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Plane wave holonomies in quantum gravity. I. A model

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This is the first of two papers which study the behavior of the SU(2) holonomies of canonical quantum gravity, when they are acted upon by a unidirectional, plane gravity wave. The present paper constructs a model based on holonomies and fluxes having support on a lattice (LHF denotes lattice-holonomy flux, rather than loop quantum gravity, since there are no loops). Initially, the flux-holonomy variables are treated as classical, commuting functions rather than quantized operators, in a limit where variation from vertex to vertex is small and fields are weak. We impose symmetries and fix gauges at the classical level. Despite the weakness of the fields, the field equations are not linear. Also, the theory can be quantized, and the expectation values of the quantum operators behave like their classical analogs. Exact LHF theories may be either local or nonlocal. The present paper argues that a wide class of nonlocal theories share nonlocal features which survive to the semiclassical limit, and these nonlocal features are included in the near-classical limit studied here. An appendix computes the surface term required when the propagation direction is the real line rather than S_1. Paper II introduces coherent states, constructs a damped sine wave solution to the model, and solves for the behavior of the holonomies in the presence of the wave.

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I. INTRODUCTION

This and a succeeding paper [1] investigate the behavior of quantum gravity fluxes and holonomies in the presence of a gravitational plane wave. The behavior of traditional, metric variables in the presence of a weak gravitational wave is well known. Metric waves are discussed in most classical textbooks; quantization of the theory is straightforward. However, no corresponding discussion has been given for the flux-holonomy variables characteristic of canonical quantum gravity.

Most calculations in this paper are classical. Constraints may be imposed either at the classical level, or at the quantum level, in the Dirac manner. However, the unidirectional constraints are second-class and *must* be treated at the classical level. Second class constraints require Dirac brackets, which are messy. Computation of Dirac brackets for the most general classical theory (two polarizations, no gauges fixed) is especially complicated. We choose to fix gauges and impose symmetries at the classical level, which leads to the simplest possible Dirac brackets. Quantization (early in paper II [1]) is then straightforward, requiring a single paragraph: replace (Dirac) brackets by commutators; choose factor orderings.

After that one-paragraph foray into the quantum theory, the paper reverts to the classical side. The Hilbert space is based on coherent states, and these states turn quantum operators back into classical, commuting functions. Consider a quantum constraint which is a product of operators $O_1O_2\cdots$, acting on a coherent state $|\mathrm{coh}\rangle$:

$$(O_1O_2\cdots)|\mathrm{coh}\rangle = (O_1(cl)O_2(cl)\cdots)|\mathrm{coh}\rangle + \mathrm{SC}. \qquad (1)$$

SC denotes small correction states, down by order $1/\sqrt{\mathrm{L}}$; the coherent state is typically only an approximate eigenstate of the O_i. Since coherent states are peaked at specific values of flux and holonomy, the operators become functions $O_i(cl)$ evaluated at those peak values.

Paper II [1] constructs a sinusoidal solution to the classical constraints. One then reads Eq. (1) right to left: in a regime where coherent states are applicable, a classical solution to the constraints implies the vanishing of the quantum constraint. The classical results then carry over to the quantum theory: the expectation value of a quantum operator varies with the plane wave in the same way as the corresponding classical variable.

We refer to the model as a LHF theory (latticr, holonomy, flux) rather than LQG (loop quantum gravity), because there are no loops (and perhaps because most calculations are done at the classical level). Also, various authors define LQG differently, and calling the model LHF allows the reader to decide whether the present LHF equals his/her LQG.

The present LHF model possesses six features associated with many LQG models. The basic variables are holonomies and fluxes; they have support only on a lattice; variables are invariant under spatial diffeomorphisms; areas and volumes are quantized; the field theory limit (the limit where the lattice disappears) is correct; and the theory is adequately regulated. ("Adequately": 1/volume need not be regulated. Since spins are large in the classical limit, the volume does not vanish.)

Every *exact* theory should also possess the above six features (excepting the comment about the volume).

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Two further assumptions are introduced to make the theory easier to solve, as well as closer to the classical limit: dynamical quantities vary slowly from vertex to vertex; holonomies are small. Precise definitions of "slowly" and "small" are given in Secs. II A and II B.

From the discussion just given, the model studied here is not an exact LHF theory, but rather a near-classical limit of an exact theory. When one proposes an exact theory, it is important to check that the classical limit is correct. However, the present paper does not have as its goal to construct and test a specific, exact theory; rather, the paper investigates the behavior of holonomies in a near classical limit. For this goal, and in the present theoretical environment, where there is no universally agreed upon exact theory, the six properties listed above become not just an adequate starting point. They become a fundamental starting point. They are fundamental in the sense that, if the classical limit of an exact theory did not have one or more of the six features, presumably one would reject that exact theory as inadequate. Given our relatively modest goals, as well as the (current) fundamental nature of the six properties, we believe it is better to start from the six rather than from an exact theory.

The term "classical" is slightly ambiguous. "Classical" may refer to a theory using commuting flux-connection variables, with support on a continuum. Or "classical" may mean a theory using commuting flux-holonomy variables, with support only on a lattice. The theory defined on the continuum will be referred to as classical field theory, or simply field theory (FT). A classical theory defined on a lattice will be referred to as classical LHF.

For works which study the renormalization flows connecting exact to classical theories, see Han [2], and Giesel and Thiemann [3]. Han uses a path integral/spin foam approach, rather than the canonical approach used here. Giesel and Thiemann take the semiclassical limit of the master constraint, rather than the usual scalar and vector constraints, S and V. However, the semiclassical master constraint is essentially a sum of squares, $S^2 + V^a V_a$, so that the solutions of the follow-up paper II [1] (which are annihilated by S and V) should also be solutions to the master constraint.

Both the above treatments are quite general; there is no discussion of the planar case or gauge-fixing. Bannerjee and Date [4] construct an exact theory which is specialized to the planar case; see also Hinterleitner and Major [5]. Both papers use a Bohr quantization of the transverse degrees of freedom. There are no holonomies along transverse directions (x,y), only holonomies along the longitudinal direction (direction of propagation, z). Transverse degrees of freedom are represented by two scalars (essentially, the magnitudes of the axes of the polarization ellipse) plus an angle (nonzero, if the ellipse axes do not coincide with the x and y axes). However, if the goal is a study of the behavior of holonomies in the presence of a gravitational wave, then the theory must use holonomies along x and y, rather than Bohr quantization.

In addition, both those theories are local, meaning, holonomic loops used to define the field strengths at the nth vertex remain infinitesimally close to that vertex. A nonlocal, "nearest neighbor," theory uses finite loops which include the nearest neighbor vertices at $n \pm 1$. Motivation for including nonlocal features is given at the beginning of Sec. III.

The present work has both a primary and a secondary goal. The primary goal is to study the behavior of holonomies in the presence of a gravitational wave. The secondary goal is to assume the exact theory is nonlocal, and study the effect of nonlocal features on the theory.

In deciding which nonlocal features to include, again one should not start from one exact nonlocal theory. Rather, one should identify nonlocal features which are common to a large class of nonlocal theories and can survive to the classical limit. Consider the class of theories which treat both nearest neighbors on an equal footing. For example, let the vertices along z be indexed by integers n_z. If the model includes a nonlocal holonomic loop starting at n_z and going to $n_z + 1$, then it must also include a loop starting at n_z and going to $n_z - 1$, and with equal weight. Since there is no reason to favor one nearest neighbor over the other, this class of theories is likely to be large.

Given equal treatment of both nearest neighbors, which nonlocal features survive? Section III studies a sample nonlocal model which treats both nearest neighbors equally. Let $h_a(n_z)$ be a holonomy along transverse edges a = x or y, located at vertex numbered n_z. A local definition for the derivative of h_a would be

$$[h_a(n_z + \epsilon/2) - h_a(n_z - \epsilon/2)]/\epsilon.$$

ϵ is an infinitesimal regulator which cancels out at the end of the calculation. The model of Sec. III, when taken to the limit of small connections and slow variation, replaces the above local definition by a nearest-neighbor, nonlocal generalization,

$$[h_a(n_z + 1) - h_a(n_z - 1)]/2\Delta z. \tag{2}$$

Δz, the distance between vertices, must be small, like ϵ, so that the above ratio is a good approximation to the derivative. Unlike ϵ, Δz is not taken to zero at the end of the calculation. In a nonlocal approach the difference, and not the derivative, is fundamental.

Similarly, the exact version of the model of Sec. III is built from holonomies

$$h_z(a,b) := \exp\left[i \int_a^b \mathrm{A}_z^Z \sigma_Z/2\right],$$

where h_z is the holonomy along the propagation direction, and a and b $(b - a = 1)$ are nearest neighbor vertices.

In the slow variation limit, these are replaced by a combination which treats the two nearest neighbors symmetrically:

$$h_z(n_z) \coloneqq [h_z(n_z, n_z+1) + h_z(n_z-1, n_z)]/2 \quad \text{(nonlocal)};$$
$$h_z(\text{local}) = [h_z(n_z, n_z+\epsilon) + h_z(n_z-\epsilon, n_z)]/2. \tag{3}$$

(The local version is included for comparison.[1])

The model considered in Sec. III is relatively simple. Appendix A discusses a more complex nonlocal model, which also treats nearest neighbors equally. This model uses nonstandard grasps, as well as definitions of differences and z holonomies which do not agree with the nonlocal definitions given in Eqs. (2) and (3). Nevertheless, in the semiclassical limit the nonlocal definitions in the exact model are replaced by the definitions at (2) and (3). The two models offer strong support for the idea that a nonlocal model which treats neighbors equally will always possess a semiclassical limit with differences and holonomies given by (2) and (3).

Note the above nonlocal modifications, (2) and (3), are certainly plausible. Even before studying any exact nonlocal model, if one wished to treat nearest neighbors equally, then the above central difference and z holonomy are certainly the simplest possibilities. Presumably the reader could skip the detailed study of the nonlocal models on a first reading.

The existence of a universal classical limit is another reason why we do not start from an exact theory. The slow variation assumption is a mini version of the coarse graining assumption used to treat models in statistical mechanics, because slow variation allows contributions from several vertices to be averaged together. As in statistical mechanics, the finer details of the exact theory often are lost; what counts is the universal limit.

The present model is an effort to go beyond standard approximations in two ways. As just discussed, the model is nonlocal. Also, it is less nonlinear than exact theories, but still nonlinear. The solution constructed in paper II [1] is an expansion in a small amplitude a, and nonlinear effects appear at order a^2. (If only terms linear in a are kept, and if differences are replaced by derivatives, then the model is equivalent to weak field geometrodynamics.)

Section IV constructs the Euclidean Hamiltonian of the model. Section V discusses the flux-holonomy algebra of the model.. Section VI constructs extrinsic curvatures using a technique proposed by Thiemann. Section VII constructs the Gauss constraint and the Lorentzian Hamiltonian. Sections VIII, IX, and XI discuss single polarization, diffeomorphism, and unidirectional constraints respectively. Section X discusses boundary conditions at infinity.

[1]The nonlocal version actually transforms like a local operator under gauge transformations, but only after a slow variation assumption is invoked. See Sec. III B.

The term "planar" is a slight misnomer: the theory does not have full planar symmetry in the xy plane. With a suitable choice of coordinates, the Killing vectors become $\partial/\partial x, \partial/\partial y$, implying that all functions are independent of x and y. However, this is translational invariance, not full planar symmetry, which would require isotropy with respect to rotations in the xy plane. Isotropy is inconsistent with the presence of waves. Vibrations of the usual cloud of test particles are described by an ellipse, which picks out preferred directions. The translational invariance implies the ellipse is the same everywhere in the xy plane.

For a quantization of plane waves using geometrodynamics variables, see Mena Marugán and Montejo [6].

A. Conventions

Throughout, indices from the middle of the alphabet $i, j, \ldots$ range over coordinates x, y, z on the manifold. Indices from the beginning of the alphabet $a, b, \ldots$ range over x, y only, where z is the direction of propagation. Similarly, indices I, J, K range over coordinates X, Y, Z in the local free-fall frame. Indices $A, B\ldots$ range over transverse directions X, Y only.

When expanding 2×2 matrices, we use Hermitian sigma matrices, rather than anti-Hermitian tau matrices. A typical Lie group valued operator would be written

$$\mathbf{O}_i \coloneqq \mathrm{O}_i^I \boldsymbol{\sigma}_I. \tag{4}$$

The sigma matrices, and bold face for matrices, will be suppressed except when it is necessary to emphasize the matrix character of an equation. It should be clear from the context which quantities are sigma valued. Usually the operator in Eq. (4) will be written simply as O_i.

In LHF densitized cotriads are written as area two-forms,

$$\mathrm{E}^i_I(n) dx^j \wedge dx^k \epsilon_{ijk}/2!,$$

and connections are written as one-forms, $\mathrm{A}^J_j\, dx^j$. The area and line integrals in the definitions of triad and connection guarantee simpler transformation properties under spatial diffeomorphisms; also, the [holonomy, triad] commutator will contain enough integrations to kill the delta function. Usually, the area and line integrals will be suppressed. For example, E^i_I will be written as

$$\mathrm{E}^i_I dx^j \wedge dx^k \epsilon_{ijk}/2! \to \mathrm{E}^i_I. \tag{5}$$

B. Initial gauge fixing

Because of the planar symmetry, Husain and Smolin are able to choose gauges which simplify the $\tilde{\mathrm{E}}$ and connection fields [7]. These choices reduce the general, $3+1$ dimensional case to the planar case; they therefore precede all the gauge choices to be made in this paper.

$$\mathrm{E}_Z^a = \mathrm{E}_A^z = 0;$$
$$\mathrm{A}_a^Z = \mathrm{A}_z^A = 0. \quad (6)$$

$a = \mathrm{x}, \mathrm{y}$; $\mathrm{A} = \mathrm{X}, \mathrm{Y}$. These choices fix the SU(2) rotations around axes X,Y and the diffeomorphisms in transverse directions x,y. Three constraints survive: the scalar constraint, the vector constraint for z diffeomorphisms, and the Gauss constraint for rotations around Z: $\mathrm{SU}(2) \to \mathrm{U}(1)$.

Since the only A_z^I which survives has $\mathrm{I} = \mathrm{Z}$, holonomies along the longitudinal z direction are quite simple, involving only the rotation generator S_Z for rotations around Z.

$$\exp\left(i \int \mathrm{A}_z^Z \cdot \mathrm{S}_\mathrm{Z}\right).$$

Conversely, the transverse holonomies (those along the x and y directions of the spin network) contain no S_Z and involve $\mathrm{S}_X, \mathrm{S}_Y$ only.

C. Topology of the spin network

As a convenience for readers not familiar with the usual network used in the planar case, this section includes a description of the topology.

In the z direction (direction of propagation of the wave) the spin network has the topology of the real line. The line includes a series of vertices, labeled by integers n_z. The vertices are connected by edges, which may be labeled by their endpoints, as (n_z, n_{z+1}).

In directions transverse to propagation, there are two possible approaches. The first approach is easiest to relate to the full, three-dimensional case. We give each vertex on the original z axis three integer coordinates: $(n_x = 0, n_y = 0, n_z)$. We then construct a three-dimensional rectangular lattice by drawing a congruence of lines, all parallel to the original z axis. All lines have the identical arrangement of vertices, but differ in their x and y coordinates, $n_x = \pm 1, \pm 2, \ldots, n_y = \pm 1, \pm 2, \ldots$. We connect neighboring vertices having the same n_z with edges $(n_x, n_{x+1}), (n_y, n_{y+1})$. In this way one fills out a full, three-dimensional rectangular lattice.

Each member of the congruence is labeled by a pair of indices (n_x, n_y), and each vertex by a triplet (n_x, n_y, n_z). Because of the translational invariance, physics will be independent of (n_x, n_y). We will refer to this as the "congruence" picture. [This is a slight abuse of notation, since members of a traditional congruence are labeled by continuously varying parameters, rather than discrete integers (n_x, n_y).]

