|\mathrm{coh}\rangle \\ &\quad = (O_1(\mathrm{cl})O_2(\mathrm{cl})\ldots)|\mathrm{coh}\rangle + \mathrm{order}\ 1/\sqrt{L}.\end{aligned}$$

$O_i(\mathrm{cl})$ is a classical function, part of a classical solution to the constraint: the leading term on the right-hand side vanishes. O_i is the corresponding quantum operator. The next two sections construct such a classical solution.

IV. A GRAVITATIONAL SINE WAVE

This section constructs a sine wave solution which is periodic, but undamped. The following section adds damping.

The undamped solution is

$$\begin{aligned}\mathrm{E}^x_X(LHT; n) &= (\Delta z\Delta y)\mathrm{sgn}(x)\{1 - a\sin[(2\pi n/\mathrm{N}_\lambda)]/2! \\ &\quad - (a^2/32)[\cos(4\pi n/\mathrm{N}_\lambda) \\ &\quad + (4\pi/\mathrm{N}_\lambda)^2(n)^2/2]\}. \qquad (22)\end{aligned}$$

a is a small, dimensionless, constant amplitude. N_λ is a constant, the number of vertices in a length equal to one wavelength. $\mathrm{sgn}(x) = \pm 1$ is the sign of e^x_X, -1 if the x and X axes increase in opposite directions. The order a^2 terms are frequency doubled, a typical nonlinear effect.

From the expression for the Einstein tensor G, the expression for E^y_Y must have a linear-in-a term identical to Eq. (22), except $a \to -a$ (and $x \leftrightarrow y$).

With a slight abuse of a standard notation, one can define a k vector in n space, i.e. a vector which gives the change in phase per unit change in n.

$$(2\pi/\mathrm{N}_\lambda) := k;$$
$$kn = (k/|\Delta Z|)(n|\Delta Z|) = (2\pi/\text{wavelength})(|Z|). \tag{23}$$

The second line gives the connection to the usual k, the change in phase per unit change in length.

The above solution is approximate because an exact solution requires an infinite series, whereas the quantum solution of Eq. (22) stops at order a^2. Nonlinear effects begin at order a^2; if the theory is solved to order a only, we recover the usual weak field limit.

To check the constraint and compute curvature, one must compute $\delta^{(2)}E/E$. The second difference of the linear-in-a term, Eq. (22), is

$$\begin{aligned}&-(a/2)\mathrm{sgn}(x)\left\{\sum_{\pm}\sin[(2\pi/\mathrm{N}_\lambda)(n\pm 1)]\right.\\&\qquad\left.-2\sin(2n\pi/\mathrm{N}_\lambda)\right\}(\Delta z\Delta y)\\&\quad=-(a/2)\mathrm{sgn}(x)\{\sin(2n\pi/\mathrm{N}_\lambda)(2\cos(2\pi/\mathrm{N}_\lambda)-2)\}\\&\qquad\times(\Delta z\Delta y).\end{aligned} \tag{24}$$

The first sine was expanded using $\sin(A\pm B) = \sin A\cos B \pm \cos A \sin B$.

To estimate the size of N_λ, one can use the connection between N_λ and the classical wavelength, Eq. (23). Since that wavelength is macroscopic, whereas ΔZ, the change in z per unit change in n, is of order a few hundred Planck lengths, N_λ must be astronomically large, and $1/\mathrm{N}_\lambda$ must be negligible, except when multiplied by n. Therefore one can expand the cosine in Eq. (24), and the second difference becomes

$$(a/2)\mathrm{sgn}(x)(2\pi/\mathrm{N}_\lambda)^2\sin(2\pi n/\mathrm{N}_\lambda)(\Delta z\Delta y). \tag{25}$$

The term quadratic in a, Eq. (22), is handled similarly: trigonometric identities are used to expand functions of $n\pm 1$; functions of $1/\mathrm{N}_\lambda$ are power-series expanded. The total second difference (both linear and quadratic in a) is

$$\begin{aligned}\delta^{(2)}\mathrm{E}^x_X(\mathrm{LHF};n) &= (a/2)\mathrm{sgn}(x)(2\pi/\mathrm{N}_\lambda)^2\sin(2n\pi/\mathrm{N}_\lambda)\\&\quad\times[1-(a/2)\sin(2n\pi/\mathrm{N}_\lambda)](\Delta z\Delta y).\end{aligned} \tag{26}$$

The square bracket is $\mathrm{E}^x_X(\mathrm{FT})$, so that the curvature is

$$2\delta^{(2)}\mathrm{E}^x_X(\mathrm{LHF};n)/\mathrm{E}^x_X = 2(a/2)(k)^2\sin(kn), \tag{27}$$

where we have shifted to the new k vector $2\pi/\mathrm{N}_\lambda$.

The factor of 2 on the left, Eq. (27), takes into account the contribution from E^y_Y. The calculation for the y direction is identical to the x calculation, except $(a\to -a)$. The y contribution to curvature therefore exactly doubles the x contribution; see Eq. (20). The y contribution to the scalar constraint exactly cancels the x contribution, as at Eq. (18).

The role of the small amplitude a needs to be clarified. In deriving Eq. (27) one may assume that the curvature is *linear* in amplitude a, while transverse $\tilde{\mathrm{E}}$ are infinite series in a. For example, order a^2 and higher corrections to curvature vanish. To see how this happens, rewrite the expression for the $\tilde{\mathrm{E}}$ in a manner which emphasizes the dependence on a.

$$\begin{aligned}\delta^{(2)}\mathrm{E}^x/\mathrm{E}^x &= (\ddot{B}_1+\ddot{B}_2+\cdots)/(1+B_1+\cdots)\\&=\ddot{B}_1+\ddot{B}_2-B_1\ddot{B}_1+\cdots.\end{aligned} \tag{28}$$

B_p is order a^p; double dots indicate second differences; and $\cdots$ indicate terms which contribute cubic and higher terms to the curvature. Choose B_2 such that

$$\begin{aligned}&(\ddot{B}_1+\ddot{B}_2+\cdots)/(1+B_1+\cdots)\\&\quad=\ddot{B}_1(1+B_1+\cdots)/(1+B_1+\cdots).\end{aligned}$$

Equivalently, choose B_2 so that the order a^2 terms in Eq. (28) cancel.

$$\ddot{B}_2/B_1 = \ddot{B}_1. \tag{29}$$

Then the order a^2 contributions to curvature vanish.

One can generalize Eq. (29) to cubic and higher orders in a. Given $B_1, B_2, \ldots, B_{p-1}$, determine B_p by solving the equation

$$\ddot{B}_p/B_{p-1} = \ddot{B}_1. \tag{30}$$

Then

$$\begin{aligned}&(\ddot{B}_1+\cdots+\ddot{B}_p)/(1+B_1+\cdots+B_{p-1})\\&\quad=\ddot{B}_1(1+\cdots+B_{p-1})/(1+\cdots+B_{p-1})\\&\quad=\ddot{B}_1.\end{aligned} \tag{31}$$

The curvature is order a, to all orders.

Let Eq. (31) represent the series for $\ddot{\mathrm{E}}^x_X/\mathrm{E}^x_X$. There is another one for $\ddot{\mathrm{E}}^y_Y/\mathrm{E}^y_Y$ with $B_1\to -B_1$, in order for the Einstein tensor to vanish in order a. From the recurrence relation Eq. (30), in the y series *all* terms with odd powers of a have the opposite sign.

To make contact with the classical curvature, Eq. (19), we divide the second difference by $(\Delta\mathrm{U})^2$, then convert differences to derivatives with respect to U. From $U = (Z-T)/\sqrt{2}$,

$$\Delta\mathrm{U} = \Delta\mathrm{Z}/\sqrt{2},$$

in a formalism where T is held constant. Then

$$\begin{aligned} C^x_{uxu}(\mathrm{cl}) &= \{\delta^{(2)}E^x_X(LQG;n)/[E^x_X(LQG;n)] \\ &\quad - (x \to y)\}/(\Delta U)^2 \\ &= 2(a/2)(k)^2 \sin(kn)/(\Delta U)^2 \\ &= a(2\pi\Delta Z/\lambda)^2 \sin(kn)2/(\Delta Z)^2 \\ &= 2a(2\pi/\lambda)^2 \sin(2\pi z/\lambda), \end{aligned} \tag{32}$$

where λ is the classical wavelength.

In this section $E^a_A(\mathrm{cl})$ was chosen to start off with leading term $+1$. This choice, together with the gauge choice $e^Z_z = +1$, implies $\mathrm{sgn}(e) = +1$. To obtain the opposite choice, $\mathrm{sgn}(e) = -1$, change one E^b_B to $-E^b_B$. The new solution leaves the Weyl tensor unchanged and continues to satisfy the $\tilde{H} = 0$ constraint.

A. A second solution

Since we are dealing with second order difference equations, there should be a second solution, in addition to the solution given at Eq. (22). In the theory of second-order differential equations, the two series solutions around $z = 0$ have leading powers 1 and z. The first series fits the function at $z = 0$; the second series fits the first derivative.

By analogy, one would expect two solutions to the difference equation, with leading powers $B_0 = 1$ and $B_0 = (2\pi n/4q) := kn$. With this hint, plus

$$\delta^{(2)}[B_0 + B_1]/[B_0 + \cdots] = a \sin(kn),$$

one can construct a second solution. It has B_0 and B_1 terms

$$kn - a[(kn)\sin(kn) + 2\cos(kn)]/k^3 + \cdots. \tag{33}$$

This solution would be needed if the difference of E^a were nonzero at infinity. Since the difference vanishes, this solution can be ignored.

V. INCLUSION OF DAMPING

The solution Eq. (32) is infinite in length. The solution may be made into a packet by including damping factors.

$$\begin{aligned} E^x_X(\mathrm{LHF};n) &= (\Delta z \Delta y)\mathrm{sgn}(x) \\ &\quad \times \{1 - (a/2)\exp(\mp \rho n)\sin[kn \mp \phi] \\ &\quad + (-a^2/32)[\exp(\mp 2\rho n)\cos(2kn \mp 2\phi) \\ &\quad + (\exp[\mp 2\rho n] \pm 2\rho n - 1)(f^2/\rho^2)\cos\phi]\}; \\ f^2 &:= (k^2 + \rho^2). \end{aligned} \tag{34}$$

Upper (lower) sign refers to $n > 0$ ($n < 0$). For simplicity in what follows, The discussion to follow will consider only the case $n > 0$ (upper sign); the $n < 0$ follows by changing

$$\rho \to -\rho; \qquad \phi \to -\phi. \tag{35}$$

The expression for $E^y_Y(\mathrm{LHF};n)$ is Eq. (34) with $x \leftrightarrow y$ and $a \to -a$.

The exponential damping factors have discontinuities in derivative at $n = 0$; and from Eq. (35) the angle ϕ is undefined at $n = 0$. A discontinuity by itself is not a problem because the damping function is defined only at discrete points. The problems at $n = 0$ turn out to be minor; the value $n = 0$ is treated in Sec. V A.

If the curvature is to remain a sine wave with zero phase, then $\tilde{E}$ must include a constant phase ϕ. When one solves the differential equation $F = ma$ for the damped oscillator, one finds that each derivative shifts the phase by more than the usual $\pi/2$.

$$\begin{aligned} &(d/dt)[\exp(-\rho t)\sin(\omega t - \phi)] \\ &\quad = \sqrt{\omega^2 + \rho^2}\exp(-\rho t)\cos(\omega t - \phi + \psi); \\ &\cos\psi = \omega/\sqrt{\omega^2 + \rho^2}. \end{aligned} \tag{36}$$

Exactly the same phenomenon occurs in the difference case. One may choose a nonzero phase ϕ for $\tilde{E}$, ϕ to be determined. The differences shift this phase, until the curvature becomes a sine wave with zero phase.

Computation of the damped second difference is straightforward. As before, sinusoidal functions of $n \pm 1$ are expanded using trigonometric identities. As before, k is assumed small and functions $\sin k$, $\cos k$ are power-series expanded. A new feature: the damping parameter ρ is assumed small compared to wavelength,

$$\rho/k \ll 1,$$

so that functions $\exp(-\rho)$ may be power-series expanded, whenever ρ is not multiplied by n. Since $1/\rho$ measures the length of the packet, small ρ/k implies the packet contains many wavelengths.

The second difference of the term linear in a is

$$\begin{aligned} &(a/2)\exp(-\rho n)\{\sin(kn - \phi)[k^2 - \rho^2] \\ &\quad + 2k\rho\cos(kn - \phi)\}(\Delta Z)^2[1 + \text{order } k^2, \rho^2, k\rho]; \\ &n \neq 0. \end{aligned} \tag{37}$$

We choose ϕ so that the linear-in-a term (and ultimately, the curvature) collapses to a $\sin(kn)$ times a damping factor.

$$\begin{aligned} \cos\phi &= (k^2 - \rho^2)/f^2; \\ \sin\phi &= 2k\rho/f^2; \\ f^2 &= k^2 + \rho^2 \quad (n > 0). \end{aligned} \tag{38}$$

(For $n < 0$ one must replace ρ by $-\rho$.) The second difference of the linear-in-a term reduces to

$$(a/2)\exp(-\rho n)f^2\sin(kn)(\Delta Z)^2.$$

The term quadratic in a, Eq. (34), requires one extra trigonometric identity. After the usual expansions, that second difference becomes

$$\begin{aligned}&-(a^2/8)\exp(-2\rho n)\{-(k^2-\rho^2)\cos(2kn-2\phi)\\&\quad+(2\rho k)\sin(2kn-2\phi)+f^2\cos\phi\}(\Delta Z)^2\\&=-(a^2/8)f^2\exp(-2\rho n)\{-\cos(2nk-\phi)+\cos\phi\}(\Delta Z)^2\\&=-(a^2/4)f^2\exp(-2\rho n)\{\sin(kn-\phi)\sin(kn)\}(\Delta Z)^2.\end{aligned}\tag{39}$$

The last line uses the identity

$$2\sin A\sin B=\cos(A-B)-\cos(A+B).$$

One can now factor out

$$\begin{aligned}E^x_X&=(\Delta z\Delta y)\mathrm{sgn}(x)[1-(a/2)\exp(-\rho n)\sin(kn-\phi)]\\&\quad+\text{order }a^2\end{aligned}$$

from the total second difference. The final curvature contribution is then

$$\delta^{(2)}E^x_X(\mathrm{LHF};n)/E^x_X=(a/2)f^2\exp(-\rho n)\sin(kn);\quad n\neq0.\tag{40}$$

Again, there are no order a^2 corrections.

A. Curvature at $n = 0$

To dampen the discontinuities at $n = 0$, we assume the ratio ρ/k is small. This minimizes the discontinuity in the slope of the exponent $\exp(-\rho|n|)$ at $n = 0$, as well as the discontinuity in the phase ϕ. From Eq. (38), the leading terms in an expansion in powers of ρ/k are

$$\begin{aligned}\phi&=+2\rho/k+\cdots\quad(n>0);\\&=-2\rho/k+\cdots\quad(n<0).\end{aligned}\tag{41}$$

Since $\rho/\Delta z =$ order 1/(length of the packet) and $k/\Delta z = 2\pi$/wavelength, the ratio ρ/k gives an estimate of the number of wavelengths in the central, not strongly damped part of the packet.

$$\rho/k=\text{order wavelength}/(\text{packet length})\ll1.\tag{42}$$

The packet contains many wavelengths. The relative magnitudes are

$$\rho\ll k\ll2\rho/k=\phi.\tag{43}$$

Because of the discontinuity in ϕ, E^x_X at $n = 0$ is undefined. We parametrize it as

$$E^x_X(n=0)=(1+a_1+a_2)\Delta x\Delta z,\tag{44}$$

where a_p is of order a^p in the small amplitude a. The E^x_X at $n = \pm1, \pm2$ follow from Eq. (34).

$$\begin{aligned}&\{E^x_X(\pm2)=E^x_X(\pm1)\}\\&\qquad=\{1\pm a(\sin\phi)/2-(a^2/32)\cos(2\phi)\}\Delta x\Delta z.\end{aligned}$$

We have kept leading order in the smaller quantities ρ and k, and (temporarily) all orders in ϕ.

The a_i in Eq. (44) can be determined by requiring the order a^2 corrections to curvature to vanish, as at Eq. (28). The a_i contribute to curvature only at $n = \pm1$ and $n = 0$.

$$\begin{aligned}&\delta^{(2)}E^x_X(\pm1)/E^x_X(\pm1)\\&\quad=\{E^x_X(\pm2)-2E^x_X(\pm1)+E^x_X(0)\}/E^x_X(\pm1)\\&\quad=a_1+a_2+(1+a_1)[\mp a(\sin\phi)/2]\\&\qquad+a^2[\cos(2\phi)/32+(1/4)\sin^2\phi];\\&\delta^{(2)}E^x_X(0)/E^x_X(0)\\&\quad=\{E^x_X(+1)-2E^x_X(0)+E^x_X(-1)\}/E^x_X(0)\\&\quad=-a^2/16\cos(2\phi)-(a_1+a_2)+2(a_1)^2,\end{aligned}\tag{45}$$

to order a^2. Setting a^2 curvature terms to zero gives

$$\begin{aligned}a_1&=0;\\a_2&=-a^2/32,\end{aligned}\tag{46}$$

neglecting terms of second order in ϕ. The surviving contributions to curvature are now

$$\begin{aligned}\delta^{(2)}E^x_X/E^x_X(\pm1)&=\mp a(\sin\phi)/2;\\\delta^{(2)}E^x_X/E^x_X(0)&=0.\end{aligned}\tag{47}$$

The discontinuities in the triads are also minimized. Compare

$$\begin{aligned}E^x_X(0)&=1-a^2/32;\\E^x_X(\pm1)&=1\mp(a/2)\sin(k-\phi)-(a^2/32).\end{aligned}\tag{48}$$

In a theory so fundamentally discrete as LHF, some traditional notions of continuity may have to be abandoned; however, the above values for curvature and triad establish a smooth extrapolation through $n = 0$.

