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Polarized gravitational waves from cosmological phase transitions

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We estimate the degree of circular polarization for the gravitational waves generated during the electroweak and QCD phase transitions from the kinetic and magnetic helicity generated by bubble collisions during those cosmological phase transitions.

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I. INTRODUCTION

Gravitational-wave (GW) astronomy opens a new window to study the physical processes in the very early Universe: relic GWs propagate almost freely throughout the Universe's expansion, and thus they retain the information about the physical conditions and physical processes at the moment of their generation (see for reviews, Refs. [1–3] and references therein). There are various mechanisms that might generate such GWs. In the present paper we focus on the generation of GWs during cosmological electroweak (EW) and quantum chromodynamic (QCD) phase transitions (PTs) through the turbulent helical sources which can arise and follow the PT bubble collisions. The GW generation mechanism associated with bubble collisions during the first-order PTs has been widely discussed in the literature, starting from the pioneering works [4–7] and readdressed later [8–25].

For a cosmological phase transition to produce strong enough turbulent motions and magnetic fields, which will result in the detectable signal of GWs, they must be first-order PTs, with bubble formation and bubble collisions. For the EWPT, with the standard EW Lagrangian plus a top squark, the supersymmetric partner of the top quark, called the minimal supersymmetric standard model EW Lagrangian, the EWPT is a first-order PT [26]. The QCDPT has been shown to be a first-order PT by lattice gauge calculations [27,28].

Both turbulent motions and magnetic fields can produce relic GWs through their anisotropic stresses; see Refs. [29–39]. It has been pointed that the GWs generated by magnetic fields can be detected through the Laser Interferometer Space Antenna (LISA) [40–44]. As opposed to the GWs sourced solely by PT bubble collisions, the presence of turbulent (kinetic and magnetic) sources increases the detection prospects [45] not only from EWPT but from QCDPT too [46].[1] One of the main goals of European Space Agency (ESA)-NASA planned joint mission LISA [48], was the detection of low-frequency GWs (sub-Hz region). The new development of this program is the European-only ESA mission, the so-called New Gravitational wave Observatory (NGO)—aka eLISA (evolved LISA) [49]. One of the major parts of its science program consists of the direct detection of GWs from cosmological PTs; see Refs. [50–53] for details.

In the present paper we extend our previous study [38], and we investigate the degree of polarization of GWs generated via cosmological PTs through helical hydro and magnetized turbulent sources using the formalism given in Ref. [32]. We adjust the previous formalism to determine the polarization degree of GWs from helical kinetic turbulence to a more complex scenario of magnetohydrodynamics (MHD) turbulence present during the cosmological PTs. More precisely, we use the recent results of numerical simulations [54–56] to set the statistical properties of helical MHD turbulence. Another difference from the formalism of Ref. [32] consists in computing the energy density and peak frequency of GWs using the analogy with acoustic-wave production by hydrodynamical turbulence [34] (which we can call the *aeroacoustic* approach [57–59]).

Charge-conjugation-parity (CP) violation is necessary for the production of magnetic helicity via bubble collisions, [60]. EWPT and QCDPT bubble collisions result in the development of helical (kinetic or/and magnetic) turbulence, due in part to CP violation, which will lead to a circularly polarized GW background. In the case of strong enough helical sources [35,36], the degree of polarization is potentially detectable [61–67].[2]

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[1]GWs from QCDPT are potentially detectable through pulsar timing; see Ref. [47] and references therein.

[2]The indirect tool to detect circularly polarized GWs consists of searching parity-violating signals on cosmic microwave background maps; see Refs [68–71] for original studies and Ref. [72] for a review.

In our present study we follow the helical (chiral) magnetic field generation scenarios (during PTs through bubble collisions) presented in Refs. [73,74] (EWPT) and Refs. [60,75] (QCDPT) (see also Ref. [38] for a brief review of these models). Upon generation the magnetic field starts to interact with primordial plasma that leads to the development of magnetically dominant MHD and secondary kinetic turbulence; for pioneering studies see Refs. [76–79]. In what follows we adopt the results of numerical simulations of Refs. [54–56],[3] and their phenomenological interpretation given in Refs. [80,81].

The structure of the paper is as follows. In Sec. II we review the GW generation formalism and define the circular polarization degree of GWs. We discuss the hydro and MHD helical turbulence modeling in Sec. III and compute the GW signal and its polarization in Sec. IV. We give our results for both EWPT- and QCDPT-generated GWs in Sec. IV, and we conclude in Sec. V. We use natural ($\hbar = 1 = c$) Lorentz-Heaviside units.