The second method for handling the transverse directions is simpler topologically, but a little harder to relate to the three-dimensional case. We construct a small cubic box surrounding each vertex and equip each face with an outward normal. We call a face positive (negative) if its normal points in the positive (negative) coordinate direction. The holonomy with support on edge (n_x, n_{x+1}) leaves a cube at position n_x, passing through the positive x face, then enters the nearest neighbor cube at n_{x+1}, passing through a negative x face. Because of the translational symmetry, the holonomy entering the negative x face of cube n_{x+1} must be identical to the holonomy entering the negative x face of cube n_x. Therefore one could give the edge (n_x, n_{x+1}) the topology of a circle. The holonomy leaves cube n_x through the positive x face, travels along (n_x, n_{x+1}) (now a circle, rather than a straight line) and reenters n_x through the negative x face.

The congruence has now disappeared. There is only a real line R in the z direction, and two S_1 edges leaving each vertex in the x and y directions. We will refer to this as the "S_1 picture." The $\mathrm{R} \times \mathrm{S}_1 \times \mathrm{S}_1$ topology is simpler for calculations; but for thinking, it is perhaps better to use the congruence: one has more assurance the results will generalize to three dimensions.

In the congruence picture, it is natural to refer to the smallest rectangular area enclosed by x and y edges as an "xy plane." We use this terminology, even though in the S_1 picture this area has the topology of a torus. Similarly, an area bounded by two neighboring edges in the z direction and two neighboring x edges will be called the "xz plane." In the S_1 picture this area has the topology of a cylinder.

II. APPROXIMATIONS

This section proposes specific small field and slow variation assumptions. These assumptions simplify calculations; they also bring the theory close to the limit where quantum behavior goes over to classical behavior.

A. The small field (small sine) approximation

One can obtain the field theory limit of LHT (lattice → continuum) by expanding the holonomy as

$$h_i = \exp\left(i \int \mathrm{A}_i \cdot \mathrm{S}\right) \to 1 + i \int \mathrm{A}_i \cdot \mathrm{S} \quad (\mathrm{FT}). \quad (7)$$

This expansion is too drastic for present purposes. It replaces a bounded expression by an unbounded one. The following, small sine approximation is less drastic, in that the bounded expression is replaced by another bounded expression, because the connection remains inside a holonomy. We expand the basic spin $1/2$ holonomy in sigma matrices:

$$\mathbf{h}_i = \exp(i\boldsymbol{\sigma} \cdot \hat{n}^{(i)} \theta_i/2)$$
$$= \mathbf{1} \cos(\theta_i/2) + i \sin(\theta_i/2) \hat{n}^{(i)} \cdot \boldsymbol{\sigma}. \quad (8)$$

h_i is a rotation through θ around an axis given by $\hat{n}$. We now expand the expression in powers of sine, keeping out to linear in sine.

$$\mathbf{h}_i = \mathbf{1}[1 - \sin^2(\theta_i/2)/2 + \cdots] + i\sin(\theta_i/2)\hat{n}_i \cdot \boldsymbol{\sigma}$$
$$= \mathbf{1} + i\sin(\theta_i/2)\hat{n}_i \cdot \boldsymbol{\sigma} \quad \text{(SS)}. \tag{9}$$

SS denotes a relation which holds only in the small sine limit, when terms of order $\sin^2(\theta_i/2)$ are neglected. The function on the right remains bounded.

When carrying out this expansion, it is a little simpler to write each holonomy as

$$h \coloneqq \bar{h} + \hat{h};$$
$$\bar{h} = (h + h^{-1})/2 = \mathbf{1}\cos(\theta_i/2);$$
$$\hat{h} = (h - h^{-1})/2 = i\sin(\theta_i/2)\hat{n}^{(i)} \cdot \boldsymbol{\sigma}. \tag{10}$$

Then

$$h = 1 + \hat{h}, \quad \text{(SS)} \tag{11}$$

which is a more compact notation not involving explicit factors of $\sin(\theta/2)$. In this notation, the passage from small sine to field theory is [compare (11) and (7)]

$$-2i\hat{h}_i \to \int A_i \cdot \sigma \quad \text{(FT)}. \tag{12}$$

When expanding the Hamiltonian in sines, how many terms should be kept? When taking LHF to the field theory limit, one must keep terms out to order A^2, in order to recover the usual field theory Hamiltonian. Therefore, in the small sine expansion of the constraints, one must keep terms out to order $\hat{h}^2 = \text{order}\sin^2$. This will guarantee that the small sine limit has the same FT limit as full LHF.

The small sine replacement is simply a recognition that certain terms in the scalar constraint are negligible in the weak field limit. *SS need not be used everywhere in the theory.* If a given constraint or a Hilbert space state is already tractable, in its exact form, there is no need to simplify further. In particular, the follow-up paper [1] constructs a Hilbert space of states. Those states are products of exact spin 1/2 holonomies (no SS expansion). The states are coherent, so that their behavior (when acted upon by holonomy or flux operators) is already simple; a SS expansion of states would be pointless.

Since the basic holonomy is just an SU(2) rotation matrix, the products of holonomies at each vertex form representations of SU(2). One might question the validity of the SS approximation, because the kinematic dot product based on SU(2) Haar measure integrates over all values of θ_i; therefore over all values of $\sin(\theta_i/2)$, not just small values.

Here the coherent states come to the rescue. Coherent states are designed to be peaked simultaneously at both a coordinate and a conjugate momentum (θ_a and typical spin L_a, $a = \mathrm{x}, \mathrm{y}$; or θ_z and typical z component of spin m). If peak values of θ_i are chosen small, then matrix elements will be dominated by small values of $\sin(\theta_i/2)$, and the small sine approximation will be valid.

The wave functional can be peaked at small $\sin(\theta_i/2)$, only if typical angular momenta L_a (and z components m) are moderately large. As is typical for coherent states, the standard deviations of θ_a and its conjugate momentum L_a are inverses of each other. The standard deviations are order $1/\sqrt{\mathrm{L}_a}$ and $\sqrt{\mathrm{L}_a}$ respectively. Sharp θ_a therefore requires moderately large L_a, $1/\sqrt{\mathrm{L}_a} \ll \pi$. The small sine approximation breaks down if the representations of the rotation group occurring at a given vertex have too small values of total angular momentum.

The small sine assumption, discussed above, does not explicitly mention large quantum numbers. Nevertheless, it is clear from the discussion of coherent states that the small sine assumption will not work unless quantum numbers are large.

B. The slow variation assumption

In the classical limit one expects slow variation of dynamical quantities from one vertex to the next [8]. Slow variation implies that a plot of the quantity versus vertex index n_z looks like a smooth curve, rather than a union of piecewise smooth segments.

To make this idea more precise, define central and forward differences by

$$\delta_c f(n) = (f(n+1) - f(n-1))/2; \tag{13}$$

$$\delta_f f(n) = f(n+1) - f(n). \tag{14}$$

The slow variation assumption is

$$(\delta f/f) \ll 1, \tag{15}$$

where δ may be either difference.

The slow variation assumption also applies to higher differences. We define second differences by

$$\delta_c^{(2)} f(n) \coloneqq [\delta_c f(n+1) - \delta_c f(n-1)]$$
$$= [f(n+2) - 2f(n) + f(n-1)]/4;$$
$$\delta_f^{(2)} f(n-1) \coloneqq [\delta_f f(n) - \delta_f f(n-1)]$$
$$= [f(n+1) - 2f(n) + f(n-1)]. \tag{16}$$

If $\delta f/f$ is negligible, $(\delta f/f)(\delta g/g)$ is more so. Let $\mathrm{g} = \delta f$.

$$(\delta f/f)(\delta g/g) = (\delta f/f)(\delta(\delta f)/(\delta f)$$
$$= \delta^{(2)} f/f \ll 1. \tag{17}$$

The second difference is of second order in small differences.

The slow variation assumption may be thought of as a consequence of the small sine assumption. Sines contain time derivatives $\partial/\partial ct$, since, from Eq. (12),

$$2\sin(\theta_i/2) = -2i\hat{h} \to \int A_i \cdot \sigma,$$

and A contains the extrinsic curvature K. The differences δf correspond to space derivatives $\partial/\partial z$. Since the excitations are massless, time and space derivatives should be comparable: both are small if one is small. Since the two assumptions are closely connected, for brevity sometimes we will refer to small sine, slow variation simply as small sine.

Since the space derivatives are of the same order as the sines, and we are keeping out to order $\sin^2$ in the Hamiltonian, we must keep differences out to order $(\delta E/E)^2$. The spin connections Γ_i^I are of order $\delta E/E$, since they contain one derivative and are homogeneous of degree zero in the triads. Therefore Γ terms must be kept out to order Γ^2.

III. LOCAL VS NONLOCAL

Initial formulations of LQG used local field strengths [9]. Smolin used the renormalization group to argue that a local formulation does not allow propagation of information from one vertex to the next [10,11]. Thiemann [12,13] proposed his "master constraint" program, which allows nonlocal field strengths, while preserving a constraint algebra free of anomalies. (The two issues, anomalies and nonlocality, are closely connected, because the original, local formulation is anomaly free.)

A large quantum number calculation is not suited for checking the master constraint program, or equivalently checking for the presence of anomalies. In quantum geometrodynamics, anomalies arise when a constraint commutator produces a metric component to the right of a constraint. The corresponding result in LHT would be triad components to the right of a constraint. Triads have matrix elements of order the spin of the state, i.e. if a coherent state has angular momentum peaked at L, the matrix element will be order L. If the triad is moved to the left of a basic spin 1/2 holonomy in the constraint, the holonomy will change the spin of the state by order unity, and therefore change the matrix element of the triad, in its new position, by order unity. The fractional change in the matrix element, on moving the triad to the left, is then order $\Delta L/L \sim 1/L$, which is negligible in the limit of large quantum numbers. (See also the further comments on anomalies in Sec. X.)

A. A nonlocal model

If a classical nonlocal model includes contributions from both nearest neighbors, and includes them with equal weights, then the weak field limit will involve central differences and averaged z holonomies, as at (2) and (3). The following, specific model shows how this happens.

The model employs holonomic loops, nearest neighbor nonlocal.

$$\begin{aligned} F_{xy}(n_z) &= 2ih_x(n_z)^{-1}h_y(n_z)^{-1}h_x(n_z)h_y(n_z)/\Delta x\Delta y; \\ F_{za}(n_z, n_z+1) &= 2ih_a(n_z)h_z(n_z, n_z+1)^{-1}h_a(n_z+1)^{-1} \\ &\quad \times h_z(n_z, n_z+1)/\Delta x^a \Delta z; \\ F_{za}(n_z, n_z-1) &= -2ih_a(n_z)h_z(n_z-1, n_z)h_a(n_z-1)^{-1} \\ &\quad \times h_z(n_z-1, n_z)^{-1}/\Delta x^a \Delta z. \end{aligned} \tag{18}$$

On each line, add the Hermitian conjugate. The loop for F_{ij} is a finite rather than infinitesimal rectangle in plane ij. The loops $F_{za}(n, n')$ run from vertex n to nearest neighbor vertex n′, then return to vertex n. The field strength $F_{xy}(n_z)$ traverses a loop in the xy plane, and hence remains entirely at z value n_z; there is no need for two arguments.

These field strengths may be taken to the small sine limit by systematically replacing one or two $h_i^{\pm 1}$ by $\pm\hat{h}_i$, then replacing the remaining h_i by unity. A $[\hat{h}_a(n_z), \hat{h}_a(n_z+1)]$ term in F_{za} cancels because the commutator of transverse sigma matrices can give only unity or σ_z, and this vanishes when F_{za} is traced with the triad-dependent factor in the Hamiltonian, Σ^{za}. That factor contains no σ_Z

$$\Sigma^{za} = [E_Z^z \sigma_Z, E_A^a \sigma_A]/|e| \propto \sigma_B, \qquad A, B \neq Z. \tag{19}$$

The small sine limits are then

$$\begin{aligned} F_{xy}(n_z) &= [\hat{h}_x, \hat{h}_y]2i/\Delta x\Delta y; \\ F_{za}(n_z, n_z+1) &= (-2i)[\hat{h}_a(n_z+1) - \hat{h}_a(n_z)]/\Delta x^a \Delta z + (2i)[\hat{h}_z(n_z, n_z+1), \hat{h}_a(n_z+1)]/\Delta x^a \Delta z; \\ F_{za}(n_z, n_z-1) &= (-2i)[\hat{h}_a(n_z) - \hat{h}_a(n_z-1)]/\Delta x^a \Delta z + (2i)[\hat{h}_z(n_z-1, n_z), \hat{h}_a(n_z-1)]/\Delta x^a \Delta z \quad \text{(SS)}. \end{aligned} \tag{20}$$

The factors of $(-2i)$ lead to the correct FT limit; cf. Eq. (12). (Another factor of $2i$ is generated by the commutators of the sigma matrices.) There is only one possible F_{xy} because only holonomies at n_z are available for its construction. There are two F_{za} because n_z has two nearest neighbors.

We now apply slow variation to the model. Both loops $F_{za}(n_z, n_z \pm 1)$ start from the same vertex n_z and are

multiplied by the same triad factor $\Sigma^{za}(n_z)$. Nearest neighbor contributions from both n_z-1 and n_z+1 are included, and with equal weights, because there seems no reason to favor one nearest neighbor over the other.

Treating the neighbors equally has significant consequences. Consider first the forward difference terms.

$$\begin{aligned} H_e &= \cdots + (\mathrm{Tr}/2)[\mathrm{F}_{za}(n_z, n_z+1) + \mathrm{F}_{za}(n_z, n_z-1)]\Sigma^{za}(n_z) \\ &= \cdots + (\mathrm{Tr}/2)[\hat{h}_a(n_z+1) - \hat{h}_a(n_z) + \hat{h}_a(n_z) \\ &\quad - \hat{h}_a(n_z-1) + \text{order}\,\hat{h}^2]\Sigma^{za}(n_z) \\ &= \cdots + (\mathrm{Tr}/2)[2\delta_{(c)}\hat{h}_a(n_z) + \text{order}\,\hat{h}^2]\Sigma^{za}(n_z). \end{aligned} \tag{21}$$

$\mathrm{Tr}/2$ means take $1/2$ the trace over the sigma matrices in the expression to the right. The two forward differences in $\mathrm{F}_{za}(n_x, n_z \pm 1)$ have combined into one central difference, Eq. (13).

There are also consequences for the commutator terms (order $\hat{h}^2$ terms).

$$\begin{aligned} \mathrm{H}_e = \cdots + (2i)(\mathrm{Tr}/2)\{&[\hat{h}_z(n_z, n_z+1), \hat{h}_a(n_z+1)] \\ &+ [\hat{h}_z(n_z-1, n_z), \hat{h}_a(n_z-1)]\}\Sigma^{za}(n_z). \end{aligned} \tag{22}$$

Each operator may be split up into an average plus a difference.

$$\begin{aligned} \hat{h}_z(n_z, n_z \pm 1) &= [\hat{h}_z(n_z, n_z+1) + \hat{h}_z(n_z-1, n_z)]/2 \\ &\quad \pm [\hat{h}_z(n_z, n_z+1) - \hat{h}_z(n_z-1, n_z)]/2 \\ &\coloneqq \hat{h}_z(n_z) \pm \tilde{\delta}\hat{h}_z(n_z); \\ \hat{h}_a(n_z \pm 1) &= \hat{h}_a(n_z) \pm \delta_f \hat{h}(n_z) \quad \text{(SV)}. \end{aligned} \tag{23}$$

The last line uses slow variation (SV). From Eq. (16), one can work out an exact formula for $\hat{h}_a(n_z-1)$:

$$\hat{h}_a(n_z-1) = \hat{h}_a(n_z) - \delta_f \hat{h}(n_z) + \delta_f^{(2)}\hat{h}_a(n-1),$$

The slow variation assumption is used to drop the second difference, leading to the expression given in Eq. (23). These expansions may be inserted into Eq. (22) for the commutator.

$$\begin{aligned} \mathrm{H}_e = \cdots + (\mathrm{Tr}/2)\sum_{\pm}(2i)[&\hat{h}_z(n_z) \pm \tilde{\delta}\hat{h}_z/2, \hat{h}_a(n_z) \\ &\pm \delta_f \hat{h}(n_z)]\Sigma^{za}(n_z). \end{aligned} \tag{24}$$

We now expand in the small differences. Because the sum is even under $(+ \leftrightarrow -)$, terms with an odd number of differences vanish. The leading term involves the average, $\hat{h}_z$ times the *local* holonomy $\hat{h}_a(n_z)$; the $\hat{h}_a(n_z \pm 1)$ have disappeared. The term linear in differences vanishes. The term quadratic in differences is down by $(\delta f/f)^2$ and may be dropped.