B. The ADM energy

The expression Eq. (34) contains undamped terms involving

$$2\rho|n| - 1.$$

The quantum triads diverge at infinity.

Initially, these divergent terms were included to make the solution analytic in ρ, in the limit $\rho \to 0$. With these terms included, the damped form also reduces correctly to the undamped form, Eq. (22).

These terms also have a fundamental significance, however. Because the rest of Eq. (34) is damped, these are the only terms which survive at large |n|, and therefore the only terms which contribute to the surface term in the Hamiltonian. Some terms must survive, or the ADM energy will vanish. The ADM energy is computed in Appendix A.

VI. COHERENT STATES

This problem requires both U(1) coherent states (for longitudinal holonomies, along z) and SU(2) coherent states (for transverse holonomies, along x and y). This difference [U(1) vs SU(2)] is a consequence of the initial gauge fixing which reduces the full $3+1$ dimensional problem to the planar problem. The connection reduces to 1×1 and 2×2 subblocks [12].

$$\mathrm{A}^Z_{x,y} = \mathrm{A}^{X,Y}_z = 0.$$

Longitudinal holonomies

$$\exp\left[i\int \mathrm{A}^Z_z \mathrm{S}_Z\right]$$

involve only A^Z_z and are U(1) rotations around Z. Transverse holonomies

$$\exp\left[i\int(\mathrm{A}^X_a\mathrm{S}_X + \mathrm{A}^Y_a\mathrm{S}_Y)\right], \qquad a = x, y$$

involve no S_Z (the axis of rotation lies in the XY plane) but are otherwise full SU(2) rotations.

Longitudinal coherent states are parametrized by a peak rotation angle and its conjugate variable, the component of angular momentum along z. The longitudinal coherent states have been constructed elsewhere [11], and will not be discussed here.

Construction of the transverse, SU(2) coherent states required an entire paper [2]. However, the basic structure of these states should not be surprising to anyone familiar with coherent states for a free particle. The next subsection reviews construction of the free particle coherent states. A follow-up section reviews the construction of the SU(2) coherent states, emphasizing the close parallel between the free particle and SU(2) cases.

A. The free particle analogy

The recipe for constructing a coherent state for the free particle starts from a wave function which is a delta function.

$$\delta(x - x_0) = \int \exp[ik(x - x_0)]dk/2\pi.$$

This wave function is certainly strongly peaked, but it is not normalizable. Also, it is peaked in position, but it needs to be peaked in both momentum and position. To make the packet normalizable, one inserts a Gaussian operator $\exp(-p^2/2\sigma^2)$. (Choosing the Gaussian form is a "cheat," because we know the answer; but for future reference note that all the eigenvalues k^2 of p^2 must be positive, so that the Gaussian damps for all k.) To produce a peak in momentum, we complexify the peak position: $x_0 \to x_0 + ip_0/\sigma^2$. With these changes, the packet becomes

$$\begin{aligned} \mathrm{N}&\int \exp[-p^2/(2\sigma^2)]\exp[ik(x - x_0) + kp_0/\sigma^2]dk/2\pi \\ &= \mathrm{N}\int \exp[-k^2/(2\sigma^2) + ik(x - x_0) + kp_0/\sigma^2]dk/2\pi \\ &= (\mathrm{N}\exp(p_0^2/2\sigma^2)/\sqrt{2\pi})\cdot \exp[ip_0(x - x_0) \\ &\quad - (x - x_0)^2\sigma^2/2]. \end{aligned} \tag{49}$$

The last line, which follows after completing the square on the exponential, exhibits the characteristic coherent state form.

There is not just one coherent state, but a family of coherent states, characterized by the parameter σ. The shape of the wave function is highly sensitive to σ; but the peak values (x_0, p_0) are independent of σ, as is the minimal uncertainty relation $\Delta x\Delta p = \hbar/2$. The coherent states constructed below contain a parameter t which is analogous to $1/\sigma^2$.

We now apply the above recipe to the SU(2) planar case. The free particle states are parametrized by peak values of two conjugate variables (x, p), whereas the SU(2) states are parametrized by peak values of conjugate angles and angular momentum. Both conjugate variables may be thought of as vectors, since the angles determine the rotation vector for the holonomy (directed along the axis of rotation, with magnitude the angle of rotation). Because angles are peaked, the holonomies are peaked. Because angular momentum is peaked, the $\tilde{\mathrm{E}}$ are peaked.

The first step in the recipe requires construction of a delta function (in angle, since angle is the new coordinate replacing position x). One might start from the simplest holonomy, which is

$$\begin{aligned} h^{(1/2)} &= \exp[i\hat{m}\cdot\vec{\sigma}\theta/2] \\ &= h^{(1/2)}(-\phi + \pi/2, \theta, \phi - \pi/2); \\ \hat{m} &= (\cos\phi, \sin\phi, 0). \end{aligned} \tag{50}$$

$h^{(1/2)}$ has rotation axis along $\hat{m}$, magnitude of rotation θ, and angular momentum $1/2$. A hat denotes a unit vector. $\hat{m}$ has no component along z because the gauge fixing has eliminated the $A^Z_{x,y}$. The middle line is the usual Euler angle decomposition. A complete set of rotation matrices on the group manifold (LHF) replaces the complete set of plane waves on the real line (free particle). The matrices have the same Euler angle decomposition as the simplest holonomy.

$$\begin{aligned}
&\delta(\chi-\beta)\delta(\theta-\alpha)\delta(\phi-\beta)/\sin(\alpha)\\
&\quad=\sum_{J,m,m'}((2J+1)/4\pi)\mathrm{D}^{(J)}(h)_{mm'}\mathrm{D}^{(J)}(u)^*_{mm'};\\
&\mathrm{D}(h)=\mathrm{D}(-\chi+\pi/2,\theta,\phi-\pi/2);\\
&\mathrm{D}(u)=\mathrm{D}(-\beta+\pi/2,\alpha,\beta-\pi/2). \qquad (51)
\end{aligned}$$

Note the need for an extra variable χ. Haar measure requires three angles for full orthogonality. If χ appeared in a final formula, presumably one should take the limit $\chi\to\beta$. Proceeding along these lines, one would arrive at states very similar to those constructed by Thiemann and Winkler for the general case of full local SU(2) symmetry [11,13,14].

The planar case, however, has an unusual holonomy-flux algebra based on anticommutators.

$$\{\mathrm{E}^a_A, h_a\} = i(\gamma\kappa/2)[\sigma_A/2, h_a]_+. \qquad (52)$$

The $\tilde{\mathrm{E}}$ are double grasp: they grasp both incoming and outgoing holonomies at the vertex. Because the transverse topology is S_1, one and the same holonomy is both incoming and outgoing. It is grasped on both sides, leading to the anticommutator. Note this seldom happens in the full SU(2) case, where the holonomy usually connects two different vertices. Even if an $\tilde{\mathrm{E}}$ is double grasp, an $\tilde{\mathrm{E}}$ can grasp only one side of a given holonomy. Equation (52) can be generalized to the higher spin holonomies $D(h)$ present in Eq. (51)

$$\begin{aligned}
&\{\mathrm{E}^a_A, D^{(J)}(h)_{mm'}\}\\
&\quad= i(\gamma\kappa/2)[\langle m|\mathrm{S}_A|n\rangle D_{nm'} + D_{mn}\langle n|\mathrm{S}_A|m'\rangle.] \qquad (53)
\end{aligned}$$

S_A is the $2J+1$ dimensional representation of the rotation generator.

Presumably an approach based on Eqs. (51) and (53) would work. We did not pursue that approach, because it is possible to construct a more convenient basis where χ is not needed and a grasp produces only a single rotation generator, rather than the two of Eq. (53) [15].

To discover the new basis, we note that the basic anticommutator, Eq. (52), maps the three matrix elements of h into themselves. (There are only three independent elements of h, not four. Because the axis of rotation lies in the XY plane, the two diagonal elements of h are equal.) The action of the $\tilde{\mathrm{E}}$ on the three h is isomorphic to the action of the generators of the rotation group O(3) on the three-dimensional representation of O(3), the spherical harmonic Y^M_L with $\mathrm{L}=1$.

The matrix elements of h happen to be proportional to spherical harmonics, although spherical harmonics with unusual, half-angle angular dependence. From Eq. (50),

$$\begin{aligned}
(\mathcal{N}/\sqrt{2})h_{\mp,\pm} &= \mp\mathcal{N}\sin(\theta/2)\exp[\pm(i\phi-i\pi/2)]/\sqrt{2}\\
&= Y^{\pm}_1(\theta/2,\phi-\pi/2);\\
\mathcal{N}h_{++} &= \mathcal{N}h_{--} = \mathcal{N}\cos(\theta/2)\\
&= \mathrm{Y}^0_1(\theta/2,\phi-\pi/2).
\end{aligned}$$

The subscripts on h abbreviate the spin values; e.g. h_{+-} is the element in row $m=+1/2$ and column $m=-1/2$. It is interesting that the h are proportional to spherical harmonics; but the essential feature is that grasps of triads map (h_{-+},h_{++},h_{+-}) into h in the same way that the O(3) generators S_A map $(\mathrm{Y}^+_1,\mathrm{Y}^0_1,\mathrm{Y}^-_1)$ into Y_1. For a sample anticommutator calculation which illustrates this mapping, see appendix C. The unconventional half-angle $\theta/2$ is a reminder that the Y's are constructed from holonomies $h^{(1/2)}$ depending on a half-angle.

To obtain higher spin representations of this O(3) symmetry, one couples together $\mathrm{L}=1$ representations in the usual manner to form the $\mathrm{L}>1$ representations Y^M_L. The action of the $\tilde{\mathrm{E}}$ is given by a *single* matrix element of an O(3) generator.

$$\begin{aligned}
(\gamma\kappa/2)^{-1}\mathrm{E}^x_{\pm}\mathrm{Y}^M_L &= \Sigma_N \mathrm{Y}_{LN}\langle L,N|\mathrm{S}_{\pm}|L,M\rangle;\\
f_{\pm} &:= (f_x \pm i f_y)/\sqrt{2}. \qquad (54)
\end{aligned}$$

$\mathrm{Y}_{LM}=\mathrm{Y}_{LM}(\theta/2,\phi-\pi/2)$. For $\mathrm{L}=1$ the Y_{1M} reduce to matrix elements of h, and the anticommutator Eq. (52) gives the expansion on the right-hand side of Eq. (54).

Equation (54) gives the two $\tilde{\mathrm{E}}$ isomorphic to $\mathrm{S}_{\pm}$. What is the operator isomorphic to S_0? It cannot be E^x_Z since that field has been gauged to zero. If one applies the commutator of the $\mathrm{E}^x_{\pm}$ to h, one finds an operator

$$(\gamma\kappa/2)^{-1}\mathrm{E}^a_0 h_a := [h_a,\sigma_z/2]_-. \qquad (55)$$

Note the commutator. One can verify directly that this commutator is isomorphic to the action of S_0: the diagonal elements of h (isomorphic to Y_0) are mapped into zero; off-diagonal elements (isomorphic to $\mathrm{Y}_{\pm}$) are multiplied by factors of $\pm1/2$. The action on a general Y is Eq. (54) with $\mathrm{S}_{\pm}$ replaced by S_0.

$$(\gamma\kappa/2)^{-1}\mathrm{E}^x_0\mathrm{Y}^M_L = M\mathrm{Y}^M_L. \qquad (56)$$

When the O(3) symmetry is taken into account, the formula for the delta function is

$$\delta(\theta/2-\alpha/2)\delta(\phi-\beta)/\sin(\alpha/2) = \sum_{L,M} \mathrm{Y}_{LM}(h)\mathrm{Y}_{LM}(u)^*;$$
$$\mathrm{Y}(h) := \mathrm{Y}(\theta/2, \phi-\pi/2);$$
$$\mathrm{Y}(u) := \mathrm{Y}(\alpha/2, \beta-\pi/2). \tag{57}$$

This delta function may also be expressed in terms of rotation matrices, since Y is just a rotation matrix.

$$\mathrm{Y}_{LM}(u) = \sqrt{(2\mathrm{L}+1)/4\pi}\,\mathrm{D}^{(L)}_{0M}(-\beta+\pi/2, \alpha/2, \beta-\pi/2). \tag{58}$$

The axis of rotation for u must lie in the xy plane, since u is the peak value of h. This dictates the Euler angle decomposition.

It is useful to compare old and new delta functions, Eqs. (51) and (57). The two expansions use different basis functions. For example, for $\mathrm{D}(u)$,

$$\{\mathrm{D}^{(J)}_{mm'}(-\beta+\pi/2, \alpha, \beta-\pi/2)\}$$
$$\to \{\mathrm{D}^{(L)}_{0M}(-\beta+\pi/2, \alpha/2, \beta-\pi/2)\}.$$

The sum over L excludes the half integers, because of the shift from SU(2) to O(3). The $\mathrm{D}^{(J)}(u)$ are products of the matrix elements of a basic $j=1/2$, SU(2) matrix $u^{(1/2)}(\alpha,\beta)$. The $\mathrm{D}^{(L)}$ are products of the same matrix elements $u^{(1/2)}(\alpha,\beta)$, but arranged so as to form $\mathrm{D}^{(1)}$, the $\mathrm{L}=1$ representation of O(3). The new basis harmonics $\mathrm{D}^{(L)}$ depend on the half angle $\alpha/2$ rather than α, but they have the same axis of rotation (same β) as the old basis. Because $\mathrm{D}^{(1)}$ is just a reshuffling of $u^{(1/2)}(\alpha,\beta)$, one could label a coherent state using the symbol u, rather than the more awkward $\mathrm{D}^{(1)}$; and we will generally do this.

We continue with the recipe for constructing the coherent state: we dampen the sum using a Gaussian

$$\exp[-tL(L+1)/2].$$

The parameter t is the analog of the parameter $1/\sigma^2$ in the free particle case. We complexify by extending the angles in u to complex values, replacing u by a matrix g in the complex extension of O(3). The coherent state has the general form

$$|u,\vec{p}\rangle = \mathrm{N}\sum_{L,M} \exp[-tL(L+1)/2]\mathrm{Y}(h)_{LM}\mathrm{Y}(g)^*_{LM}. \tag{59}$$

Every matrix g in the complex extension of O(3) can be decomposed into a product of a Hermitian matrix times a unitary matrix (polar decomposition; see for example [16]).

$$g = \text{Hermitian } x \text{ unitary}. \tag{60}$$

It is useful to compare this to the free particle case, where the complexification is also a product of factors. The "matrices" in that case are 1×1.

$$\exp[-ikx_0] \to \exp[-ikx_0 + kp_0/\sigma^2].$$

Here, $\exp[-ikx_0]$ plays the role of the unitary factor. The free particle analogy suggests that the Hermitian factor should produce the damping and should contain a vector related to (angular) momentum.

There are a lot of matrices in the complex extension. Some trial and error is needed to obtain the desired peak properties. The natural first choice for the unitary factor in Eq. (60) is u, the value of g in the limit the damping disappears (Hermitian matrix $\to$ 1). This choice leads to the simplest proofs.

The Hermitian factor ($:= \mathcal{H}$) may be parametrized by a vector $\vec{p} = p\hat{p}$. In the $\mathrm{L}=1$ representation,

$$g = \mathcal{H}u = \exp(\vec{\mathrm{S}}^{(1)}\cdot\vec{p})u. \tag{61}$$

The vector $\vec{p}$ gives the matrix $\mathcal{H}$ an axis $\hat{p}$, analogous to axes $\hat{m}$ and $\hat{n}$ for matrices h and u.

The higher order representations Y_{LM} may be complexified similarly.

$$\mathrm{D}^{(L)}_{0M}(u) \to \exp[\vec{\mathrm{S}}^{(L)}\cdot\vec{p}]_{0R}\mathrm{D}^{(L)}_{RM}(u) = \mathcal{H}^{(L)}u^{(L)}. \tag{62}$$

This formula replaces the Y's by the corresponding rotation matrices, in order to clarify the matrix multiplication.

$\mathcal{H}^{(L)}$ is expected to diverge as $\exp(p\mathrm{L})$ for large L, because of its $\exp[\vec{\mathrm{S}}\cdot\vec{p}]$ form. We multiply this exponential by the damping factor:

$$\exp[-tL(L+1)/2]\exp[pL]$$
$$= \exp\{-(t/2)[L+1/2-p/t]^2 + f(t,p)\}. \tag{63}$$

The exponent has a maximum at an $\langle \mathrm{L}\rangle$ given by

$$\langle L\rangle + 1/2 = p/t.$$

The $1/2$ looks a bit peculiar until one realizes

$$\sqrt{L(L+1)} \cong \mathrm{L}+1/2.$$

Evidently the coherent states tend to maximize $\sqrt{\mathrm{L}(\mathrm{L}+1)}$ rather than L. Usually the $1/2$ will be dropped.

