

# Gravitational, lensing, and stability properties of Bose-Einstein condensate dark matter halos

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The possibility that dark matter, whose existence is inferred from the study of galactic rotation curves, and from the mass deficit in galaxy clusters, can be in the form of a Bose-Einstein condensate has been extensively investigated lately. In the present work, we consider a detailed analysis of the astrophysical properties of the Bose-Einstein condensate dark matter halos that could provide clear observational signatures that help discriminate between different dark matter models. In the Bose-Einstein condensation model, dark matter can be described as a nonrelativistic, gravitationally confined Newtonian gas, whose density and pressure are related by a polytropic equation of state with index  $n = 1$ . The mass and gravitational properties of the condensate halos are obtained in a systematic form, including the mean logarithmic slopes of the density and of the tangential velocity. The lensing properties of the condensate dark matter are investigated in detail. In particular, a general analytical formula for the surface density, an important quantity that defines the lensing properties of a dark matter halos, is obtained in the form of series expansions. This enables arbitrary-precision calculations of the surface mass density, deflection angle, deflection potential, and of the magnification factor, thus giving the possibility of the comparison of the predicted lensing properties of the condensate dark matter halos with observations. The stability properties of the condensate halos are also investigated by using the scalar and the tensor virial theorems, respectively, and the virial perturbation equation for condensate dark matter halos is derived. As an application of the scalar virial theorem, we consider the problem of the stability of a slowly rotating and slightly disturbed galactic dark matter halo. For such a halo, the oscillation frequencies and the stability conditions are obtained in the linear approximation.

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## I. INTRODUCTION

The polytropic gas model, in which the pressure  $P$  has a simple power law dependence of the matter energy density  $\rho$ , given by  $P = K\rho^\gamma$ , plays an important role in the modeling of self-gravitating systems of compact astrophysical objects, such as white dwarfs and neutron stars, as well as in the study of the equation of state of nuclear matter. Usually the polytropic equation of state is represented as  $P = K\rho^{1+1/n}$ , where  $n$  is called the polytropic index. The Poisson and the hydrostatic equilibrium equations for a self-gravitating, spherically symmetric polytropic fluid can be combined to provide the basic equation describing the equilibrium properties of the system as

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) + \theta^n = 0, \quad (1)$$

where the dimensionless quantities  $\theta$  and  $\xi$  are defined as  $\rho = \rho_c \theta^n$  and  $\xi = \sqrt{4\pi G \rho_c^2 / (n+1) P_c} r$ , with  $\rho_c$  and  $P_c$  the

central energy density and pressure, respectively. Equation (1) is known as the Lane-Emden equation, and its physical, astrophysical and mathematical properties have been intensively studied by using both analytical and numerical methods. It admits only three exact solutions, corresponding to  $n = 0$ ,  $n = 1$  and  $n = 3$ , respectively [1].

From these three solutions, the unusual solution corresponding to  $n = 1$ , having the form  $\theta = \sin \zeta / \zeta$ , does not seem at first sight to have many realistic physical properties, and consequently it has been less investigated, as compared to the other solutions. However, somehow unexpectedly, the  $n = 1$  polytropic equation of state appears in a fundamental field of research apparently unrelated to astrophysics, namely, the theory of the condensed Bose-Einstein cold atomic gases. A basic result in quantum statistical mechanics is that at very low temperatures, in a dilute Bosonic system (Bose gas), all particles condense to the same quantum ground state, forming a so-called Bose-Einstein condensate (BEC). From a physical point of view, a BEC is characterized by a sharp peak in both coordinate and momentum space distribution. The BEC phase transition takes place when the particles are correlated with each other quantum mechanically. This

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happens when their wavelengths overlap; that is, the thermal de Broglie wavelength  $\lambda_T$  becomes greater than the mean interparticle distance  $l$ . Hence, the BEC transition occurs at a temperature

$$T < \left( \frac{2\pi\hbar^2}{mk_B} \right) n^{2/3}, \quad (2)$$

where  $m$  is the mass of the particle in the condensate,  $n$  is the number density, and  $k_B$  is Boltzmann's constant [2–9]. A coherent quantum condensate state develops in two physical situations: a) when the particle density is high enough, or b) the system temperature is sufficiently low. The Bose-Einstein condensation has been observed in laboratory experiments [10–12]. A large number of quantum degenerate gases have been created by a combination of laser, magnetic and evaporative cooling techniques. The experimental investigations have opened several new directions of research, involving atomic, statistical and condensed matter physics [2–12].

The purpose of this paper will be applications of the BEC to the dark matter problem. Indeed, the existence of dark matter is primarily supported by the rotation curve of galaxies [13–15], and also by the difference between the masses derived from luminosity and from the virial theorem in clusters of galaxies [16]. Furthermore, gravitational lensing data also indicates that light deflection is more accentuated than would be to visible matter alone [17]. Thus, by introducing a new matter component, denoted dark matter, it is possible to explain all of these observational phenomena. However, the nature of dark matter, and its constituent particle(s), still remain essentially unknown, despite several decades of intensive research. For reviews on the dark matter problem, see [17,18].

In this context, the possibility that dark matter could be in the form of a Bose-Einstein condensate was analyzed in detail in [19] (for earlier work in this field, see [20–23]). In particular, the equation of state of the condensate dark matter was obtained as a polytropic equation of state of index  $n = 1$ , and the corresponding density profile was discussed. The theoretical rotation curves were fitted with the observational data from a sample of Low Surface Brightness (LSB) galaxies, and it was pointed out that the lensing properties may discriminate between the standard pressureless and the condensate dark matter models. The physical and astrophysical properties of Bose-Einstein condensate dark matter halos were investigated in [24].

Recently, the rotation curves in the Bose-Einstein condensate dark matter model were analyzed in [25], and a good agreement with observations was found. A comparison of the predicted rotation curves with the observational data for eight dwarf galaxies was performed in [26], and it was shown that the presence of the condensate dark matter can solve the core/cusp problem faced by the standard  $\Lambda$ CDM (Acold dark matter) model. The effects of the

angular momentum and on the vortices on the equilibrium of self-gravitating, rotating BEC halos which satisfy the Gross-Pitaevskii-Poisson equations were considered in [27–31]. In a Bose-Einstein condensate vortices form as long as the self-interaction is strong enough, which is not the case for axionic dark matter. The galactic masses in the framework of the condensate model were discussed in [32,33], and the effects of the finite temperature of the condensate on the density profiles were studied in [34]. The equilibrium properties of Newtonian self-gravitating Bose-Einstein condensates with short-range interactions were investigated in detail in [35,36]. The study of the cosmological implications of the Bose-Einstein condensation has also become an active field of research [37–46]. In particular, in [44] it was shown that condensate dark matter effects can be seen in the CMB matter power spectrum if the mass  $m_\chi$  of the condensate particle lies in the range  $15 \text{ meV} < m_\chi < 700 \text{ meV}$ , leading to a small, but perceptible, excess of power at large scales. The possibility of the existence of the BEC stars was also considered [47], while BEC string models were investigated in [48].

One of the central issue in the study of Bose-Einstein condensate dark matter halos is the nature, and properties, of the dark matter particle. One of the most interesting dark matter particle candidate is the axion, a (still) hypothetical elementary particle postulated by the Peccei-Quinn theory [49] to resolve the strong  $CP$  problem in quantum chromodynamics. It was shown in [50] that dark matter axions do form a BEC, as a result either of their self-interactions, or as a result of their gravitational interactions. It was also proven, from the study of the cosmological perturbations, that axion Bose-Einstein condensate dark matter halos differ from standard cold dark matter ones on small scales only. Unlike vortices in superfluid  $\text{He}^4$  and dilute gases, the vortices in the axion Bose-Einstein condensate dark matter are attractive [51]. Therefore, a large fraction of the vortices in the axion BEC may form a single big vortex along the rotation axis of the galaxy. The resulting enhancement of caustic rings could explain the rises in the Milky Way rotation curve, usually attributed to caustic rings. The properties of the axionic dark matter have been extensively investigated in [52].

It is the purpose of the present paper to extend the previous investigations of the  $n = 1$  polytropic Bose-Einstein dark matter halos, by considering in detail and in a systematic way the observationally relevant physical parameters. As a first step in this study we consider the mass and gravitational properties of the dark matter halos in the condensate model. The gravitational potential and energy of the halo, its kinetic energy, as well as the observationally extremely important logarithmic slopes of the density and velocity profiles are obtained. One of the most important method which could discriminate between the condensate model and other dark matter models is gravitational lensing. We obtain and present in

detail, in an analytical form, the physical parameters (surface mass density, deflection angle and magnification etc.) necessary for the analysis and comparison of the observational data with the predictions of the model. Finally, the formulations of the virial theorem are obtained for rotating and nonrotating dark matter halos, the virial perturbation equation is derived, and some general stability conditions are formulated. As an application of the scalar virial theorem we consider in detail the problem of the stability of a slowly rotating and slightly disturbed galactic dark matter halo. By introducing a mathematical description based on the Lagrangian displacements, from the scalar virial theorem we obtain the equation describing the time evolution of the perturbations. The oscillations frequencies and the stability conditions of the Bose-Einstein condensate dark matter galactic halo are obtained in the linear approximation.

The present paper is organized as follows. In Sec. II, the basic equations describing the Bose-Einstein condensate dark matter halos are presented. In Sec. III, the general gravitational properties of the condensate halos are investigated. The gravitational lensing by dark matter halos is considered in Sec. IV. The virial theorem for Bose-Einstein condensate dark matter halos and their stability properties are presented in Sec. V. As an application of the scalar virial theorem we consider the problem of the stability of a slowly rotating and slightly disturbed galactic dark matter halo is considered in Sec. VI. We discuss and conclude our results in Sec. VII.