We have now arrived at (2) and (3)—central, rather than forward differences—and an averaged z holonomy. This outcome is a consequence of the small sine, slow variation assumptions and the decision to include both nearest neighbor field strengths with equal weights. A nonlocal model which weights nearest neighbors equally will yield a limit with central differences, local xy holonomies, and averaged z holonomies.

In this limit one could replace differences by derivatives, because differences approach derivatives when variation from vertex to vertex is small. However, if the nonlocal approach has any validity, the future of LQG will involve differences. It is therefore helpful to retain some nonlocal features in the present calculation. It is reassuring that use of differences and averaged holonomies causes no problems, at least at this SS, SV level.

B. Brackets involving $\hat{h}_z$

The $\hat{h}_z(n_z)$ defined at Eq. (3) is nonlocal: it does not commute with $\mathrm{E}^z_Z(n_z \pm 1)$. Assuming the h have the same Poisson brackets as $\exp[i\int A\cdot\sigma/2]$, the nonlocality comes from the basic bracket

$$\begin{aligned} &\{h_z(n_z, n_z+1), \mathrm{E}^z_Z(n_z \text{ or } n_z+1)\} \\ &\quad = i(\kappa\gamma/2)(\sigma_z/2)h_z(n_z, n_z+1). \end{aligned} \tag{25}$$

$\kappa = 8\pi$ G; γ is the Immirzi parameter.

$$\mathrm{A}^A_a \coloneqq \gamma \mathrm{K}^A_a + \Gamma^A_a. \tag{26}$$

There is a factor of $1/2$ on the right in Eq. (25) because the grasps occur at endpoints of the integration ranges; therefore integrals are over only half a delta function.

Despite the nonlocality, in practice $\hat{h}_z(n_z)$ commutes like a local variable. Once again slow variation comes to the rescue. Typically, $h_z(n_z)$ occurs in a sum or is commuted with a sum. For example,

$$\begin{aligned} &\left\{h_z(n_z), \sum_m g(m)\mathrm{E}^z_Z(m)\right\} = i(\kappa\gamma/2)(\sigma_z/2)\{h_z(n_z, n_z+1)[g(n_z+1)+g(n_z)] + h_z(n_z-1, n_z)[g(n_z-1)+g(n_z)]\}/2; \\ &\quad g(n_z+1) + g(n_z) = 2g(n_z) + \delta_f g(n_z); \\ &\quad g(n_z-1) + g(n_z) = 2g(n_z) - \delta_f g(n_z) + \delta_f^{(2)} g(n_z). \end{aligned} \tag{27}$$

After neglect of terms of order $(\delta f/f)^2$, the commutator collapses to

$$i(\kappa\gamma/2)(\sigma_z/2)h_z(n_z)g(n_z)\times 2 \quad \text{(SV)},$$

which is just the local result.

The bracket, Eq. (27), involves the full holonomy h rather than $\hat{h}$; but one can extend the proof to $\hat{h}$ by using the basic Eq. (25).

$$\begin{aligned}\{\hat{h}(n_z, n_z\pm 1), \mathrm{E}^z_Z(n_z)\} &= \{[h-h^{-1}, \mathrm{E}^z_Z\}/2\\ &= i(\kappa\gamma/2)(\sigma_z/2)[[h+h^{-1}]/2\\ &= i(\kappa\gamma/2)(\sigma_z/2)\bar{h}. \end{aligned} \quad (28)$$

If this commutator occurs in a context where it is multiplied by a term of order sine, one can approximate the final $\bar{h}$ by unity.

We now discuss the gauge variance of $\hat{h}_z(n_z)$. The gauge transformation is a U(1) rotation around Z,

$$\Lambda(n) = \exp(i\chi\sigma_z/2),$$

characterized by an angle of rotation $\chi(n)$. Since rapidly varying χ would lead to transformed fields which violate slow variation, we assume $\sin\chi$ is small, of the same order as the small sines of holonomic angles.

We consider first the $\hat{h}(n, n+1)$ half of $\hat{h}_z(n)$. The small sine F_{za} field strengths are linear in $\hat{h}_z(n, n+1)$.

$$\mathrm{H}_e = \cdots + 2i(\mathrm{Tr}/2)[\hat{h}_z(n), \hat{h}_a(n)]\Sigma^{za}(n) \quad \text{(SS, SV)}.$$

However, the full Hamiltonian before the SS limit, Eq. (18), contains one factor of $h_z(n, n+1)$ and one factor of $h_z^{-1}(n, n+1)$. The two transform differently:

$$\begin{aligned} &h_z(n, n+1) \to \Lambda(n+1)^{-1}h_z(n, n+1)\Lambda(n);\\ &h_z^{-1}(n, n+1) \to \Lambda(n)^{-1}h_z^{-1}(n, n+1)\Lambda(n+1).\end{aligned}$$

In the small sine limit both $h_z(n, n+1)$ and $h_z^{-1}(n, n+1)$ reduce to $\hat{h}_z(n, n+1)$, but with different transformation behavior:

$$\begin{aligned} &h_z(n, n+1) \to \Lambda(n+1)^{-1}\hat{h}_z(n, n+1)\Lambda(n);\\ &h_z^{-1}(n, n+1) \to \Lambda(n)^{-1}(-\hat{h}_z(n, n+1))\Lambda(n+1) \quad \text{(SS)}.\end{aligned}$$

It does not matter which of these transformation rules we use for the $\hat{h}_z(n, n+1)$ in the small sine limit. Either rule can be related to a local transformation rule:

$$\begin{aligned} &\Lambda(n+1)^{-1}\Lambda(n)\Lambda^{-1}(n)\hat{h}_z(n, n+1)\Lambda(n)\\ &\quad = \exp[i\chi(n+1) - i\chi(n)]\Lambda^{-1}(n)\hat{h}_z(n, n+1)\Lambda(n);\\ &\Lambda(n)^{-1}\hat{h}_z(n, n+1)\Lambda(n)\Lambda^{-1}(n)\Lambda(n+1)\\ &\quad = \Lambda(n)^{-1}\hat{h}_z(n, n+1)\Lambda(n)\exp[-i\chi(n+1) + i\chi(n)].\end{aligned}$$

In both cases, the local transformation law is multiplied by exponential factors. These factors are negligible in the small sine limit, because they occur in a term in the Hamiltonian which is already order $\sin^2$.

The $\hat{h}(n, n-1)$ half of $\hat{h}_z(n)$ is discussed similarly, with $n+1$ replaced by $n-1$. The remaining factors in the small sine Hamiltonian, $\hat{h}_a(n)$ and $\Sigma(n)$, also transform locally, leaving the Hamiltonian invariant.

IV. A SMALL SINE, NONLOCAL H_e

We now propose the following small sine LHF Euclidean Hamiltonian.

$$\begin{aligned} -\mathrm{NH}_e + \mathrm{ST} &= \sum_{nz}\mathrm{N}(n_z)\{F^Z_{xy}(n_z)\mathrm{E}^x_J\mathrm{E}^y_K\epsilon_{ZJK}\\ &\quad + F^A_{za}\mathrm{E}^z_Z\mathrm{E}^a_B\epsilon_{AZB}\}/\kappa|e|) + \mathrm{ST};\\ F_{xy}(n_z) &= F^Z_{xy}(n_z)\sigma_Z = 2i[\hat{h}_x(n_z), \hat{h}_y(n_z)];\\ F_{za}(n_z) &= F^A_{za}(n_z)\sigma_A = 2i[\hat{h}_z(n_z), \hat{h}_a(n_z)]\\ &\quad + (-2i)\delta_c\hat{h}_a(n_z) \quad \text{(SS, SV)}. \end{aligned} \quad (29)$$

ST stands for "surface term," required because the z axis is the real line rather than S_1 (as in the Gowdy model). The surface term will allow us to calculate the energy of the plane wave in the follow-up paper [1]. The surface term is calculated in Appendix B.

The field strengths are given by the leading-order $(\sin + \sin^2)$ terms in the small sine approximation. The exact theory is assumed to treat nearest neighbors symmetrically, and the Hamiltonian is modeled after the weak field limits of the nearest neighbor models considered in Sec. III and Appendix A. Consequently, the above Hamiltonian involves central differences, rather than derivatives or forward differences. Also, the exact F_{za} may contain $\hat{h}_a(n_z \pm 1)$, but the small sine limit contains only $\hat{h}_a(n_z)$, the nearest neighbor nonlocal average defined at Eq. (3).

The triads

$$\Sigma^{ijK} := \mathrm{E}^i_I\mathrm{E}^j_J\epsilon^{IJK}/|e| \quad (30)$$

are moved to the right, a standard choice. The triads are "double grasp." For example, triad $\mathrm{E}^z(n)$ has support on xy areas on both the incoming and outgoing sides of vertex n, so that $\mathrm{E}^z(n)$ grasps both the incoming and the outgoing z holonomy at vertex n. The model considered in Appendix A employs triads which grasp only incoming or only outgoing holonomies, but not both. However, in the small sine limit,

these single grasp triads are replaced by double grasp triads. The volume e need not be regulated in this limit, since it does not vanish for large quantum numbers.

V. FLUX-HOLONOMY ALGEBRA

The flux-holonomy algebra for these quantities is determined by the assumption that the $\tilde{\mathrm{E}}$ grasp both incoming and outgoing holonomies at a given vertex. In the transverse case, the double area grasp produces an anticommutator.

$$\{\mathrm{E}^a_A, h_a\} = i(\gamma\kappa/2)[\sigma_A/2, h_a]_+;$$
$$\{\mathrm{E}^z_Z, h_z\} = i(\gamma\kappa/2)[\sigma_Z/2, h_z]_+. \tag{31}$$

The area integrations in the E's (suppressed) combine with the line integration in the holonomy to cancel the delta functions. For comparison, second line of Eq. (31) exhibits the grasp of the longitudinal holonomy. That result agrees with Eq. (27), because

$$[\sigma_Z/2, h_z]_+ = (\sigma_Z/2)h_z \times 2.$$

The algebra for the $\hat{h}$ was derived at Eq. (28).

$$\{\mathrm{E}^a_A, \hat{h}\} = i(\gamma\kappa/2)[\sigma_A/2, \bar{h}]_+$$
$$= i(\gamma\kappa/2)\sigma_A \quad (\mathrm{SS}). \tag{32}$$

VI. SMALL SINE EXPRESSION FOR THE EXTRINSIC CURVATURE

Thiemann [9] has proposed a two-step process for constructing a regulated extrinsic curvature. His procedure uses the Poisson brackets

$$\{|e|, H_e\} \propto K \cdot E \coloneqq K^I_i E^i_I;$$
$$h_i\{h_i^{-1}, K \cdot E\} \propto K_i;$$
$$|e| \coloneqq \sqrt{\mathrm{sgn}(e)\det E}. \tag{33}$$

sgn(e) is the sign of e and det E. If one inserts the SS Hamiltonian Eq. (29) on line one above, the result for extrinsic curvatures and spin connections is

$$\gamma \mathrm{K}_i = -2i\hat{h}_i - \Gamma_i;$$
$$2\Gamma^I_j \mathrm{E}^j_I = -\mathrm{sgn}(e)(\delta_{(c)}\Sigma^{mzM})\Sigma^{ni}_M \epsilon_{mni}/2! \coloneqq 2\Gamma \cdot \mathrm{E};$$
$$\Gamma^I_j \mathrm{E}^j_M = \mathrm{sgn}(e)(\delta_{(c)}\Sigma^{mz}_M)\Sigma^{niI}\epsilon_{mni}/2!$$
$$+ \Gamma \cdot \mathrm{E}\delta^I_M. \tag{34}$$

The E^j_J are left in place because the Γ_j in the Hamiltonian typically occur contracted with a triad.

The functions $\Gamma = \Gamma[\tilde{\mathrm{E}}]$ in Eq. (34) are identical to the classical ones, except z derivatives of the $\tilde{\mathrm{E}}$ are replaced by central differences. To understand how this happens, we start from the SS H_e, Eq. (29). It has terms linear and quadratic in the $\hat{h}$. The $\{|e|, \mathrm{H}_e\}$ bracket removes one $\hat{h}$ [compare Eq. (32)]. $\mathrm{K} \cdot \mathrm{E}$ then contains terms linear in the $\hat{h}$ and terms independent of $\hat{h}$. Since the subsequent bracket with h does not remove an $\hat{h}$, the Γ come from the terms in $\mathrm{K} \cdot \mathrm{E}$ independent of $\hat{h}$; equivalently, they come from the terms in H_e linear in $\hat{h}$.

The terms in H_e linear in $\hat{h}$ are the central difference terms.

$$-\mathrm{H}_e + \mathrm{ST} = +\cdots\delta_{(c)}\hat{h}_a\Sigma^{za} + \cdots + \mathrm{ST}$$
$$= -\hat{h}_a\delta_{(c)}\Sigma^{za} + \cdots + \text{no ST}. \tag{35}$$

The last line, the difference analog of an integration by parts, produces a surface term which is canceled by the surface term ST. It also produces the $\delta_{(c)}\Sigma$ terms present in Eq. (34). The additional Σ factor (the factor with no $\delta_{(c)}$) comes from the bracket of $|e|$ with the $\hat{h}_a$ in Eq. (35). □

Proof that an "integration by parts" maneuver is possible when dealing with differences rather than derivatives: we start from the following formula, which is exactly true:

$$\delta_{(c)}(A\Sigma)(n) = \delta_{(c)}A(n)(1/2)[\Sigma(n+1) + \Sigma(n-1)]$$
$$+ (1/2)[A(n+1) + A(n-1)]\delta_{(c)}\Sigma(n). \tag{36}$$

Equation (36) contains averages such as $(1/2)[\mathrm{A(n+1)} + \mathrm{A(n-1)}]$, whereas the distributive law for derivatives has just A(n). From Eq. (16) the sum $\mathrm{A(n+1)} + \mathrm{A(n-1)}$ equals 2 A(n) plus a forward second derivative. The slow variation assumption, Eq. (17), can be used to drop the second derivative. Then

$$\delta_{(c)}(A\Sigma)(n) = \delta_{(c)}A(n)\Sigma(n) + A(n)\delta_{(c)}\Sigma(n) \quad (\mathrm{SV}). \tag{37}$$

This formula more closely resembles the corresponding relation for derivatives. If the $A(n)\delta_{(c)}\Sigma(n)$ term is moved to the left-hand side of the equation, the result is an integration by parts identity for the central difference. □

A number of FT brackets have closely similar LHF analogs. The following example is given without proof (FT bracket first; then analogous SV LHT bracket).

$$\{\mathrm{KE}(z), \partial'_z \mathrm{E}(z')/\mathrm{E}(z')\} = [\partial'_z\delta(z-z')]\mathrm{E}(z)/\mathrm{E}(z')$$
$$- \delta(z-z')\partial'_z\mathrm{E}(z')/\mathrm{E}(z')$$
$$= \partial'_z\delta(z-z') \quad (\mathrm{FT});$$
$$\{\mathrm{KE}(q), \delta_{(c)}\mathrm{E}(m)/\mathrm{E}(m)\} = \delta_{(c)}(m)\delta(q,m)\mathrm{E}(q)/\mathrm{E}(m)$$
$$- \delta(q,m)\delta_{(c)}\mathrm{E}(m)/\mathrm{E}(m)$$
$$= \delta_{(c)}(m)\delta(q,m) \quad (\mathrm{SV}). \tag{38}$$

K and E stand for any K_j^I and its conjugate E_I^j. The quantity

$$\delta_{(c)}(m)\delta(q,m) \coloneqq [\delta(q,m+1)-\delta(q,m-1)]/2$$

is the difference of a Kronecker delta $\delta(q,m)$. This difference is the discrete analog of the derivative of a Dirac delta function.