All three axes of rotation are assumed to lie in the xy plane: $\hat{p}$, $\hat{m}$, and $\hat{n}$ for $\mathcal{H}$, h, and u respectively.

$$\hat{p} = (\cos(\beta+\mu), \sin(\beta+\mu), 0);$$
$$\hat{m} = (\cos\phi, \sin\phi, 0);$$
$$\hat{n} = (\cos\beta, \sin\beta, 0). \tag{64}$$

μ is the angle between the peak axis of rotation $\hat{n}$ and $\hat{p}$. Of course the axis of u should lie in the XY plane, because u is the peak value of h, and the axis of h is in the XY plane.

Placing $\hat{p}$ in the xy plane is a bit worrisome, because it seems to suggest the angular momentum is restricted to the xy plane. However, we shall see in the next section that the angular momentum is not along $\hat{p}$ but rather along $\hat{p}$ rotated through u.

B. Basic matrix elements

$$
\begin{aligned}
&(2/\gamma\kappa)\mathrm{E}^a_A(\mathrm{LHF})|u(n),\vec{p}(n)\rangle \\
&\qquad = \langle L_a(n)\rangle \hat{p}_B \mathrm{D}^{(1)}(u)_{BA}|u,\vec{p}\rangle + \text{order}\sqrt{\langle L\rangle};\\
&\langle L_a(n)\rangle = p_a(n)/t;\\
&\hat{\mathbf{h}}|u,\vec{p}\rangle = \mathrm{i}\boldsymbol{\sigma}\cdot\hat{n}\sin(\alpha/2)|u,\vec{p}\rangle + \text{order}\sqrt{\langle L\rangle};\\
&\bar{\mathbf{h}}|u,\vec{p}\rangle = \mathbf{1}\cos(\alpha/2)|u,\vec{p}\rangle + \text{order}\sqrt{\langle L\rangle}.
\end{aligned} \tag{65}
$$

There are two transverse directions, $a = x, y$. Therefore each of the above equations is actually two equations, one for x and one for y. The brackets around $\langle \mathrm{L}\rangle$ are of course designed to distinguish the peak value from the variable L which is summed over in e.g. Eq. (57).

The direction of angular momentum is given by a rotated version of $\hat{p}$ (first line). The last two lines give the matrix elements for the two parts of the holonomy.

$$
\begin{aligned}
\hat{\mathbf{h}} &= (\mathbf{h} - \mathbf{h}^{-1})/2;\\
\bar{\mathbf{h}} &= (\mathbf{h} + \mathbf{h}^{-1})/2.
\end{aligned}
$$

Only $\hat{\mathbf{h}}$ occurs in the small sine Hamiltonian. The explicit dependence of $\hat{p}_B \mathrm{D}^{(1)}(u)_{BA}$ on the angles μ, α, β will be derived at a later point, Eq. (74) of Sec. VII. For completeness, here is the result.

$$
\begin{aligned}
\hat{P}^a_A(\alpha) &\coloneqq \hat{p}^a_B \mathrm{D}^{(1)}(u_a)_{BA} = \cos\mu_a(\hat{n}_a)_A\\
&\quad + \sin\mu_a[\cos(\alpha_a/2)(\hat{Z}\times\hat{n}_a)_A + \sin(\alpha_a/2)\hat{Z}].\\
\hat{n}_a &= (\cos\beta_a, \sin\beta_a, 0); \qquad a = x, y.
\end{aligned}
$$

u, the peak value of the holonomy, is a rotation through $\alpha/2$ around the axis $\hat{n}$. μ is the angle between $\hat{n}$ and $\hat{p}$.

There are now two $\hat{p}$ vectors. The original $\hat{p}$, introduced in Eq. (61), characterizes the complex extension of O(3), and lies in the XY plane. The new $\hat{P}$, just introduced, is the original $\hat{p}$ after a rotation by $\mathrm{D}^{(1)}$ (rotation through $\alpha/2$ around axis along $\hat{n}$). From Eq. (65), the new $\hat{P}$ gives the direction of angular momentum.

Longitudinal matrix elements resemble the transverse ones.

$$
\begin{aligned}
&(2/\gamma\kappa)\mathrm{E}^z_Z(\mathrm{LHF})|\langle\theta_z\rangle,\langle m_Z\rangle\rangle = \langle m_Z\rangle|\langle\theta_z\rangle,\langle m_z\rangle\rangle;\\
&\hat{\mathbf{h}}_{\mathbf{z}}|\langle\theta_z\rangle,\langle m_z\rangle\rangle = \mathrm{i}\boldsymbol{\sigma}_z\sin(\langle\theta_z\rangle/2)|\langle\theta_z\rangle,\langle m_z\rangle\rangle;\\
&\bar{\mathbf{h}}_{\mathbf{z}}|\langle\theta_z\rangle,\langle m_z\rangle\rangle = \mathbf{1}\cos(\langle\theta_z\rangle/2)|\langle\theta_z\rangle,\langle m_z\rangle\rangle.
\end{aligned} \tag{66}
$$

Again, each Eq. (66) is really a pair of equations. If the holonomy is outgoing (respectively, incoming), then the peak angle is labeled $\theta_z(n, n+1)$ [respectively, $\theta_z(n-1, n)$], and the peak z component of angular momentum is m_f (respectively, m_i).

Table I lists the various parameters occurring in the coherent state, together with a brief definition. Occasionally, where there is no danger of confusion, the parameters will be written without their characteristic transverse label $a = x$ or y.

C. The Δx^i should be simple

The LHF formulas for the triads will contain factors of Δx^i. These parameters are largely arbitrary, and to keep formulas simple, We choose them to be positive and independent of n_z. We respect the symmetry by choosing

TABLE I. Parameters occurring in the coherent state. $a =$ x or y.

Parameter	Definition
$u_a^{(1/2)}$	Peak value of $\mathrm{j} = 1/2$ transverse SU(2) holonomy $h_a^{1/2}$
$\mathrm{D}^{(1)}(u_a)$	Peak value of $\mathrm{L} = 1$ O(3) holonomy $\mathrm{D}^{(1)}_{0M}$; matrix elements proportional to matrix elements of $u_a^{(1/2)}$
$\hat{n}_a$	Axis of rotation for $\mathrm{D}^{(1)}(u_a)$ and $u_a^{(1/2)}$; lies in XY plane
α_a, β_a	$u^a = u^a(\alpha_a, \beta_a)$; $\alpha_a/2 =$ angle of rotation around $\hat{n}_a$; $\beta_a =$ angle between $\hat{n}_a$ and X axis
$\mathrm{D}^{(L)}(u_a)_{0M}$	O(3) rotation matrix, a product of L copies of u_a; rotation angle $\alpha_a/2$; axis of rotation $\hat{n}_a$
$\vec{p}^a$	Vector in XY plane characterizing the complex rotation $\exp[\vec{\mathrm{S}}\cdot\vec{p}]$ multiplying each term in the coherent superpostion; $\vec{p}^a = p^a\hat{p}^a$
$\hat{P}^a(\alpha_a)$	$= \hat{p}^a_B \mathrm{D}^{(1)}(u_a)_{BA}$. $\hat{p}^a$ after rotation through $\alpha/2$; gives direction of angular momentum $\vec{\mathrm{L}}^a$
μ_a	Angle between $\hat{p}^a$ and $\hat{n}^a$
M^a	Peak value of Z component of transverse angular momentum
m_f, m_i	Peak value of Z component of angular momentum carried by Z axis holonomies entering (m_i) or leaving (m_f) vertex n
$\langle\theta_z\rangle$	Peak value of angle for the Z axis holonomy

$$\Delta x = \Delta y.$$

Global and local Lorentz coordinates x^i and X^I are related by

$$x^i = X^I e^i_I = X^I \mathrm{E}^i_I/|e|.$$

If the two coordinates increase in opposite directions, then the corresponding $\mathrm{E}^i_I(\mathrm{cl})$ has leading term -1 and $\Delta X^i/\Delta x^i$ is negative. Since the Δx^i have been chosen always positive,

$$\Delta X^I = |\Delta X^I|\mathrm{sgn}(i);$$
$$\mathrm{E}^i_I(\mathrm{FT}) = \mathrm{sgn}(i) + \cdots. \tag{67}$$

$\mathrm{sgn}(i)$ is the sign of E^i_I and e^i_I.

In the present gauge ($e^Z_z = \pm 1$), $\Delta Z = \pm\Delta z$. Only the $\Delta X, \Delta Y$ can vary with n_z; ΔZ is a constant.

VII. DETERMINING THE COHERENT STATE PARAMETERS

The Hamiltonian, Eq. (18), is correct for one specific set of gauge conditions and symmetries. However, the coherent states just constructed above are general. They are not gauge-fixed and do not reflect the symmetries. Imposition of symmetries and gauges determines the peak values u_a and $\vec{p}^a$.

The states must obey nine constraints: four single polarization constraints (which constrain the four off-diagonal transverse $\tilde{\mathrm{E}}$ and transverse K to vanish); two unidirectional constraints; two diffeomorphism constraints; and the Gauss constraint.

A coherent state "obeys" a constraint when the peak values satisfy the constraint. The state is usually not an eigenfunction of the constraint.

Equation (65) expresses the basic quantities $\tilde{\mathrm{E}}$ and $\hat{h}$ in terms of coherent state parameters. The Hamiltonian depends only on $\tilde{\mathrm{E}}$; but the constraints depend on extrinsic curvature K and spin connection Γ as well. The next section relates K and Γ to the basic quantities.

A. K and Γ

From paper I, the connection A becomes $-2i\hat{h}^I_i$ in the small sine (SS) limit. K becomes

$$\begin{aligned}\gamma \mathrm{K}^I &= \mathrm{A}^I - \Gamma^I(\mathrm{FT})\\ &\to -2i\hat{h}^I - \Gamma^I(\mathrm{SS})\\ &= 2\hat{n}^I \sin(\alpha/2) - \Gamma^I.\end{aligned} \tag{68}$$

The last line expresses $\hat{h}$ in terms of the peak values for the angle of rotation α, and axis of rotation

$$\hat{n} = (\cos\beta, \sin\beta, 0).$$

For longitudinal fields, the $\hat{h}(n)$ on the second line of Eq. (68) is replaced by the average of the two z holonomies at vertex n:

$$\hat{h}_z(n) \coloneqq [\hat{h}_z(n, n+1) + \hat{h}_z(n-1, n)]/2,$$

where $\hat{h}(\mathrm{n}, \mathrm{n}+1)$ is the holonomy on edge $(n, n+1)$.

Now consider Γ. From [1] the products $\Gamma \cdot \mathrm{E}$ are given by

$$\begin{aligned}\Gamma^Y_x \mathrm{E}^x_X + \Gamma^X_y \mathrm{E}^y_Y &= [\delta_{(c)}\mathrm{E}^y_Y/\mathrm{E}^y_Y - \delta_{(c)}\mathrm{E}^x_X/\mathrm{E}^x_X]\mathrm{E}^z_Z;\\ \Gamma^Y_x \mathrm{E}^x_X - \Gamma^X_y \mathrm{E}^y_Y &= \delta_{(c)}\mathrm{E}^z_Z.\end{aligned} \tag{69}$$

In the present gauge we may use Eq. (16) to replace $\delta_{(c)}\mathrm{E}^z_Z$ on the last line by

$$[\delta_{(c)}\mathrm{E}^x_X/\mathrm{E}^x_X + \delta_{(c)}\mathrm{E}^y_Y/\mathrm{E}^y_Y]\mathrm{E}^z_Z,$$

then solve for the individual $\Gamma \cdot \mathrm{E}$.

$$\begin{aligned}\Gamma^X_y \mathrm{E}^y_Y &= -\delta_{(c)}(\mathrm{E}^x_X)\mathrm{E}^y_Y/\mathrm{C(LHF)};\\ \Gamma^Y_x \mathrm{E}^x_X &= +\delta_{(c)}(\mathrm{E}^y_Y)\mathrm{E}^x_X/\mathrm{C(LHF)};\\ \mathrm{C(LHF)} &= \mathrm{sgn}(e)(\Delta z)^2.\end{aligned} \tag{70}$$

The two Γ in Eq. (70) are the only ones which occur in the constraints. The single polarization constraints force all other Γ to vanish.

All nonbasic variables (K, Γ) are now expressed in terms of basic variables ($\hat{h}$, $\tilde{\mathrm{E}}$). The latter in turn have been expressed in terms of coherent state parameters at Eq. (65).

B. Evaluation of the β_a

It is a bit easier to work with the combinations $\mathrm{U}_1 \pm \mathrm{U}_3$ of unidirectional constraints from paper I. Using Eq. (68), the K's may be replaced by combinations of the $\tilde{\mathrm{E}}$ and holonomies, quantities with known action on coherent states.

$$\begin{aligned}0 &= [\mathrm{K}^Y_y \mathrm{E}^y_Y + \mathrm{E}^z_Z\delta_{(c)}(\mathrm{E}^x_X)/\mathrm{E}^x_X]/\sqrt{\mathrm{E}^z_Z}\\ &= \{2\sin\beta_y \sin(\alpha_y/2)/\gamma + \delta_{(c)}(\mathrm{E}^x_X)/\mathrm{C(LHF)}\}\mathrm{E}^y_Y/\sqrt{\mathrm{E}^z_Z};\\ 0 &= [\mathrm{K}^X_x \mathrm{E}^x_X + \mathrm{E}^z_Z\delta_{(c)}(\mathrm{E}^y_Y)/\mathrm{E}^y_Y]/\sqrt{\mathrm{E}^z_Z}\\ &= [2\cos\beta_x \sin(\alpha_x/2)/\gamma + \delta_{(c)}(\mathrm{E}^y_Y)/\mathrm{C(LHF)}]\mathrm{E}^x_X/\sqrt{\mathrm{E}^z_Z}.\end{aligned} \tag{71}$$

Similarly, the single polarization constraints $\mathrm{K}^X_y = \mathrm{K}^Y_x = 0$ may be expressed in terms of the $\tilde{\mathrm{E}}$ and holonomies, using Eqs. (68) and (70).

$$\begin{aligned} 0 &= \gamma \mathrm{K}_y^X \\ &= 2\cos\beta_y \sin(\alpha_y/2) + \delta_{(c)}(\mathrm{E}_X^x)/\mathrm{C(LHF)}; \\ 0 &= \gamma \mathrm{K}_x^Y \\ &= 2\sin\beta_x \sin(\alpha_x/2) - \delta_{(c)}(\mathrm{E}_Y^y)/\mathrm{C(LHF)}. \end{aligned} \tag{72}$$

The unidirectional constraints have an additional $\tilde{\mathrm{E}}/\sqrt{\mathrm{E}_Z^z}$ on the right. However, this additional factor merely produces a constant, when acting on a coherent state. Therefore this factor may be commuted to the left. The two sets of constraints, unidirectional and single polarization, arise from different physical effects, but display similar mathematical structure. Unidirectional constraints are linear in the $\delta_{(c)}$E because time plus space derivatives equal zero, and $\delta_{(c)}$E supplies the space derivative. The single polarization constraints K = 0 are linear in $\delta_{(c)}$E because the K's are holonomy minus spin connection, and the spin connection is linear in $\delta_{(c)}$E. If one eliminates the $\delta_{(c)}$E, one finds the first two equations below:

$$\begin{aligned} \tan\beta_x &= -1/\gamma; \\ \tan\beta_y &= +\gamma; \\ \cos\beta_x &= \mathrm{sgn}(\hat{n}_x)\gamma/\sqrt{1+\gamma^2}; \\ \sin\beta_x &= -\mathrm{sgn}(\hat{n}_x)1/\sqrt{1+\gamma^2}; \\ \cos\beta_y &= \mathrm{sgn}(\hat{n}_y)/\sqrt{1+\gamma^2}; \\ \sin\beta_y &= \mathrm{sgn}(\hat{n}_y)\gamma/\sqrt{1+\gamma^2}. \end{aligned} \tag{73}$$

The remaining four equations follow by solving the first two for cosine and sine. There is a sign ambiguity: $\mathrm{sgn}(\hat{n}_a) = \pm 1$, $\hat{n}_a$ the axis of rotation for the peak holonomy u_a, because the first two lines determine β_a only mod π. Since β_a is the angle between $\hat{n}_a$ and the X axis, $\hat{n}_a$ is determined only up to an overall sign (equivalently, only up to a reflection through the origin). Independent of signs, the two rotation axes $\hat{n}_x$ and $\hat{n}_y$ are 90° apart. This can be seen by computing

$$\tan\beta_x = -1/\tan\beta_y = \tan(\beta_y \pm \pi/2).$$

The unidirectional and single polarization constraints are now equivalent. One can drop the unidirectional constraints and focus on the single polarization constraints; the number of independent constraints has dropped to seven.