## II. BOSE-EINSTEIN CONDENSATE DARK MATTER HALOS

The transition of the dark bosonic component to Bose-Einstein condensed state is assumed to have taken place during the cosmological expansion of the Universe at a redshift of around  $z \approx 1400$ – $1500$  [43]. At that moment the temperature and the matter density of the Universe did satisfy the condition given by Eq. (2). Due to the expansion of the Universe, its temperature and density further decreased. In the following we make the fundamental assumption that the dark matter halos are composed of a strongly coupled cold dilute Bose-Einstein condensate, at absolute zero temperature. Hence, this assumption implies that almost all of the dark matter particles are in the condensate. Since the dark matter halo is dilute and cold, only binary collisions at low energy between dark matter particles are relevant. Therefore, independently of the details of the two-body potential, it follows that the collisions can be characterized by a single physical parameter, the  $s$ -wave scattering length  $l_a$  [2,8]. Therefore, with an excellent approximation, one can substitute the interaction potential with an effective interaction term of the form  $V_I(\vec{r}' - \vec{r}) = U_0 \delta(\vec{r}' - \vec{r})$ , where the coupling constant  $U_0$  is related to the scattering length  $l_a$  through the relation  $U_0 = 4\pi\hbar^2 l_a / m_\chi$ , where  $m_\chi$  is the mass of the dark

matter particle [2,8]. In this approach the ground state properties of the dark matter are described by the mean-field Gross-Pitaevskii (GP) equation [2,8,9]. The GP equation for the dark matter halos can be derived from the GP energy functional [2,8,9,53],

$$\begin{aligned} E[\psi] &= \int \left[ \frac{\hbar^2}{2m_\chi} |\nabla\psi(\vec{r})|^2 + \frac{U_0}{2} |\psi(\vec{r})|^4 \right. \\ &\quad \left. - m_\chi \psi^*(\vec{r}) \vec{\Omega} \cdot (\vec{r} \times \vec{p}) \psi(\vec{r}) \right] d^3\vec{r} \\ &\quad - \frac{1}{2} G m_\chi^2 \int \int \frac{|\psi(\vec{r})|^2 |\psi(\vec{r}')|^2}{|\vec{r} - \vec{r}'|} d^3\vec{r} d^3\vec{r}' \\ &= E_K + E_{\text{int}} + E_{\text{rot}} + E_{\text{grav}}, \end{aligned} \quad (3)$$

where  $\psi(\vec{r})$  is the wave function of the condensate [2,5–9]. The first term in the energy functional is the quantum pressure (kinetic energy  $E_K$ ), the second is the interaction energy  $E_{\text{int}}$ , the third term is the rotational energy  $E_{\text{rot}}$ , with  $\vec{\Omega}$  the angular velocity, and  $\vec{p}$  the condensate momentum, and the fourth term is the gravitational potential energy, respectively. The mass density of the condensate is defined as

$$\rho(\vec{r}) = m_\chi |\psi(\vec{r})|^2, \quad (4)$$

and the normalization condition is  $N = \int |\psi(\vec{r})|^2 d^3\vec{r}$ , where  $N$  is the total number of dark matter particles. The variational procedure,

$$\delta E[\psi] - \mu \delta \int |\psi(\vec{r})|^2 d^3\vec{r} = 0, \quad (5)$$

provides the GP equation as [2,5–9]

$$\begin{aligned} -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + m_\chi V_{\text{grav}}(\vec{r}) \psi(\vec{r}) \\ + U_0 |\psi(\vec{r})|^2 \psi(\vec{r}) - \Omega L_z \psi(\vec{r}) = \mu \psi(\vec{r}), \end{aligned} \quad (6)$$

where the Lagrangian multiplier  $\mu$  is the chemical potential, and the confining gravitational potential  $V_{\text{grav}}$  satisfies the Poisson equation

$$\nabla^2 V_{\text{grav}} = 4\pi G \rho. \quad (7)$$

The term  $\Omega L_z = i\hbar(x\partial_y y\partial_x)$  is due to the rotation of the system about the  $z$  axis with an angular frequency  $\Omega$ . For a rotating Bose-Einstein condensate dark matter halo, its rotational energy may be approximated as [9]

$$E_{\text{rot}} = \frac{1}{2} m_\chi \int_V \omega^2(\Omega)(x^2 + y^2) |\psi(\vec{r})|^2 d^3\vec{r}, \quad (8)$$

where  $\omega$  is some function of the angular velocity of the rotation  $\Omega$ .

In the physically important case when the number of particles in the condensate becomes large enough, the quantum pressure term makes a significant contribution to the thermodynamical parameters only near the boundary, and it is much smaller than the interaction energy term. Thus, for this case the quantum pressure term can be neglected (the Thomas-Fermi approximation). When  $N \rightarrow \infty$ , it can be proven explicitly that the Thomas-Fermi approximation becomes exact [2,5–9]. Therefore, we obtain

$$\rho(\vec{r}) = \frac{m_\chi}{U_0} \left[ \mu - m_\chi V_{\text{grav}}(\vec{r}) - \frac{1}{2} m_\chi \omega^2 (x^2 + y^2) \right]. \quad (9)$$

In the nonrotating case, besides the conservation of the total energy  $E = E_0 + \int m_\chi V_{\text{grav}} |\psi|^2 d^3\vec{r}$ , where

$$E_0 = \int \left[ \frac{\hbar^2}{2m_\chi} |\nabla\psi(\vec{r})|^2 + \frac{U_0}{2} |\psi(\vec{r})|^4 \right] d^3\vec{r}, \quad (10)$$

the GP energy functional provides two more conservation laws, the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0, \quad (11)$$

where the momentum  $\vec{j}$  is given by

$$\vec{j} = \frac{i\hbar}{2} (\psi \nabla \psi^* - \psi^* \nabla \psi), \quad (12)$$

and the momentum conservation equation [54,55],

$$\frac{\partial j_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = -\rho \frac{\partial V_{\text{grav}}}{\partial x_i}, \quad i, k = 1, 2, 3, \quad (13)$$

with

$$\Pi_{ik} = \frac{\hbar^2}{4m_\chi} \left( \frac{\partial \psi}{\partial x_i} \frac{\partial \psi^*}{\partial x_k} - \psi \frac{\partial^2 \psi^*}{\partial x_i \partial x_k} + \text{c.c.} \right) + p \delta_{ik}, \quad (14)$$

where c.c. denotes complex conjugation, and the quantum pressure  $p$  is given by

$$p = \frac{U_0}{2m_\chi} \rho^2 = \frac{2\pi \hbar^2 l_a}{m_\chi^2} \rho^2. \quad (15)$$

By representing the wave function as  $\psi(\vec{r}, t) = \rho(\vec{r}, t) \exp[i\varphi(\vec{r}, t)]$ , the velocity of the condensate can be defined as  $v(\vec{r}, t) = (\hbar/m_\chi) \nabla \varphi(\vec{r}, t)$ . Then Eq. (11) leads to the mass conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \quad (16)$$

and taking into account the Thomas-Fermi approximation, Eq. (13) gives the basic evolution equation of the Bose-Einstein condensate dark matter halos,

$$\frac{\partial}{\partial t} \rho v_i + \frac{\partial}{\partial x_k} (\rho v_i v_k + p \delta_{ik}) = -\rho \frac{\partial V_{\text{grav}}}{\partial x_i}. \quad (17)$$

Formally, Eq. (17) is identical to the Euler equation in standard hydrodynamics, however, the distinctive feature of the Bose-Einstein condensate is that its flow is, in general, irrotational,  $\nabla \times \vec{v} = 0$  [9,55]. After applying the  $\nabla^2$  operator on both sides of Eq. (9), the Poisson equation becomes

$$\nabla^2 \rho + k^2 \rho = 0, \quad (18)$$

where for simplicity we have denoted

$$k = \sqrt{\frac{4\pi G m_\chi^2}{U_0}}. \quad (19)$$

Note that if the condensate is rotating at an angular velocity  $\vec{\Omega}$  that is greater than the critical one  $\vec{\Omega}_{\text{cr}}$ , its energy is minimized via a creation of vortices. Hence, it follows that the phase of the condensate order parameter changes by  $2\pi$  around a path that includes vortex lines,

$$\nabla \times \vec{v} = \frac{\hbar}{m_\chi} \nabla \times \nabla \varphi(\vec{r}) = \frac{2\pi \hbar}{m_\chi} \vec{n} \sum_j \delta^{(2)}(\vec{r} - \vec{r}_j), \quad (20)$$

where  $\vec{n}$  is a unitary vector along a vortex line,  $\vec{r}_j$  is the radius vector of a vortex line in the plane orthogonal to the unit vector  $\vec{n}$ , and  $\delta^{(2)}$  is a two-dimensional delta function in the corresponding plane [9,55].

In order to write the Euler equation Eq. (17) in a frame rotating uniformly with the angular velocity  $\vec{\Omega}$ , we need to add to the right-hand side of the Euler equation the centrifugal potential  $|\vec{\Omega} \times \vec{r}|/2$ , and the Coriolis acceleration  $2\vec{u} \times \vec{\Omega}$ , respectively, where  $\vec{u}$  is the condensate velocity in the rotating frame. Therefore, the equations of motion of an inviscid Bose-Einstein condensate, rotating at a constant angular velocity  $\vec{\Omega}$  in the presence of a confining gravitational field, are given, in Cartesian coordinates, by [9,55]

$$\rho \frac{du_j}{dt} - 2\rho \epsilon_{jkl} \Omega_k u_l = -\frac{\partial p}{\partial x_j} - \rho \frac{\partial}{\partial x_j} \left( V_{\text{grav}} - \frac{1}{2} |\vec{\Omega} \times \vec{r}|^2 \right), \quad (21)$$



where  $\varepsilon_{jkl}$  is the completely antisymmetric unit tensor of rank three.

### III. MASS AND GRAVITATIONAL PROPERTIES OF STATIC CONDENSED DARK MATTER HALOS

For static (nonrotating) condensates  $\omega = 0$ , the general solution of Eq. (18), describing the density distribution  $\rho$  of the static gravitationally bounded single component dark matter Bose-Einstein condensate is given by [19]

$$\rho(r) = \rho_c \frac{\sin kr}{kr}, \quad (22)$$

where  $\rho_c$  is the central density of the condensate,  $\rho_c = \rho(0)$ . Therefore, the Bose-Einstein condensate dark matter profile is given by the  $n = 1$  polytropic density profile.

The  $n = 1$  polytropic Bose-Einstein condensate dark matter density profile has a well defined boundary radius  $R$ , at which the dark matter density vanishes, so that  $\rho(R) = 0$ . From this condition we obtain  $R$  in the form  $kR = \pi$ , which fixes the radius of the condensate dark matter halo as

$$R = \frac{\pi}{k} = \pi \sqrt{\frac{\hbar^2 l_a}{Gm_\chi^3}}. \quad (23)$$

For  $r > R$  the density is smaller than zero, so that the  $n = 1$  polytropic density profile cannot be extended to infinity, that is, beyond its sharp boundary. For small values of  $r$  the condensate dark matter profile can be written as

$$\rho = \rho_c \left( 1 - \frac{\pi^2}{6R^2} r^2 + \frac{\pi^4}{120R^4} r^4 + \dots \right). \quad (24)$$

The logarithmic slope  $\alpha_{\text{BE}}$  of the Bose-Einstein profile is given by [56]

$$\alpha_{\text{DM}}(r) = -\frac{d \ln \rho}{d \ln r} = 1 - kr \cot(kr). \quad (25)$$

Note that Eq. (25) is not of the power law form  $\alpha_{\text{DM}}(r) \sim r^n$ , where  $n$  is a positive number defining the steepness of the power law, as are most of the density profiles used in the current astrophysical research. For  $r \rightarrow 0$ ,  $\alpha_{\text{DM}}(0) = 0$ , while at the surface of the dark halo, where  $kR = \pi$ ,  $\alpha_{\text{DM}}$  diverges, so that  $\lim_{kr \rightarrow \pi} \alpha_{\text{DM}} = -\infty$ . For small values of  $r$  the logarithmic slope can be written as  $\alpha_{\text{DM}}(r) \approx k^2 r^2/3 + k^4 r^4/45 + O(r)^6$ .