VII. CONSTRAINTS IN THE SMALL SINE LIMIT

A. The Gauss constraint

In classical or quantum field theory, the Gauss identity

$$0 = \partial_i E_I^i + \epsilon_{IJK} A_m^J E_K^m \tag{39}$$

can be broken into two parts:

$$\begin{aligned} 0 &= \partial_i E_I^i + \epsilon_{IJK}\Gamma_m^J E_K^m; \\ 0 &= \epsilon_{IJK} K_m^J E_K^m. \end{aligned} \tag{40}$$

The breakup is possible because the first line vanishes by itself: the covariant divergence of a density one triad vanishes, and involves no Christoffel symbols.

In the plane wave case only U(1) gauge rotations corresponding to rotations around the Z axis survive. Line one above becomes

$$0 = \delta_{(c)} E_Z^z + \epsilon_{ZAB}\Gamma_m^A E_B^m. \tag{41}$$

The derivative has been replaced by a central difference. Equation (41) may be derived from the SS Γ, Eq. (34). Main steps in a direct proof: relabel M, I → J, K on the last line of Eq. (34), and replace the Σ by triads, using

$$\begin{aligned} \Sigma^{ijK} &\coloneqq E_I^i E_J^j \epsilon^{IJK}/|e| \\ &= e_I^i e_J^j |e| \\ &= \mathrm{sgn}(e) e_k^K \epsilon^{ijk}. \end{aligned} \tag{42}$$

Then use the antisymmetry of the Levi-Civita tensors to replace

$$\begin{aligned} \epsilon^{zmr}\delta_{(c)} e_r^J e_m^I &\to (\epsilon^{zmr}/2)[(\delta_{(c)} e_r^J)e_m^I + (\delta_{(c)} e_m^I)e_r^J] \\ &= (\epsilon^{zmr}/2)\delta_{(c)}(e_r^J e_m^I) \quad \text{(SV)} \\ &= (1/2)\mathrm{sgn}(e)\delta_{(c)} E_Z^z. \quad \square \end{aligned} \tag{43}$$

The second line of Eq. (40) must be imposed as a constraint on the Hilbert space of coherent states.

B. The Lorentzian Hamiltonian

The Lorentzian Hamiltonian H equals minus the Euclidean Hamiltonian, plus terms quadratic in the extrinsic curvature.

$$\mathrm{H} = \sum_n [-(1+\gamma^2)/2\kappa](K_i^I K_j^J \epsilon_{IJK} e^{ijK} N)(n) - H_e \quad \text{(FT)}. \tag{44}$$

The Hamiltonian of Eq. (44) contains three variables: K_i, $\hat{h}_i$, and Σ^{ijK}. They are not independent, and one must decide which variable to eliminate. From Eq. (34), one can eliminate either K_i or $\hat{h}_i$. Either choice introduces a new, and complicated field, the Γ_i.

There is no way of avoiding the Γ_i. However, the unidirectional constraints will allow K to be replaced by a function of the Σ. Anticipating this, we eliminate the $\hat{h}_i$.

$$\begin{aligned} \mathrm{H} + \mathrm{ST} = \sum_n [1/\kappa]\{ & -K_x^I K_y^J \epsilon_{IJK}\epsilon^{KMN} E_M^x E_N^y N(n)/|e| \\ & - K_z^Z K_a^A E_A^a E_Z^z N(n)/|e| \\ & + \Gamma_x^I \Gamma_y^J \epsilon_{IJK} E_M^x E_N^y \epsilon^{MNK} N(n)/|e| \\ & + \Gamma_z^Z \Gamma_a^A E_A^a E_Z^z N(n)/|e| \\ & - \Gamma_a^A \epsilon_{BA}\delta_{(c)}[N E_Z^z E_B^a(n)/|e|]\}. \end{aligned} \tag{45}$$

C. Identities obeyed by the Γ

In Eq. (45), terms linear in K have canceled out, because of Gauss, second line of Eq. (40), plus an identity obeyed by the Γ,

$$0 = \delta_{(c)}\Sigma^{mzI} + \epsilon_{IJK}\Gamma_i^J \Sigma^{miK}. \tag{46}$$

This is the LHF analog of a FT identity: with $\delta_{(c)}$ replaced by ∂_z, Eq. (46) states that the covariant divergence of a function of triads must vanish. The relation involves no Christoffel symbols because of the antisymmetry in indices z, a. The FT version of this identity guarantees that the FT Hamiltonian contains no linear in K terms; the LHF version functions similarly.

Equation (46) can be used to *define* the Γ, since it can be solved for the Γ. The Γ obtained in this manner are the same as the Γ obtained from Thiemann's procedure, Eq. (34). Main steps in the proof: multiply the above equation by the triad e_m^M. In the second, ΓΣ term, the $\Sigma^{miK} e_m^M$ gives an $\tilde{E}$. In the first, $\delta_{(c)}\Sigma$ term, replace e_m^M by a Σ, by inverting the relation between Σ and triad:

$$\begin{aligned} \Sigma^{imM} &\coloneqq E_I^i E_N^m \epsilon^{INM}/|e| \\ &= e_r^M \epsilon^{rim}\mathrm{sgn}(e); \\ e_m^M &= \mathrm{sgn}(e)\Sigma^{inM}\epsilon^{\min}/2!. \quad \square \end{aligned}$$

The following relations are also useful for simplifying the Hamiltonian.

$$\Gamma_z^A = \Gamma_a^Z = 0; \tag{47}$$

$$\Gamma^A_a \mathrm{E}^a_A = 0; \qquad \text{a, A transeverse.} \tag{48}$$

For example, Γ, like $\tilde{\mathrm{E}}$, is block diagonal, with Γ^Z_z in the 1×1 block, and the transverse trace vanishes. Both these results follow from Eq. (34) by taking M = Z, A and I = Z, A in turn.

The Γ obey various relationships in FT, and *most of these remain valid in SS LHT*, despite replacement of derivatives by differences. This is so, because the relationships are proved using only algebra. The proofs do not involve calculus or the properties of the derivative.

D. The vector constraint H_z

Since the wave functional to be constructed is not based on closed loops, the diffeomorphism constraint is not satisfied automatically. It must be treated as an additional constraint.

The classical field theory constraint is

$$\mathrm{N}^z\mathrm{H}_z = (1/\kappa\gamma)\int d^3x \mathrm{N}^z \mathrm{F}^A_{za}\mathrm{E}^a_A \quad \text{(FT)}. \tag{49}$$

We introduce the lattice and make the same assumptions as for the Hamiltonian: the nonlocal version includes both nearest neighbors with equal weight; small sine and slow variation approximations apply. The outcome is as for the Hamiltonian: the $\hat{h}_z(n_z, n_z \pm 1)$ are replaced by an average; the $\hat{h}_a(n_z \pm 1)$ are replaced by $\hat{h}_a(n_z)$; derivatives are replaced by central differences.

$$\kappa\gamma \mathrm{N}^z\mathrm{H}_z = \sum_n \mathrm{N}^z(\mathrm{Tr}/2)\{(-2i)\delta_{(c)}\hat{h}_a(n)\mathrm{E}^a(n) + 2i[\hat{h}_z, \hat{h}_a(n)]\mathrm{E}^a(n)\} \quad \text{(SS, SV)}. \tag{50}$$

As in the FT case, one must add in a term proportional to the Gauss constraint to make H_z into the generator of z diffeomorphisms. The second line of Eq. (50) equals

$$(-2i)^2\hat{h}^Z_z\epsilon_{AB}\hat{h}^A_a(n)\mathrm{E}^a_B(n). \tag{51}$$

One can replace $(-2i)\hat{h} \to \gamma\mathrm{K} + \Gamma$. From Gauss, Eq. (40), the term involving K vanishes, and the term involving Γ equals one half of the Gauss constraint, Eq. (40). Equation (50) becomes

$$\kappa\gamma \mathrm{N}^z\mathrm{H}_z = \sum_n \mathrm{N}^z\{(-2i)\delta_{(c)}\hat{h}^A_a(n)\mathrm{E}^a_A(n) - (-2i)\hat{h}_z(n)\delta_{(c)}\mathrm{E}^z_Z\} \quad \text{(SS, SV)}. \tag{52}$$

The next few sections simplify the Hamiltonian by choosing gauges and imposing constraints.

VIII. SINGLE POLARIZATION CONSTRAINTS

The single polarization constraints are

$$\mathrm{E}^x_Y = \mathrm{E}^y_X = 0. \tag{53}$$

These constraints must obey the consistency conditions

$$\{\mathrm{H} + \mathrm{ST}, \mathrm{E}^x_Y\} = \{\mathrm{H} + \mathrm{ST}, \mathrm{E}^y_X\} = 0,$$

which of course just require the vanishing of the conjugate coordinates.

$$\mathrm{K}^x_Y = \mathrm{K}^y_X = 0. \tag{54}$$

For brevity these conditions will be denoted simply as "single polarization" constraints; but they not only specialize to a single polarization; they also fix the U(1) gauge. If one wishes to specialize to single polarization without fixing the U(1) gauge, one may impose $\mathrm{E}^x_I\ \mathrm{E}^y_I = 0$.

The triad and $\tilde{\mathrm{E}}$ matrices are now diagonal. Additionally, from Eq. (42), all three indices of Σ^{ijK} must be unequal. For example,

$$\Sigma^{mzZ} = \Sigma^{xyX} = 0 \quad \text{(single pol)},$$

while Σ^{xyZ} is finite. This follows from Eq. (42) and the diagonal nature of the triads. Also, from Eq. (34), the only surviving Γ are the two off-diagonal Γ^X_y, Γ^Y_x.

$$\Gamma^X_x = \Gamma^Y_y = \Gamma^Z_z = 0 \quad \text{(single pol)}. \tag{55}$$

IX. THE DIFFEOMORPHISM CONSTRAINT

In FT, LHF, and LQG the usual gauge choice which fixes the Lorentz boosts, reducing the full Lorentz group to SU(2), is

$$e^t_{X,Y,Z} = 0 = e^{x,y,z}_T. \tag{56}$$

This gauge still allows transformations

$$t' = t'(t); \qquad z' = z'(z,t). \tag{57}$$

The transverse triads vary with this change in the z coordinate, despite their lack of an explicit z index, because e, the volume factor, contains an implicit z subscript. Conversely, the longitudinal triad (has an explicit z index but) does not change with change in z coordinate.

$$\mathrm{E}^a_A \propto |e| = \mathrm{sgn}(e)e^{Z(2)}_z e;$$
$$\mathrm{E}^z_Z = |e|e^z_Z = \mathrm{sgn}(e)({}^{(2)}e); \tag{58}$$

${}^{(2)}e$ is the determinant of the 2×2 transverse triad matrix, an invariant. Therefore E^z_Z is a scalar, while the E^a_A are rank

one covariant tensors. Therefore a gauge fixing constraint must involve at least some transverse triads. We use a gauge fixing function constructed from the two simplest triad functions which are U(1) scalars, ${}^{(2)}\tilde{\mathrm{E}}$ and E^z_Z.

$$0 = \ln[{}^{(2)}\tilde{\mathrm{E}}/(\mathrm{CE}^z_Z)^{p+1/2}] \coloneqq \mathrm{D}_1;$$
$$0 = 2\mathrm{K}_z\mathrm{E}^z + \mathrm{K}_a\mathrm{E}^a/2 - p\mathrm{K}_a\mathrm{E}^a \coloneqq \mathrm{D}_2. \tag{59}$$

C is a constant. ${}^{(2)}\tilde{\mathrm{E}}$ is the determinant of the 2×2 transverse cotriad matrix. The single polarization constraints imply

$${}^{(2)}\tilde{\mathrm{E}} = \mathrm{E}^x_X\mathrm{E}^y_Y.$$

Equation (59) is a family of gauge choices, depending on a parameter p. D_1 depends on $\mathrm{p}+1/2$, rather than p, because at a later point the value $\mathrm{p}=0$ will prove to be special. The second line is the consistency condition, the result of demanding $\{\mathrm{H},\mathrm{D}_1\}=0$.

The first line proposes a gauge choice involving a logarithm, rather than a simpler choice

$${}^{(2)}\tilde{\mathrm{E}} - (\mathrm{CE}^z_Z)^{p+1/2} = 0. \tag{60}$$

To see the reason for this, recall fixing the diffeomorphism gauge is equivalent to first transforming to new canonical coordinates (π,q), then discarding one pair of (π,q)'s. Write

$$\mathrm{E}^i_I(\mathrm{dK}^I_i/\mathrm{d}t) = -\mathrm{dE}^i_I/\mathrm{d}t\mathrm{K}^I_i + \text{total derivative},$$

then expand:

$$\begin{aligned}
-\mathrm{K}^i_I\mathrm{dE}^i_I/\mathrm{d}t &= -\mathrm{K}_i\mathrm{E}^i\mathrm{d}(\ln\mathrm{E}^i)/\mathrm{d}t \\
&= -\mathrm{K}_z\mathrm{E}^z\mathrm{d}(\ln\mathrm{E}^z)/\mathrm{d}t - (1/2)\mathrm{K}_a\mathrm{E}^a\mathrm{d}(\ln{}^{(2)}\tilde{\mathrm{E}})/\mathrm{d}t \\
&\quad - (1/2)(\mathrm{K}_y\mathrm{E}^y - \mathrm{K}_x\mathrm{E}^x)\mathrm{d}[\ln(\mathrm{E}^y/\mathrm{E}^x)]/\mathrm{d}t \\
&= +\{(2\mathrm{K}_z\mathrm{E}^z + \mathrm{K}_a\mathrm{E}^a/2 - p\mathrm{K}_a\mathrm{E}^a) \\
&\quad \times \mathrm{d}[\ln{}^{(2)}\tilde{\mathrm{E}} - (\ln\mathrm{E}^z)/2 - p\ln\mathrm{E}^z]/\mathrm{d}t \\
&\quad - (p\to -p)\}(1/4p) \\
&\quad - (1/2)(\mathrm{K}_y\mathrm{E}^y - \mathrm{K}_x\mathrm{E}^x)\mathrm{d}[\ln(\mathrm{E}^y/\mathrm{E}^x)]/\mathrm{d}t.
\end{aligned} \tag{61}$$

The fourth and fifth lines are D_2 times the derivative of D_1. One can drop this (π,q) pair completely from the theory, without altering the canonical brackets of the other (π,q) pairs. The constant C of Eq. (60) arises as a constant of integration. Note the special case $\mathrm{p}=0$ has a singularity.

In practice the gauge choice Eq. (59) does not introduce logarithms into the constraints; only Eq. (60) and its first difference are needed when simplifying the Hamiltonian.

The popular choice for C and p, in the classical literature, is $\mathrm{C}=\mathrm{sgn}(\mathrm{e})$, $\mathrm{p}=1/2$, which implies $\mathrm{g}_{zz}=1$. C and p will be determined in the succeeding paper.

The case $\mathrm{p}=0$ clearly requires a special discussion. Because the classical literature favors the gauge choice $\mathrm{p}=1/2$, presumably the $\mathrm{p}=0$ case will not be needed. This paper does not discuss it.