II. GRAVITATIONAL-WAVE GENERATION OVERVIEW

We assume that GWs are generated through kinetic and MHD turbulence which follow the PT bubble collisions [7,29–31,82]. To be as general as possible we present the common description for EWPTs and QCDPTs, defining the PT temperature as $T_\star$ ($T_\star = 100$ GeV for EWPT and $T_\star = 0.15$ Gev for QCDPT), and the typical *proper* length scale through l_0 which can be associated with the PT bubble size l_b (the assumption $l_0 \simeq l_b$ is well justified because in our theory bubble collisions during PT generate a magnetic field at bubble walls, and this initial field starts to interact with primordial plasma resulting in the development of kinetic and MHD turbulence with a typical length scale that corresponds to the magnetic field injection scale) [83]. We also define $g_\star$ as a number of relativistic degrees of freedom: for the standard model we have $g_* = 106.75$ as $T \to \infty$. ($g_\star = 100$ for EWPT and $g_\star = 15$ for QCDPT). In our further consideration we assume that the duration of the turbulent sources, τ_T are short compared to the Universe expansion time scale at PTs, i.e. $\tau_T \leq H_\star^{-1}$ where $H_\star^{-1}$ is the Hubble radius at the PT. This assumption makes it possible to neglect the expansion of the Universe, although it limits our consideration of the GW background generation *only* from PT, and completely neglects GWs arising from decaying turbulence (which might last log-enough after the end of PTs).

GWs (the tensor metric perturbations above the standard Friedmann-Lemaître-Robertson-Walker homogeneous and isotropic background) are generated from turbulence (including kinetic and magnetic fluctuations) through the presence of anisotropic stresses as

$$\nabla^2 h_{ij}(\mathbf{x},t) - \frac{\partial^2}{\partial t^2} h_{ij}(\mathbf{x},t) = -16\pi G \Pi_{ij}^{(T)}(\mathbf{x},t), \qquad (1)$$

where $h_{ij}(\mathbf{x},t)$ is the tensor metric perturbation, t is physical time, i and j are spatial indices (repeated indices are summed), and G is the gravitational constant. We have neglected the term $\propto \partial h_{ij}(\mathbf{x},t)/\partial t$ due to our assumption of the short duration of turbulence. $P_{ij}^{(T)}$ (the script "T" indicates that we are interested in the tensor part of the turbulent source) is the traceless part of the stress-energy tensor $T_{ij}(\mathbf{x},t)$, which is constructed from kinetic (K) or magnetic (M) turbulence normalized vector fields[4] (as we will show below the equipartition is established between kinetic and magnetic turbulent motions which simply doubles the source term, i.e. $\Pi_{ij}^{(K)} + \Pi_{ij}^{(M)} \simeq 2\Pi_{ij}^{(T)}$) given by [84]

$$\Pi_{ij}^{(T)}(\mathbf{x},t) = T_{ij}(\mathbf{x},t) - \frac{1}{3}\delta_{ij} T(\mathbf{x},t), \qquad (4)$$

where $T \equiv [T]_k^k$ is the trace of the T_{ij} tensor.

As we can expect the kinetic and magnetic turbulent fluctuations generate stochastic GWs, which can be characterized by the wave-number space two-point function as,

$$\begin{aligned}\langle h_{ij}^\star(\mathbf{k},t) h_{lm}(\mathbf{k}',t+\tau)\rangle &= (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k}') \\ &\quad \times [\mathcal{M}_{ijlm}(\hat{\mathbf{k}}) H(k,\tau) \\ &\quad + i\mathcal{A}_{ijlm}(\hat{\mathbf{k}})\mathcal{H}(k,\tau)]. \qquad (5)\end{aligned}$$

Here we use the Fourier transform pair of the tensor perturbation as $h_{ij}(\mathbf{k},t) = \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} h_{ij}(\mathbf{x},t)$ and $h_{ij}(\mathbf{x},t) = \int d^3k e^{-i\mathbf{k}\cdot\mathbf{x}} h_{ij}(\mathbf{k},t)/(2\pi)^3$. The brackets $\langle \ldots \rangle$ denote an ensemble average over the realization of the stochastic source. The spectral functions $H(k,t)$ and $\mathcal{H}(k,t)$ determine

[3] We underline the nature of the secondary character of fluid motions, because the bubble collision itself might lead to the development of purely hydrodynamical turbulence during PTs (see Ref. [31]), while here we note that bubble collisions result in the generation of magnetic fields [60,73–75].