C. Evaluation of the μ_a

We can determine the μ_a from Eq. (65). μ_a occurs in $\hat{p}^a$, and Eq. (65) relates the $\tilde{\mathrm{E}}$ to $\hat{P}^a$, the rotated version of $\hat{p}^a$. First we need an explicit expression for $\hat{P}^a$. In order to study the rotation of $\hat{p}^a$ into $\hat{P}^a$, it is convenient to use as basis $\hat{n}$, plus two vectors perpendicular to $\hat{n}$, since that vector is invariant under rotations. For arbitrary polarization,

$$\begin{aligned} \hat{p}^a &= \cos\mu_a \hat{n}^a + \sin\mu_a \hat{Z} \times \hat{n}^a; \\ \hat{P}^a &= \cos\mu_a \hat{n}^a + \sin\mu_a[\cos(\alpha_a/2)\hat{Z} \\ &\quad \times \hat{n}^a + \sin(\alpha_a/2)\hat{Z}]. \end{aligned} \tag{74}$$

Proof: The unrotated $\hat{p}^a$ lies in the xy plane, and therefore has components along $\hat{n}^a$ (rotation axis for u, so also in the xy plane) and $\hat{Z} \times \hat{n}^a$. The angle between $\hat{p}$ and $\hat{n}$ is μ, which gives the first line of Eq. (74). After $\hat{p}$ is rotated through $\alpha/2$ around axis $\hat{n}$, the angle between $\hat{n}$ and $\hat{p}$ remains μ, which explains the $\hat{n}$ term on the second line. The vector $\hat{Z} \times \hat{n}^a$ becomes the square bracket on the second line. □

When Eq. (74) is substituted into the first Eq. (65), we obtain four equations which may be written in matrix form.

$$\begin{pmatrix} \mathrm{E}_X^a/\Delta x \Delta z \\ \mathrm{E}_Y^a/\Delta x \Delta z \end{pmatrix} = (\langle L_a(n)\rangle/L_0) \times \begin{bmatrix} \cos\beta_a & -\sin\beta_a\cos(\alpha_a/2) \\ \sin\beta_a & \cos\beta_a\cos(\alpha_a/2) \end{bmatrix} \times \begin{pmatrix} \cos\mu_a \\ \sin\mu_a \end{pmatrix}, \tag{75}$$

where $a = x, y$ and $\mathrm{L}_0 = (2/\gamma\kappa)\Delta x \Delta z$.

$\cos(\alpha_a/2)$ is near 1, since α is small.

$$\cos(\alpha_a/2) = 1 - \sin(\alpha_a/2)^2/2 + \mathrm{O}\ \sin(\alpha_a/2)^4.$$

In the following section we shall see that $\sin(\alpha_a/2)$ is order a in the small amplitude a. Initially we replace the cosine by unity, and obtain the μ correct to order a. The matrix on the right in Eq. (75) becomes orthogonal and can be evaluated using Eq. (73) for the β_a. The vector on the left becomes

$$\mathrm{sgn}(x)\begin{pmatrix} 1 - E_1 \\ 0 \end{pmatrix} \quad \text{or} \quad \mathrm{sgn}(y)\begin{pmatrix} 0 \\ 1 + E_1 \end{pmatrix},$$

for $a = x$ or y respectively. To suppress distracting detail, we have abbreviated the order a parts of $\tilde{\mathrm{E}}$ by E_1. Inversion of Eq. (75) yields

$$(\langle L_x\rangle/L_0)\cos\mu_x(0)=\mathrm{sgn}(x)\mathrm{sgn}(\hat{n}_x)\left(\gamma/\sqrt{1+\gamma^2}\right)(1-\mathrm{E}_1);$$
$$(\langle L_x\rangle/L_0)\sin\mu_x(0)=\mathrm{sgn}(x)\mathrm{sgn}(\hat{n}_x)\left(1/\sqrt{1+\gamma^2}\right)(1-\mathrm{E}_1);$$
$$(\langle L_y\rangle/L_0)\cos\mu_y(0)=\mathrm{sgn}(y)\mathrm{sgn}(\hat{n}_y)\left(1/\sqrt{1+\gamma^2}\right)(1+\mathrm{E}_1);$$
$$(\langle L_y\rangle/L_0)\sin\mu_y(0)=\mathrm{sgn}(y)\mathrm{sgn}(\hat{n}_y)\times\left(-\gamma/\sqrt{1+\gamma^2}\right)(1+\mathrm{E}_1)+\mathrm{O}\,a^2. \quad (76)$$

The (0) indicates zero order in a. This is not quite the whole story, however, because we have not yet evaluated the $\langle \mathrm{L}_a\rangle$. If we impose $(\cos\mu_a)^2+(\sin\mu_a)^2=1$, we find

$$(\langle L_x\rangle/L_0)=1-\mathrm{E}_1;$$
$$(\langle L_y\rangle/L_0)=1+\mathrm{E}_1+\mathrm{O}\ a^2. \quad (77)$$

The E_1 dependence cancels out of Eq. (76); the μ_a are constants to order a^2.

$$\cos\mu_x(0)=\mathrm{sgn}(x)\mathrm{sgn}(\hat{n}_x)\left(\gamma/\sqrt{1+\gamma^2}\right);$$
$$\sin\mu_x(0)=\mathrm{sgn}(x)\mathrm{sgn}(\hat{n}_x)\left(1/\sqrt{1+\gamma^2}\right);$$
$$\cos\mu_y(0)=\mathrm{sgn}(y)\mathrm{sgn}(\hat{n}_y)\left(\gamma/\sqrt{1+\gamma^2}\right);$$
$$\sin\mu_y(0)=\mathrm{sgn}(y)\mathrm{sgn}(\hat{n}_y)\left(1/\sqrt{1+\gamma^2}\right)+\mathrm{O}\ a^2; \quad (78)$$

The $\mathrm{sgn}(a)\mathrm{sgn}(\hat{n}_a)$ factors mean that $\hat{p}_a$, like $\hat{n}_a$, is determined only up to a reflection through the origin. (Equivalently, μ_a, the angle between $\hat{p}_a$ and $\hat{n}_a$, is determined only up to π.)

One may choose $\langle \mathrm{L}_a\rangle$ any convenient size by adjusting the Δx_a in $\mathrm{L}_0=(2/\gamma\kappa)\Delta x\Delta z$. Equation (77) is a precise statement of a scaling behavior: the amplitudes L_a scale with the Δx_a.

To zeroth order in a, $\hat{p}_x$ lies along the X axis, while $\hat{p}_y$ lies along the Y axis. This may be seen from

$$\tan\mu_x(0)=-\tan\beta_x;$$
$$\tan\mu_y(0)=1/\tan\beta_y=-\tan(\beta_y\pm\pi/2),$$

which implies

$$\mu_x(0)=-\beta_x \mathrm{mod}\,\pi; \qquad \mu_y(0)=-(\beta_y\pm\pi/2)\mathrm{mod}\,\pi.$$

$\hat{p}_a$ makes an angle $\mu_a+\beta_a$ with the X axis. Hence $\hat{p}_x$ is along X and $\hat{p}_y$ is along Y. □

Actually, $\hat{p}_y$ (for example) is not *exactly* along the Y axis; it has a small X component of order a^2. If one solves Eq. (75) exactly, without expanding $\cos(\alpha_a/2)$, one finds

$$\sin\mu_a\cos(\alpha_a/2)=\cos\mu_a/\gamma. \quad (79)$$

One can expand each of the factors in Eq. (79).

$$\mu_a=\mu_a(0)+\mu_a(2); \qquad \cos(\alpha_a/2)=1-\sin^2(\alpha_a/2)/2,$$

correct to order a^2. The solution for $\mu(2)$ is

$$\mu(2)=\sin^2(\alpha_a/2)/2[\gamma/(1+\gamma^2)];$$
$$\cos\mu(0)\to\cos[\mu(0)+\mu(2)]$$
$$=\cos\mu(0)-\sin\mu(0)\mu(2)+\mathrm{O}\ a^4. \quad (80)$$

The second line shows a sample correction to the formulas of Eq. (78). The next section calculates $\sin(\alpha_a/2)$, which is found to be order a and oscillating, with $|\sin(\alpha_a/2)|$ independent of a. Therefore $\mu(2)$ is order a^2 and oscillating; and it needs no x, y subscript.

The nonzero transverse components of rotated $\hat{P}^a$ and unrotated $\hat{p}^a$ can now be written to order a^2.

$$\hat{P}^a_A=\mathrm{sgn}(a)\cos[\mu(0)+\mu(2)]/\cos\mu(0)]; \qquad a=A;$$
$$\hat{p}^x=\{\cos[0\ \mathrm{mod}\ \pi+\mu(2)],\sin[0\ \mathrm{mod}\ \pi+\mu(2)]\};$$
$$\hat{p}^y=\{\cos[\pi/2\ \mathrm{mod}\ \pi+\mu(2)],\sin[\pi/2\ \mathrm{mod}\ \pi+\mu(2)]\}, \quad (81)$$

i.e., apart from correction factors of order a^2, transverse $\hat{P}^a$ and $\hat{p}$ are unit vectors along $\hat{A}$.

Note the quantity $\mathrm{sgn}(\vec{n}^a)$ drops out of the physically significant result, the direction of transverse angular momentum $\hat{P}^a_A$, a = A. Each factor in the expression for transverse momentum is a product of a trigonometric function of β times a trigonometric function of μ. Since both sets of trig functions contain $\mathrm{sgn}(\vec{n}^a)$, the sign occurs squared. This is the first example of a general feature: $\mathrm{sgn}(\vec{n}^a)$ does not occur in physically significant results.

The unidirectional constraints and two of the single polarization constraints are now satisfied. The remaining constraints are the two single polarization constraints, two diffeomorphism constraints, and Gauss.

D. Determination of $\sin(\alpha/2)$

Section V constructed a set of transverse $\tilde{\mathrm{E}}$ which satisfy the scalar constraint. One can insert those $\tilde{\mathrm{E}}$ into the remaining single polarization constraints Eq. (72), and thereby determine $\sin(\alpha/2)$, $\alpha/2$ the peak angle of rotation.

$$\begin{aligned} 0 &= \gamma \mathrm{K}_a^B, \qquad a \neq B, \\ &= 2\sin\beta_x \sin(\alpha_x/2) - \delta_{(c)}\mathrm{E}_Y^y(\mathrm{FT})(\Delta x/\Delta z)\mathrm{sgn}(e) \\ &= 2\cos\beta_y \sin(\alpha_y/2) + \delta_{(c)}\mathrm{E}_X^x(\mathrm{FT})(\Delta y/\Delta z)\mathrm{sgn}(e). \end{aligned} \tag{82}$$

From Eq. (34),

$$\begin{aligned} \delta_{(c)}\mathrm{E}_X^x(\mathrm{FT};n) &= \mathrm{sgn}(x)\{-(a/2)f\exp(-\rho n)\cos(kn-\phi/2) \\ &\quad + (-a^2/16)[f\exp(-2\rho n)\cos(2kn-3\phi/2) \\ &\quad + [-\rho\exp(-2\rho n)+\rho](f/\rho^2)\cos\phi]\}; \\ f^2 &= (k^2+\rho^2). \end{aligned} \tag{83}$$

We insert Eqs. (83) and (73) into Eq. (82).

$$\begin{aligned} &-2\mathrm{sgn}(\hat{n}_x)\sin(\alpha_x/2) \\ &\quad = +\sqrt{1+\gamma^2}\delta_{(c)}\mathrm{E}_Y^y(\mathrm{FT})(\Delta x/\Delta z)\mathrm{sgn}(e) \\ &\quad = \mathrm{sgn}(y)\mathrm{sgn}(e)\sqrt{1+\gamma^2}(a/2)f\exp(-\rho n) \\ &\qquad \times\cos(kn-\phi/2)(\Delta x/\Delta z); \\ &2\mathrm{sgn}(\hat{n}_y)\sin(\alpha_y/2) = -\sqrt{1+\gamma^2}\delta_{(c)}\mathrm{E}_X^x(\mathrm{FT})(\Delta y/\Delta z)\mathrm{sgn}(e) \\ &\quad = +\mathrm{sgn}(x)\mathrm{sgn}(e)\sqrt{1+\gamma^2}(a/2)f \\ &\qquad \times\exp(-\rho n)\cos(kn-\phi/2)(\Delta y/\Delta z), \end{aligned} \tag{84}$$

neglecting terms of order a^2. As advertised, the $\sin(\alpha_a/2)$ are order a.

Previously we mentioned that $\mathrm{sgn}(\hat{n}_a)$ drops out of final formulas for the transverse components of $\hat{P}^a$. We now know that $\sin(\alpha_a/2)$ is proportional to $\mathrm{sgn}(\hat{n}_a)$, and from this we can conclude that $\mathrm{sgn}(\hat{n}_a)$ also drops out of final formulas for the longitudinal components of $\hat{P}^a$. From Eq. (74), longitudinal components are given by

$$\hat{P}_Z^a = \sin\mu_a \sin(\alpha_a/2)\hat{Z}.$$

Both factors in this expression contain $\mathrm{sgn}(\hat{n}_a)$.

There is a geometrical reason why $\mathrm{sgn}(\hat{n}_a)$ always drops out. The basic holonomy is

$$u_a = \cos(\alpha_a/2) + i\sigma\cdot\hat{n}_a\sin(\alpha_a/2).$$

This rotation depends on the product of axis of rotation times rotation angle. If one changes $\mathrm{sgn}(\hat{n}_a)$ (reflects the axis, changes $\hat{n}_a \to -\hat{n}_a$) and simultaneously changes the sign of $\sin(\alpha_a/2)$, one obtains the same physical rotation. Only the product $\mathrm{sgn}(\hat{n}_a)\sin(\alpha_a/2)$ is geometrically significant. This means $\sin(\alpha_a/2)$ must contain a factor $\mathrm{sgn}(\hat{n}_a)$. Also, if a final formula does not contain $\sin(\alpha_a/2)$, then it cannot contain $\mathrm{sgn}(\hat{n}_a)$.

E. K_z^Z, E_Z^z, and Gauss

The four single polarization constraints are now satisfied. Gauss and two diffeomorphism constraints remain.

The diffeomorphism constraints from paper 1 are (for $p = 1/2$)

$$\begin{aligned} 1 &= {}^{(2)}\tilde{\mathrm{E}}/(\mathrm{C(LHF)}\mathrm{E}_Z^z); \\ \mathrm{C(LHF)} &= (\Delta z)^2\mathrm{sgn}(e); \\ 0 &= \mathrm{K}_z. \end{aligned}$$

The last line yields

$$\begin{aligned} 0 &= \gamma\mathrm{K}_z^Z(n) \\ &= -2i[\hat{h}_z^Z(n,n+1) + \hat{h}_z^Z(n-1,n)]/2 - \Gamma_z^Z \\ &= 2[\sin(\langle\theta_z\rangle/2)(n,n+1) + \sin(\langle\theta_z\rangle/2)(n-1,n)]/2 - 0. \end{aligned} \tag{85}$$

Either all peak θ_z are zero, or θ_z alternates between two values having opposite sign. Since holonomic angles should go to zero in the flat space in front of the packet, $\langle\theta_z\rangle = 0$.

The remaining diffeomorphism constraint may be used to show E_Z^z, ${}^{(2)}\tilde{\mathrm{E}}$, and $(m_f + m_i)$ are constants, to order a^2. Since E_Z^z grasps on both sides of the vertex, its expectation value depends on $m_f + m_i$, the expectation values of S_z on the ingoing plus outgoing sides of the vertex.

$$\mathrm{E}_Z^z(\mathrm{LHF}) = (\kappa\gamma/2)(m_f + m_i). \tag{86}$$

The LHF values are related to classical values by the diffeomorphism constraint Eq. (4).

$$\begin{aligned} \mathrm{E}_Z^z(\mathrm{LHF}) &= (\mathrm{E}_X^x\mathrm{E}_Y^y)(\mathrm{LHF})/\mathrm{C(LHF)} \\ &= (\mathrm{E}_X^x\mathrm{E}_Y^y)(\mathrm{cl})\mathrm{sgn}(e)\Delta x\Delta y. \end{aligned}$$

The classical $\tilde{\mathrm{E}}$ have the form

$$\begin{aligned} \mathrm{E}_X^x(\mathrm{FT}) &= (1 - E_1 + E_2)\mathrm{sgn}(x); \\ \mathrm{E}_Y^y(\mathrm{FT}) &= (1 + E_1 + E_2)\mathrm{sgn}(y), \end{aligned}$$

where $E_p = \mathrm{O}\ a^p$. Therefore

$$\mathrm{E}_Z^z(\mathrm{LHF}) = \mathrm{sgn}(z)[1 - E_1^2 + 2E_2 + \mathrm{O}\ a^3]\Delta x\Delta y. \tag{87}$$

Comparison of Eqs. (86) and (87) gives

$$\begin{aligned} (2/\kappa\gamma)\mathrm{E}_Z^z(\mathrm{LHF}) &= (m_f + m_i) \\ &= \mathrm{sgn}(z)(2/\kappa\gamma)\Delta x\Delta y(1 + \mathrm{O}\ a^2). \end{aligned} \tag{88}$$

$E^z_Z(\mathrm{FT})$, ${}^{(2)}\tilde{E}(\mathrm{FT})$, and $(m_f + m_i)$ are constants, to order a^2. □

Equation (88) is another example of scaling behavior. The overall amplitudes (but not the fluctuating factors) scale with the Δx^i.