X. THE SWITCH FROM N TO $\underline{\mathrm{N}}$

A. Boundary conditions at infinity

In Newtonian static planar gravity the gravitational potential at infinity does not die off as some power of z, but rather grows linearly. Attempts to generalize the Newtonian static result to full general relativity have not been successful [14]. Reasonable restrictions on the stress-energy of the planar matter source presumably lead to instability.

If the source is a time-varying wave packet, it should be safe to assume flatness at infinity, because of causality: the packet has not yet reached infinity. (Note the Newtonian static plane result does not rule out flat space at infinity, because an observer in a large free-fall elevator would detect the *same* force at the top and bottom of the elevator.) However, for now there is no loss of generality if one makes the more conservative assumption, conformal flatness at infinity. The (z,t) portion of the metric at infinity is assumed to take the conformal form

$$\begin{aligned}
&[-\mathrm{N}^2 + (\mathrm{N}^z)^2 g_{zz}]dt^2 + 2\mathrm{N}^z g_{zz}dzdt + g_{zz}dz^2 \\
&\quad = g_{zz}(-dt^2 + dz^2),
\end{aligned} \tag{62}$$

where N and N^z are the ADM lapse and shift. This requires the boundary conditions

$$\begin{aligned}
&\mathrm{Limit}(|z|\to\infty)\mathrm{N}^z = 0; \\
&\qquad\qquad \underline{\mathrm{N}}^2 \coloneqq \mathrm{N}^2/g_{zz}; \\
&\mathrm{Limit}(|z|\to\infty)\underline{\mathrm{N}}^2 = 1.
\end{aligned} \tag{63}$$

$\underline{\mathrm{N}}$, rather than N, goes to ± 1. The underlining is needed because $\underline{\mathrm{N}}$ is density weight -1. We have not used tildes or overbars to indicate the density weight of the triads. They are familiar to most readers, and it is understood the triads are weight 1. $\underline{\mathrm{N}}$, however, is an unfamiliar quantity, and its density weight will play a role in Sec. XI, when the unidirectional constraints are imposed.

For plane waves, the boundary conditions require a shift from N to $\underline{\mathrm{N}}$. In cotriad notation,

$$\underline{\mathrm{N}}(n) = (\mathrm{NE}^z_Z/|e|)(n). \tag{64}$$

The lapse N is a scalar under spatial diffeomorphisms. Therefore from Eq. (64) $\underline{\mathrm{N}}$ is a rank one contravariant tensor. N has no factors of Δx^i, but $\underline{\mathrm{N}}$ has a factor $1/\Delta z$.

This shift in lapse generates a shift in the Hamiltonian.

$$\mathrm{NH} \coloneqq \underline{\mathrm{N}}\,\tilde{\mathrm{H}};$$
$$\tilde{\mathrm{H}} = \mathrm{H}|e|/\mathrm{E}^z_Z.$$

When the Σ are expressed in terms of the $\tilde{\mathrm{E}}$, and N is replaced by $\underline{\mathrm{N}}$, the $1/|e|$ singularities disappear, but some terms acquire a $1/\mathrm{E}^Z_z$ singularity.

$$\begin{aligned}\mathrm{NH}_e &= \underline{\mathrm{N}}\tilde{\mathrm{H}}_e \\ &= \sum_n \underline{\mathrm{N}}\{\mathrm{F}^Z_{xy}\epsilon^{ZAB}\mathrm{E}^x_A\mathrm{E}^y_B/\mathrm{E}^z_Z + \mathrm{F}^A_{za}\epsilon^{AZB}\mathrm{E}^a_B\}/\kappa + \mathrm{ST}.\end{aligned} \tag{65}$$

The Lorentzian Hamiltonian, Eq. (45), becomes

$$\begin{aligned}\underline{\mathrm{N}}\,\tilde{\mathrm{H}} + \mathrm{ST} = \sum_n (1/\kappa)\{&-(\mathrm{K}^A_a\mathrm{E}^a_A)^2\underline{\mathrm{N}}(n)/(4\mathrm{E}^z_Z) \\ &+ (\mathrm{K}^Y_y\mathrm{E}^y_Y - \mathrm{K}^X_x\mathrm{E}^x_X)^2\underline{\mathrm{N}}(n)/(4\mathrm{E}^z_Z) \\ &- \mathrm{K}^Z_z(n)\mathrm{E}^z_Z\mathrm{K}^A_a(n)\mathrm{E}^a_A\underline{\mathrm{N}}(n)/\mathrm{E}^z_Z \\ &- (\Gamma^Y_x\mathrm{E}^x_X + \Gamma^X_y\mathrm{E}^y_Y)^2(n)\underline{\mathrm{N}}(n)/(4\mathrm{E}^z_Z) \\ &+ (\Gamma^Y_x\mathrm{E}^x_X - \Gamma^X_y\mathrm{E}^y_Y)^2(n)\underline{\mathrm{N}}(n)/(4\mathrm{E}^z_Z) \\ &- \Gamma^A_a(n)\epsilon_{BA}\delta_{(c)}(\underline{\mathrm{N}}\mathrm{E}^{\mathrm{a}}_{\mathrm{B}})\}.\end{aligned} \tag{66}$$

This equation incorporates the single polarization constraints (K and E entirely diagonal, Γ entirely off diagonal), but not the diffeomorphism constraints. However, the equation anticipates those constraints and pairs each K with an E. (The reshuffled E's are still to the right of their conjugate K's.)

Equation (66) also switches to combinations of Γ and E which are relatively simple to express in terms of $\tilde{\mathrm{E}}$.

$$\begin{aligned}\Gamma^Y_x\mathrm{E}^x_X + \Gamma^X_y\mathrm{E}^y_Y &= [\delta_{(c)}\mathrm{E}^y_Y/\mathrm{E}^y_Y - \delta_{(c)}\mathrm{E}^x_X/\mathrm{E}^x_X]e^X_x e^Y_y \mathrm{sgn} \\ &= [\delta_{(c)}\mathrm{E}^y_Y/\mathrm{E}^y_Y - \delta_{(c)}\mathrm{E}^x_X/\mathrm{E}^x_X]\mathrm{E}^z_Z; \\ -\Gamma^X_y\mathrm{E}^y_Y + \Gamma^Y_x\mathrm{E}^x_X &= \delta_{(c)}\mathrm{E}^z_Z.\end{aligned} \tag{67}$$

Proof of Eq. (67): from (34) and (42),

$$\Gamma^X_y\mathrm{E}^y_Y = \mathrm{sgn}(e)\delta_{(c)}e^Y_n\epsilon^{mzn}e^X_m \quad \text{(single pol)},$$

plus an additional formula with $\mathrm{X} \leftrightarrow \mathrm{Y}$.

If the equation $e^Y_y\ \mathrm{E}^y_Y = |e|$ is differenced, then divided by $e^Y_y\ \mathrm{E}^y_Y$, one gets

$$\delta_{(c)}e^Y_y/e^Y_y + \delta_{(c)}\mathrm{E}^y_Y/\mathrm{E}^y_Y = \delta_{(c)}|e|/|e|,$$

plus a similar equation for $\mathrm{x} \to \mathrm{y}$. These equations imply the first line of Eq. (67). The last line is one half of Gauss, Eq. (41).

In quantum geometrodynamics for the plane wave case, the shift from N to $\underline{\mathrm{N}}$ modifies the constraint algebra, making it anomaly free. Usually, a commutator of constraints produces a metric component to the right of a constraint, causing an anomaly. When N is replaced by $\underline{\mathrm{N}}$, the dangerous metric component is absorbed into $\underline{\mathrm{N}}$ and disappears from the commutator.[2]

XI. THE UNIDIRECTIONAL CONSTRAINTS

A. Are Dirac brackets necessary?

Since unidirectional constraints typically are second class, it is necessary to replace Poisson by Dirac brackets. Dirac brackets often are not pretty. Are there ways of avoiding the introduction of Dirac brackets?

To develop some intuition, we consider the simplest case: a real, scalar, free field ϕ. One can expand this field in plane waves, with coefficients the usual creation and destruction operators. There are now two ways to impose a unidirectional constraint. (Either) strip out all terms in the expansion depending on $\mathrm{z} + \mathrm{ct}$; (or) construct Dirac brackets, starting from the unidirectional constraint $\mathrm{U} = \pi + \delta_{(c)}\phi = 0$ (time derivative plus z derivative vanishes).

We consider first the expansion approach. With half the degrees of freedom now missing, it is obvious that $[\pi, \phi]$ and the other commutators can no longer be canonical.

Similarly, the Dirac bracket approach starts by computing the commutator of the constraint with itself.

$$\begin{aligned}[\mathrm{U}(z_1), \mathrm{U}(z_2)] &= [\pi + \partial_{z1}q(z_1), \pi + \partial_{z_2}q(z_2)] \\ &= (-i\hbar)[+\partial_{z2}\delta(z_1 - z_2) - \partial_{z1}\delta(z_1 - z_2)] \\ &= -2i\hbar\partial_{z2}\delta(z_1 - z_2).\end{aligned} \tag{68}$$

The calculation is done in FT for the convenience of the reader, but the result in the presence of a lattice is similar: replace (z_1, z_2) by (n_1, n_2), and replace $\partial_{z2}\delta(z_1 - z_2)$ by

$$\delta_{(c)}(n_2)\delta(n_1, n_2) \coloneqq [\delta(n_1, n_2 + 1) - \delta(n_1, n_2 - 1)]/2. \tag{69}$$

The derivative of a Dirac delta is replaced by the difference of a Kronecker delta. The commutator of a unidirectional constraint with itself does not vanish. The constraint is second class, and one must construct Dirac brackets.

The Dirac approach is rather formal, but the expansion approach gives the intuitive explanation for the unorthodox brackets: one cannot strip out half the degrees of freedom, and expect commutators to remain invariant. It is not possible to avoid Dirac brackets.

[2]The author would appreciate help from readers in locating the original source of this result: no anomalies in the planar case.

B. The unidirectional operator

In conventional wave theory a solution is unidirectional if all fields depend only on z-ct. In general relativity those coordinates are arbitrary, and one must use local free-fall coordinates Z-cT instead.

In terms of derivatives

$$\sqrt{2}\partial_U = \partial_Z - \partial_T;$$
$$\sqrt{2}\partial_V = \partial_Z + \partial_T.$$

The constraint (no V dependence) is $(\partial_Z + \partial_T) = 0$.

This constraint can be rewritten in terms of (z,t) derivatives.

$$\begin{aligned} 0 &= (\partial_Z + \partial_T) f(Z - cT) \\ &= (e^z_Z \partial_z + e^t_Z \partial_t + e^z_T \partial_z + e^t_T \partial_t) f \\ &= (e^z_Z \partial_z + 0 + (-\mathrm{N}^z/\mathrm{N})\partial_z + (1/\mathrm{N})\partial_t) f. \end{aligned} \quad (70)$$

This equation invokes the usual gauge which fixes the Lorentz boosts and reduces the full Lorentz group to SU(2): $e^t_{X,Y,Z} = 0$.

One can replace the derivatives ∂_z and ∂_t in Eq. (70) by Poisson brackets with H_z and $\underline{\mathrm{N}}\tilde{\mathrm{H}}(z) + \mathrm{ST} + \mathrm{N}^z\mathrm{H}_z$ respectively.

$$\begin{aligned} 0 &= [e^z_Z - \mathrm{N}^z/\mathrm{N}(z)]\{f, \mathrm{H}_z(z)\} \\ &\quad + (1/\mathrm{N})\{f, (\underline{\mathrm{N}}\tilde{\mathrm{H}}(z) + \mathrm{ST}) + \mathrm{N}^z\mathrm{H}_z(z)\} \\ &\quad \propto (\mathrm{N}\mathrm{E}^z_Z/|e|)\{f, \mathrm{H}_z(z)\} + \{f, (\underline{\mathrm{N}}\tilde{\mathrm{H}}(z) + \mathrm{ST})\} \\ &= \{f, \underline{\mathrm{N}}\mathrm{H}_z(z) + (\underline{\mathrm{N}}\tilde{\mathrm{H}}(z) + \mathrm{ST})\}. \end{aligned} \quad (71)$$

The third line is multiplied by N. this anticipates a later result: given the unidirectional constraints and the diffeomorphism gauge choice, N cannot vanish.

Strictly speaking H_z is not the z derivative operator unless its Lagrange multiplier $\underline{\mathrm{N}}$ is a constant. If $\underline{\mathrm{N}}$ is not a constant, $\{f, \mathrm{H}_z\}$ does take the z derivative of f; but it also generates gauge transformations proportional to $\partial_z\underline{\mathrm{N}}$. In the present case this is not a problem; once unidirectional and diffeomorphism constraints are imposed, $\underline{\mathrm{N}}$ will turn out to be a constant.

The constraint Eq. (71) may be written out explicitly, using (52) and (87) for H_z and $\tilde{\mathrm{H}}$.

$$\begin{aligned} &\underline{\mathrm{N}}\mathrm{H}_z(z) + \underline{\mathrm{N}}\tilde{\mathrm{H}}(z) + \mathrm{ST} \\ &= (1/\kappa)\sum_n \underline{\mathrm{N}}\{(-2i)\delta_{(c)}\hat{h}^A_a(n)\mathrm{E}^a_A(n) - (-2i)\hat{h}_z(n)\delta_{(c)}\mathrm{E}^z_Z\} \\ &\quad + \sum_n (1/\kappa)\{-\mathrm{K}^X_x\mathrm{K}^Y_y\mathrm{E}^x_X\mathrm{E}^y_Y\underline{\mathrm{N}}(n)/\mathrm{E}^z_Z \\ &\quad - \mathrm{K}^Z_z(n)\mathrm{K}^A_a(n)\mathrm{E}^a_A(n)\underline{\mathrm{N}}(n) - (\Gamma^Y_x\Gamma^X_y)(n)\underline{\mathrm{N}}(n)^{(2)}\tilde{\mathrm{E}}(n)/\mathrm{E}^z_Z \\ &\quad - \Gamma^A_a(n)e^{zaA}(n)\delta_{(c)}\mathrm{N}\}. \end{aligned} \quad (72)$$

C. The unidirectional constraints

In a unidirectional theory, Eq. (72) commutes with every dynamical variable. Therefore one can construct unidirectional constraints by commuting Eq. (72) with any set of independent functions f_i, then setting the commutators equal to zero. We choose the f_i to be three independent functions of the three triads: E^z_Z, ${}^{(2)}\tilde{\mathrm{E}}$, and $\ln[\mathrm{E}^y_Y/\mathrm{E}^x_X]$. Commutation of these three yields the constraints

$$\begin{aligned} 0 &= \{\mathrm{K}^A_a\mathrm{E}^a_A + \delta_{(c)}\mathrm{E}^z_Z\}/\sqrt{\mathrm{E}^z_Z} := \mathrm{U}_1; \\ 0 &= \{\mathrm{K}^A_a\mathrm{E}^a_A + 2\mathrm{K}^Z_z\mathrm{E}^z_Z + \mathrm{E}^z_Z\delta_{(c)}{}^{(2)}\tilde{\mathrm{E}}/{}^{(2)}\tilde{\mathrm{E}} + 2\mathrm{E}^z_Z\delta_{(c)}\underline{\mathrm{N}}/\underline{\mathrm{N}}\}/\sqrt{\mathrm{E}^z_Z} \\ &\quad := \mathrm{U}_2; \\ 0 &= \{\mathrm{K}^Y_y\mathrm{E}^y_Y - \mathrm{K}^X_x\mathrm{E}^x_X - \mathrm{E}^z_Z[\delta_{(c)}\mathrm{E}^y_Y/\mathrm{E}^y_Y - \delta_{(c)}\mathrm{E}^x_X/\mathrm{E}^x_X]\}/\sqrt{\mathrm{E}^z_Z} \\ &\quad := \mathrm{U}_3. \end{aligned} \quad (73)$$

The appearance of $\delta_{(c)}\underline{\mathrm{N}}$ in the second constraint may be a bit surprising. This comes from a bracket

$$\begin{aligned} &\{\underline{\mathrm{N}}(-2i)\delta_{(c)}\hat{h}\mathrm{E}, {}^{(2)}\tilde{\mathrm{E}}\} \\ &\quad = \left\{\sum_n \underline{\mathrm{N}}\{(-2i)\delta_{(c)}\hat{h}^A_a(n)\mathrm{E}^a_A(n), {}^{(2)}\tilde{\mathrm{E}}(m)\right\} \\ &\quad = -\delta_{(c)}(\underline{\mathrm{N}}\mathrm{E}^y_Y)\mathrm{E}^x_X + (x \leftrightarrow y, X \leftrightarrow Y), \end{aligned}$$

followed by multiplication by $\sqrt{\mathrm{E}^z_Z}/\underline{\mathrm{N}}{}^{(2)}\tilde{\mathrm{E}}$. In effect, the $\delta_{(c)}$ has been integrated by parts off the holonomy and onto the $\underline{\mathrm{N}}$ and $\tilde{\mathrm{E}}$. Similarly, an integration by parts produces the $\delta_{(c)}\tilde{\mathrm{E}}/\tilde{\mathrm{E}}$ terms in the third constraint; in that case a $\delta_{(c)}\underline{\mathrm{N}}$ term cancels out.