[4] The kinetic and magnetic perturbation stress-energy tensors are

$$T_{ij}^{(K)}(\mathbf{x},t) = \mathrm{w} u_i(\mathbf{x},t) u_j(\mathbf{x},t), \qquad (2)$$

$$T_{ij}^{(M)}(\mathbf{x},t) = \mathrm{w} b_i(\mathbf{x},t) b_j(\mathbf{x},t), \qquad (3)$$

where $\mathrm{w} = \rho + p$ is the enthalpy of the fluid with density energy, ρ, and pressure p, $\mathbf{u}(\mathbf{x},t)$ is the kinetic motion velocity field and $\mathbf{b}(\mathbf{x},t)$ is the normalized magnetic field, $\mathbf{b} = \mathbf{B}/\sqrt{4\pi \mathrm{w}}$, that represents the Alfvén velocity, v_A of the magnetic field. The normalized energy of the magnetic field is then $\mathcal{E}_M(\eta) = \langle \mathbf{b}^2(t)\rangle/2$, while the normalized kinetic energy density is given through $\mathcal{E}_K(t) = \langle \mathbf{u}^2(t)\rangle/2$. The advantage of such a representation consists in eliminating the expansion of the Universe, since physical and comoving values of the normalized magnetic field amplitude are the same.

the GW amplitude and polarization, $4\mathcal{M}_{ijlm}(\hat{\mathbf{k}})\equiv P_{il}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}})+P_{im}(\hat{\mathbf{k}})P_{jl}(\hat{\mathbf{k}})-P_{ij}(\hat{\mathbf{k}})P_{lm}(\hat{\mathbf{k}})$, and $8\mathcal{A}_{ijlm}(\hat{\mathbf{k}})\equiv\hat{\mathbf{k}}_q[P_{jm}(\hat{\mathbf{k}})\epsilon_{ilq}+P_{il}(\hat{\mathbf{k}})\epsilon_{jmq}+P_{im}(\hat{\mathbf{k}})\epsilon_{jlq}+P_{jl}(\hat{\mathbf{k}})\epsilon_{imq}]$ are tensors, with the projection tensor $P_{ij}(\hat{\mathbf{k}})=\delta_{ij}-\hat{k}_i\hat{k}_j$ (with the Kronecker delta δ_{ij}, $\hat{k}_i=k_i/k$ and $k=|\mathbf{k}|$), and ϵ_{ijl} is the totally antisymmetric symbol. We choose a GW propagation direction pointing alon the unit vector $\hat{\mathbf{e}}_3$, and we use the usual circular polarization basis tensors $e^{\pm}_{ij}=-(\mathbf{e}_1\pm i\mathbf{e}_2)_i\times(\mathbf{e}_1\pm i\mathbf{e}_2)_j/\sqrt{2}$. We define two states h^+ and h^- corresponding to right- and left-handed circularly polarized GWs, $h_{ij}=h^+e^+_{ij}+h^-e^-_{ij}$. Through the above notations the circular polarization degree was derived for GWs from gamma-ray bursts in Ref. [85] and in the context of cosmological GWs it is reproduced as [32],

$$\mathcal{P}^{\mathrm{GW}}(k)=\frac{\langle h^{+\star}(\mathbf{k})h^+(\mathbf{k}')-h^{-\star}(\mathbf{k},)h^-(\mathbf{k}')\rangle}{\langle h^{+\star}(\mathbf{k})h^+(\mathbf{k}')+h^{-\star}(\mathbf{k})h^-(\mathbf{k}')\rangle}=\frac{\mathcal{H}(k)}{H(k)}. \tag{6}$$

Here we omit the time dependence of the polarization degree $\mathcal{P}^{\mathrm{GW}}(k)$.

As we already underlined we are interested in GW generation only from short-duration sources acting during PTs. After generation the GWs propagate almost freely, and we account for the expansion of the Universe by a simple rescaling of the frequency and the amplitude by a factor equal to

$$\frac{a_\star}{a_0}\simeq 8\times10^{-16}\left(\frac{100\ \mathrm{GeV}}{T_\star}\right)\left(\frac{100}{g_\star}\right)^{1/3}. \tag{7}$$

This factor is safely canceled when computing $\mathcal{P}^{\mathrm{GW}}(k)$, although the more complex consideration of decaying turbulence (long-lasting sources) will make $\mathcal{P}^{\mathrm{GW}}(k)$ a time-dependent function.