The mass profile  $m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'$  of the Bose-Einstein condensate galactic halo is

$$\begin{aligned} m(r) &= \frac{4\pi\rho_c}{k^2} r \left[ \frac{\sin(kr)}{kr} - \cos(kr) \right] \\ &= \frac{4}{\pi} R^2 r \rho(r) \alpha_{\text{DM}}(r). \end{aligned} \quad (26)$$

The total mass  $M$  of the condensate is

$$M(R) = \frac{4\pi^2\rho_c}{k^3} = \frac{4}{\pi} \rho_c R^3, \quad (27)$$

which represents a simple cubic proportionality between mass and radius. The mass of the condensate is around three times smaller than the mass of a constant  $\rho_c$  density sphere. The central density of the condensate is determined by the total mass  $M(R)$  and the radius of the condensate as

$$\rho_c = \frac{\pi M(R)}{4R^3}. \quad (28)$$

The mean density  $\langle \rho \rangle = 3M/4\pi R^3$  of the condensate can be obtained as

$$\langle \rho \rangle = \frac{3\rho_c}{k^2 R^2} = \frac{3\rho_c}{\pi^2}. \quad (29)$$

Alternatively, we can define the mean density of the dark matter halo as

$$\bar{\rho} = \frac{1}{R} \int_0^R \rho(r) dr = \frac{\text{Si}(\pi)}{\pi} \rho_c = 0.5895 \rho_c, \quad (30)$$

where  $\text{Si}(x) = \int_0^x \sin(x)/x dx$ , and  $\text{Si}(\pi) = 1.8519$ .

The gravitational potential  $V_{\text{grav}}(r)$  of the condensed dark matter distribution is determined by the equation

$$\begin{aligned} V_{\text{grav}}(r) &= G \int_r^R \frac{m(r')}{r'^2} dr' \\ &= \frac{4G\rho_c R^3}{\pi^2 r} \sin\left(\frac{\pi r}{R}\right), \quad r \leq R. \end{aligned} \quad (31)$$

At small radii the potential behaves as

$$V_{\text{grav}}(r) \approx \frac{4G\rho_c R^2}{\pi} - \frac{2\pi G\rho_c}{3} r^2 + \frac{\pi^3 G\rho_c}{30R^2} r^4 + O(r)^6, \quad (32)$$

for  $r \leq R$ .

The gravitational potential energy  $U(r)$  per unit mass and inside radius  $r$  of the condensed dark matter halo is given by

$$\begin{aligned} U(r) &= -4\pi G \int_0^r \frac{\rho(r') m(r')}{r'} r'^2 dr' = -\frac{2GR^4 \rho_c^2}{\pi^3} \\ &\times \left\{ 2\pi r \left[ 2 + \cos\left(\frac{2\pi r}{R}\right) \right] - 3R \sin\left(\frac{2\pi r}{R}\right) \right\}. \end{aligned} \quad (33)$$

The total potential energy of the halo is given by

$$U(R) = -\frac{12\pi^3 G}{k^5} \rho_c^2 = -\frac{12G\rho_c^2}{\pi^2} R^5. \quad (34)$$

The intrinsic velocity dispersion for an isotropic model can be obtained from the definition

$$\langle v_r^2(r) \rangle = \frac{1}{\rho(r)} \int_r^R \frac{GM(r')\rho(r')dr'}{r'^2} \quad (35)$$

and is given by

$$\langle v_r^2(r) \rangle = \frac{2GR^3\rho_c}{\pi^2 r} \sin\left(\frac{\pi r}{R}\right) = \frac{2GR^2}{\pi} \rho(r). \quad (36)$$

One can define a kinetic energy  $K$  of the halo in terms of the average velocity dispersion  $\sigma_V$  as

$$K(r) = \frac{3}{2} \int \rho(r) \sigma_V^2(r) dV. \quad (37)$$

If the velocity dispersion is a constant, we obtain

$$K(r) = \frac{6\pi\rho_c}{k^2} \sigma_V^2 r \left[ \frac{\sin(kr)}{kr} - \cos(kr) \right], \quad (38)$$

and

$$K(R) = \frac{6\pi^2\rho_c}{k^3} \sigma_V^2 = \frac{6\rho_c}{\pi} \sigma_V^2 R^3. \quad (39)$$

For the ratio of the kinetic and potential energy, we obtain

$$\frac{K(R)}{|U(R)|} = \frac{k^2}{2\pi G\rho_c} \sigma_V^2 = \frac{\pi}{2G\rho_c R^2} \sigma_V^2. \quad (40)$$

The tangential velocity of a test particle moving in the condensed dark halo can be represented as [19]

$$\begin{aligned} V^2(r) &= \frac{Gm(r)}{r} = \frac{4\pi G\rho_c}{k^2} \left[ \frac{\sin(kr)}{kr} - \cos(kr) \right] \\ &= \frac{4GR^2}{\pi} \rho(r) \alpha_{\text{DM}}(r). \end{aligned} \quad (41)$$

The velocity at the vacuum boundary of the halo has the maximum value

$$V^2(R) = \frac{4\pi G\rho_c}{k^2} = \frac{4G\rho_c R^2}{\pi} = \frac{GM(R)}{R}. \quad (42)$$

Therefore,

$$\frac{K(R)}{|U(R)|} = 2 \frac{\sigma_V^2}{V^2(R)} = \frac{2R}{GM(R)} \sigma_V^2. \quad (43)$$

Similarly to the case of the density we can define a mean value of the tangential velocity as  $\bar{V}^2 = (1/R) \int_0^R V^2(r) dr$ , and which is given by

$$\bar{V}^2 = \frac{4G}{\pi^2} \text{Si}(\pi) \rho_c R^2. \quad (44)$$

The logarithmic slope of the tangential velocity is defined as  $\beta_V = -d \ln V / d \ln r$ , and is given by

$$\begin{aligned} \beta_V &= \frac{1}{2} \left[ 1 + \frac{1}{R} \frac{\pi^2 r^2}{\pi r \cot(\pi r/R) - R} \right] \\ &= \frac{1}{2} \left[ 1 - \frac{\pi^2}{\alpha_{\text{DM}}(r)} \left( \frac{r}{R} \right)^2 \right]. \end{aligned} \quad (45)$$

When  $r = 0$ ,  $\beta_V = -1$ , while  $\beta_V = 1/2$  at the vacuum boundary of the dark matter halo.

One can define the core (inner) radius  $R_{\text{core}}$  of the Bose-Einstein condensate dark matter halos as the radius which satisfies the condition  $\alpha_{\text{DM}}(R_{\text{core}}) = 1$  [25]. By taking into account the definition of the logarithmic density slope, we obtain for the core radius the relation  $kR_{\text{core}} = \pi/2$ , or

$$R_{\text{core}} = \frac{R}{2}. \quad (46)$$

The mean value of the logarithmic density slope within the radius  $0 \leq r \leq R_{\text{core}}$  can be defined as [56]

$$\begin{aligned} \langle \alpha_{\text{DM}} \rangle &= \frac{1}{R_{\text{core}}} \int_0^{R_{\text{core}}} \alpha_{\text{DM}}(r) \\ &= 1 - \frac{2}{\pi} \int_0^{\pi/2} x \cot x dx, \end{aligned} \quad (47)$$

and it has the universal value

$$\langle \alpha_{\text{DM}} \rangle = 0.3068. \quad (48)$$

For the mean of the logarithmic slope of the tangential velocity, we obtain

$$\begin{aligned} \langle \beta_V \rangle &= \frac{1}{R_{\text{core}}} \int_0^{R_{\text{core}}} \beta_V(r) dr \\ &= \frac{1}{R} \int_0^{R/2} \left[ 1 - \pi^2 \left( \frac{r}{R} \right)^2 \right] dr \\ &= \frac{1}{2} \left( 1 - \frac{\pi^2}{12} \right) = 0.088. \end{aligned} \quad (49)$$

The density corresponding to the inner radius of the condensed dark matter halo is

$$\rho(R_{\text{core}}) = \frac{2}{\pi} \rho_c. \quad (50)$$

Since the radius  $R$  (and consequently also the inner core boundary  $R_{\text{core}}$ ) of the dark matter halo is a universal constant, depending only on the fundamental physical constants and the mass and scattering length of the dark matter particles, it follows that for a given central density the product  $\rho_c R_{\text{core}}$  is a universal constant,

$$\rho_c R_{\text{core}} = \frac{\pi M(R)}{8 R^2} = \text{constant}. \quad (51)$$

#### IV. LENSING PROPERTIES OF CONDENSED DARK MATTER HALOS

One of the important ways one could test the Bose-Einstein condensate galactic dark matter model is by studying the light deflection by galaxies, and in particular by studying the deflection of photons passing through the region where the rotation curves are flat. In the present section, we consider the basic lensing properties of the condensate dark matter halos. For a discussion of the gravitational lens properties of scalar field dark matter haloes, see [57].