The usual lapse N is a scalar, but the new lapse $\underline{\mathrm{N}}$ is a contravariant tensor. Both differences, $\delta_{(c)}{}^{(2)}\tilde{\mathrm{E}}/{}^{(2)}\tilde{\mathrm{E}}$ and $\delta_{(c)}\underline{\mathrm{N}}/\underline{\mathrm{N}}$ in Eq. (73), therefore have inhomogeneous terms in their diffeomorphism transformation laws. However, the inhomogeneous terms cancel out in the sum. The combination which occurs in the unidirectional constraint,

$$\delta_{(c)}{}^{(2)}\tilde{\mathrm{E}}/{}^{(2)}\tilde{\mathrm{E}} + 2\delta_{(c)}\underline{\mathrm{N}}/\underline{\mathrm{N}},$$

transforms like a tensor.

The factors of $1/\sqrt{\mathrm{E}^z_Z}$ have been added to split up the $1/\mathrm{E}^z_Z$ singularity into two parts. (The Hamiltonian is of the form $\sum \mathrm{U}_i\mathrm{U}_j$, with $\sqrt{1/\mathrm{E}^z_Z}$ absorbed into each U_i.)

These constraints have the right form. The K dependent terms represent time derivatives; the $\delta_{(c)}$ terms the corresponding space derivatives.

D. Eliminating one unidirectional constraint

Because of the diffeomorphism gauge fixing, the unidirectional constraints $\mathrm{U}_1, \mathrm{U}_2$ are not independent. For $\mathrm{p} \neq 0$,

one can solve Eq. (59) or Eq. (60) to eliminate a (π, q) pair. The surviving (π, q) pair is

$$\Pi := (2\mathrm{K}_z\mathrm{E}^z + \mathrm{K}_a\mathrm{E}^a/2 + p\mathrm{K}_a\mathrm{E}^a)/4p;$$
$$Q := \ln {}^{(2)}\tilde{\mathrm{E}} + (p-1/2)\ln(\mathrm{CE}^z_Z). \tag{74}$$

This is the pair indicated schematically by $(\mathrm{p} \to -\mathrm{p})$ in Eq. (61). The factor $1/4\mathrm{p}$ of that equation is absorbed into the Π (an arbitrary choice); this gives bracket $\{\Pi, \mathrm{Q}\}$ the correct norm.

Several quantities simplify when the diffeomorphism constraints D_i are dropped from the Hamiltonian. In particular,

$$\mathrm{K}^A_a\mathrm{E}^a_A = (4p\Pi - \mathrm{D}_2)/2p \to 2\Pi;$$
$$2\mathrm{K}^Z_z\mathrm{E}^z_Z + \mathrm{K}^a_A\mathrm{E}^a_A/2 = (4p\Pi + \mathrm{D}_2)/2 \to 2p\Pi;$$
$$2\mathrm{K}^Z_z\mathrm{E}^z_Z \to (2p-1)\Pi. \tag{75}$$

Similarly for the triads,

$$\ln {}^{(2)}\tilde{\mathrm{E}} = Q(p+1/2)/2p + \mathrm{D}_1(p-1/2)/2p \to Q(p+1/2)/2p;$$
$$\ln(\mathrm{CE}^z_Z) = (Q - \mathrm{D}_1)/2p \to Q/2p;$$
$$\ln({}^{(2)}\tilde{\mathrm{E}}/\mathrm{CE}^z_Z) \to Q(p-1/2)/2p. \tag{76}$$

When (75) and (76) are inserted into the unidirectional constraints, the two constraints $\mathrm{U}_1, \mathrm{U}_2$ in Eq. (73) collapse to the same constraint, except U_2 has an extra term proportional to $(\delta_{(c)}\underline{\mathrm{N}})/\underline{\mathrm{N}}$. Therefore that expression must vanish. In place of Eq. (73) one gets [15]

$$0 = (\delta_{(c)}\underline{\mathrm{N}})/\underline{\mathrm{N}};$$
$$0 = \{\mathrm{K}^A_a\mathrm{E}^a_A + \delta_{(c)}\mathrm{E}^z_Z\}/\sqrt{\mathrm{E}^z_Z} = \{2\Pi + \delta_{(c)}\mathrm{E}^z_Z\}/\sqrt{\mathrm{E}^z_Z} = \mathrm{U}_1;$$
$$0 = \{\mathrm{K}^Y_y\mathrm{E}^y_Y - \mathrm{K}^X_x\mathrm{E}^x_X - [\delta_{(c)}\mathrm{E}^y_Y/\mathrm{E}^y_Y - \delta_{(c)}\mathrm{E}^x_X/\mathrm{E}^x_X]\mathrm{E}^z_Z\}/\sqrt{\mathrm{E}^z_Z} = \mathrm{U}_3. \tag{77}$$

Because of the conformal boundary conditions, $\underline{\mathrm{N}}$ must equal unity. Note the unidirectional constraints force us to choose a specific lapse. The scalar constraint is now fixed.

E. Dirac brackets

The two surviving unidirectional constraints are still second class. The Dirac bracket matrix is (rows and columns in order $\mathrm{U}_1, \mathrm{U}_3$)

$$\{\mathrm{U}_i(m), \mathrm{U}_j(n)\} = \begin{Bmatrix} -(2/p)\mathcal{A} & \mathcal{C} \\ -\mathcal{C} & -4\mathcal{A} \end{Bmatrix}$$
$$\mathcal{A} = \delta_{(c)}(n)\delta(m,n) := (\delta(m,n+1) - \delta(m,n-1))/2;$$
$$\mathcal{C} = (1/2p)[\delta_{(c)}\mathrm{E}^y_Y/\mathrm{E}^y_Y - \delta_{(c)}\mathrm{E}^x_X/\mathrm{E}^x_X]\delta(m,n). \tag{78}$$

The inverse bracket matrix is

$$\{\mathrm{U}_j(n), \mathrm{U}_k(r)\}^{-1} = \begin{Bmatrix} \mathcal{K}^{-1} & \mathcal{K}^{-1}\mathcal{C}\mathcal{A}^{-1}/4 \\ -\mathcal{A}^{-1}\mathcal{C}\mathcal{K}^{-1}/4, & -\mathcal{A}^{-1}/4 - \mathcal{A}^{-1}\mathcal{C}\mathcal{K}^{-1}\mathcal{C}\mathcal{A}^{-1}/16 \end{Bmatrix};$$
$$\mathcal{K} = -(2/p)\mathcal{A} - \mathcal{C}\mathcal{A}^{-1}\mathcal{C}/4;$$
$$\mathcal{A}^{-1}(n,r) = -\Theta(n-r). \tag{79}$$

The theta function is a discrete analog of the usual step function.

$$\Theta(n-r) = \begin{cases} 0 & \text{for } n-r \text{ even, including } 0 \\ +1 & \text{for } n-r \text{ odd} > 0, \\ -1 & \text{for } n-r \text{ odd} < 0. \end{cases} \tag{80}$$

$$\delta_{(c)}(n)\Theta(n-r) := [\Theta(n+1-r) - \Theta(n-1-r)]/2 = \delta(n,r). \tag{81}$$

The last line is reminiscent of Eq. (69), where the derivative of a Dirac delta is shown to have a discrete analog, the difference of a Kronecker delta. Similarly here, the continuous formula

$$\partial_2\Theta(z_1 - z_2) = \delta(z_1 - z_2)$$

has a discrete analog, Eq. (81).

The solution for Θ, Eq. (80), is determined only up to a solution to the homogeneous version of Eq. (81).

$$\Theta(n-r)_h = \begin{cases} a & \text{for } n-r \text{ even, including } 0 \\ b & \text{for } n-r \text{ odd}, \end{cases} \tag{82}$$

a and b are constants. In a scalar free field theory, if one drops all the $\mathrm{k} < 0$ Fourier components from the field ϕ, the resulting commutator $[\phi(x), \phi(y)]$ is a step function which changes sign at $\mathrm{x} = \mathrm{y}$. We have chosen Θ_h so that the function $\Theta(n-r)$ also changes sign at $\mathrm{n} - \mathrm{r} = 0$.

E^z_Z and K^Z_z have disappeared at this point, replaced by Q and Π. However, it is perhaps better to retain the more compact and familiar E^z_Z, rather than Q. The $\{E^z_Z \Pi\}$ bracket is [recall $(E^z_Z \propto \exp[Q/2p])$]

$$\{E^z_Z, \Pi\} = (1/2p)E^z_Z. \tag{83}$$

Again, the case p = 0 requires separate discussion.

The matrix of constraints contains a field-dependent quantity

$$\mathcal{C} = (1/2p)[\delta_{(c)}E^y_Y/E^y_Y - \delta_{(c)}E^x_X/E^x_X]\delta(1,2).$$

This is very unusual. In most field theories brackets between unidirectional constraints are field independent. In a weak field limit, $\mathcal{C}$ disappears from the Dirac brackets because $E^z_Z \to 1$, and the off-diagonal bracket

$$\{U_1, U_3\} \propto \{\Pi, \sqrt{E^z}\}$$

vanishes. The presence of $\mathcal{C}$ is therefore a consequence of the nonlinearity of the theory, as represented by the area factors E^z_Z.

The field dependence prohibits an exact solution for $\mathcal{K}^{-1}$. However, an integral equation for $\mathcal{K}^{-1}$ has a power series solution.

$$\begin{aligned}\delta(1,3) &= \sum_2 \mathcal{K}(1,2)\mathcal{K}^{-1}(2,3)\\ &= +(2/p)\delta_{(c)}(1)\mathcal{K}^{-1}(1,3)\\ &\quad - \sum_2 \mathcal{C}(1)(\mathcal{A}^{-1}/4)(1,2)\mathcal{C}(2)\mathcal{K}^{-1}(2,3).\end{aligned} \tag{84}$$

F. Determining lapse and shift

Since the diffeomorphism constraints are a (π, q) pair, further commutation of these constraints produces (almost) nothing new. However, the bracket of D_2 with H gives a Laplace-like equation for $\underline{N}$.

$$\begin{aligned}0 = {}& 2(E^z_Z)\delta_{(c)}(\delta_{(c)}\underline{N}) + \underline{N}\{2\epsilon_{AB}K^A_x K^{B(2)}_y \tilde{E}\\ &+ 2K^Z_z(n)E^z_Z K^A_a(n)E^a_A(n)\\ &+ (1/2)(\delta_{(c)}E^y_Y/E^y_Y - \delta_{(c)}E^x_X/E^x_X)^2\\ &- (\delta_{(c)}{}^{(2)}\tilde{E}/{}^{(2)}\tilde{E})(\delta_{(c)}E^z_Z) + (1/2)(\delta_{(c)}E^z_Z)^2\}/E^z_Z.\end{aligned} \tag{85}$$

The unidirectional constraints, Eq. (73), force every term in this equation to cancel, except $\delta_{(c)}\underline{N}$ terms. We have rederived $\delta_{(c)}\underline{N} = 0$, the first of the unidirectional plus diffeomorphism constraints, Eq. (77).

Also, the bracket of D_1 with H_z gives

$$\delta_{(c)}N^z = 0. \tag{86}$$

XII. FINAL FORM OF THE HAMILTONIAN

Using the unidirectional constraints, one may eliminate KE products from the Hamiltonian, Eq. (66). The Γ may be replaced by functions of $\tilde{E}$, using Eq. (67).

$$\begin{aligned}\underline{N}\tilde{H} + ST = \sum_n (1/\kappa)\{&(\underline{N}/2)E^z_Z(\delta_{(c)}E^y_Y/E^y_Y - \delta_{(c)}E^x_X/E^x_X)^2\\ &+ \delta_{(c)}E^z_Z[-\underline{N}(\delta_{(c)}{}^{(2)}\tilde{E})/2^{(2)}\tilde{E} - \delta_{(c)}\underline{N}]\}.\end{aligned} \tag{87}$$

On the last line of Eq. (66) if the Σ are replaced by their values in terms of $\tilde{E}$, then

$$\begin{aligned}\text{last line} &= [-\Gamma^A_a(n)\epsilon^{ZBA}\delta_{(c)}[E^a_B(n)\underline{N}]\\ &= -\Gamma^A_a(n)E^a_B\epsilon^{ZBA}\underline{N}[\delta_{(c)}E^a_B(n)]/E^a_B\\ &\quad -\Gamma^A_a(n)\epsilon^{ZBA}E^a_B(n)\delta_{(c)}\underline{N}\\ &= (\underline{N}/2)\{[\Gamma^Y_x E^x_X + \Gamma^X_y E^y_Y][-\delta_{(c)}E^x_X/E^x_X + \delta_{(c)}E^y_Y/E^y_Y]\\ &\quad - [\Gamma^Y_x E^x_X - \Gamma^X_y E^y_Y][+\delta_{(c)}E^x_X/E^x_X + \delta_{(c)}E^y_Y/E^y_Y]\}\\ &\quad - \delta_{(c)}E^z_Z\delta_{(c)}\underline{N}.\end{aligned} \tag{88}$$

The final line uses Gauss, Eq. (40). The Γ × E products may be simplified using Eq. (67).

The above calculations use the original three unidirectional constraints, Eq. (73), rather than the constraints which survive diffeomorphism gauge fixing, Eq. (77). Consequently the diffeomorphism gauge is not yet imposed. (This will be done in the following paper.)

In Eq. (87) there is a ST (surface term) on the left, but no ST on the right. The $\delta_{(c)}\hat{h}\Sigma$ term in the Euclidean Hamiltonian has been integrated by parts, and the surface term from that integration by parts cancels the ST.

A fine point: $\tilde{H} + ST$ is not a constraint; it is the true Hamiltonian. To get the constraint ($\tilde{H}$ only; no ST) one must undo the integration by parts, which restores the ST, then discard the ST. Undoing the integration by parts changes only the $\delta_{(c)}\underline{N}$ term in Eq. (87):

$$-\delta_{(c)}E^z_Z\delta_{(c)}\underline{N} \to +\underline{N}\delta_{(c)}(\delta_{(c)}E^Z_z). \tag{89}$$

One can also simplify H_z, again using the unidirectional constraints to eliminate K. Then the scalar and vector constraints are the same, except for a sign.

$$H_z = -\tilde{H}.$$

The minus sign is reasonable, since $\partial/\partial Z = -\partial/\partial T$.

The number of surviving equations now equals the number of surviving unknowns. After the single

polarization constraints are introduced and the triad matrix becomes diagonal, three diagonal triads remain, plus their associated momenta, plus lapse and shift. The (π, q) pair

$$(\mathrm{K}_y^Y \mathrm{E}_Y^y - \mathrm{K}_x^X \mathrm{E}_X^x, \ln(\mathrm{E}_Y^y/\mathrm{E}_x^X))$$

represents the physical degree of freedom and must be fixed using initial conditions.