The quantity $m_f - m_i$ occurs in Gauss's law. Gauss requires a vanishing net flow of z momentum through all six sides of the cube surrounding a given vertex. Equivalently, if the β_x, β_y, and θ_z at a given vertex are all subjected to the same rotation, the product of holonomies at the vertex must be invariant. This requires

$$M_x + M_y + (m_f - m_i) = 0.$$

The first two terms are the net outflow of Z angular momentum contributed by the transverse directions; the last parenthesis is net outflow contributed by the z holonomies.

The expectation value of M_a is given by the operator E^a_0, Eqs. (65) and (74).

$$\begin{aligned} M_a &= \langle L_a \rangle \hat{p}^a_B D^{(1)}(u_a)_{B0} \\ &= \langle L_a \rangle \sin\mu_a \sin(\alpha_a/2). \end{aligned} \tag{89}$$

Gauss then requires

$$\begin{aligned} 0 &= \langle L_x \rangle \sin\mu_x \sin(\alpha_x/2) + \langle L_y \rangle \sin\mu_y \sin(\alpha_y/2) \\ &\quad + (m_f - m_i). \end{aligned} \tag{90}$$

From Eq. (84) $\sin(\alpha_x/2)$ is a power series in a of the form

$$\sin(\alpha_x/2) = \mathrm{sgn}(x)\mathrm{sgn}(\hat{n}_x)[-A_1 + A_2 + \cdots]\Delta x/\Delta z, \tag{91}$$

where $A_p = \mathrm{O}\ a^p$. From Eq. (77),

$$\langle L_x \rangle = L_0[1 - E_1 + E_2 + \cdots]. \tag{92}$$

From Eq. (80), μ is a constant plus corrections of order a^2; therefore the power series for $\sin\mu$ goes as

$$\sin(\mu_x) = \mathrm{sgn}(\hat{n}_x)\mathrm{sgn}(x)[B_0 + B_2 + \cdots], \tag{93}$$

where $B_p = \mathrm{O}\ a^p$.

We insert these expansions into Eq. (90) (as usual, changing the sign of odd powers of a for the y term). Equation (90) then collapses to (for $\Delta x = \Delta y$)

$$2(\Delta x/\Delta z)[B_0 A_2 + E_1 A_1 + \mathrm{O}\ a^3]L_0 + (m_f - m_i) = 0. \tag{94}$$

From $L_0 \propto \Delta x \Delta z$ plus Eq. (88),

$$m_f = m_i = \mathrm{sgn}(z)(\kappa\gamma)^{-1}(\Delta x)^2(1 + \mathrm{O}\ a^2). \tag{95}$$

In the above calculation Eq. (90) was used for Gauss, rather than its small sine approximation,

$$0 = \delta_{(c)} E^z_Z + (-2i)\hat{h}^A_a E^a_A.$$

The latter is not quite as accurate. For example, $\delta_{(c)} E^z_Z = m_f - m_i$, but only after using slow variation. The small sine version is fine when Gauss occurs multiplied by factors of sine, as in the Hamiltonian and vector constraints. When Gauss is standalone, Eq. (90) is more accurate.

VIII. THE METRIC AT SPATIAL INFINITY

To this point the calculation has been carried out to order a^2 in the small amplitude a. This is fine, except for the undamped part of the amplitude, which diverges at infinity.

$$\begin{aligned} E^a_A(\mathrm{LHF}) &= (\Delta z \Delta x^b)\mathrm{sgn}(x)\{1 + \cdots \\ &\quad + (a^2/32)(\mp 2\rho n + 1)(f/\rho^2)\cos\phi\}, \end{aligned} \tag{96}$$

from Eq. (34), $z = \pm|z|$. If there are divergent corrections of higher order in n, they will be needed to compute the ADM energy.

It is safe to assume that the space outside the wavepacket is flat. The present solution is time varying, and the wave has not yet reached spatial infinity, which must be flat therefore. In flat space both the scalar constraint and the Riemann tensor must vanish. From Eq. (19),

$$\partial^2_u E^x_X = \partial^2_u E^y_Y = 0. \tag{97}$$

The variable $\sqrt{2}u = (z - ct)$ corresponds to the discrete variable n. In the present small sine LQG approach, derivatives with respect to $\sqrt{2}u$ become differences with respect to n. The $\tilde{E}$ are therefore linear functions of n at infinity. Equation (34) for the $\tilde{E}$ diverges linearly at infinity, and therefore is correct as it stands. There are no higher order corrections in n (though there may be higher order corrections in a).

The surviving terms at $n \to \pm\infty$ may be read off from Eq. (34).

$$\begin{aligned} E^x_X = E^y_Y &\to (\Delta z \Delta y)\mathrm{sgn}(x) \\ &\quad \times \{1 + (-a^2/32)(\pm 2\rho n - 1)(f^2/\rho^2)\cos\phi\} \\ &:= (\Delta z \Delta y)\mathrm{sgn}(x)\{1 \pm Dn + D_0\}. \end{aligned} \tag{98}$$

D_0 and D are constants of order a^2. If terms down by $\rho/k \ll 1$ are dropped,

$$D_0 = (a^2/32)(f/\rho^2)\cos\phi = 1 + (a^2/32)(k/\rho)^2;$$
$$Dn = -(a^2/16)(k^2/\rho)n. \tag{99}$$

E^z_Z follows from the gauge choice $E^z_Z \propto E^x_X E^y_Y$, Eq. (4).

$$\begin{aligned} E^z_Z(\mathrm{LHF}) &= \mathrm{sgn}(e)(\Delta z\Delta y)[1 \pm Dn + D_0]^2 \\ &= \mathrm{sgn}(e)(\Delta z\Delta y)[1 \pm 2Dn + 2D_0 + \mathrm{O}\ a^4]; \\ \delta_{(c)}E^z_Z(\mathrm{LHF}) &= \mathrm{sgn}(e)(\Delta z\Delta y)[\pm 2D + \mathrm{O}\ a^4]. \end{aligned} \tag{100}$$

The order a^4 terms should be dropped, because the E^a_A are known only to order a^2.

Since $\delta_{(c)}\, E^z_Z$ appears in the surface term for the energy, it is useful to check the above result by a second method: solve the constraint $\tilde{H} = 0$. From paper I, the section on the final form of the Hamiltonian,

$$\begin{aligned} \underline{N}\,\tilde{H} = \sum_n (1/\kappa)\{\cdots + \delta_{(c)}E^z_Z[-(\delta_{(c)}{}^{(2)}\tilde{E})/{}^{(2)}\tilde{E} \\ + \delta_{(c)}E^z_Z/2E^z_Z] + \delta_{(c)}(\delta_{(c)}E^Z_z)\} = 0, \end{aligned} \tag{101}$$

where we emphasize the terms which survive to infinity. We have dropped a term proportional to

$$(\delta_{(c)}E^y_Y/E^y_Y - \delta_{(c)}E^x_X/E^x_X)^2.$$

This expression vanishes at infinity for the present explicit solution, and also generally, because it represents the physical degree of freedom, which should be absent in flat space. We use the diffeomorphism gauge Eq. (4) to replace

$$\begin{aligned} (\delta_{(c)}{}^{(2)}\tilde{E})/{}^{(2)}\tilde{E} &= \delta_{(c)}E^z_Z/E^z_Z \\ &= \delta_{(c)}E^y_Y/E^y_Y + \delta_{(c)}E^x_X/E^x_X \\ &\to 2\delta_{(c)}E^x_X/E^x_X. \end{aligned}$$

Equation (101) becomes

$$-\delta_{(c)}E^x_X/E^x_X + \delta_{(c)}(\delta_{(c)}E^Z_z)/\delta_{(c)}E^z_Z = 0.$$

The solution is

$$\delta_{(c)}E^Z_z \to AE^x_X, \tag{102}$$

A a constant. We know E^x_X is linear in n at infinity, from the argument at Eq. (97). We now know that $\delta_{(c)}E^Z_z$ is also linear in n. Equation (102) agrees with our previous result for $\delta_{(c)}E^Z_z$ if we take $A = \pm 2D$, use Eq. (98) for E^x_X, and drop order a^4.

IX. SIGNS, FOR $\mathbf{sgn}(i) = +1$

It is useful to examine the pattern of signs for the simplest and most natural case: axes x^i and X^I running in the same direction; right-handed coordinate system: $\mathrm{sgn}(i) = \mathrm{sgn}(e) = +1$. For this case, only three signs are left in the problem: the sign of the small amplitude a; and $\mathrm{sgn}(\hat{n}_a)$.

The last two have limited physical significance. Only the signs of the products $\mathrm{sgn}(\hat{n}_a)$ times $\sin(\alpha_a/2)$ are physically significant, since a rotation through α around axis $\hat{n}$ is equivalent to a rotation through $-\alpha$ around $-\hat{n}$.

However, presumably the relative sign

$$\mathrm{sgn}(\hat{n}_x)\sin(\alpha_x/2)/\mathrm{sgn}(\hat{n}_y)\sin(\alpha_y/2)$$

has some significance. From Eq. (84), this ratio is

$$\begin{aligned} &(-a)f\exp(-\rho n)\cos(kn - \phi/2)/(+a)f \\ &\quad\times\exp(-\rho n)\cos(kn - \phi/2) + \mathrm{O}\ a^2 = -1. \end{aligned} \tag{103}$$

Meaning, for a given value of n, only one of numerator and denominator will be positive, and which one depends on the sign of a.

Given the high degree of symmetry between the x and y directions, and the lack of a screw sense (no circular polarization) one would expect solutions to occur in pairs differing by $x \leftrightarrow y$. The two solutions differing by $a \leftrightarrow -a$ form such a pair. As a check: performing both changes simultaneously ($a \leftrightarrow -a,\ x \leftrightarrow y$) leaves the curvature unchanged.

TABLE II. Variables to order a, for $\mathrm{sign(i)} = +1$. $f^2 = k^2 + \rho^2 \cong k^2; \mathrm{L}_0 = (2/\gamma\kappa)\Delta x\Delta z$.

Variable	Behavior	Reference
$\mathrm{E(FT)}^x_X$	$1 - (a/2)\exp(\mp\rho n)\sin(kn - \phi)$	Eq. (34)
$\mathrm{E(FT)}^y_Y$	$1 + (a/2)\exp(\mp\rho n)\sin(kn - \phi)$	Eq. (34)
$\mathrm{E(FT)}^z_Z$	$1 + \mathrm{O}\ a^2$	Eq. (87)
$\mathrm{L}(n)_x\hat{P}^x_X$	$\mathrm{L}_0\{1 - (a/2)\exp(\mp\rho n) \times\sin[kn \mp \phi]\}$	Eqs. (77), (81)
$\mathrm{L}(n)_y\hat{P}^y_Y$	$\mathrm{L}_0\{1 + (a/2)\exp(\mp\rho n) \times\sin[kn \mp \phi]\}$	Eqs. (77), (81)
$\mathrm{M}^x = \mathrm{L}_0\hat{P}^x_Z$	$-(\mathrm{L}_0 fa/4)\exp(\mp\rho n) \times\cos(kn - \phi/2)$	Eqs. (84), (89)
$\mathrm{M}^y = \mathrm{L}_0\hat{P}^y_Z$	$+(\mathrm{L}_0 fa/4)\exp(\mp\rho n) \times\cos(kn - \phi/2)$	Eqs. (84), (89)
β_a	fixed	Eq. (73)
μ_a	fixed	Eq. (80)
m_f, m_i	fixed > 0	Eq. (95)
$\langle\theta_z\rangle$	0	Eq. (85)

Table II gives the order a behavior of the dynamical variables, for the choice $\mathrm{sgn}(i) = 1$ and the reader's choice of signs for $\hat{n}_a$ and a.

X. DISCUSSION

The oscillations in transverse coordinates follow from Eqs. (6) and (34).

$$\Delta Y = \mathrm{sgn}(y)\Delta y\{1 - (a/2)\exp(\mp\rho n)\sin[kn \mp \phi] + \mathrm{O}\ a^2\};$$
$$\Delta X = \mathrm{sgn}(x)\Delta x\{1 + (a/2)\exp(\mp\rho n)\sin[kn \mp \phi] + \mathrm{O}\ a^2\}. \tag{104}$$

For example, a light beam traveling between markers separated by Δx would travel a standard meter stick distance given by ΔX.

To study the movement of test particles in the field of the wave, one can calculate geodesic deviation. Alternatively, one can reverse the interpretation of coordinates used in Eq. (104). The coordinates $(\Delta X, \Delta Y)$ now become fixed, because they are the coordinates of a particle in free fall. The $(\Delta x, \Delta y)$ become the variables. We invert Eq. (104), obtaining

$$\Delta y = \mathrm{sgn}(y)\Delta Y\{1 + (a/2)\exp(\mp\rho n)\sin[kn \mp \phi] + \mathrm{O}\ a^2\};$$
$$\Delta x = \mathrm{sgn}(x)\Delta X\{1 - (a/2)\exp(\mp\rho n)\sin[kn \mp \phi] + \mathrm{O}\ a^2\}. \tag{105}$$

We interpret $(\Delta X, \Delta Y)$ as the position vector of a test particle near the origin. Operationally, the (x, y) coordinates are constructed so as to be inertial coordinates before arrival of the wave. After the wave arrives, we may interpret the (x, y) coordinates geometrically, as a noninertial system. Alternatively, we can continue to interpret the coordinates as inertial. Then $(\Delta x, \Delta y)$ become the position of the particle under the effect of the "force" of gravity. Equation (105) yields

$$[\Delta x(t)/\Delta x(-\infty)(1 + a\sin\omega t)]^2 + [\Delta y(t)/\Delta y(-\infty)(1 - a\sin\omega t)]^2 = 2. \tag{106}$$

We have replaced $kn \to kn - \omega t$ and evaluated at $n = 0$. $(\Delta X, \Delta Y)$ are also the $(\Delta x, \Delta y)$ coordinates of the particle before the wave arrives, $(\Delta x(-\infty), \Delta y(-\infty))$. Equation (106) is the usual elliptical picture: a circle of test particles becomes an ellipse with time varying major and minor axes.

The standard formula for the area,

$$\mathrm{area} = \kappa\gamma\sqrt{j(j+1)},$$

might suggest that spins are input and areas are output. However, when deriving the classical limit, it is perhaps better to think of area as input, and (average or peak) spin as output. For example, from Eq. (88),

$$(2/\gamma\kappa)\Delta x\Delta y[1 + \mathrm{O}\ a^2] = \mathrm{sgn}(z)(m_f + m_i).$$

One adjusts the left-hand side, until the right-hand side is large enough to be semiclassical. Classical field theory variables $\tilde{\mathrm{E}}(\mathrm{FT})$ can be near unity, even though LHF eigenvalues are far from unity, because of the area elements in

$$\tilde{\mathrm{E}}(\mathrm{LHF}) = \tilde{\mathrm{E}}(\mathrm{FT})\Delta x^i\Delta x^j d.$$

Also, because of the area elements in the LHF triads, fixing the diffeomorphism gauge fixes ΔZ, the linear spacing between vertices.

The z components of angular momentum m_i, m_f are not related to the helicity operator for the wave. The helicity is given by [17]

$$-2\mathrm{i}\sum_n(\mathrm{E}^+_+\mathrm{K}^-_- - \mathrm{E}^-_-\mathrm{K}^+_+), \tag{107}$$

where $f_\pm = (f_x \pm \mathrm{i}f_y)/\sqrt{2}$. Equation (107) counts $+2\hbar$ times the number of spin 2 E^+_+ minus $2\hbar$ times the number of spin -2 E^-_-. If the E and K are expanded in terms of more familiar fields,

$$\mathrm{E}^+_+ = [\mathrm{E}^x_X + \mathrm{i}\mathrm{E}^x_Y + \mathrm{i}(\mathrm{E}^y_X + \mathrm{i}\mathrm{E}^y_Y)]/2, \tag{108}$$

etc., one can show that the helicity operator vanishes, as it should [18]. From the discussion in Appendix A, $m_f + m_i$ is closely related to energy, rather than helicity.

The behavior of the transverse holonomies is relatively simple. Each holonomy is characterized by an axis of rotation $\hat{n}_a$ and an angle of rotation $\alpha_a/2$. $\hat{n}_a$ can be reflected through the origin, but otherwise cannot change: the angle it makes with the X axis, β_a, is fixed by the Immirzi parameter. $\sin(\alpha_a/2)$ is proportional to $\delta_{(c)}\mathrm{E}^a_A$, from Eq. (84), and therefore oscillates with the frequency of the wave, the amplitude of oscillation being order k a. We can visualize this oscillation by drawing a unit vector along the Z axis and imagining that a rotation around $\hat{n}_a$ is applied to this vector. The tip of this vector oscillates along a small arc, approximately a straight line centered on the Z axis and lying in the plane perpendicular to $\hat{n}_a$.

The directions of the axes of rotation must be fixed in order for the unidirectional and single polarization constraints to agree. If the single polarization constraint is relaxed, the rotation axes are no longer fixed; see Appendix D.

In contrast to the transverse holonomies, longitudinal holonomies are trivial: $\langle\theta_z\rangle = 0$. To order a, the longitudinal momenta and angles (m_z and $\langle\theta_z\rangle$) do not oscillate.