##### A. Surface mass density

The surface mass density of a spherically symmetric lens is obtained by integrating along the line of sight of the three-dimensional density profile,  $\Sigma(\xi) = \int_{-\infty}^{+\infty} \rho(\xi, r) dz$ , where  $\xi$  is the radius measured from the center of the lens and  $r = \sqrt{\xi^2 + z^2}$ . The surface mass density can be written as an Abel transform [15], so that

$$\Sigma(\xi) = 2 \int_{\xi}^{+\infty} \frac{\rho(r) r dr}{\sqrt{r^2 - \xi^2}}. \quad (52)$$

By inserting the density profile of the Bose-Einstein condensate dark matter halos, we find

$$\Sigma(\xi) = 2\rho_c \frac{R}{\pi} \int_{\xi}^R \frac{\sin(kr) dr}{\sqrt{r^2 - \xi^2}}. \quad (53)$$

Now, introducing a new variable  $\vartheta$  by means of the transformation  $r = \xi \cosh \vartheta$ , we obtain

$$\Sigma(\xi) = 2\rho_c \frac{R}{\pi} \int_0^{\text{arccosh}(R/\xi)} \sin[k\xi \cosh \vartheta] d\vartheta. \quad (54)$$

By taking into account the expansion

$$\begin{aligned} \sin(k\xi \cosh \vartheta) &= \sum_{s=0}^{\infty} \frac{(-1)^s 2^{-(2s+1)}}{(2s+1)!} k^{2s+1} \xi^{2s+1} \\ &\times \sum_{l=0}^{2s+1} C_l^{2s+1} e^{(2s-2l+1)\vartheta}, \end{aligned} \quad (55)$$

where  $C_l^s = s! / l!(s-l)!$ , we obtain

$$\begin{aligned} \frac{\Sigma(\xi)}{2\rho_c} &= \frac{R}{\pi} \sum_{s=0}^{\infty} \sum_{l=0}^{2s+1} \frac{(-1)^s 2^{-(2s+1)} C_l^{2s+1}}{(2s+1)!(2s-2l+1)!} k^{2s+1} \xi^{2s+1} \\ &\times \left[ \left( \frac{R}{\xi} \right)^{2s-2l+1} \left( 1 + \sqrt{1 - \frac{\xi^2}{R^2}} \right)^{2s-2l+1} - 1 \right]. \end{aligned} \quad (56)$$

In the following, for the sake of clarity and simplicity, instead of showing the general results that can be derived from Eq. (56), we will present only the results obtained for some particular values of  $s$ . More exactly, we will restrict our study of lensing to the specific case  $s=7$ . The expansion of the  $\sin x$  function for  $s=7$  gives a very good approximation of the function for  $x$  in the range  $0 \leq x \leq \pi$ . The series expansions can be easily generalized and obtained for larger values of  $s$ , thus giving the possibility of obtaining the lensing parameters of Bose-Einstein condensate dark matter halos at any prescribed level of precision. In fact, since the condition  $x \leq \pi$  can be formulated as  $\pi \xi \cosh \vartheta \leq \pi$ , it follows that the adopted approximation gives a very good description of the condensate dark matter halos for all  $r \leq R$ .

For  $s=7$  the surface mass density of the dark matter halo is given by

$$\begin{aligned} \frac{\Sigma(\xi)}{2\rho_c} &= \sqrt{R^2 - \xi^2} \left[ \left( 1 - \frac{\pi^2}{18} + \frac{\pi^4}{600} - \frac{\pi^6}{35280} \right) \right. \\ &- \frac{\pi^2}{9} \left( 1 - \frac{\pi^2}{50} + \frac{3\pi^4}{9800} \right) \left( \frac{\xi}{R} \right)^2 \\ &\left. + \frac{\pi^4}{225} \left( 1 - \frac{\pi^2}{98} \right) \left( \frac{\xi}{R} \right)^4 - \frac{\pi^6}{11025} \left( \frac{\xi}{R} \right)^6 \right], \end{aligned} \quad (57)$$

or, in numerical form,

$$\begin{aligned} \Sigma(\xi) &= \sqrt{R^2 - \xi^2} \left[ 1.17357 - 1.82572 \left( \frac{\xi}{R} \right)^2 \right. \\ &\left. + 0.778658 \left( \frac{\xi}{R} \right)^4 - 0.174402 \left( \frac{\xi}{R} \right)^6 \right] \rho_c. \end{aligned} \quad (58)$$

In this approximation the central surface mass density can be evaluated as

$$\frac{\Sigma(0)}{2\rho_c} \approx R \left( 1 - \frac{\pi^2}{18} + \frac{\pi^4}{600} - \frac{\pi^6}{35280} \right) = 0.5867R. \quad (59)$$

The total mass of the Bose-Einstein condensate model can be calculated by integrating the surface mass density over the plane of the sky,  $M_{\xi}(R) = 2\pi \int_0^R \Sigma(\xi) \xi d\xi$ , and is given by

$$M_\xi(R) \approx \frac{4\pi}{3} \left( 1 - \frac{\pi^2}{10} + \frac{\pi^4}{280} - \frac{\pi^6}{15120} \right) \rho_c R^3 \\ = 1.2455 \rho_c R^3, \quad (60)$$

which gives an excellent approximation for the exact mass formula  $M(R) = (4/\pi)\rho_c R^3 = 1.2739\rho_c R^3$ .

An important quantity for gravitational lensing studies is the cumulative surface mass density, i.e., the total mass contained in a infinite cylinder with radius  $\xi$ ,  $M(\xi) = 2\pi \int_0^\xi \Sigma(\xi') \xi' d\xi'$  [58–60], which for the Bose-Einstein condensate dark halo is given by

$$M_\xi(\xi) = \rho_c \pi \xi^2 \sqrt{R^2 - \xi^2} \left[ \left( 2 - \frac{\pi^2}{9} + \frac{\pi^4}{300} - \frac{\pi^6}{17640} \right) \right. \\ \left. - \pi^2 \left( \frac{2}{9} - \frac{\pi^2}{225} + \frac{\pi^4}{14700} \right) \left( \frac{\xi}{R} \right)^2 \right. \\ \left. + \pi^4 \left( \frac{2}{225} - \frac{\pi^2}{11025} \right) \left( \frac{\xi}{R} \right)^4 - \frac{2\pi^6}{11025} \left( \frac{\xi}{R} \right)^6 \right], \quad (61)$$

or, equivalently,

$$M_\xi(\xi) = 3.68689 \xi^2 \sqrt{R^2 - \xi^2} \left[ 1 - 1.5557 \left( \frac{\xi}{R} \right)^2 \right. \\ \left. + 0.66349 \left( \frac{\xi}{R} \right)^4 - 0.14861 \left( \frac{\xi}{R} \right)^6 \right] \rho_c. \quad (62)$$

### B. The lens equation and the deflection of light

In the following, we consider a simplified gravitational lens scenario involving a point source and a circular symmetrical lens. The three basic “planes” in thin lens approximation are the source  $S$ , the lens  $L$ , and the observer  $O$ . Light rays emitted from the source are detected by the lens. For a pointlike lens, there will always be (at least) two images  $S_1$  and  $S_2$  of the source. With external shear (due to the tidal field of objects outside but near the light bundles) there can be more images. The observer sees the images in directions corresponding to the tangents to the real incoming light paths [58–60].

In the thin lens approximation, the lens equation for axially symmetric lens is [58–60]

$$\eta = \frac{D_S}{D_L} \xi - D_{LS} \tilde{\alpha}, \quad (63)$$

where the quantities  $\eta$  and  $\xi$  are the physical positions of the source in the source plane and an image in the image plane, respectively,  $\tilde{\alpha}$  is the deflection angle, and  $D_L$ ,  $D_S$  and  $D_{LS}$  are the angular distances from observer to lens, from

observer to source, and from lens to source, respectively. By introducing the dimensionless position  $\beta = \eta/D_S$ ,  $\theta = \xi/D_L$  and the dimensionless angle  $\alpha = (D_{LS}/D_S)\tilde{\alpha}$ , the thin lens equation can be written as

$$\beta = \theta - \alpha(\theta). \quad (64)$$

In the circular-symmetric case the deflection angle is given as

$$D_L \alpha(\xi) = \frac{2}{\xi} \int_0^\xi \frac{\Sigma(\xi')}{\Sigma_{\text{crit}}} d\xi' = \frac{2}{\xi} \int_0^\xi \xi' \kappa(\xi') d\xi' \\ = \frac{1}{\pi \Sigma_{\text{crit}}} \frac{M_\xi(\xi)}{\xi}, \quad (65)$$

where the convergence  $\kappa$  is defined as  $\kappa(\xi) = \Sigma(\xi)/\Sigma_{\text{crit}}$ , where  $\Sigma_{\text{crit}}$  is the critical surface density defined as  $\Sigma_{\text{crit}} = c^2 D_S / 4\pi G D_L D_{LS}$ . The central convergence,  $\kappa_c = \kappa(0)$  [58–60], a parameter that determines the lensing properties of the Bose-Einstein condensate dark matter halo profiles, is given by the relation

$$\kappa_c = \frac{1.17357 \rho_c R}{\Sigma_{\text{crit}}} = \frac{1.17357 \pi M(R)}{4R^2 \Sigma_{\text{crit}}}. \quad (66)$$

The dimensionless surface mass density can be given by

$$\kappa(D_L \theta) = \kappa_c \frac{\sqrt{R^2 - D_L^2 \theta^2}}{1.17357 R} \left[ 1.17357 - 1.82572 \left( \frac{D_L \theta}{R} \right)^2 \right. \\ \left. + 0.778658 \left( \frac{D_L \theta}{R} \right)^4 - 0.174402 \left( \frac{D_L \theta}{R} \right)^6 \right]. \quad (67)$$

For a spherically symmetric lens that is capable of forming multiple images of the source, one sufficient condition is  $\kappa_c > 1$ . In the case  $\kappa_c < 1$ , only one image of the source is formed. Similarly to the case of other nonsingular profiles, such as the Einasto profile [60], the Bose-Einstein condensate dark matter density profiles are not capable of forming multiple images for any mass. Instead, the condition  $\kappa_c > 1$  sets a minimum value for lens mass required to form multiple images.

The deflection potential  $\psi(\xi)$  for a spherically symmetric lens is given by [58–60]

$$\psi(\xi) = 2 \int_0^\xi \xi' \kappa(\xi') \ln \left( \frac{\xi}{\xi'} \right) d\xi', \quad (68)$$

and can be computed to give



$$\begin{aligned} \psi(\xi) = \rho_c \left\{ \left( \frac{16}{9} - \frac{4\pi^2}{25} + \frac{4\pi^4}{735} - \frac{\pi^6}{10206} \right) R^3 + \sqrt{R^2 - \xi^2} \left[ - \left( \frac{16}{9} - \frac{4\pi^2}{25} + \frac{4\pi^4}{735} - \frac{\pi^6}{10206} \right) R^2 \right. \right. \\ \left. \left. + \left( \frac{4}{9} - \frac{2\pi^2}{25} + \frac{\pi^4}{7350} - \frac{\pi^6}{714420} \right) \xi^2 - \frac{\pi^2}{225} \xi^2 \left( 4 - \frac{2\pi^2}{49} + \frac{\pi^4}{2646} \right) \left( \frac{\xi}{R} \right)^2 + \frac{4\pi^4}{11025} \xi^2 \left( 1 - \frac{\pi^2}{162} \right) \left( \frac{\xi}{R} \right)^4 \right. \right. \\ \left. \left. - \frac{4\pi^6}{893025} \xi^2 \left( \frac{\xi}{R} \right)^6 \right] - \frac{1}{3} \left( 4 - \frac{2\pi^2}{5} + \frac{\pi^4}{70} - \frac{\pi^6}{3780} \right) R^3 \ln \frac{2R}{R + \sqrt{R^2 - \xi^2}} \right\}. \end{aligned} \quad (69)$$

In a numerical form we have

$$\begin{aligned} \psi(\xi) = \rho_c \left\{ \sqrt{R^2 - \xi^2} R^2 \left[ -0.63456 + 0.368622 \left( \frac{\xi}{R} \right)^2 - 0.159404 \left( \frac{\xi}{R} \right)^4 + 0.0331881 \left( \frac{\xi}{R} \right)^6 \right. \right. \\ \left. \left. - 0.00430621 \left( \frac{\xi}{R} \right)^8 \right] + 0.63456 R^3 \left( 1 - 0.62478 \ln \frac{2.R}{R + \sqrt{R^2 - \xi^2}} \right) \right\}. \end{aligned} \quad (70)$$