One of the remaining (π, q) pairs was fixed by the diffeomorphism gauge conditions D_i. Requiring consistency of those constraints fixes lapse and shift.

The two constraints, $\mathrm{H}_z = -\tilde{\mathrm{H}} = 0$, have collapsed to a single constraint; nevertheless, this single constraint plus a unidirectional constraint are enough to determine the remaining nondynamical pair. This pair is the (Π, Q) introduced in Sec. XI D. $\mathrm{H} = 0$ determines Q, which is essentially E_z^Z; and Π is related to Q by a unidirectional constraint.

XIII. DISCUSSION

The theory constructed here includes nonlocal features. The good news is that the nonlocal features which survive in small sine (SS) limit cause no difficulties. If the nonlocal exact theory treats nearest neighbors symmetrically, then the Euclidean Hamiltonian will contain the central difference of a holonomy. If terms linear in extrinsic curvature are required to vanish from the LHF Hamitonian, as they do from the FT Hamiltonian, then every derivative in the spin connection must be replaced by a central difference, leading to a theory with central differences everywhere. Similarly, the other nonlocal feature, the z holonomies, conceivably could cause problems with commutators, but (see Sec. III B) the commutators are reasonable because of the slow variation assumption.

The bad news is that other nonlocal features disappear in this limit. Small sine calculations are not a good way to distinguish between various nonlocal versions, since they tend to possess the same, universal, small sine limit. The nonlocal example discussed in Sec. III A uses forward differences and unaveraged z holonomies, whereas the SS limit contains central differences and the averaged holonomies $\hat{h}_z(n_z)$ defined at Eq. (23). Similarly, the nonlocal theory constructed in Appendix A starts from single grasp $\tilde{\mathrm{E}}$ operators, i.e. operators which grasp only ingoing holonomies at the vertex, or only outgoing holonomies. In the SS limit, the single grasp $\tilde{\mathrm{E}}$ disappear, replaced by $\tilde{\mathrm{E}}$ which grasp both holonomies.

The calculation required various identities involving the spin connection Γ. Those identities hold in classical continuum field theory (FT), as well as nearest neighbor, nonlocal LHF. The proofs use algebra, rather than calculus or properties of the derivative. The identities therefore continue to hold even after derivatives are replaced by differences.

In the small sine limit, it is easy to separate extrinsic curvature from spin connection. Although the Thiemann procedure generates only the combination $\gamma\mathrm{K} = \mathrm{A} - \Gamma$, rather than Γ, the symmetries force off-diagonal A_b^A and on-diagonal Γ_a^A to vanish. This circumstance allows us to separate out the Γ: on-diagonal Thiemann K's are pure extrinsic curvature; off-diagonal Thiemann K's are pure Γ.

It may be possible to separate out the Γ, even when higher powers of sine are included. The Γ are odd under $\mathrm{n}+1 \leftrightarrow \mathrm{n}-1$ (because they contain a central difference), while the K's are even under this interchange, and contain one higher power of sine. If this pattern persists to higher orders in sine, it should be possible to split off the spin connection (odd) part of the K tensor, and check whether the SS identities continue to hold in higher order.

Readers who are familiar with the relation between geometrodynamical variables, Szekeres variables, and $(\mathrm{K}, \tilde{\mathrm{E}})$ variables, will recognize numerous points where the theory shifts to combinations of the K and $\tilde{\mathrm{E}}$ which equal Szekeres or geometrodynamical variables [16]. For example, the ADM π^{ij} are linear combinations of $\mathrm{K} \times \tilde{\mathrm{E}}$ products. The triad combinations involving logarithms are Szekeres variables. Whatever the superiority of K and $\tilde{\mathrm{E}}$ at short distances, the traditional combinations hold the edge in the SS limit.

Of course $\mathrm{K} \cdot \mathrm{E}$ is not really a geometrodynamical variable, because the K is a holonomy, not a field. This is the fundamental change which leads to quantization of areas and volumes. Nevertheless, the combination holonomy times triad seems to be more appropriate than holonomy alone.

In this paper, the small sine approximation was used to simplify the Hamiltonian. Suppose, however, one retains the small sine assumption, even near $\mathrm{e} = 0$. (Near $\mathrm{e} = 0$ one must abandon slow variation, of course, and regulate the cotriads.) Then one has a model which retains the most desirable features of full LQG: geometrical quantities are quantized, connections and triads are bounded, and the model has a simpler Hamiltonian.

The small sine model is especially convenient in the plane wave case. Given the shift from N to $\underline{\mathrm{N}}$,

$$\mathrm{N}\mathrm{E}_J^j \mathrm{E}_K^k \epsilon^{IJK}/|e| = \underline{\mathrm{N}}\mathrm{E}_J^j \mathrm{E}_K^k \epsilon^{IJK}/\mathrm{E}_Z^z,$$

one needs to regulate only $1/\mathrm{E}_Z^z$. Bannerjee and Date [4] replace the $1/\mathrm{E}_Z^z$ by two factors of

$$\begin{aligned}(8/\kappa\gamma) h_z\left[h_z(n)^{-1}, \sqrt{\mathrm{sgn}(z)\mathrm{E}_Z^z(n)}\right] \quad \text{(LQG)} \\ \to \mathrm{sgn(z)}\sigma_Z/\sqrt{\mathrm{sgn}(z)\mathrm{E}_Z^z}. \quad \text{(QFT)}\end{aligned} \tag{90}$$

sign(z) is the sign of E_Z^z. For a similar maneuver in a cosmological context, see Bojowald [17]. Because E_Z^z is already diagonal, its operator square root is immediate.

In a SS model, the spin connections also need to be regulated. From FT formulas for $\Gamma \cdot \mathrm{E}$, Eq. (34), with derivatives replaced by differences, plus FT formulas for the cotriads, we get

$$\begin{aligned}\Gamma_y^X \mathrm{E}_Y^y + \Gamma_x^Y \mathrm{E}_X^x &= \mathrm{sgn}(e)[-(\delta_{(c)} e_y^Y) e_x^X + e_y^Y \delta_{(c)} e_x^X] \\ &= \mathrm{sgn}(e)(\mathrm{E}_Z^z/|e|)^2[(\delta_{(c)}\mathrm{E}_Y^y)\mathrm{E}_X^x - \mathrm{E}_Y^y(\delta_{(c)}\mathrm{E}_X^x)].\end{aligned} \tag{91}$$

The dangerous overall factor of $(\mathrm{E}_Z^z/|e|)^2$ can be removed by an appropriate choice of gauge. For $\mathrm{p} = 1/2$, that factor becomes a constant. The other linear combination follows from Gauss and is free of singularities.

$$\delta_{(c)}\mathrm{E}_Z^z = -\Gamma_y^X \mathrm{E}_Y^y + \Gamma_x^Y \mathrm{E}_X^x.$$

One could argue the small sine model should always be the first model tried, when testing LQG. Small sine keeps the order $(\sin + \sin^2)$ terms in the constraints. These are the only terms we are sure of, because they supply the correct FT limit.

APPENDIX A: A NONLOCAL MODEL

This appendix constructs an exact LHF Euclidean Hamiltonian. It starts from field strengths which are nearest neighbor nonlocal, and z holonomies integrated from vertex to vertex, like the nonlocal model discussed in the main body of the paper. However, the $\tilde{\mathrm{E}}$ are single grasp: they grasp at only one of the six surfaces of the cube surrounding each vertex. The main result of this appendix is, in the semiclassical limit, single grasp $\tilde{\mathrm{E}}$ are replaced by double grasp, i.e., $\tilde{\mathrm{E}}$ which grasp both the incoming and outgoing holonomy at each vertex.

This appendix uses the "congruence" rather than S_1 picture for the topology of the spin network. (Topology is discussed at the beginning of Sec. II.) The congruence picture is better for determining the number of loops which contribute to each exact field strength.

Recall the quantum field theory expression for the Euclidean Hamiltonian.

$$\begin{aligned}-\mathrm{NH}_e + \mathrm{ST} &= \int d^3x \mathrm{N}(F_{jk}^I \mathrm{E}_J^j \mathrm{E}_K^k \epsilon_{IJK}/2\kappa|e|) + \mathrm{ST}; \\ \mathrm{ST} &= -(\mathrm{NA}_a^I \mathrm{E}_Z^z \mathrm{E}_K^a \epsilon_{IZK}/\kappa|e|)|_{z=-\infty}^{+\infty}; \\ \kappa &= 8\pi\mathrm{G}; \\ |e| &:= \sqrt{\det \mathrm{E}}\mathrm{sgn}(e).\end{aligned} \tag{A1}$$

The LHF formula for field strengths in Eq. (A1) generalizes the classical formula

$$\mathrm{F}_{ij} = \lim_{\Delta A \to 0}\left(\prod h\right)\Big/\Delta A(ij), \tag{A2}$$

where $\prod h$ is the product of holonomies around the edges of infinitesimal area Δ A(ij). A local treatment of the Hamiltonian would construct each F using small areas at a single vertex n_z, plus triads at that vertex; then sum over vertices. For example, the basic modular unit would be the vertex.

However, for a nonlocal Hamiltonian, one must use nonlocal modular units. Since Eq. (A2) contains an area, the appropriate basic modular units should be areas. The areas are bounded by nearest neighbor vertices: i.e. holonomies along each edge of the area run from one vertex to the next, nearest neighbor vertex. For example, the LHF contribution to F_{xy} from the area bounded by vertices (n_x, n_y, n_z), $(n_x, n_y + 1, n_z)$, $(n_x + 1, n_y + 1, n_z)$, $(n_x + 1, n_y, n_z)$ is

$$\mathrm{F}_{xy} = \mathrm{i}\,\mathrm{const}\, h_y^{-1} h_x^{-1} h_x h_y + \mathrm{H.c.} \quad \text{(LHF)}. \tag{A3}$$

Each h_i traverses edge i from n_i to $n_i + 1$; the h^{-1} traverse in the reverse direction. Reading from right to left, the explicitly written term circulates the xy area in a counterclockwise direction. The Hermitian conjugate (H.c.) term circulates in the clockwise direction.

The above two terms are not the only possibilities, even though we restrict ourselves to circuits starting from (n_x, n_y, n_z) and continuing in the xy plane to nearest neighbors only. In fact there are eight such terms. The eight correspond to the four vertices in the xy plane which are nearest neighbors to the vertex (n_x, n_y, n_z), times two for clockwise or counterclockwise circuit. The eight may be grouped into four sets of two terms each, after we impose the requirement that the field strength is Hermitian. The full LHF expression for F_{xy} therefore contains four adjustable constants analogous to the "const" in Eq. (A3). One could determine them by carrying out a small sine expansion, and demanding that the expansion have the minimum number of powers of sine beyond those needed to recover the quadratic limit. The discussion would be straightforward but lengthy, and will be omitted in order to focus on the semiclassical limit.

One can small-sine expand the contribution Eq. (A3), using Eq. (11). Keeping up to order $(\sin)^2$, one gets

$$\mathrm{F}_{xy}(n_z) = 2\mathrm{i}[\hat{h}_x(n_z), \hat{h}_y(n_z)] \quad \text{(SS)}. \tag{A4}$$

All eight loops give the same small sine limit. The $\hat{h}_a$ need only a single argument n_z, since the loop remains in an xy plane where every holonomy has the same n_z, and all variables are independent of x and y.

Equation (A4) has no linear-in-sine terms. These would spoil the FT limit

$$F_{ij} \to (\partial_i \mathrm{A}_j^I - \partial_j \mathrm{A}_i^I + \mathrm{A}_i^I \mathrm{A}_j^J \epsilon_{IJK})\sigma^I \Delta x^i \Delta x^j \quad \text{(FT)}. \tag{A5}$$

For $ij = xy$, the linear in A terms are absent, because all fields are independent of x and y.

The remaining two field strengths, F_{za} with $a = x, y$ may be constructed in a similar fashion. For F_{zx}, for example, the relevant plane is zx rather than xy; but there are still eight loops, corresponding to four vertices which are nearest neighbors to the vertex (n_x, n_y, n_z), times two, for clockwise or anticlockwise. Four of the holonomy loops travel forward to $n_z + 1$; the remaining four travel backward to $n_z - 1$. The four forward loops are

$$\begin{aligned} F_{za}(n_z, n_z + 1) = \mathrm{i}\,\mathrm{const}[&h_z(n_z, n_z, +1)^{-1} h_a^{-1}(n_z + 1) \\ &\times h_z(n_z, n_z, +1) h_a(n_z) \\ &- (h_a \leftrightarrow h_a^{-1}) + \mathrm{H.c.}] \end{aligned} \tag{A6}$$

The $(h_a \leftrightarrow h_a^{-1})$ term is the second forward loop; "H.c." includes the remaining two loops, which have clockwise $\leftrightarrow$ anticlockwise. Since the h_z link two vertices, F_{za} requires two n_z arguments.

In the xy case all four areas had the same small sine limit. In the za case, forward and rearward loops have different SS limits and cannot be combined, because they involve different x holonomies $h_x(n_z \pm 1)$ on different x edges.

The first two forward loops in Eq. (A6) each contain unwanted $h_x(n_z) h_x(n_z + 1)$ quadratic terms in the SS limit; the minus sign between these two loops ensures that the unwanted terms cancel. Explicitly, the small sine limits are

$$\begin{aligned} 2F_{za}(n_z, n_z + 1) = (-2i)\{&-[\hat{h}_z(n_z, n_z + 1), \hat{h}_a(n_z + 1)] \\ &+ \hat{h}_a(n_z + 1) - \hat{h}_a(n_z) \\ &+ [\hat{h}_a(n_z + 1), \hat{h}_a(n_z)]\} \\ &- (\hat{h}_a \leftrightarrow -\hat{h}_a) \quad \text{(SS)}; \\ 2F_{za}(n_z, n_z - 1) \to (-2i)\{&-[\hat{h}_z(n_z - 1, n_z), \hat{h}_a(n_z - 1)] \\ &+ \hat{h}_a(n_z) - \hat{h}_a(n_z - 1) \\ &+ [\hat{h}_a(n_z - 1), \hat{h}_a(n_z)]\} \\ &- (\hat{h}_a \leftrightarrow -\hat{h}_a) \quad \text{(SS)}. \end{aligned} \tag{A7}$$

The $\hat{h} \leftrightarrow -\hat{h}$ terms represent the second loop; the unwanted $(\hat{h}_a)^2$ terms cancel out when the two loops are summed.

The classical xy field strength distorts a spherical cloud of test particles into an ellipsoid. This distortion mimics the behavior of the sphere under free fall in a *static* gravitational field. The za field strengths produce the distortions typical of time-varying waves. It is not surprising, therefore, that the za field strengths possess all the nonlocality.

We now consider the triads. Since triads are associated with areas and volumes, which are local, we assume the LHF triads are local, with support at the vertices. More precisely: we draw a small cube around each vertex, and the triads live on the six faces of the cube. The triads are associated with area two-forms via

$$E^a_A \epsilon_{abc} dx^b \wedge dx^c,$$

When defining the volume operator, typically one assumes each E^a_A grasps at both faces having outward normals in the a direction, the positive face (with normal pointing in the positive a direction) and the negative face. This is natural; the volume operator is not directional, and one would expect contributions from all six faces. The volume operator therefore involves the product of three operators, each grasping at two faces:

$$E^i_I := E^i_I(+) + E^i_I(-), \tag{A8}$$

where $\pm$, the sign of the normal, indicates the area where the $\tilde{E}$ grasps. However, if one considers $\tilde{E}$ other than those involved in the volume operator, the double grasp $\tilde{E}$ is not especially natural.

For example, in the operator expression for Gauss's law, one needs the *difference* between the + and − operators, $E^z_Z(+)$-$E^z_Z(-)$, because Gauss's law involves the difference between the ingoing and outgoing Z component of spin. It is not enough, therefore, to specify the vertex n_z where the triad has its support; one must also supply an argument $\pm$ specifying which face is grasped.