Turning from holonomies to fluxes and angular momenta, the unrotated $\hat{p}^a$ must be very close to the $\hat{A}$

axis, so that a small rotation through $\alpha_a/2$ can remove the component perpendicular to $\hat{A}$, as required by the single polarization constraints. The rotated $\hat{P}^a$ is also close to the $\hat{A}$ axis, therefore.

The peak values of the $\hat{P}^a$, Eq. (74), have Z components. This is a bit surprising, since from Eq. (65) $\hat{P}^a$ is the peak value of a transverse operator E^a_A, and the triads E^a_Z have been gauged to zero. If one imagines a rectangular volume ΔXΔYΔZ surrounding each vertex, E^a_A supposedly gives the area of the side having normal $\hat{A} \neq \hat{Z}$.

One might try to understand the need for Z components by noting that change in area with normal $\hat{A}$ produces change in area with normal $\hat{Z}$. The triad E^y_Y (for example) changes because the associated area ΔXΔZ changes. That area changes because length ΔX changes. (ΔZ is gauge fixed.) The changes in ΔX in turn induce changes in the area ΔXΔY with normal $\hat{Z}$. Unfortunately, we must try harder: this interpretation predicts Z components $\hat{P}^a_Z$ going as ΔX, i.e. as sine rather than cosine.

At Eq. (55) we sketched the construction of the two operators E^a_0 which produce the Z components of $\hat{P}^a$. These operators do not act in the same manner as the dynamical variables (E^x_X, E^y_Y, E^z_Z). The latter take the *sum* of the areas at front and back of a small cube surrounding the vertex; for example, the grasp of E^z_Z is proportional to $m_f + m_i$. In contrast, (E^x_0, E^y_0) are not fundamental but emerge in the course of constructing the coherent states. They take the *difference* between front and back areas. In particular, Gauss's law, Eq. (90), is

$$0 = \langle L_x\rangle \sin\mu_x \sin(\alpha_x/2) + \langle L_y\rangle \sin\mu_y \sin(\alpha_y/2) + (m_f - m_i).$$

The first two terms are eigenvalues of the E^a_0 operators, and Gauss's law involves differences.

The action of (E^x_X, E^y_Y) on the basic holonomy is given by an anticommutator, Eq. (52), whereas the action of the E^a_0 is given by a commutator, Eq. (55). Presumably this is the reason for the shift from sum to difference.

The explicit expressions for momentum in Table II are consistent with the foregoing qualitative discussion (largest component of $\hat{P}^a$ along $\hat{A}$; smaller component along $\hat{Z}$, proportional to a difference). For example $\vec{L}^y$ to order a is

$$\begin{aligned}\vec{L}^y_Y &= L^y(n)\hat{P}^y_Y\\ &= L_0[1 + (a/2)\exp(\mp\rho n)\sin(kn - \phi)]\hat{Y};\\ M^y &= \hat{Z}L_0\exp(\mp\rho n)(fa/4)\cos(kn - \phi/2).\end{aligned} \tag{109}$$

The largest component is along Y and is a measure of area ΔXΔZ. Hence this component tracks the variation of ΔX, Eq. (104), which varies as a sine. The smaller, Z component tracks the first difference of the area ΔXΔZ, and therefore varies as the first difference of the sine, namely $f\cos \cong k\cos$. We have listed transverse and longitudinal components on separate lines to emphasize their conceptual difference: area sum vs area difference.

The above results are largely unaffected by spatial diffeomorphisms, since the holonomies and $\tilde{E}$ (LHF) are constructed to be invariant. Even the classical E^a_A(FT) are largely invariant. Change occurs only in order a^2. From Eq. (2), the classical gauge is characterized by a power p. The following gauge transformation changes p to p'.

$$z' = \int^z [\mathrm{sgn}(z)E^z_Z]^{(1-2p')/4}dz. \tag{110}$$

The above integrand, expanded in powers of a, is unity plus order a^2. Therefore the order a terms in $\tilde{E}$ (FT) are invariant. A corollary: the order a oscillations of $\Delta X, \Delta Y$ are invariant.

Although the treatment given in this paper is primarily classical, the results carry over to the quantum theory because of the use of coherent states. From Eq. (21) the expectation value of the quantum constraint vanishes, if the classical constraint vanishes. Also, the expectation value of a quantum operator varies in the same manner as the corresponding classical operator.

Although this paper uses O(3) harmonics Y_L rather than SU(2) harmonics, the latter are not wrong, merely less convenient. In particular, despite the use of an O(3) basis, we do not lose information about SU(2).

$$\begin{aligned}&D(-\phi + \pi/2, \theta, \phi - \pi/2)^{(j)}|u, \vec{p}\rangle\\ &\quad = D(-\beta + \pi/2, \alpha, \beta - \pi/2)^{(j)}|u, \vec{p}\rangle,\end{aligned}$$

where $D^{(j)}$ is a representation of SU(2). The O(3) and SU(2) harmonics have identical axes of rotation, with

$$\langle L\rangle = \langle 2j\rangle.$$

As a check: for L = 1, the O(3) harmonics are combinations of $j = 1/2$ SU(2) harmonics.

We list two concerns. We dampened the sine wave using an exponential. This choice works well to order a^2, and we have suggested a procedure for extending the calculation to higher powers in a. The choice of exponential is motivated by mathematical simplicity, rather than dynamics, however, and the procedure we suggest may break down in higher orders.

Also, Eq. (A3), and the discussion in the last few paragraphs of Appendix B suggests that the energy of the wave is quantized, because the frontal area of the wave is quantized. In weak field limit the energy is quantized in units of $\hbar\omega$, but the present calculation goes beyond the weak field, and perhaps a new quantization rule is to be expected.

However, the packet is three-dimensional. Should not energy be quantized because volume is quantized, rather than frontal area only? This question may be connected to the previous one about damping. Presumably an improved damping mechanism would quantize the length of the packet.

APPENDIX A: THE ADM ENERGY

From [1], the surface term is given by $-\underline{\mathrm{N}}\delta_{(c)}E^z_Z(\mathrm{LHF})/\kappa$. From Eqs. (100) and (99) for E^z_Z at infinity,

$$\begin{aligned}\text{ADM energy} &= -(\underline{\mathrm{N}}/\kappa)\mathrm{sgn}(z)\Delta x\Delta y(\mp a^2/8)(k^2/\rho)|^{+\infty}_{-\infty}\\ &= (1/\Delta Z\kappa)\Delta x\Delta y\mathrm{sgn}(z)2(a^2/8)(k^2/\rho),\end{aligned} \tag{A1}$$

where $\underline{\mathrm{N}}(\mathrm{LHF}) = 1/\Delta\mathrm{Z}$. $\Delta Z\mathrm{sgn}(z) = \Delta z$ is positive [Eq. (67)].

The factor of $\Delta Z\mathrm{sgn}(z) = \Delta z$ looks gauge variant. However, we can introduce a $k(\mathrm{cl})$ and $\rho(\mathrm{cl})$, defined by

$$\begin{aligned}\exp[-\rho n]\sin[kn] &= \exp[-\rho(\mathrm{cl})n\Delta z]\sin[k(\mathrm{cl})n\Delta z];\\ n\Delta z &= z;\\ \rho(\mathrm{cl}) &= \rho/\Delta z;\\ k(\mathrm{cl}) &= k/\Delta z.\end{aligned} \tag{A2}$$

If we shift to the classical quantities in Eq. (A1), the factor of $\Delta Z\mathrm{sgn}(z) = \Delta z$ disappears.

$$\text{ADM energy} = \hbar c(\Delta x\Delta y/\kappa)(a^2/4)(k^2/\rho)(\mathrm{cl}). \tag{A3}$$

The first term in parentheses is dimensionless because we have given κ the dimension of length squared.

The energy should be proportional to the volume occupied by the wave. After the shift to classical k and ρ, the ADM energy contains a factor

$$\Delta x\Delta y/\rho(\mathrm{cl}) \sim \Delta x\Delta y \text{ length}.$$

The above expression is a measure of volume. Since the packet is proportional to $\exp[-\rho(\mathrm{cl})z]$, $1/\rho(\mathrm{cl})$ is a measure of the length of the packet.

Rough estimates of the energy give results similar to Eq. (A1). The energy in weak field approximation is of order

$$\begin{aligned}\int(\partial_z\tilde{\mathrm{E}})^2 &\sim (\mathrm{xy\ area})(k(\mathrm{cl})a)^2\\ &\quad\times\int dz[\sin(k(\mathrm{cl})z)\exp(-\rho(\mathrm{cl})z)]^2\\ &= (\mathrm{xy\ area})(k(\mathrm{cl})a)^2[k(\mathrm{cl})^2/(2\rho(\mathrm{cl}))]\\ &\quad\times\{1/[k(\mathrm{cl})^2-\rho(\mathrm{cl})^2]\}\\ &\approx (\mathrm{xy\ area})(ka/\Delta z)^2(\Delta z/2\rho).\end{aligned} \tag{A4}$$

The last line of Eq. (A4) neglects terms down by $\rho(\mathrm{cl})/k(\mathrm{cl})(=\rho/k \ll 1)$. This back-of-the-envelope estimate contains the same factors as Eq. (A1).

One might suppose the energy is not quantized, because periodic boundary conditions were not used, and k in Eq. (A1) can be anything. However, see the next section.

APPENDIX B: SPREADING OF A COHERENT WAVE PACKET

The extent of wave packet spreading was estimated elsewhere [19]; the present appendix modifies that discussion for the planar case. We first argue that all packets approach (nonspreading) simple harmonic oscillator (SHO) packets in the limit of large quantum numbers.

For minimal spreading, the spacing between energy levels of the system should be as constant as possible, resembling the spacing between levels of the usual oscillator [20,21]. Suppose, for example, the energy goes as L^p, p some power other than linear, L a quantum number. (For example, the spherical harmonics making up the coherent state of the earth have energy going as L^2.) The spacing between levels is

$$\delta\mathrm{E} = \mathrm{const}\,pL^{p-1}\delta L, \tag{B1}$$

which is no longer in SHO form: a constant times the change of an integer.

Although the factor multiplying the integer is now a function, rather than a constant, the variation of this factor across the packet is very small.

$$\begin{aligned}\delta(\mathrm{factor})/\mathrm{factor} &= (p-1)\delta L/L\\ &= (p-1)\sigma(L)/L.\end{aligned} \tag{B2}$$

On the second line we have estimated the spread of L values in the packet, $\delta\mathrm{L}$, using $\sigma(\mathrm{L})$, the standard deviation of the L values in the classical limit $\sigma(\mathrm{L})$ is expected to be $\ll \mathrm{L}$. All packets approach a SHO packet in the limit of large quantum numbers.

Now consider the present case. We will need some assumptions to make the case that lifetime of the packet is infinite.

In weak field geometrodynamics, the packet can be Fourier transformed. The dynamics guarantees every Fourier component has the same velocity c, and the wave does not disperse. The present dynamics is nonlinear and this simple option is not available.

From Eq. (A3) the energy is proportional to the frontal area of the wave $\Delta x \Delta y$. This area is quantized, by equation Eq. (95), and the quantum number is an integer (m, rather than $\sqrt{j(j+1)}$).

The quoted m is a peak value; the wave is actually a superposition of area eigenvalues m. This resembles the SHO situation, a superposition of occupation number eigenvalues n.

In order to obtain constant energy spacing between area eigenstates, we must assume the remaining factor in the ADM energy ($a^2 k(\mathrm{cl})^2/\rho(\mathrm{cl})$) is constant. Presumably this factor is determined by the matter source, and investigation of the source is beyond the scope of the present work. The constants (k, ρ, a) are unlikely to contain hidden dependence on other gravitational quantum numbers, however, because those constants occur in expressions such as

$$\delta_{(c)} \mathrm{E}^x_X / \mathrm{E}^x_X, \delta_{(c)} \mathrm{E}^y_Y / \mathrm{E}^y_Y$$

which are independent of area.

When higher orders in a are included, the area $\Delta \mathrm{X} \Delta \mathrm{Y}$ and hence m fluctuate at finite values of n. The ADM energy, however, is determined by long-range “tails” which extend beyond the packet. These presumably do not fluctuate. We assume the higher orders in a correct the $a^2 k(\mathrm{cl})^2/\rho(\mathrm{cl})$ factor; the spacing between levels changes, but the spacing remains uniform.

A three-dimensional wave has a frontal area determined by $j(j+1)$ area eigenvalues. Integer eigenvalues m are special to the planar case. We would expect the three-dimensional packet to be long lived, by the argument at the beginning of this section. But it is not clear that the lifetime would be infinite.

APPENDIX C: THE PLANAR HILBERT SPACE

It is possible, but inconvenient, to use coherent states with full SU(2) symmetry [13,14] as a basis for the Hilbert space. The flux-holonomy algebra unique to the planar case allows one to construct a Hilbert space with simpler matrix elements for the transverse $\tilde{\mathrm{E}}$.

We recall first the structure of the transverse holonomies. The x and y holonomies involve only generators $\mathrm{S}_X, \mathrm{S}_Y$, since the A^Z_a have been gauged to zero. Each transverse holonomy $h^{(1/2)}$ therefore has an axis of rotation with no Z component.

$$h^{(1/2)} = \exp[i\hat{m} \cdot \vec{\sigma}\theta/2]; \qquad \hat{m} = (\cos\phi, \sin\phi, 0), \tag{C1}$$

for some angle ϕ. There is one holonomy for each transverse direction x, y; and one ϕ for each transverse direction, ϕ_x and ϕ_y. Since the two directions are treated equally, it is sufficient to discuss only the x holonomies; the subscript x will be suppressed. When expanded out, the spin 1/2 holonomy $h^{(1/2)}$, Eq. (C1), becomes

$$h^{(1/2)} = \begin{bmatrix} \cos(\theta/2) & i\exp(-i\phi)\sin(\theta/2) \\ i\exp(+i\phi)\sin(\theta/2) & \cos(\theta/2) \end{bmatrix} \tag{C2}$$

The usual Euler angle decomposition for this rotation is

$$\begin{aligned} h^{(1/2)} &= \exp[-i\sigma_Z(\phi - \pi/2)/2]\exp(i\sigma_Y\theta/2) \\ &\quad \times \exp[+i\sigma_Z(\phi - \pi/2)/2] \\ &= h^{(1/2)}(-\phi + \pi/2, \theta, \phi - \pi/2). \end{aligned} \tag{C3}$$

The flux-holonomy algebra is somewhat unusual. From the discussion at Eq. (52), $\tilde{\mathrm{E}}$ produces an anticommutator.

$$\begin{aligned} \mathrm{E}^x_A h^{(1/2)} &= \mathrm{E}^x_A \exp\left[i\int \mathrm{A}^B_x \mathrm{S}_B dx\right] \\ &= (\gamma\kappa/2)[\sigma_A/2, h^{(1/2)}]_+. \end{aligned} \tag{C4}$$

Fortunately, the anticommutator reshuffles the elements of h in a relatively simple way. We introduce the operators $\mathrm{E}^x_\pm$, where

$$f_\pm \coloneqq (f_x \pm i f_y)/\sqrt{2}. \tag{C5}$$

The operators $\mathrm{E}^x_\pm$ reshuffle the components of h in the same way that the familiar angular momentum operators $\mathrm{L}_\pm$ reshuffle the $\mathrm{L} = 1$ Legendre polynomials Y^M_1. For example, we write out the action of the anticommutator in Eq. (C4), for index $\mathrm{A} = +$.

$$\begin{aligned} &[\sigma_+/2, h^{(1/2)}]_+ \\ &\quad = \sqrt{1/2}\begin{bmatrix} i\exp(-i\phi)(\sin\theta/2) & 2\cos)\theta/2) \\ 0 & i\exp(+i\phi)\sin(\theta/2) \end{bmatrix} \end{aligned} \tag{C6}$$

When we compare this matrix to the original matrix, Eq. (C2), we find E^x_+ has reshuffled the matrix elements as

$$\begin{aligned} &(i/\sqrt{2})\exp(-i\phi)\sin\theta/2 \to \cos\theta/2; \\ &\qquad \cos\theta/2 \to (i\sqrt{2})\exp(+i\phi)\sin\theta/2; \\ &(i/\sqrt{2})\exp(+i\phi)\sin\theta/2 \to 0. \end{aligned} \tag{C7}$$

This is isomorphic to the action of the operator L_+ on the $\mathrm{L} = 1$ Legendre polynomials. The isomorphism is

$$\begin{aligned}
L_{\pm} &\leftrightarrow 2\mathrm{E}^x_{\pm}/\gamma\kappa;\\
L_0 &\leftrightarrow 2\mathrm{E}^x_0/\gamma\kappa;\\
\mathrm{Y}^{\pm}_1 &\leftrightarrow (\mathcal{N}/\sqrt{2})h_{\mp,\pm}\\
&= \mp\,\mathcal{N}\sin(\theta/2)\exp[\pm(i\phi - i\pi/2)]/\sqrt{2}\\
&= Y^{\pm}_1(\theta/2, \phi-\pi/2);\\
\mathrm{Y}^0_1 &\leftrightarrow \mathcal{N}h_{++} = \mathcal{N}h_{--}\\
&= \mathcal{N}\cos(\theta/2)\\
&= \mathrm{Y}^0_1(\theta/2,\phi-\pi/2).
\end{aligned} \tag{C8}$$

Because of the half angles, normalization of the Y's requires integrating θ from 0 to 2π. $\mathcal{N} = \sqrt{4\pi/3}$.