### C. The magnification factor

Gravitational lensing effect preserves the surface brightness but it causes variations in shape and solid angle of the source. Therefore, the source luminosity is amplified by a magnification factor  $\mu$ , given by [58–60]

$$\mu = \frac{1}{(1 - \kappa)^2 - \gamma^2}, \quad (71)$$

where  $\gamma = \gamma(\xi)$  is the shear. For a spherically symmetric lens the shear is given by

$$\gamma(\xi) = \bar{\kappa} - \kappa = \frac{\bar{\Sigma}(\xi) - \Sigma(\xi)}{\Sigma_{\text{crit}}}, \quad (72)$$

where  $\bar{\kappa} = \bar{\Sigma}(\xi)/\Sigma_{\text{crit}}$ , and  $\bar{\Sigma}(\xi)$  is the average surface mass density within  $\xi$  given by

$$\bar{\Sigma}(\xi) = \frac{2}{\xi^2} \int_0^\xi \xi' \Sigma(\xi') d\xi'. \quad (73)$$

By using Eq. (57) we obtain for  $\bar{\Sigma}(\xi)$  the expression

$$\begin{aligned} \bar{\Sigma}(\xi) = 2R\rho_c \left( \frac{\xi}{R} \right)^{-2} \left\{ 0.198228 + \sqrt{1 - \left( \frac{\xi}{R} \right)^2} \right. \\ \times \left[ 0.487671 \left( \frac{\xi}{R} \right)^2 - 0.384069 \left( \frac{\xi}{R} \right)^4 \right. \\ \left. \left. + 0.114005 \left( \frac{\xi}{R} \right)^6 - \frac{0.019378\xi^8}{R^8} - 0.198228 \right] \right\}. \end{aligned} \quad (74)$$

Therefore, for the shear  $\gamma(\xi)$  of a BEC dark matter halo, we obtain the expression

$$\begin{aligned} \gamma(\xi) = \frac{R\rho_c}{\Sigma_{\text{crit}}} \sqrt{1 - \left( \frac{\xi}{R} \right)^2} \left[ -0.68590 \right. \\ \left. - 0.198228 \left( \frac{\xi}{R} \right)^{-2} + 0.155028 \left( \frac{\xi}{R} \right)^6 \right. \\ \left. - 0.66465 \left( \frac{\xi}{R} \right)^4 + 1.4417 \left( \frac{\xi}{R} \right)^2 \right. \\ \left. + \frac{0.198228}{(\xi/R)^2 \sqrt{1 - (\xi/R)^2}} \right]. \end{aligned} \quad (75)$$

The magnification by a spherically symmetric lens can be written as [58–60]

$$\begin{aligned} \mu = \frac{1}{(1 - \bar{\kappa})(1 + \bar{\kappa} - 2\kappa)} \\ = \frac{1}{(1 - \bar{\Sigma}(\xi)/\Sigma_{\text{crit}})(1 + \bar{\Sigma}(\xi)/\Sigma_{\text{crit}} - 2\Sigma(\xi)/\Sigma_{\text{crit}})}. \end{aligned} \quad (76)$$

The magnification may be divergent for some image positions. The loci of the diverging magnification in the image plane are called the critical curves. From Eq. (76) we see that the lensing profile has two critical curves. The first curve,  $1 - \bar{\kappa} = 0$  is the tangential critical curve, which corresponds to an Einstein ring with a given Einstein radius. The second curve,  $1 + \bar{\kappa} - 2\kappa = 0$  is the radial critical curve, which also defines a ring, and its corresponding radius [58–60].

## V. THE VIRIAL THEOREM AND THE PERTURBATION EQUATION FOR BOSE-EINSTEIN CONDENSATES

One of the basic relations in theoretical physics, the virial theorem, represents a very powerful method for the study of

the equilibrium and perturbation properties of fluids, including the quantum ones. The virial theorem was originally proved, and used for the study of the equilibrium and of the stability of rotating fluid bodies, bound by self-gravitation [61–63]. One of the important forms of the virial theorem, the tensor-virial theorem, reduces the local hydrodynamical Euler equations into global virial equations, which give the essential information on the structure and stability of the whole gravitating system. An important application of the virial theorem is for the study of the small perturbations of incompressible uniform ellipsoids when perturbed from an initial equilibrium state. In this case, in the absence of viscous dissipation, each perturbed virial equation provides a different set of normal modes [55]. Moreover, in the physically very interesting case when the equilibrium of the considered system is sustained by an external confining potential, the perturbations of the fluid obviously do not modify the confining potential itself. Therefore, it follows that the tensor virial methods, and the virial theorems, can be extended to nonuniform compressible flows. The virial theorem represent an extremely powerful method for the study of gases with polytropic equations of state [61–66]. It has also been extensively used in the study of Bose-Einstein condensates [53–55].

### A. The scalar virial theorem

In order to derive the scalar virial theorem we consider the behavior of the physical parameters of the dark matter halos under the scaling transformation  $\vec{r} \rightarrow \alpha\vec{r}$ , where  $\alpha$  is a constant. Then the normalization condition giving the total number of particles,  $N = \int |\psi(\vec{r})|^2 d^3\vec{r}$  requires that  $\psi(\vec{r}) \rightarrow \alpha^{-3/2}\psi(\vec{r})$  [2,8]. Thus the total energy scales as

$$E[\alpha] = \alpha^{-2}E_K + \alpha^2E_{\text{rot}} + \alpha^{-3}E_{\text{int}} + \alpha^{-1}E_{\text{grav}}. \quad (77)$$

Since the energy is stationary for any variation of the wave function  $\psi$  around the exact solution of the Gross-Pitaevskii equation, by requiring the energy variation to vanish at first order in  $\alpha$ , that is,  $(\delta E[\alpha]/\delta\alpha)|_{\alpha=1} = 0$ , we obtain the virial theorem in the form [2,8]

$$2E_K - 2E_{\text{rot}} + 3E_{\text{int}} + E_{\text{grav}} = 0. \quad (78)$$

By multiplying both sides of Eq. (9) with  $\rho(\vec{r})$  and integrating over  $\vec{r}$  we obtain

$$\mu N = E_{\text{rot}} + 2E_{\text{int}} + 2E_{\text{grav}}, \quad (79)$$

where the interaction energy  $E_{\text{int}}$  and the gravitational energy  $E_{\text{grav}}$  are defined as  $E_{\text{int}} = (U_0/2) \int_V \rho^2(\vec{r}) d^3\vec{r}$ , and  $V_{\text{grav}} = (m_\chi^2/2) \int_V V_{\text{grav}}(\vec{r}) \rho(\vec{r}) d^3\vec{r}$ , respectively. On the other hand from Eq. (9) we obtain

$$\mu = \frac{U_0}{m_\chi} \rho_c + m_\chi V_{\text{grav}}(0). \quad (80)$$

By using Eqs. (79) and (80), we obtain for the total energy of the dark matter halo the expression

$$E = \frac{1}{2} \left[ \frac{U_0}{m_\chi} \rho_c + m_\chi V_{\text{grav}}(0) \right] + \frac{1}{2} E_{\text{rot}}. \quad (81)$$

By using the virial theorem we can easily find the condition for the validity of the Thomas-Fermi approximation. By using the density for the static condensate given by Eq. (22), it is easy to estimate  $E_K \approx \bar{2}M(R)k^2/m_\chi^2$  and  $E_{\text{int}} \approx U_0 M(R)^2 k^3/m_\chi^2$ , respectively. The Thomas-Fermi approximation requires  $E_K \ll E_{\text{int}}$ , giving

$$N = \frac{M}{m_\chi} \gg \frac{1}{kl_a} = \frac{R}{\pi l_a}, \quad R \gg \sqrt{\frac{m_\chi}{4l_a \rho_c}}. \quad (82)$$

Hence, for systems with enough high particle numbers the Thomas-Fermi approximation is always valid.

### B. The tensor virial theorem

In the following we assume that the dark matter gravitational condensate rotates like a rigid body, and its rotation is supported by an array of vortices. Moreover, it is confined by a gravitational field with potential  $V_{\text{grav}}$ , assumed to satisfy the Poisson equation Eq. (7). To obtain the tensor virial equation for the BEC halo we multiply both sides of Eq. (21) by  $x_i$  and we integrate over the volume  $V$  of the dark matter halo. Thus we first obtain [55,61–63]

$$\frac{d}{dt} \int_V \rho x_i u_j d^3\vec{r} = \int_V \rho x_i \frac{du_j}{dt} d^3\vec{r} + \int_V \rho u_i u_j d^3\vec{r}. \quad (83)$$

The last term on the right-hand of Eq. (21) can be rewritten as

$$\frac{1}{2} \int_V \rho x_i \frac{\partial}{\partial x_j} |\vec{\Omega} \times \vec{r}|^2 d^3\vec{r} = \vec{\Omega}^2 I_{ij} - \Omega_j I_{ik} \Omega_k, \quad (84)$$

where

$$I_{ij} = I_{ji} = \int_V \rho x_i x_j d^3\vec{r}, \quad (85)$$

is the moment of inertia tensor of the dark matter condensate. Assuming that the pressure becomes a constant  $p_0$  outside the dark matter distribution, one can write

$$\begin{aligned} - \int_V x_i \frac{\partial p}{\partial x_j} d^3\vec{r} &= - \int_S x_i p_0 d\Sigma_j + \int_V \delta_{ij} p d^3\vec{r} \\ &= \delta_{ij} \int_V (p - p_0) d^3\vec{r} = \delta_{ij} \bar{\Pi}, \end{aligned} \quad (86)$$

where  $\delta_{ij}$  is the Kronecker delta symbol. By taking into account the explicit form of the condensate self-gravitational potential,  $\phi(\vec{r}, t) = -G \int_V d^3\vec{r}' \rho(\vec{r}', t) / |\vec{r} - \vec{r}'|$ , one can show that

$$\int_V \rho x_i \frac{\partial \phi}{\partial x_j} d^3\vec{r} = -\frac{1}{2} \int_V \rho B_{ij} d^3\vec{r} = -\Phi_{ij}, \quad (87)$$

where

$$B_{ij}(\vec{r}, t) = -G \int_V \rho(\vec{r}', t) \frac{(x_i - x'_i)(x_j - x'_j)}{|\vec{r} - \vec{r}'|^3} d^3\vec{r}', \quad (88)$$

is the potential tensor of Chandrasekhar [61–63]. By defining the kinetic energy tensor as

$$T_{ij} = \frac{1}{2} \int_V \rho u_i u_j d^3\vec{r}, \quad (89)$$

the tensor virial theorem for a rotating Bose-Einstein condensate dark matter halo in a gravitational field can be written as [61–64]

$$\begin{aligned} \frac{d}{dt} \int_V \rho x_i u_j d^3\vec{r} &= 2T_{ij} + \delta_{ij} \bar{\Pi} + \Phi_{ij} + \bar{\Omega}^2 I_{ij} \\ &- \Omega_j I_{ik} \Omega_k + 2\epsilon_{jkl} \Omega_k \int_V \rho x_l u_i d^3\vec{r}. \end{aligned} \quad (90)$$

For a dark matter halo in hydrostatic equilibrium  $u_i = 0$ , and from Eq. (90) we find

$$\Phi_{ij} + \delta_{ij} \bar{\Pi} + \bar{\Omega}^2 I_{ij} - \Omega_j I_{ik} \Omega_k = 0. \quad (91)$$

For  $i \neq j$  we have  $\Phi_{ij} + \bar{\Omega}^2 I_{ij} - \Omega_j I_{ik} \Omega_k$ , while contracting over  $i$  and  $j$  gives

$$\Phi + 3\bar{\Pi} + \bar{\Omega}^2 I - \Omega_i I_{ij} \Omega_j = 0, \quad (92)$$

where  $\Phi$  represents the gravitational potential energy.