To estimate the polarization degree of GWs from PT-generated helical fields we need to compute two spectral functions $\mathcal{H}(k)$ and $H(k)$ at the moment of PT, which are determined by the helical anisotropic sources $\Pi^{(K)}_{ij}$ and $\Pi^{(M)}_{ij}$. In our previous work we have computed the typical amplitude and frequency of GWs generated during cosmological PTs [38]. In Ref. [32] the polarization degree of GWs from kinetic (hydro) turbulence has been estimated. It has been shown that fully helical turbulence leads to $\mathcal{P}^{\mathrm{GW}}(k)\to 1$. In the present work we follow the GW generation formalism from helical magnetized sources presented in Refs. [35,36], and apply it to the EWPT and QCDPT cases.

III. KINETIC AND MHD TURBULENCE MODELING

The magnetic field amplitude (i.e. total magnetic field energy density) is strongly limited by the big bang nucleosynthesis bound requesting that the total magnetic field energy density cannot exceed 10% of the radiation density at the moment of the magnetic field generation. In terms of the *effective* comoving magnetic field value, $B_{\mathrm{eff}}\simeq 8.4\times10^{-7}(100/g_\star)^{1/6}$ G or in the terms of Alfveń velocity $v_A\equiv|\mathbf{b}|\leq 0.4$ [83].

As we noted above the magnetic field generated at one scale [the magnetic field initial spectrum can be approximated as being peaked at the typical wave number $k_0=2\pi/l_0$, so in Fourier $\mathbf{k}$ space it is described by the $\delta^{(3)}(\mathbf{k}-\mathbf{k}_0)$ function] after interactions with plasma leads to the development of turbulence,[5] and the sharply peaked initial spectrum is redistributed respectively. For isotropic *stationary* turbulence the normalized magnetic vector field two-point correlation function is

$$\langle b^*_i(\mathbf{k})b_j(\mathbf{k}')\rangle=(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')F^M_{ij}(\mathbf{k}) \tag{8}$$

where

$$F^M_{ij}(\mathbf{k})=P_{ij}(\hat{\mathbf{k}})S_M(k)+i\epsilon_{ijl}\hat{k}_lA_M(k). \tag{9}$$

The power-law spectral functions $S_M(k)=S_0k^{n_S}$ and $A_M(k)=A_0k_0^{n_S-n_A}k^{n_A}$ determine the energy density and current helicity of the magnetic field[6] and n_S and n_A are the magnetic field and helicity spectral indices which determine the spatial distribution of the magnetic field and its helicity. The establishment of stationary turbulence with the stationary (time-independent) two-point correlation function given through Eqs. (8)–(9) requires the presence of long-lasting sources. To account for the short-acting PT turbulent source (the turbulence duration time τ_{T} is short enough compared to the Universe expansion time scale $H_\star^{-1}$) we have to modify the spectra $S_M(k)$ and $A_M(k)$ making them time dependent; see below.

Following the description of hydro and MHD turbulence generated during PTs, we distinguish three spatial spectral subregimes of turbulent fluctuations: (i) the large-scale decay range $k_{H_\star}<k<k_0$ (where physical wave numbers $k_{H_\star}=2\pi/H_\star^{-1}$ and k_0 correspond to the PT Hubble length scale and the largest PT length size); the minimal wave number corresponds to the Hubble scale $H_\star^{-1}$ beyond which the causally generated magnetic field is *frozen in* and any interactions are forbidden due to the causality requirement;

[5] Primordial plasma is a perfect conductor with extremely high values of kinetic and magnetic Reynolds numbers, and current numerical simulations are still unable to approach necessary resolutions and time scales to describe adequately physical conditions and processes in the early Universe.

[6] The magnetic helicity defined as $\langle\mathbf{A}(\mathbf{x})\cdot\mathbf{B}(\mathbf{x})\rangle$ is a gauge-dependent quantity, while the normalized [or regular expressed through $\mathbf{B}(\mathbf{x})$] current helicity $\langle\mathbf{b}(\mathbf{x})\cdot[\nabla\times\mathbf{b}(\mathbf{x})]\rangle$ is gauge independent (see Ref. [86] for details), and also it allows for a direct analogy with the kinetic helicity $\langle\mathbf{u}(\mathbf{x})\cdot[\nabla\times\mathbf{u}(\mathbf{x})]\rangle$, and thus to consider both helical sources in a common formalism.