Because the $Y^M_1(\theta/2, \phi-\pi/2)$ transform more simply than matrix elements of $h^{(1/2)}$ under the action of $\tilde{\mathrm{E}}$, one obtains a more convenient basis by using O(3) 3J coefficients and products of Y_1's, rather than SU(2) coefficients and products of $h^{(1/2)}$'s. The resultant basis is just the set of spherical harmonics $\mathrm{Y}^M_L(\theta/2, \phi-\pi/2)$ for O(3).

One can take into account the y edges as well as the x edges, by constructing two bases, Y^{Mx}_{Lx} and Y^{My}_{Ly} for holonomies along the x and y directions, respectively. These harmonics transform simply under the action of the $\tilde{\mathrm{E}}$:

$$(\gamma\kappa/2)^{-1}\mathrm{E}^x_P \mathrm{Y}^M_L = \Sigma_N \mathrm{Y}_{LN}\langle L,\mathrm{N}|\mathrm{S}_P|L,\mathrm{M}\rangle, \tag{C9}$$

where $\mathrm{Y}_{LM} = \mathrm{Y}_{LM}(\theta/2,\phi-\pi/2)$. The unconventional half-angle reminds us of the origin of these objects in a holonomy $h^{(1/2)}$ depending on half-angles.

The transverse coherent states constructed here do not have unique values for M_x and M_y. These states are superpositions of $\mathrm{D}^{(La)}_{0Ma}$ matrices ($a = x, y$); and the superpositions will contain a range of values M_a. (Similarly, coherent states in the longitudinal direction will not have definite m_z.) The superpositions are sharply peaked at central values of the M's, however, so that M-values which violate U(1) are suppressed.

The relation between h and the Y^M_1 is

$$\mathcal{N}\mathbf{h}^{(1/2)} = \mathbf{1}\mathrm{Y}^0_1 + i\mathrm{Y}^+_1\mathbf{S}_- + i\mathrm{Y}^-_1\mathbf{S}_+, \tag{C10}$$

where boldface denotes a 2×2 matrix. Equation (C10) demonstrates that the Y's are as complete a set as the elements of $h^{1/2}$, since the three independent elements of $h^{(1/2)}$ can be expressed in terms of the three $\mathrm{Y}^M_1(\theta/2,\phi-\pi/2)$.

APPENDIX D: CIRCULAR POLARIZATION

When both polarizations are present, we solve the equations in weak field approximation. Equivalently, all equations are solved only to order a in the small amplitude a. We also take $\mathrm{sgn}(a) = \mathrm{sgn}(e) = +1$ and neglect damping. Working to order a means some $\tilde{\mathrm{E}}$ (LHF) must be evaluated to order unity. For example, $\mathrm{E}^x_X(\mathrm{LHF}) \to \Delta x\Delta z$. Intermediate formulas are then peppered with unhelpful factors of Δx^i. We suppress these factors but restore them in final formulas. For a weak field treatment using both connection fields and a loop representation, see Ashtekar, Rovelli, and Smolin [22].

We use the following formula from paper I for the Γ; the formula is correct for all polarizations.

$$\begin{aligned}
2\Gamma^I_j\mathrm{E}^j_I &= -\mathrm{sgn}(e)(\delta_{(c)}\Sigma^{mzM})\Sigma^{ni}_M\epsilon_{mni}/2! \coloneqq 2\Gamma\cdot\mathrm{E};\\
\Gamma^I_j\mathrm{E}^j_M &= \mathrm{sgn}(e)(\delta_{(c)}\Sigma^{mz}_M)\Sigma^{niI}\epsilon_{mni}/2!\\
&\quad + \Gamma\cdot\mathrm{E}\delta^I_M.
\end{aligned}$$

The off-diagonal fields E^x_Y and E^y_X, and the on-diagonal Γ^X_x, Γ^Y_y are now nonzero. Dropping all corrections of order a^2,

$$\begin{aligned}
&\Gamma^X_x = \delta_{(c)}\mathrm{E}^x_Y; \qquad \Gamma^Y_y = -\delta_{(c)}\mathrm{E}^y_X;\\
&\Gamma^X_y = \delta_{(c)}[-\mathrm{E}^z_Z - \mathrm{E}^x_X + \mathrm{E}^y_Y]/2;\\
&\Gamma^Y_x = \delta_{(c)}[+\mathrm{E}^z_Z - \mathrm{E}^x_X + \mathrm{E}^y_Y]/2;\\
&\Gamma^Z_z = \Gamma\cdot\mathrm{E} = 0.
\end{aligned} \tag{D1}$$

The unidirectional constraints are the (linearized) single polarization constraints from paper I [see also Eq. (71)], plus one additional constraint.

$$\begin{aligned}
0 &= \mathrm{K}^X_x + \mathrm{K}^Y_y + \delta_{(c)}\mathrm{E}^z_Z;\\
0 &= 2\mathrm{K}^Z_z + \mathrm{K}^Y_y + \mathrm{K}^X_x + \delta_{(c)}\mathrm{E}^x_X + \delta_{(c)}\mathrm{E}^y_Y + 2\delta_{(c)}\underline{\mathrm{N}}/\underline{\mathrm{N}};\\
0 &= \mathrm{K}^Y_y - \mathrm{K}^X_x - \delta_{(c)}\mathrm{E}^y_Y + \delta_{(c)}\mathrm{E}^x_X;\\
0 &= \mathrm{K}^X_y + \mathrm{K}^Y_x - \mathrm{K}^Z_z(\mathrm{E}^x_Y + \mathrm{E}^y_X) - \delta_{(c)}\mathrm{E}^x_Y - \delta_{(c)}\mathrm{E}^y_X.
\end{aligned} \tag{D2}$$

The U(1) gauge is fixed by making the cotriad matrix symmetric.

$$\begin{aligned}
\mathrm{E}^x_Y - \mathrm{E}^y_X &= 0;\\
\mathrm{K}^x_Y - \mathrm{K}^y_X &= 0.
\end{aligned} \tag{D3}$$

The second line is also the linearized Gauss constraint.

To quadratic order, the scalar constraint is now

$$\begin{aligned}
\tilde{\mathrm{H}} &= [\delta_{(c)}\mathrm{E}^x_X - \delta_{(c)}\mathrm{E}^y_Y]^2/2 - \delta_{(c)}\mathrm{E}^z_Z\delta_{(c)}(\mathrm{E}^x_X\mathrm{E}^y_Y)/2\\
&\quad + \delta_{(c)}\delta_{(c)}\mathrm{E}^z_Z + [\delta_{(c)}\mathrm{E}^x_Y + \delta_{(c)}\mathrm{E}^y_X]^2/2.
\end{aligned} \tag{D4}$$

We adopt the same diffeomorphism gauge (linearized) as for the single polarization case:

$$\delta_{(c)}\mathrm{E}^{\mathrm{z}}_{\mathrm{Z}} = \delta_{(c)}\mathrm{E}^{\mathrm{x}}_{\mathrm{X}} + \delta_{(c)}\mathrm{E}^{\mathrm{y}}_{\mathrm{Y}}; \quad \mathrm{K}^{\mathrm{Z}}_{\mathrm{z}} = 0. \tag{D5}$$

This choice removes the first three terms in the scalar constraint, leaving us with

$$\tilde{\mathrm{H}} = \delta_{(c)}\delta_{(c)}\mathrm{E}^{\mathrm{x}}_{\mathrm{X}} + \delta_{(c)}\delta_{(c)}\mathrm{E}^{\mathrm{y}}_{\mathrm{Y}} + [\delta_{(c)}\mathrm{E}^{\mathrm{x}}_{\mathrm{Y}} + \delta_{(c)}\mathrm{E}^{\mathrm{y}}_{\mathrm{X}}]^2/2. \tag{D6}$$

We would have to modify the diffeomorphism gauge to remove the last term, and we do not know how to do this. This is why we work only to order a and drop the last, order a^2 term in Eq. (D6). As in the single polarization case, $\tilde{\mathrm{H}} = -\mathrm{H}^{z}$.

We focus on circular polarization, which is simpler than elliptical polarization but nevertheless instructive, because quite different from the single polarization case. In a circular polarized wave, the two physical degrees of freedom are coherent and equally weighted.

$$\begin{aligned} \mathrm{E}^{\mathrm{x}}_{\mathrm{X}} &= [1 - a\cos(kn)/2]\Delta x\Delta z; \\ \mathrm{E}^{\mathrm{y}}_{\mathrm{Y}} &= [1 + a\cos(kn)/2]\Delta x\Delta z; \\ \mathrm{E}^{\mathrm{x}}_{\mathrm{Y}} &= \mathrm{E}^{\mathrm{y}}_{\mathrm{X}} = [\pm a\sin(kn)/2]\Delta \mathrm{x}\Delta \mathrm{z}, \end{aligned} \tag{D7}$$

to order a. This ansatz satisfies the scalar constraint, Eq. (D6), and the first U(1) constraint, Eq. (D3). From the first diffeomorphism constraint, Eq. (D5), plus Eq. (D7),

$$\delta_{(c)}\mathrm{E}^{\mathrm{z}}_{\mathrm{Z}} = \delta_{(c)}\mathrm{E}^{\mathrm{x}}_{\mathrm{X}} + \delta_{(c)}\mathrm{E}^{\mathrm{y}}_{\mathrm{Y}} = \mathrm{O}\ a^2.$$

Terms involving these quantities may be dropped from the unidirectional constraints.

The longitudinal sector is now identical to the longitudinal sector of the single polarization case. The second diffeomorphism constraint, $\mathrm{K}^{\mathrm{Z}}_{\mathrm{z}} = 0$, implies all longitudinal holonomic angles $\langle\theta_z\rangle$ vanish. The arguments of Sec. VII E go through unchanged: incoming and outgoing z components of angular momentum are equal, and are constant to order a^2.

$$m_f = m_i = (\kappa\gamma)^{-1}(\Delta x)^2(1 + \mathrm{O}\ a^2).$$

We now have four equations involving K's: the three unidirectional constraints, Eq. (D2), plus the second U(1) constraint Eq. (D3). One may eliminate the K's, as at Eq. (68). After this step, the new unidirectional constraint (for example) becomes

$$0 = 2[\cos\beta_y \sin(\alpha_y/2) + \sin\beta_x \sin(\alpha_x/2)] + ak\sin(kn) \mp ak\gamma\cos(kn).$$

The four equations involving K's can be solved for the four unknowns $\tan\beta_a$ and $\sin(\alpha_a/2)$.

$$\begin{aligned} &\tan\beta_x(0) = \tan(\beta_y + \pi/2); \\ &\sin\beta_x(0) = \pm\mathrm{sgn}(\hat{n}_x)\cos(kn \pm \lambda); \\ &\cos\beta_x(0) = \mathrm{sgn}(\hat{n}_x)\sin(kn \pm \lambda); \\ &\sin\beta_y(0) = \mp\,\mathrm{sgn}(\hat{n}_y)\sin(kn \pm \lambda); \\ &\cos\beta_y(0) = \mathrm{sgn}(\hat{n}_y)\cos(kn \pm \lambda); \\ &\quad \cos\lambda \coloneqq \gamma/\sqrt{1+\gamma^2}; \qquad \sin\lambda = 1/\sqrt{1+\gamma^2}; \\ &\sin(\alpha_x/2) = ka\ \mathrm{sgn}(\hat{n}_x)\gamma\sqrt{1+\gamma^2}/4; \\ &\sin(\alpha_y/2) = \pm ka\ \mathrm{sgn}(\hat{n}_y)\gamma\sqrt{1+\gamma^2}/4. \end{aligned} \tag{D8}$$

The (0) indicates a result valid to zeroth order in a; there may be linear in a corrections.

In some respects the results of Eq. (D8) resemble the results for single polarization. Each axis of rotation β_a and angle of rotation $\alpha_a/2$ are defined only up to sign. One can reflect the axis of rotation through the origin, while simultaneously changing the sign of the angle of rotation, and obtain the same physical rotation. Also, (first line) the two axes of rotation remain perpendicular.

However, in other respects circular polarization is dramatically different. The single polarization constraint forces each axis to remain fixed. If one imagines a unit vector initially along Z, then the oscillations of (α_a) cause this vector to oscillate along a fixed arc perpendicular to $\hat{n}_a$. In the circular case, the axes $\hat{n}_a$ are not fixed, but rotate through a circle in the xy plane, at constant angular velocity. It is the angles α_a which now remain fixed. The imaginary unit vector perpendicular to $\hat{n}_a$ makes a fixed angle $\alpha_a/2$ with the $\hat{Z}$ axis and precesses with constant angular velocity, on a cone of half angle $\alpha_a/2$.

Equation (65) yields four equations which determine the μ_a. They may be written in matrix form as

$$\begin{pmatrix} \mathrm{E}^a_X/\Delta y\Delta z \\ \mathrm{E}^a_Y/\Delta y\Delta z \end{pmatrix} = (\langle L_a(n)\rangle/L_0)\begin{bmatrix} \cos\beta_a - \sin\beta_a \\ \sin\beta_a \cos\beta_a \end{bmatrix}\begin{pmatrix} \cos\mu_a \\ \sin\mu_a \end{pmatrix}, \tag{D9}$$

where $a = x, y$ and $\mathrm{L}_0 = (2/\gamma\kappa)\Delta x\Delta z$. We insert Eq. (D7) on the left in Eq. (D9), and use Eq. (D8) on the right.

$$\begin{aligned} &(\langle L_x\rangle/L_0)\cos\mu_x = \mathrm{sgn}(\hat{n}_x)[\sin(kn\pm\lambda) - (a/2)\sin(\pm\lambda)]; \\ &(\langle L_x\rangle/L_0)\sin\mu_x = \mp\,\mathrm{sgn}(\hat{n}_x)[\cos(kn\pm\lambda) - (a/2)\cos(\pm\lambda)]; \\ &(\langle L_y\rangle/L_0)\cos\mu_y = \mp\,\mathrm{sgn}(\hat{n}_y)[\sin(kn\pm\lambda) + (a/2)\sin(\pm\lambda)]; \\ &(\langle L_y\rangle/L_0)\sin\mu_y = \mathrm{sgn}(\hat{n}_y)[\cos(kn\pm\lambda) + (a/2)\cos(\pm\lambda)]; \\ &\quad \langle L_x(n)\rangle/L_0 = 1 - (a/2)\cos kn; \\ &\quad \langle L_y(n)\rangle/L_0 = 1 + (a/2)\cos kn. \end{aligned} \tag{D10}$$

The $\langle \mathrm{L}_a(n)\rangle/\mathrm{L}_0$ follow from the requirement that $(\cos\beta_a)^2 + (\sin\beta_a)^2 = 1$.

The vector $\hat{p}^a$ makes an angle $\mu + \beta$ with the X axis, and both angles vary sinusoidally. Nevertheless, $\hat{p}^a$ does not rotate. Neglecting the order a correction terms, we have

$$\tan\beta_x = -\tan\mu_x;$$
$$\tan\beta_y = 1/\tan\mu_y = -\tan(\mu_y + \pi/2),$$

which implies

$$\beta_x = -\mu_x + \mathrm{O}\ a(\mathrm{mod}\ \pi);$$
$$\beta_y = -(\mu_y + \pi/2) + \mathrm{O}\ a(\mathrm{mod}\ \pi).$$

Since $\beta_a + \mu_a$ is the angle $\hat{p}_a$ makes with the X axis, $\hat{p}^x$ lies along X, while $\hat{p}^y$ lies along Y. (This is also true for single polarization.) The two vectors $\hat{p}^a$ and $\hat{n}^a$ counterrotate; to lowest order their angular velocities cancel rather than add.

Since $\hat{p}^a$ is nonrotating to zeroth order in a, the rotated version, $\hat{P}^a$, will also be nonrotating to zeroth order. If we insert the formulas for μ and β into Eq. (74) for $\hat{P}_a$ and include the terms of order a, we get angular momenta (basis vectors now $[\hat{X},\hat{Y},\hat{Z}]$)

$$\begin{aligned}
&L_0[1-(a/2)\cos kn]\hat{P}^x \\
&\quad = L_0[1-(a/2)\cos kn, \pm(a/2)\sin kn, \\
&\qquad \mp \left(ka\gamma/4\sqrt{1+\gamma^2}\right)\cos(kn\pm\lambda)]; \\
&L_0[1+(a/2)\cos kn]\hat{P}^y \\
&\quad = L_0[\pm(a/2)\sin kn, 1+(a/2)\cos kn, \\
&\qquad \pm \left(ka\gamma/4\sqrt{1+\gamma^2}\right)\cos(kn\pm\lambda)].
\end{aligned} \tag{D11}$$

The counterrotating μ and β produce leading terms which are stationary. The X and Y components are the original ansatz for E^a. The Z components cancel to order a, as they do in the single polarization case.