### 1. Gravitational energy of a rotating dark matter halo

As an application of the tensor virial theorem we determine in the following the gravitational energy of a rotating dark matter halo. If the  $z$ -axis is chosen as axis of rotation, Eq. (92) takes the form

$$\Phi + 3\bar{\Pi} + \bar{\Omega}^2 I_\Omega = 0, \quad (93)$$

where  $I_\Omega = \int_V \rho(x^2 + y^2) d^3\vec{r}$ . For the case of the rotating Bose-Einstein condensate dark matter halo the equation of the hydrostatic equilibrium takes the form

$$2\nabla \frac{p}{\rho} = \nabla \left( V_{\text{grav}} + \frac{1}{2} |\vec{\Omega} \times \vec{r}|^2 \right), \quad (94)$$

and can be integrated to give

$$2p = \rho \left( V_{\text{grav}} + \frac{1}{2} |\vec{\Omega} \times \vec{r}|^2 - V_{\text{grav}}^{(0)} \right), \quad (95)$$

where  $V_{\text{grav}}^{(0)}$  is the gravitational potential at the pole of the dark matter halo. Integrating Eq. (95) over the dark matter halo volume, we obtain

$$2\bar{\Pi} = -2\Phi + \frac{1}{2} \bar{\Omega}^2 I_\Omega - V_{\text{grav}}^{(0)} M, \quad (96)$$

where  $\Phi = -(1/2) \int \rho V_{\text{grav}} d^3\vec{r}$ . Hence, from Eq. (93) we obtain [65]

$$\Phi = \frac{1}{4} \left( \frac{7}{2} \bar{\Omega}^2 I_\Omega - 3V_{\text{grav}}^{(0)} M \right). \quad (97)$$

For a static halo  $\vec{\Omega} = 0$ ,  $V_{\text{grav}}^{(0)} = GM/R$ , and we reobtain the well-known result  $\Phi = -(3/4)GM^2/R$ .

### C. The perturbation equation

We consider now the small oscillations about equilibrium of a Bose-Einstein condensate dark matter halo. We assume that in the stationary state there are no fluid motions. In order to obtain the perturbation equation we use the tensor virial theorem. Considering only periodic oscillations with frequency  $\omega$ , representing the most interesting case from a physical point of view, the Lagrangian displacement of a mass element  $dm$  can be written as  $\vec{\xi}(\vec{r}, t) = \vec{\xi}(\vec{r}) \exp(i\omega t)$  [55,61–65]. Due to the equation of continuity for such Lagrangian displacements  $dm$  is a constant, and  $\delta\rho/\rho + \nabla \cdot \vec{\xi} = 0$ . By taking the Eulerian variation of the tensor virial equation for the condensate dark matter halo we obtain

$$\begin{aligned} \delta \frac{d}{dt} \int_V \rho x_i u_j d^3\vec{r} &= 2\delta T_{ij} + \delta_{ij} \delta \bar{\Pi} + \delta \Phi_{ij} + \bar{\Omega}^2 V_{ij} \\ &- \Omega_j \Omega_k V_{kj} + 2\epsilon_{jkl} \Omega_k \delta \int_V \rho x_l u_i d^3\vec{r}, \end{aligned} \quad (98)$$

where

$$V_{ij} = \delta I_{ij} = \int_V \rho (\xi_i x_j + \xi_j x_i) d^3\vec{r}, \quad (99)$$

is a tensor symmetric in its indices. By defining its nonsymmetric part as  $V_{i,j} = \int_V \rho \xi_i x_j d^3\vec{r}$ , we have  $V_{ij} = V_{i,j} + V_{j,i}$ . For the variation of  $\Phi_{ij}$  we find the equation

$$\begin{aligned}
\delta\Phi_{ij} &= -G \int_V \int_V dmdm' \xi_k \frac{\partial}{\partial x_k} \frac{(x_i - x'_i)(x_j - x'_j)}{|\vec{r} - \vec{r}'|^3} \\
&= -G \int_V d^3\vec{r} \rho(\vec{r}) \xi_k \frac{\partial}{\partial x_k} \int_V d^3\vec{r}' \rho(\vec{r}') \\
&\quad \times \xi_k \frac{\partial}{\partial x_k} \frac{(x_i - x'_i)(x_j - x'_j)}{|\vec{r} - \vec{r}'|^3} - \int_V d^3\vec{r} \rho(\vec{r}) \xi_k \frac{\partial B_{ij}}{\partial x_k},
\end{aligned} \tag{100}$$

which gives the important first order change in the gravitational potential energy, due to a small perturbation of the matter in the condensate dark matter system. Therefore, when there are no condensate dark matter motions in the unperturbed frame, the second order virial equation takes the form [55,61–63]

$$\frac{d^2 V_{i;j}}{dt^2} = 2\epsilon_{ikl} \Omega_l \frac{dV_{k;j}}{dt} + \delta\Phi_{ij} - \Omega_i \Omega_k V_{kj} + \delta_{ij} \delta\bar{\Pi}. \tag{101}$$

For small periodic perturbations we obtain [55]

$$\sigma^2 V_{i;j} = 2\epsilon_{ikl} \Omega_l \sigma V_{k;j} + \delta\Phi_{ij} - \Omega_i \Omega_k V_{kj} + \delta_{ij} \delta\bar{\Pi}. \tag{102}$$

Equation (102) contains all the second-harmonic modes of the rotating condensate dark matter halo confined by its own gravitational field [55,61–63].

## VI. APPLICATIONS OF THE VIRIAL THEOREM TO BEC DARK MATTER HALOS

As a simple example of the application of the virial theorem for the study of the stability of the Bose-Einstein condensate dark matter structures we consider the case of the slowly rotating and slightly distorted galactic halos. By employing the Lagrangian variables and the scalar virial theorem, and by using the linear approximation, we obtain the angular frequency  $\sigma$  of the lowest radial mode, as well as the condition for dynamical instability, depending on the numerical value of the adiabatic exponent  $\gamma = 1 + 1/n$ , where  $n$  is the polytropic index.

### A. The scalar virial theorem for axisymmetric BEC dark matter halos

We consider the BEC dark matter halo as a uniformly rotating, homogeneous and compressible spheroid. The galactic halo rotates at  $t=0$  with an angular velocity  $\Omega(t_0) = \Omega_0$ . We restrict our analysis to axisymmetric pulsations, and work in a Cartesian coordinate system  $(x_1, x_2, x_3)$ , which rotates at any instant  $t$  with the instantaneous angular velocity  $\Omega(t)$  of the galactic halo. Moreover, we assume that  $\Omega$  is oriented along the  $x_3$  axis. We will also ignore dissipative and electromagnetic effects. Under such assumptions, the equation of motion of the galactic halo, Eq. (21), can be written as [61–66]

$$\begin{aligned}
\frac{du_i}{dt} + 2\epsilon_{ik3} \Omega u_k + \epsilon_{ik3} \frac{d\Omega}{dt} x_k \\
= -\frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{\partial V_{\text{grav}}}{\partial x_i} + (1 - \delta_{i3}) \Omega^2 x_i,
\end{aligned} \tag{103}$$

where  $i = 1, 2, 3$  and we have taken into account that the angular velocity  $\Omega$  explicitly depends on time. For axisymmetric motions, all terms in Eq. (103) describe motions in meridional planes, except the last two terms on the left-hand side of the equation [64,65]. Therefore, Eq. (103) can be split as

$$\frac{du_i}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{\partial V_{\text{grav}}}{\partial x_i} + (1 - \delta_{i3}) \Omega^2 x_i, \tag{104}$$

where  $i = 1, 2, 3$  and

$$\epsilon_{ik3} \left( 2\Omega u_k + \frac{d\Omega}{dt} x_k \right) = 0, \tag{105}$$

respectively. Equation (105) can be immediately integrated to give

$$\frac{d}{dt} [\Omega(x_1^2 + x_2^2)] = 0. \tag{106}$$

Equation (106) is the equation of conservation of the angular momentum for each fluid element, a property which is true for axisymmetric motions only. Now we multiply Eq. (104) by  $x_i$ , we integrate over the total mass  $M$  of the galactic halo, and add the resulting equations. Hence, we obtain the scalar virial theorem for the BEC halo in the form [64,65]

$$\begin{aligned}
\frac{1}{2} \frac{d^2}{dt^2} \int_M x_k x_k dm = \int_M \frac{dx_k}{dt} \frac{dx_k}{dt} dm + \Phi \\
+ 3 \int_M \frac{p}{\rho} dm + \int_M \Omega(x_1^2 + x_2^2) dm,
\end{aligned} \tag{107}$$

where

$$\Phi = -\frac{1}{2} G \int_M \int_M \frac{dmdm'}{|\vec{r} - \vec{r}'|} \tag{108}$$

is the gravitational potential energy of the dark matter halo. At time  $t=0$  the dark matter halo is in relative equilibrium, and the virial theorem Eq. (107) gives

$$\Phi_0 + 3 \int_M \frac{p_0}{\rho_0} dm + \int_M \Omega_0 (a_1^2 + a_2^2) dm = 0, \tag{109}$$

where the subscript zero indicates the initial values, and  $a_1$  and  $a_2$  represents the  $t=0$  values of the coordinates  $x_1$  and  $x_2$ , respectively.