In Eq. (A1) there are two triads and a volume e associated with each field strength. A given field strength is a holonomy loop passing through four vertices, but at each vertex the holonomies do not pass through all six faces. One holonomy enters at one face, and a different holonomy leaves at another face. Therefore one can associate a face with each holonomy as follows. If a holonomy h_i at n_z passes through a face having sign + (−), then the triad E^i_A multiplying h_i in the Hamiltonian is given the sign + (−).

For example, consider the xy holonomy loop beginning at vertex (n_x, n_y, n_z) and passing through the + x face to $(n_x + 1, n_y, n_z)$. The loop then continues to $(n_x + 1, n_y + 1, n_z), \ldots$, finally returning down the y axis through the + y face. Both holonomies pass through + faces; the triads multiplying this contribution would be $E^x_A(+)$ and $E^y_A(+)$. The same triads occur in the H.c. loop.

The remaining three pairs of loops involve the remaining three sign pairs: $(\pm, \mp)$ and $(-, -)$. The F_{xy} contribution remains a sum of four terms. (We continue to group each term and its H.c. together as a single contribution.) Now each term contains a different sign pair.

A z triad E^z_Z also contributes to each F_{xy} term in H_e. The z triad is contained in the factor of volume, e. It is not obvious what sign to assign to the z triad, since no xy holonomy passes through a z face. However, a given xy area bounds two volumes, each containing a different z line $(n_z \pm 1, n_z)$. One can get either sign, ± 1, accordingly as the

xy area is interpreted as bounding the volume containing (n_z, n_z+1), or (n_z-1, n_z). There is no reason to favor one interpretation over the other, and we therefore make the z triad the sum of the two z signs: $\mathrm{E}^z_Z = \mathrm{E}^z_Z(+) + \mathrm{E}^z_Z(-)$.

In the small sine, slow variation limit, the triad functions in H_e lose their dependence on the individual $\mathrm{E}^i_I(\pm)$ and depend only on the sums $\mathrm{E}^i_I := \mathrm{E}^i_I(+) + \mathrm{E}^i_I(-)$. Proof: begin with the four loops contributing to F_{xy}. All field strengths have the same small sine limit, Eq. (A4). One can factor this out, leaving a sum over four triad functions.

$$\mathrm{H}_e(xy) = \mathrm{F}_{xy}(\mathrm{SS})\sum_\eta f[\mathrm{E}^x(\eta_x), \mathrm{E}^y(\eta_y), e(\eta_x, \eta_y)]$$

where $\eta_a = \pm 1$ indicate the areas grasped by the $\tilde{\mathrm{E}}$; and the E^z_Z, not indicated explicitly, are already in the desired form.

Now expand each transverse $\tilde{\mathrm{E}}$:

$$\begin{aligned}\mathrm{E}^a(\pm) &= [(\mathrm{E}(+) + \mathrm{E}(-)]^a/2 \pm [(\mathrm{E}(+) - \mathrm{E}(-)]^a/2 \\ &= \mathrm{E}^a/2 \pm \tilde{\delta}\mathrm{E}^a/2 \\ &= (\mathrm{E}^a/2)[1 \pm \tilde{\delta}\mathrm{E}^a/\mathrm{E}^a]. \\ \tilde{\delta}\mathrm{E}^a(n) &:= [(\mathrm{E}^a(+) - \mathrm{E}^a(-)]. \end{aligned} \tag{A9}$$

The usual difference δf denotes the difference between values of f evaluated at two different vertices. The tilde difference denotes the difference between two values of f located at the same vertex, but on opposite faces of the vertex. I insert the expansions of Eq. (A9) into Eq. (A9), and power series expand around $\tilde{\delta}\mathrm{E} = 0$, assuming the tilde differences are small because of the slow variation assumption.

The expansion contains no terms having an odd number of tilde differences, since the sum in Eq. (A9) is even under $(+ \leftrightarrow -)$. The expansion of an arbitrary symmetric function of the $\mathrm{E}^a_A(\pm)$ begins with the terms

$$f[\mathrm{E}(+)] + f[\mathrm{E}(-)] = 2f(\mathrm{E}/2) + (1/2!)(\partial^2 f/\partial \mathrm{E}^2)(\tilde{\delta}\mathrm{E})^2.$$

The leading term in the expansion contains two more factors of E than the second order term, and is therefore larger than the second order term by a factor

$$(\tilde{\delta}\mathrm{E})^2/(\mathrm{E}^a)^2 = \mathrm{order}(\delta f/f)^2.$$

The second order term can be neglected. Tilde differences have disappeared from the F_{xy} terms.

Now consider the F_{za} terms, for example the forward areas involving z holonomies on edge (n_z, n_z+1), and $za = zx$. At Eq. (A7) the sign between two terms was adjusted so as to cancel an unwanted commutator term. This cancelation must be reconsidered; the two terms are now multiplied by different triads $\mathrm{E}^x(\pm)$. The unwanted brackets now have a contribution of the form

$$\begin{aligned}&[\hat{h}_x(n_z+1), \hat{h}_x(n_z)] \times (f[\mathrm{E}^x(+), \mathrm{E}^z(+)] - f[\mathrm{E}^x(-), \mathrm{E}^z(+)]) \\ &\quad = [\delta_f \hat{h}_x(n_z), \hat{h}(n_z)] 2(\partial f/\partial \mathrm{E}^x)\tilde{\delta}\mathrm{E}^x \quad (\mathrm{SV}). \end{aligned} \tag{A10}$$

This term is second order in differences and can be dropped.

The remaining terms, those without the unwanted commutators, give the correct QFT limit and are even under $\mathrm{E}^a(+) \leftrightarrow \mathrm{E}^a(-)$. By the same argument as for the F_{xy} terms, the expansion in powers of $\tilde{\mathrm{E}}^a$ may be terminated at the leading term which is independent of $\tilde{\mathrm{E}}^x$. Similarly for the rearward loops.

At this point the F_{za} loops have the desired E^a dependence, but forward loops are multiplied by a function of $\mathrm{E}^z(+)$, while the rearward loops depend on $\mathrm{E}^z(-)$. From the SS limits, Eq. (A7), both loops contain forward difference terms $\delta_f \hat{h}_a$ and commutator terms $[\hat{h}_z, \hat{h}_a]$.

Consider first the difference terms. As in Eq. (A9), expand in sums plus differences and drop the term linear in differences.

$$\begin{aligned}&\delta_f h_a(n_z) f[\mathrm{E}^z(+)] + \delta_f h_a(n_z - 1) f[\mathrm{E}^z(-)] \\ &\quad [\delta_f h_a(n_z) + \delta_f h_a(n_z - 1)] f[\mathrm{E}^z] \\ &\quad + [\delta_f h_a(n_z) - \delta_f h_a(n_z - 1)](\partial f/\partial \mathrm{E}^z)\tilde{\delta}\mathrm{E}^z \quad (\mathrm{SV}). \end{aligned} \tag{A11}$$

The bracket on the middle line is twice $h_z(n_z)$, from definitions (14) and (16), and slow variation. The last line is down by two factors of $\delta f/f$ and may be dropped: the square bracket on the last line is the second forward difference of h.

For the commutator terms one must expand, not only $f[\mathrm{E}^z(+)]$, but also $\hat{h}_z$ and $\hat{h}_a$.

$$\begin{aligned}\hat{h}_z(n_z, n_z \pm 1) &= [\hat{h}_z(n_z, n_z+1) + \hat{h}_z(n_z - 1, n_z)]/2 \\ &\quad \pm \tilde{\delta}\hat{h}_z(n_z - 1, n_z)/2 \\ &:= \hat{h}_z(n_z) \pm \tilde{\delta}\hat{h}_z(n_z - 1, n_z)/2; \\ \hat{h}_a(n_z \pm 1) &= \hat{h}_a(n_z) \pm \delta_f \hat{h}_a(n_z \pm 1). \end{aligned} \tag{A12}$$

The commutator terms are then

$$\begin{aligned}&\sum_\pm [\hat{h}_z(n_z, n_z \pm 1), \quad \hat{h}_a(n_z \pm 1)] f[\mathrm{E}^z(\pm)] \\ &\quad = [\hat{h}_z(n_z), \hat{h}_a(n_z)] 2f(\mathrm{E}^z) + \mathrm{order}(\delta_{(c)} f/f)^2. \end{aligned} \tag{A13}$$

The two terms linear in $\tilde{\delta}\hat{h}_z(n_z - 1, n_z)$ and $\tilde{\delta}\mathrm{E}^z$ have opposite signs and cancel. The two terms linear in $\delta_f \hat{h}_a$ have the form

$$[\hat{h}_z(n_z), \delta_f \hat{h}_a(n_z) - \delta_f \hat{h}_a(n_z - 1)] f[\mathrm{E}^z].$$

This difference of differences is a second forward difference, which is order $(\delta f/f)^2$.

All four F_{za} loops now depend only on $E^i = E^i(+) + E^i(-)$. Dependence on the $E^i(\pm)$ has disappeared. □

APPENDIX B: THE SURFACE TERM

The $\delta_{(c)}(\hat{h}_a)e^{za}$ terms in H_e contain a second derivative, since $\hat{h}_a$, when expressed in terms of tetrads, contains a time derivative. The $\delta_{(c)}$ must be integrated by parts (IBP) onto the cotriad. The IBP brings the Hamiltonian (and Lagrangian) into a standard form with only first derivatives. The IBP generates a total derivative, which becomes a surface term $\hat{h}_a e^{za}$. This term must be canceled, which means a ST must be added to perform the cancelation: If

$$\begin{aligned} H_e = (\delta_{(c)}\hat{h}_a)e^{za} + \cdots &= \delta_{(c)}(\hat{h}_a e^{za}) - \hat{h}_a\delta_{(c)}e^{za} + \cdots \\ &\coloneqq -\mathrm{ST} - \hat{h}_a\delta_{(c)}e^{za} + \cdots, \end{aligned}$$

then

$$H_e + \mathrm{ST} = -\hat{h}_a\delta_{(c)}e^{za} + \cdots \text{ plus no ST.}$$

The $\cdots$ denote terms which are derivative free and do not contribute to the ST. The integration by parts shifts the difference onto the triads and changes the sign of a term in the Hamiltonian.

Although the surface term produces only a rather simple change in H_e, the ST must be calculated in detail, because ST is the physical Hamiltonian. H_e is a constraint, and vanishes when acting on physical states. The ST does not vanish. In the follow-on paper, the ST is used to compute the total energy of the solution constructed in that paper.

From the preceding discussion, the surface term comes entirely from the $\delta_{(c)}\hat{h}$ terms in F_{za}. Insert Eq. (29) for F_{za} into Eq. (65) for $\tilde{H}_e$:

$$\begin{aligned} -\underline{N}\tilde{H}_e + \mathrm{ST} &= (1/2\kappa)\sum_n\{\cdots + (-2i)(\delta_c\hat{h}_a^C)E_B^a\epsilon^{ZBC}\underline{N}(n)\} \\ &\quad + \mathrm{ST} \\ &= (1/2\kappa)\sum_n\{\cdots - (-2i)\hat{h}_a^C\delta_c[E_B^a\epsilon^{ZBC}\underline{N}(n)] \\ &\quad + (-2i)\delta_c[\hat{h}_a^C E_B^a\epsilon^{ZBC}\underline{N}(n)]\} + \mathrm{ST}. \end{aligned} \tag{B1}$$

Equation (37) was used to carry out the difference analog of integration by parts.

This is a good point to describe the labeling of the vertices at the surface. The $\sum_n$ in the Hamiltonian ranges from $n = \min$ to $n = \max$; $\min \le n \le \max$. However, the spin network itself extends to values $n < \min$ and $n > \max$. This is analogous to the situation in classical field theory, where one integrates the Lagrangian or Hamiltonian from min z to max z, but the space extends beyond these limits.

In principle, the limits (min, max) can be chosen anywhere. In practice, the limits are chosen to lie in an asymptotic region, so that surface terms generated by integration by parts can be evaluated using boundary conditions. Similarly here, the only restriction on min and max is that the system is asymptotic at those values of n; but the spin network does not vanish beyond those limits.

In particular, the Thiemann construction of the extrinsic curvatures, Sec. VI, predicts that the spin connection depends on central differences of the transverse $\tilde{E}$,

$$[\tilde{E}(n+1) - \tilde{E}(n-1)]/2.$$

At $n = \max$, $\tilde{E}$ $(n+1)$ is $\tilde{E}$ $(\max+1)$. This quantity is not assumed to vanish.

Since the $\delta_{(c)}$ connects every other vertex, the total derivative on the last line, Eq. (B1), gives rise to two surface terms, one from even n terms and one from odd n. $\tilde{H}_e$ becomes

$$\begin{aligned} -\kappa(\underline{N}\tilde{H}_e + \mathrm{ST}) &= \sum_n\{\cdots - (-2i)\hat{h}_a^C\delta_c[E_B^a\epsilon^{ZBC}\underline{N}](n)\} \\ &\quad + [\underline{N}(n)(-2i)\hat{h}_a^C E_B^a\epsilon^{ZBC}](n) \\ &\quad \times [|_{n=\min}^{\max} + |_{n=\min-1}^{\max+1}](1/2) + \mathrm{ST}. \end{aligned} \tag{B2}$$

The ST is now chosen so that the last line vanishes.

The $1/2$ in the surface term comes from the $1/2$ in the central difference. For example, use the definition of the central difference, Eq. (13), to expand each term in the sum

$$\sum_{-1}^{+1}\delta_{(c)}f(n) = (1/2)[f(+2) - f(-2) + f(+1) - f(-1)].$$

It is possible to eliminate the holonomy from the surface term. One can replace the $(-2i)\hat{h}_a^C$ by $(\gamma K + \Gamma)_a^C$ [Eq. (VI)]. The term involving K,

$$K_a^C E_B^a\epsilon^{ZBC},$$

is (one half of) the Gauss constraint, Eq. (40), and may be dropped. The term involving Γ may be simplified by using the other half of the constraint,

$$\Gamma_a^C E_B^a\epsilon^{BC} = \delta_{(c)}E_Z^z.$$

The surface term is then

$$\mathrm{ST} = -\underline{N}\delta_{(c)}E_Z^z(n)[|_{n=\min}^{\max} + |_{n=\min-1}^{\max+1}](1/2\kappa). \tag{B3}$$

APPENDIX C: NUMBER OF VERTICES

This calculation uses a lattice with a fixed number of vertices. Where does the number of vertices enter into the calculation? To obtain the classical limit, we must assume the fields vary slowly from vertex to vertex, so that the discrete structure of the spin network is not obvious. The precise value of the number of vertices is not important, provided the number of vertices is large enough to guarantee slow variation. One could replace the fixed number of vertices with a distribution in the number of vertices and nothing would change, provided the distribution were peaked at a large number.

The restriction to a fixed number of vertices may be more apparent than real, because the classical limit uses coherent states. In a coherent state, at each vertex n, the values of SU(2) angular momentum L are Gaussian distributed. This distribution includes angular momentum zero. In that sense a coherent state already includes the possibility of no vertex at n.

Since the distribution is Gaussian, the probability of no vertex is very small. Presumably spin networks with small numbers of vertices do not contribute significantly in the classical limit.

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[15] This is the point at which we break with Hinterleitner and Major (HM, Ref. [5]), who assert that the constraint $\mathrm{U}_1 (= \text{their constraint } \mathrm{U}_+)$ is first class. These authors invoke the diffeomorphism gauge, and therefore obtain two, rather than three unidirectional constraints, as we do. However, once that gauge is invoked, one must eliminate the gauge degrees of freedom and rewrite constraint U_1 in terms of Π and Q. See Eq. (77); E^z_Z is an implicit function of Q. The U_1 then becomes second class, rather than first class.

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