The effect on test particles is given by

$$\begin{aligned}
\Delta x &= e^x_X\Delta X + e^y_Y\Delta Y \\
&= \mathrm{E}^x_X\Delta X + \mathrm{E}^y_Y\Delta Y + \mathrm{O}\ a^2 \\
&= \Delta X + (a/2)[-\cos kn\Delta X \pm \sin kn\Delta Y] \\
&= \Delta X + (a/2)\sqrt{(\Delta X)^2+(\Delta Y)^2} \\
&\quad \times[-\cos kn\cos\Phi \pm \sin kn\sin\Phi] \\
&= \Delta X + (a/2)\sqrt{(\Delta X)^2+(\Delta Y)^2}[-\cos(kn\pm\Phi)] \\
&= \Delta X + (a/2)\sqrt{(\Delta X)^2+(\Delta Y)^2}[-\cos(\omega t\mp\Phi)].
\end{aligned} \tag{D12}$$

On the second line we have used

$$|e| = 1 + \mathrm{O}\ a^2.$$

On succeeding lines we shift to polar coordinates (R, Φ) in the inertial frame, then replace $kn \to (kn - \omega t)$ and evaluate at $n = 0$. A similar treatment for the y coordinate gives

$$\begin{aligned}
(\Delta x, \Delta y) &= (\Delta X, \Delta Y) + (a/2)\sqrt{(\Delta X)^2+(\Delta Y)^2} \\
&\quad \times[-\cos(\omega t\mp\Phi), \mp\sin(\omega t\mp\Phi)].
\end{aligned} \tag{D13}$$

As seen from the positive Z axis, a test particle at the origin rotates in a counterclockwise (clockwise) circle when we use the upper (lower) signs.

One may also form

$$\begin{aligned}
&\sqrt{\Delta x^2+\Delta y^2} \\
&\quad = \sqrt{\Delta X^2+\Delta Y^2}[1-(a/2)\cos(\omega t\mp 2\Phi)] + \mathrm{O}\ a^2.
\end{aligned} \tag{D14}$$

At $t = 0$ a circle of test particles becomes a circle, plus a small standing wave wrapped around the circle (two wavelengths long, four nodes). For $t > 0$ the standing wave rotates counterclockwise (clockwise) when we use the upper (lower) sign.

APPENDIX E: RENORMALIZATION

This section is intended for readers familiar with a coarse-graining recipe developed by a number of authors [4–7]. Readers who are not familiar but would like to learn more probably should start with Ref. [7].

The present treatment is hardly coarse-grained. The number of vertices per cycle, N_λ, is assumed to be quite large: N_λ times order 100 Planck lengths is the macroscopic wavelength. In this appendix we "coarse-grain": N vertices are replaced by a single vertex. N may be taken very large, but should be much less than N_λ, so that after the coarse-graining there are still a large number of vertices per cycle.

The coarse-graining method of the references starts by choosing a "maximal tree." This is a tree which contains no loops and passes through each of the N vertices once and only once. For a general, three-dimensional lattice, the maximal tree is not unique, and one is forced to discuss dependence on choice of tree. In the present case, the maximal tree is unique; it is just the z axis.

After the tree is chosen, one must SU(2) gauge transform each holonomy along the tree to the unit holonomy. This allows the collapse of the N vertices to a single vertex. In the planar case the holonomies along the maximal tree are z holonomies peaked at $\theta_z = 0$. The holonomies are already unit holonomies, and no gauge transformations are needed. (If the holonomies are nontrivial, further transformations are needed after the N vertices collapse to one vertex. Those additional transformations are not needed in the planar case.)

Each of the N initial vertices is the endpoint for two transverse loops, one in the x direction and one in the y direction. After collapse the final vertex has 2N loops beginning and ending at that vertex. (In the literature this is described picturesquely as a flower diagram having 2N petals.)

The wave function at the surviving vertex is quite complex. It is a product of N "x" SU(2) coherent states formerly at vertices $1, 2, \ldots, N$; and N "y" SU(2) coherent states. To estimate the new peak angular momentum, we repeat the calculation given at Eq. (63). The x coherent state (for example) now contains an exponential which is a sum over the N loops.

$$\exp[\cdots] = \exp\left[-t\sum_i L_i(L_i+1)/2 + \sum_i p_i L_i\right] = \exp\left\{-(t/2)\sum_{i=1}^{N}[L_i + 1/2 - (p_i/t)]^2 + f(p_i, t)\right\}. \tag{E1}$$

The exponent is a sum of squares, and can be minimized only by minimizing each square. From Eqs. (63) and (109)

$$p_i/t = \langle L_i + 1/2\rangle = L_0(1 + \mathrm{O}\ a) = (2/\gamma\kappa)\Delta x \Delta z(1 + \mathrm{O}\ a). \tag{E2}$$

Meaning, p_i/t is the peak value of $\mathrm{L}_i + 1/2$ before coarse-graining; and the exponent in Eq. (E1) can be minimized by retaining those peak values after coarse-graining.

Since L_i is a constant independent of i (apart from order a oscillations which are not important here), one can replace L on the first line of Eq. (E1) by an average. Neglecting terms independent of L, we get

$$[\cdots] = -(\mathrm{N}t/2)[(L+1/2)_{\mathrm{rms}} - \sum_i (p_i/t\mathrm{N})]^2;$$
$$(L+1/2)_{\mathrm{rms}} = \left[\sum_i (L_i + 1/2)^2/N\right]^{1/2}. \tag{E3}$$

The rms value is peaked; and the peak value is an average over the p_i.

One can also compute the new curvature, which goes as $\ddot{\mathrm{E}}/\mathrm{E}$, double dot denoting a second difference. Now numerator and denominator of $\ddot{\mathrm{E}}/\mathrm{E}$ become a sum of terms, one from each vertex. For simplicity we suppress the index x or y, and write $\mathrm{E}_i^{(k)}$ for the order a^k contribution from vertex i.

$$\begin{aligned}\ddot{\mathrm{E}}/\mathrm{E} &= \sum_{i=1}^{N}[\ddot{\mathrm{E}}_i^{(1)} + \ddot{\mathrm{E}}_i^{(2)}]/\sum_{i=1}^{N}(1+\mathrm{E}^{(1)}) \\ &= \sum_{i=1}^{N}[\ddot{\mathrm{E}}_i^{(1)}(1+\mathrm{E}^{(1)})/\sum_{i=1}^{N}(1+\mathrm{E}^{(1)}) \\ &= \bar{\ddot{\mathrm{E}}} + \sum_{j=1}^{N}(\ddot{\mathrm{E}}_j - \bar{\ddot{\mathrm{E}}})(\mathrm{E}_j - \bar{\mathrm{E}})/\mathrm{N}(1+\bar{\mathrm{E}}) \\ &= \bar{\ddot{\mathrm{E}}} + (-k^2)\sum_{j=1}^{N}(\mathrm{E}_j - \bar{\mathrm{E}})^2/\mathrm{N}(1+\bar{\mathrm{E}}),\end{aligned} \tag{E4}$$

where the bar denotes an average over N,

$$\bar{f} \coloneqq \sum_{i=1}^{N} f_i/N. \tag{E5}$$

Superscripts (1), (2) have been dropped; all fields are now $\mathrm{E}^{(1)}$ fields, order unity in a.

The averages may be estimated by replacing sums by integrals, for example

$$\begin{aligned}\bar{\mathrm{E}} &= (1/\mathrm{N})\sum_{j=1}^{N}\mathrm{E}_j\Delta n \\ &\cong (1/\mathrm{N})\int_{n_0}^{n}(-a/2)\sin(kn)dn \\ &= (1/\mathrm{N})(a/2k)[\cos(kn) - \cos(kn_0)] \\ &= (1/\mathrm{N})(-a/k)\sin[k(n+n_0)/2]\sin[k(n-n_0)/2],\end{aligned} \tag{E6}$$

where $n = n_0 + \mathrm{N}$. For simplicity we have ignored the damping factor.

Note we are averaging only over part of one cycle ($\mathrm{N} \ll \mathrm{N}_\lambda$) so that the averages over sinusoids are order unity, not negligible (as they would be if we were averaging over several cycles). In particular,

$$\bar{\mathrm{E}} = \mathrm{O}\ (a/(k)(1/\mathrm{N}) = \mathrm{O}\ (\mathrm{N}_\lambda/\mathrm{N})(a/2\pi).$$

Nevertheless we may drop the final sum in Eq. (E4). It is order k^2a^2 whereas the $\bar{\ddot{\mathrm{E}}}$ term is order k^2a.

For the curvature we expect

$$\ddot{\mathrm{E}}/\mathrm{E} = -(k^2a\zeta/2)\sin[k(n+n_0)/2]. \tag{E7}$$

I.e., the sine is evaluated in the middle of the interval (n, n_0); and the factor ζ takes into account the possibility of a renormalization of the amplitude a. Comparing equations (E6) and (E7), we have

$$(1/\mathrm{N})(-ka)\sin[k(n-n_0)/2] = -(k^2a\zeta/2).$$

With $(n - n_0) = \mathrm{N}$, $\mathrm{k} = 2\pi/\mathrm{N}_\lambda$, this gives

$$\zeta = \sin(\pi \mathrm{N}/\mathrm{N}_\lambda)/(\pi \mathrm{N}/\mathrm{N}_\lambda). \tag{E8}$$

For $\mathrm{N} \to 1$ (the smallest possible value of N) $\mathrm{N}/\mathrm{N}_\lambda \cong 0$, and ζ has the correct limit $\zeta \to 1$.

APPENDIX F: THE U(N) FORMALISM

A number of authors have developed a formalism which avoids explicit SU(2) rotation matrices and uses a representation of SU(2) based on holomorphic functions (Bargmann representation) [5,6,23,24]. The approach involves a number of operators which together form a representation of U(N); we will refer to this approach as the U(N) approach. In this appendix we assume the reader is already somewhat familiar with the U(N) formalism; readers who desire an introduction might try Ref. [5].

The U(N) approach shifts the focus from holonomy on edge e to spinors located at the two ends of edge e. In particular the holonomy on the transverse x edge is replaced by two spinors, a source spinor at the beginning of the edge, and a target spinor at the end:

$$\begin{pmatrix} s_+ \\ s_- \end{pmatrix}, \qquad \begin{pmatrix} t_+ \\ t_- \end{pmatrix}.$$

The U(N) formalism works with spinor operators as well as spinor peak values, when a coherent state basis is used. To be clear, the above spinors are the peak values. We suppress edge labels (x, y, or z) on the spinors. Until further notice we consider only x spinors.

In order to express the peak spinors in terms of the parameters used in the present paper, we associate each spinor with a vector according to the following theorem. The spinor χ,

$$\chi(\xi, p_z) = \begin{pmatrix} \sqrt{1+p_z} \\ \sqrt{1-p_z}\exp i\xi \end{pmatrix},$$

generates a unit vector via

$$\chi^\dagger(\vec{\sigma}/2)\chi = \left(\sqrt{1-p_z^2}\cos\xi, \sqrt{1-p_z^2}\sin\xi, p_z\right). \tag{F1}$$

The σ are the usual Pauli matrices. Conversely, the unit vector determines the spinor, up to an overall arbitrary phase.

We determine the spinors by demanding that they reproduce the correct direction for the angular momentum vector at each vertex. Normally the source and target spinors live at different vertices; here they live at the same vertex because of the S_1 topology in the transverse directions. However, angular momentum experiences no change when traveling along the z axis from vertex $n-1$ to vertex n, since the z holonomy is a unit matrix. Further, vertex n will need information about spinors at vertex $n-1$, parallel transported to vertex n, in order to construct covariant differences. We therefore take the source spinor to correspond to angular momentum at vertex $n-1$.

$$\begin{aligned} s &= u(-\beta + \pi/2, \alpha(n-1)/2, +\beta - \pi/2)s(0) \\ &:= u(n-1)s(0; n-1) \end{aligned} \tag{F2}$$

This equation is the spinor analog of Eq. (65),

$$\hat{P}^a_A(\alpha) := \hat{p}^a_B \mathrm{D}^{(1)}(u_a)_{BA}.$$

$s(0; n-1)$ corresponds to the vector $\hat{p}$; s corresponds to the rotated vector $\hat{P}$, which is along the angular momentum. u, a spin 1/2 rotation through $\alpha/2$, corresponds to $\mathrm{D}^{(1)}$, an $\mathrm{L} = 1$ rotation through $\alpha/2$. The vector

$$\hat{p} = (\cos(\mu + \beta), \qquad \sin(\mu + \beta), 0)$$

happens to be independent of n (neglecting terms of order a^2 in the small amplitude a); therefore we can drop the $n-1$ index in $s(0; n-1)$. From Eq. (F1)

$$s(0) = \begin{pmatrix} 1 \\ \exp i(\beta + \mu) \end{pmatrix}. \tag{F3}$$

The target spinor at the other end of the transverse holonomy can be written similarly

$$t = u(-\beta + \pi/2, \alpha(n)/2, +\beta - \pi/2)s(0) := u(n)s(0). \tag{F4}$$

The spinor of Eq. (F4) gives the correct direction for angular momentum at n.

$$\begin{aligned} t^\dagger(\sigma_A/2)t &= s(0)^\dagger u(n)\dagger(\sigma_A/2)u(n)s(0) \\ &= s(0)^\dagger(\sigma_B/2)s(0)D^{(1)}_{BA} \\ &= \hat{p}_B D^{(1)}_{BA} = \hat{P}. \end{aligned} \tag{F5}$$

For simplicity we have been computing with unit spinors, but strictly speaking s and t should be multiplied by $\sqrt[4]{j(j+1)}$ to give the angular momentum vectors the correct length.

The spinor t in full detail is

$$t = \begin{bmatrix} \cos(\alpha/4) + i\sin(\alpha/4)\exp(i\mu) \\ \cos(\alpha/4)\exp[i(\beta+\mu)] + i\sin(\alpha/4)\exp(i\beta) \end{bmatrix},$$

which shows that the key U(N) variables (spinors now, not holonomies) vary sinusoidally with n. In the general case the spinors produce an SU(2) result. In the planar case, the spinors must generate an O(3) symmetry; hence the half-of-half-angle cosines and sines in the $j = 1/2$ case.

The U(N) formalism contains a fundamental holonomy operator which is not simply related to the x and y holonomies used in the present paper. The U(N) operator contains a holomorphic part, one which depends only on unstarred spinors, and therefore has a peak value depending only on unstarred spinors s and t. The holonomies used in the present paper must be constructed using both starred and unstarred spinors. For example, the eigenspinors and eigenvectors $\exp(\pm i\alpha/4)$ of u can be used to construct u.

$$u(n) = \chi \exp(i\alpha/4)\chi^\dagger + C\chi^* \exp(-i\alpha/4)(C\chi^*)^\dagger;\\ \chi = \chi(0,\beta). \tag{F6}$$

χ is given by Eq. (F1). C is the usual charge conjugation matrix $-i\sigma_Y$; χ and $C\chi^*$ form a complete set. In the U(N) formalism, the matrix Eq. (F6) cannot qualify as fundamental, because both terms on the right contain starred spinors.

The following is an example of a variable which is holonomic, and therefore plays a fundamental role in the U(N) formalism. This variable is especially simple to compute, because in the planar case every transverse x holonomy has the same axis of rotation (and similarly for the y holonomies).

$$\begin{aligned} \mathrm{F}[t,s] &:= [Ct^*]^\dagger s \\ &= [Cu^*(n)s(0)^*]^\dagger u(n-1)s(0) \\ &= s(0)^{\mathrm{tr}} C^\dagger u(-\beta+\pi/2, [\alpha(n-1) \\ &\quad - \alpha(n)]/2, +\beta-\pi/2)s(0) \\ &= 2\exp[i(\beta+\mu)][\sin(\Delta)\sin(\mu)]; \\ \Delta &= [\alpha(n)-\alpha(n-1)]/4. \end{aligned} \tag{F7}$$

The overall phase can be removed by changing the arbitrary overall phases of the basic spinors.

Note all boosts have been fixed when SL(2, C) is reduced to SU(2) in the canonical approach. Ordinarily one would construct the intertwiners at each vertex from $\mathrm{F}[i,j]$ variables, because those are SL(2, C) invariant as well as SU(2) invariant. However, when boosts are fixed, one may use also $\mathrm{E}[i,j]$ variables, which are only SU(2) invariant.

$$\begin{aligned} \mathrm{E}[t,s] &:= t^\dagger s \\ &= 2\cos(\Delta) + 2i\sin(\Delta)\cos(\mu). \end{aligned} \tag{F8}$$

In the z direction one must use E, since all z angles are zero and the corresponding F[t, s] vanishes.

The formalism developed in this paper uses a matrix $\mathcal{H}$ which lies in the complex extension of SU(2). The columns of this matrix also occur in the U(N) formalism. The spin 1/2 representation of $\mathcal{H} = \exp[\vec{p}\cdot\vec{\mathrm{S}}]$ is

$$\mathcal{H}^{(1/2)} = [\exp(p/2)/2]\begin{bmatrix} 1 & \exp[-i(\beta+\mu)] \\ \exp[i(\beta+\mu)] & 1 \end{bmatrix},$$

in the limit of moderately large p ∼ 5. The two columns of $\mathcal{H}$ are essentially the spinor s(0).

The U(N) formalism is slightly less intuitive than the usual formalism, because the spinor is less intuitive than the associated vector. However, the usual formalism works directly with the vector, and is not particularly intuitive either. (A major motivation for the present paper was to build some intuition.)

Turning from intuition to computation: when one considers states of spin higher than 1/2, the U(N) expressions are easier to manipulate. One encounters factorials which are actually explicit expressions for Clebsch-Gordan coefficients. Given these expressions, usually it is easy to recouple without consulting a table of 3J symbols.

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