## B. The equations of motion in Lagrangian coordinates

It is more convenient to study the axisymmetric motion of the galactic dark matter halos in Lagrangian coordinates, in which all variables characterizing the BEC system,  $x_i$ ,  $\rho$ ,  $p$ , and  $V$  are expressed as functions of the independent quantities  $a_i$  and  $t$  [65,66]. Therefore, the Eulerian coordinates  $x_i$  can be expressed as  $x_i = x_i(a_i, t)$ . Hence, when axisymmetry is preserved, there are expanding and contracting solutions for which the coordinates of a particle are functions of their initial values and of the time only.

In Lagrangian coordinates the equations of motion of the BEC dark matter halo Eqs. (104) can be reformulated as [66]

$$\frac{\partial x_i}{\partial a_i} \frac{\partial^2 x_i}{\partial t^2} = -\frac{1}{\rho} \frac{\partial p}{\partial a_i} - \frac{\partial V_{\text{grav}}}{\partial a_i} + (1 - \delta_{i3}) \Omega^2 x_i \frac{\partial x_i}{\partial a_i}, \quad (110)$$

where  $i = 1, 2, 3$ .

The conservation of mass imposes the condition

$$\rho J = \rho_0, \quad (111)$$

where  $J$  is the Jacobian of the coordinate transformation [66],

$$J = \frac{\partial x_1}{\partial a_1} \frac{\partial x_2}{\partial a_2} \frac{\partial x_3}{\partial a_3}. \quad (112)$$

As for the pressure of dark matter halo, we will assume that it is given by Eq. (15), that is, by a polytropic equation of state with polytropic index  $n = 1$ . We assume that the pressure vanishes on the moving surface, at the vacuum boundary of the dark matter halo. Moreover, the gravitational potential must be continuous across the galactic boundary. Once these conditions are satisfied, axisymmetric motions are then entirely determined once an initial velocity distribution is prescribed at every point [65].

## C. Pseudoradial oscillations of BEC dark matter halos

In the following we restrict our analysis to the case of pseudoradial pulsations of slowly rotating and slightly distorted galactic polytropic dark matter halos. For slowly rotating spheroids in the first approximation one can adopt for the evolution of the coordinates a linear expression, so that [66]

$$x_i = a_i \zeta(t), \quad i = 1, 2, 3. \quad (113)$$

The equation governing the time evolution of  $w(t)$  can be obtained from the scalar virial theorem Eq. (107) in Lagrange coordinates. First of all, by using the adopted representations of  $x_i$  we obtain immediately the identity

$$\frac{1}{2} \frac{d^2}{dt^2} \int_M x_k x_k dm - \int_M \frac{dx_k}{dt} \frac{dx_k}{dt} dm = \zeta \frac{d^2 \zeta}{dt^2} I_0, \quad (114)$$

where  $I_0$  denotes the initial moment of inertia of the rotating galaxy, and we have used the constancy of the mass element  $dm$  as we follow the motion of the particles. In order to obtain the time variation of the gravitational potential energy, given by Eq. (108), we take into account that  $|\vec{r} - \vec{r}'| = \zeta(t) |\vec{a} - \vec{a}'|$ , in which  $\vec{a} = \vec{r}(0)$  and  $\vec{a}' = \vec{r}'(0)$  [66]. Then we obtain for the gravitational potential energy the expression

$$\Phi = \frac{\Phi_0}{\zeta^2}. \quad (115)$$

With the use of Eq. (111) and of the equation of state of the BEC dark matter we obtain

$$\frac{p}{\rho} = \frac{1}{\zeta^3} \frac{p_0}{\rho_0}. \quad (116)$$

By integrating the above equation, we obtain

$$3 \int_M \frac{p}{\rho} dm = \frac{1}{\zeta^3} (|\Phi_0| - 2K_0), \quad (117)$$

where  $K_0$  represents the initial rotational kinetic energy, and in order to eliminate the integral over pressure in Eq. (117) we have used the equilibrium condition given by Eq. (109) [66]. The conservation of the angular momentum immediately gives the equation

$$\Omega = \frac{\Omega_0}{\zeta^2}, \quad (118)$$

and, consequently,

$$\int_M \Omega^2 (x_1^2 + x_2^2) dm = \frac{2K_0}{\zeta^2}. \quad (119)$$

Therefore, with the use of the above results and of the scalar virial theorem Eq. (107) we obtain the time evolution of  $\zeta$  as [66]

$$\frac{d^2 \zeta}{dt^2} = \frac{1}{\zeta^2} \left( \frac{1}{\zeta^2} - 1 \right) \frac{|\Phi_0|}{I_0} + \frac{1}{\zeta^3} \left( 1 - \frac{1}{\zeta} \right) \frac{2K_0}{I_0}. \quad (120)$$

Equation (120) must be solved with the appropriate boundary conditions. It is also important to mention that the quantities  $\Phi_0$ ,  $K_0$  and  $I_0$  refer to the rotating galactic halo. Moreover, in obtaining Eq. (120) we have assumed that the halo rotates as a solid body.



### D. Oscillation frequency of BEC dark matter halos in the linear approximation

In the following we consider the small amplitude pseudoradial oscillations of the  $n = 1$  polytropic dark matter halos. We can see from Eq. (120) that the point  $\zeta = 1$  defines an equilibrium state of the galaxy. In order to describe linear and quasilinear oscillations about this equilibrium point, we Taylor expand  $\zeta$  near  $\zeta = 1$ . In the first approximation we obtain

$$\zeta(t) = 1 + \epsilon(t), \quad (121)$$

where  $\epsilon(t) \ll 1$ . In this approximation Eq. (120) becomes [66]

$$\frac{d^2\epsilon}{dt^2} + \sigma^2\epsilon = \lambda\epsilon^2 + O(\epsilon^3), \quad (122)$$

where [65,66]

$$\sigma^2 = 2\frac{|\Phi_0|}{I_0} - \frac{2K_0}{I_0}, \quad (123)$$

and

$$\lambda = \frac{1}{2}\left(7\sigma^2 - \frac{2K_0}{I_0}\right). \quad (124)$$

If we neglect the second and third order powers of  $\epsilon$ , the solution of Eq. (122) is obtained as

$$\epsilon(t) = \epsilon_0 \sin(\sigma t). \quad (125)$$

In terms of the Eulerian variables we obtain  $\xi_i = \epsilon_0 x_i \sin(\omega t)$ .

Equation (123) gives the stability condition for BEC dark matter halos in the linear approximation. If the halo is initially nonrotating, we have  $K_0 = 0$ , and the stability condition reduces to

$$\sigma^2 = 2\frac{|\Phi_0|}{I_0} > 0, \quad (126)$$

a condition which is always satisfied by BEC dark matter halos described by a polytropic equation of state with  $n = 1$  and  $\gamma = 2$ , respectively. With the use of Eq. (97) we can estimate the initial gravitational energy of the dark matter halo as  $|\Phi_0| = (3/4)GM^2/R$ , while for the moment of inertia we use the expression  $I_0 = 2MR^2/5$  [65]. Then the oscillation frequency of the halo is given by

$$\sigma^2 = \frac{15GM}{8R^3} = \frac{15G}{2\pi}\rho_c, \quad (127)$$

where we have taken into account Eq. (27) giving the mass-radius relation of the static BEC dark matter halo. For the period of the oscillations we obtain

$$T = \frac{2\pi}{\sigma} = \sqrt{\frac{8}{15}}\pi^{3/2}\frac{1}{\sqrt{G\rho_c}} = 1.5745 \times 10^{16} \times \left(\frac{\rho_c}{10^{-24} \text{ g/cm}^3}\right) \text{ s}. \quad (128)$$

In the case of perturbed BEC dark matter halos with an initial rotation, the stability condition reduces to

$$|\Phi_0| > K_0, \quad (129)$$

or, equivalently, with the use of Eq. (97),

$$\frac{GM^2}{R} > 2I_{\Omega_0}\Omega_0^2, \quad (130)$$

where  $\Omega_0$  is the initial angular velocity, and  $I_{\Omega_0}$  the initial moment of inertia of the galactic halo. The period of oscillations of a slowly rotating slightly disturbed BEC dark matter halo is found as

$$\begin{aligned} T &= \frac{\sqrt{2\pi}\sqrt{I_{\Omega_0}}}{\sqrt{|\Phi_0| - K_0}} \approx \frac{\sqrt{2\pi}\sqrt{I_{\Omega_0}}}{\sqrt{|\Phi_0| - K_0}} \\ &= \frac{\sqrt{32\pi}\sqrt{I_{\Omega_0}}}{\sqrt{2}\sqrt{2\Omega_0^2 I_{\Omega_0} + GM^2/R}}. \end{aligned} \quad (131)$$

By using again the mass-radius relation for BEC dark matter halos given by Eq. (27), we obtain

$$T \approx \sqrt{\frac{8}{15}}\pi^{3/2}\frac{1}{\sqrt{G\rho_c}}\frac{1}{\sqrt{1 + 15G\rho_c\Omega_0^2/\pi}}. \quad (132)$$

This equation gives the corrections to the oscillations period of the halo due to the presence of an initial rotation.

## VII. DISCUSSIONS AND FINAL REMARKS

The  $n = 1$  polytropic Bose-Einstein condensate dark matter model is the simplest existing dark matter model. All the properties of the dark matter distributions are determined by two parameters only: the mass and the radius of the dark matter halo. The central density of the dark matter is determined uniquely by  $M(R)$  and  $R$ . For halos not contaminated with baryonic matter both quantities must have the same universal value. However, the presence of the baryonic matter may increase the size of the galactic halo, thus leading to some variations in the total mass and radius of the galactic structures.

Since all the properties of the Bose-Einstein condensates are determined by two observational parameters only, once these parameters are known, the physical properties of the galactic halos can be *predicted* by the model. Thus, for

example, the tangential velocity of the test particles in circular orbits around galaxies is determined by the *universal* equation

$$V^2(r) = \frac{GM}{R} \left[ \frac{\sin(\pi r/R)}{\pi r/R} - \cos \frac{\pi R}{R} \right]. \quad (133)$$

Once the mass of the galaxy and its radius is known, the rotation curves can be obtained immediately. For example, in the case of the dwarf galaxy IC 2574, by assuming for the condensate component a mass of  $M = 1.64 \times 10^{10} M_\odot$  and a radius of  $R = 12.6$  kpc, the rotation curve is given by

$$V^2(r) = 5.6374 \times 10^3 \left[ \frac{\sin(\pi r/R)}{\pi r/R} - \cos \frac{\pi R}{R} \right] \text{ km}^2/\text{s}^2. \quad (134)$$

The comparison of the rotation curve *predicted* by the Bose-Einstein condensate dark matter model for the galaxy IC 2574 with the observational data obtained in [56] is represented in Fig. 1. The comparison of the *predicted* rotation curve for the dwarf galaxy M81 dwB with the observational data of [56] is represented in Fig. 2.

The condensate dark matter model *predicts* a mean logarithmic density slope of the inner core  $\langle \alpha_{\text{DM}} \rangle = 0.3068$ , while the observed value obtained from the study of brown galaxies is  $\langle \alpha_{\text{DM}} \rangle = -0.29 \pm 0.07$  [56] (the minus sign appears due to the opposite sign convention adopted in the present paper). In [67] it was shown, by using the mass decomposition of rotation curves using cored haloes, that the product of the central density  $\rho_c$  and the core radius  $R_{\text{core}}$  is a universal constant, independent of the galaxy mass. In the case of condensate dark matter halos the product of the central density and core can be written as

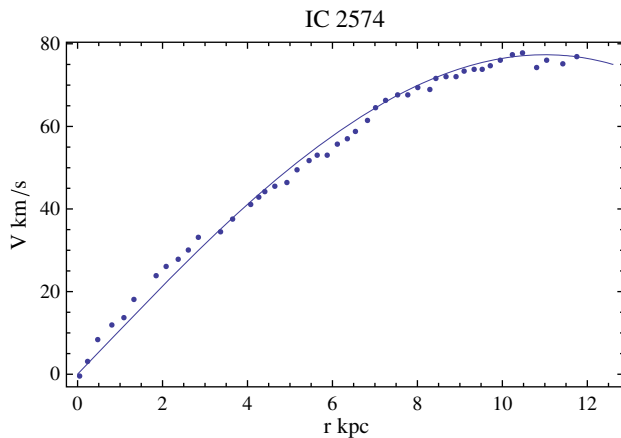


FIG. 1 (color online). Comparison of the predicted rotation curve of the dwarf galaxy IC 2574 (solid curve) with the observational data presented in [56]. The assumed mass of the galaxy is  $M = 1.64 \times 10^{10} M_\odot$  (the observationally determined value  $M = 1.462 \times 10^{10} M_\odot$ ), and the radius of the galaxy is  $R = 12.6$  kpc.

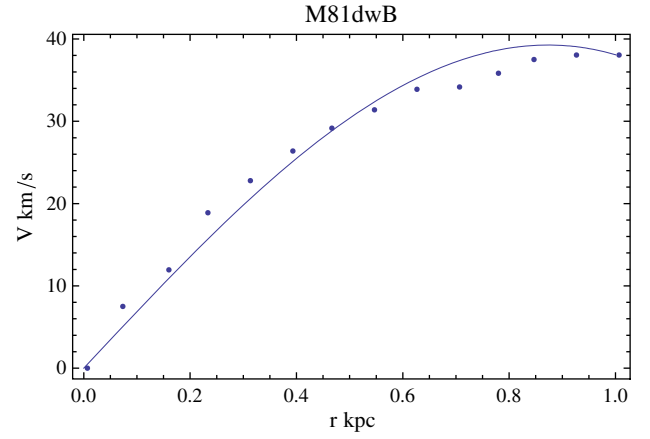


FIG. 2 (color online). Comparison of the rotation curve of the dwarf galaxy M81 dwB predicted by the condensate dark matter model (solid curve) with the observational data presented in [56]. The assumed mass of the galaxy is  $M = 0.33 \times 10^9 M_\odot$  (the observationally determined value  $M = 0.3 \times 10^9 M_\odot$ ), and the radius of the galaxy is  $R = 1$  kpc.

$$\rho_c R_{\text{core}} = 0.00827 \left( \frac{M(R)}{10^{10} M_\odot} \right) \left( \frac{R}{10 \text{ kpc}} \right)^{-2} \text{ g/cm}^2, \quad (135)$$

and for pure condensate dark matter halos this product must be a universal constant. However, the condensate dark matter model predicts a dependence *on both mass and radius* of the constant. The presence of baryonic matter may determine some modifications and variations in its numerical value (for a comparison of the theoretical predictions with the observational data see [25]). The condensate dark matter model also predicts a relation between  $\rho_c R_{\text{core}}$  and the central convergence  $\kappa_c$  of the halo of the form

$$\rho_c R_{\text{core}} = \frac{1}{7.372} \Sigma_{\text{crit}} R \kappa_c. \quad (136)$$

Therefore, the condensate dark matter model is the only existing model that makes easily testable *predictions*, without any need for fitting, and hence it can be easily falsified by comparing the theoretical predictions with the observations. However, despite its remarkable successes at the galactic scale, the Bose-Einstein condensate dark matter model needs further testing. One of the best possibilities in confirming/ruling out the model is through gravitational lensing. In the present paper we have obtained for the first time all the relevant quantities necessary for an in depth comparison of the theoretical predictions with the observational data in an analytical, easy to handle form. These general analytical formulae enable arbitrary precision calculation, as well as the study of the asymptotic behavior near the vacuum boundary of the condensate. These formulae can be used in strong and weak lensing studies

of galaxies and clusters of galaxies, where dark matter is assumed to be the dominant component. Moreover, a comparison between lensing and rotation curve predictions can be done easily.

The virial theorem is a very useful tool to investigate the general properties of the astrophysical systems. The scalar virial theorem gives a very powerful constraint for the validity of the Thomas-Fermi approximation, which can be formulated as

$$R \gg 1.581 \times 10^3 \left( \frac{m_\chi}{10^{-37} \text{ g}} \right)^{1/2} \left( \frac{l_a}{10^{-20} \text{ cm}} \right)^{-1/2} \times \left( \frac{\rho_c}{10^{-24} \text{ g/cm}^3} \right)^{-1/2} \text{ cm}. \quad (137)$$

The above constrain shows that the Thomas-Fermi approximation gives an excellent description of the properties of condensate dark matter halos. By using the tensor virial theorem we have derived the perturbation equation of the dark matter halos, which can be efficiently used to study the stability of dark matter halos under small perturbations.

Finally, we consider the problem of the mass of the dark matter particle. The use of Eq. (23) allows us to make a first estimate of the physical properties of the dark matter particle. As a function of the mass and scattering length of the particle the radius  $R$  of the condensate dark matter halo is given by  $R = \pi \sqrt{\hbar^2 l_a / G m_\chi^3}$ , the total mass of the condensate dark matter halo  $M(R)$  can be obtained as

$$M(R) = 4\pi^2 \left( \frac{\hbar^2 l_a}{G m_\chi^3} \right)^{3/2} \rho_c = \frac{4}{\pi} \rho_c R^3, \quad (138)$$

while the mean value  $\langle \rho \rangle$  of the condensate density is given by the expression  $\langle \rho \rangle = 3\rho_c / \pi^2$  [19]. Therefore, the dark matter particle mass in the condensate is given by [19]

$$m_\chi = \left( \frac{\pi^2 \hbar^2 l_a}{G R^2} \right)^{1/3} \approx 6.73 \times 10^{-2} [l_a(\text{fm})]^{1/3} [R(\text{kpc})]^{-2/3} \text{ eV}. \quad (139)$$

For  $l_a \approx 1 \text{ fm}$  and  $R \approx 10 \text{ kpc}$ , the typical mass of the condensate particle is of the order of  $m_\chi \approx 14 \text{ meV}$ . For  $l_a \approx 10^6 \text{ fm}$ , corresponding to the values of  $l_a$  observed in terrestrial laboratory experiments,  $m_\chi \approx 1.44 \text{ eV}$ .

An important method of observationally obtaining the properties of dark matter is the study of the collisions between clusters of galaxies, like the bullet cluster (1E0657-56) and the baby bullet (MACSJ0025-12). From these studies one can obtain constraints on the physical properties of dark matter, such as its interaction cross section with baryonic matter, and the dark matter-dark matter self-interaction cross section. If the ratio  $\sigma_m = \sigma / m_\chi$  of the self-interaction cross section  $\sigma = 4\pi l_a^2$  and of the dark matter

particle mass  $m_\chi$  is known from observations, with the use of Eq. (139) the mass of the dark matter particle in the Bose-Einstein condensate can be obtained as [43]

$$m_\chi = \left( \frac{\pi^{3/2} \hbar^2 \sqrt{\sigma_m}}{2G R^2} \right)^{2/5}. \quad (140)$$

By comparing results from x-ray, strong lensing, weak lensing, and optical observations with numerical simulations of the merging galaxy cluster 1E 0657-56 (the bullet cluster), an upper limit (68% confidence) for  $\sigma_m$  of the order of  $\sigma_m < 1.25 \text{ cm}^2/\text{g}$  was obtained in [68]. By adopting for  $\sigma_m$  a value of  $\sigma_m = 1.25 \text{ cm}^2/\text{g}$ , we obtain for the mass of the dark matter particle an upper limit of the order

$$m_\chi < 3.1933 \times 10^{-37} \left( \frac{R}{10 \text{ kpc}} \right)^{-4/5} \left( \frac{\sigma_m}{1.25 \text{ cm}^2/\text{g}} \right)^{1/5} \text{ g} \\ = 0.1791 \times \left( \frac{R}{10 \text{ kpc}} \right)^{-4/5} \left( \frac{\sigma_m}{1.25 \text{ cm}^2/\text{g}} \right)^{1/5} \text{ meV}. \quad (141)$$

This mass limit is consistent with the limit obtained from cosmological considerations in [69]. By using this value of the particle mass we can estimate the scattering length  $l_a$  as

$$l_a < \sqrt{\frac{\sigma_m \times m_\chi}{4\pi}} = 1.7827 \times 10^{-6} \text{ fm}. \quad (142)$$

A stronger constraint for  $\sigma_m$  was proposed in [70], so that  $\sigma_m \in (0.00335 \text{ cm}^2/\text{g}, 0.0559 \text{ cm}^2/\text{g})$ , giving a dark matter particle mass of the order

$$m_\chi \approx (9.516 \times 10^{-38} - 1.670 \times 10^{-37}) \left( \frac{R}{10 \text{ kpc}} \right)^{-4/5} \text{ g} \\ = (0.053 - 0.093) \times \left( \frac{R}{10 \text{ kpc}} \right)^{-4/5} \text{ meV} \quad (143)$$

and a scattering length of the order of

$$l_a \approx (5.038 - 27.255) \times 10^{-8} \text{ fm}. \quad (144)$$

Therefore, the galactic radii data and the bullet cluster constraints predict a condensate dark particle mass of the order of  $m_\chi \approx 0.1 \text{ meV}$ . Recent results on the self-interacting dark matter cross section have been presented in [71,72]. By using collisions between galaxy clusters as tests of the nongravitational forces acting on dark matter, from the dark matter's lack of deceleration in the bullet cluster collision one obtains a self-interaction cross section of the order of  $\sigma_{\text{DM}}/m < 1.25 \text{ cm}^2/\text{g}$  (68% C.L.) for long-ranged forces [71]. From the observation of 72 collisions, a self-interaction cross section  $\sigma_{\text{DM}}/m < 0.47 \text{ cm}^2/\text{g}$  (95% C.L.) was inferred.

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