(ii) the turbulent (or so called *inertial*) range $k_0 < k < k_D$ (where k_D is the damping scale of turbulence through viscous dissipation and magnetic resistivity, which is determined by plasma properties); (iii) the damping range $k > k_D$. All these typical wave numbers ($k_{H_\star}$, k_0, and k_D) are time dependent due to interactions of the magnetic field with plasma and the expansion of the Universe. The presence of magnetic helicity plays here a crucial role leading to the rearrangement of the helical structure at large scales [87]. The expansion of the Universe leads to additional effects: namely, the length scale l_0 determined by the PT bubble size is strengthened by a factor $a(t)/a_\star$ (which is $\propto t^{1/2}$ during the radiation-dominated epoch and $\propto t^{2/3}$ during the matter-dominated epoch), while the Hubble length scale $H_\star^{-1} \propto t$. As a result a perturbation with $k_{H_0} < k < k_{H_\star}$ (with $k_{H_0} = 2\pi/H_0$ and H_0 is the current Hubble radius) will enter the horizon at some point [88]. Since we are focused on the short-duration sources, we will completely neglect the GW signal for the large-scale decay range $k < k_0$ (where the spectral shape of the field is given through the causal Batchelor spectrum with $n_S = 2$ [89]). Obviously the GW signal from the viscous damping range $k > k_D$ is also negligibly small.

The realizability condition implies that $|A_M(k)| \leq S_M(k)$ (the modulus sign reflects a possibility of having positive or negative helicity). The spectral indices values n_S and n_A strongly depend on the turbulence model. In the inertial range ($k_0 < k < k_D$), for nonhelical turbulence the Kolmogoroff model implies $n_S = -11/3$. Some models lead to different spectral shapes such as $n_S = -7/2$ (the Iroshnikov-Kraichnan model for magnetized turbulence [90,91]), and $n_S = -4$ (the weak turbulence model [92] or magnetically dominant turbulence [93]). In the presence of helicity the consideration is even more complex, and requires careful investigation through numerical simulations which is beyond the scope of the present paper. Based on the phenomenological dimensionless description, if the process is driven by the magnetic energy dissipation at small scales, it is assumed that $n_S = -11/3$ and $n_A = -14/3$ (the so-called *helical* Kolmogoroff model) [94], while if the process is determined by helicity transfer (inverse cascade) and helicity dissipation at small scales $n_S = -13/3 = n_A$ is adopted [95].

To account for the short-duration turbulence (not enough to establish the stationary turbulent motions) we have to consider the time decorrelation of turbulent fluctuations which can be accounted for via introducing the characteristic function $f(\eta(k), \tau)$ [where $\eta(k)$ is the autocorrelation function] [96]:

$$f(\eta(k), \tau) = \exp\left[-\frac{2\pi^2}{9}\left(\frac{\tau}{\tau_0}\right)K^{4/3}\right] \tag{10}$$

where τ_0 is the largest turbulent eddy turnover time, $K \equiv k/k_0$. In this case, to determine the magnetic field two-point correlation function in real ($\mathbf{x}$) space, we have to account for magnetic field fluctuations at *different* time moments and at *different* positions, i.e. $\langle b_i(\mathbf{x}, t) b_j(\mathbf{x} + \mathbf{R}, t + \tau)\rangle$. Accordingly, in Fourier space, the two-point correlation function will be determined by the $\bar{F}^M_{ij}(\mathbf{k}, t)$ with the time-dependent spectral functions $S_M(k, t)$ and $A_M(k, t)$:

$$\bar{F}^M_{ij}(\mathbf{k}, t) = [P_{ij}(\hat{\mathbf{k}}) S_M(k, t) + i\epsilon_{ijl}\hat{k}_l A_M(k, t)] \times f(\eta(k), t). \tag{11}$$

Comparing with the stationary spectrum [see Eq. (8)], we see that formally we replace $F^M_{ij}(\mathbf{k}) \equiv F^M_{ij}(\mathbf{k}, t)$, by $\bar{F}^M_{ij}(\mathbf{k}, t) = F^M_{ij}(\mathbf{k}, t) f(\eta(k), \tau)$, To avoid a complex description of accounting for the time dependence of $S(k, t)$ and $A(k, t)$, we use the Proudman argument for kinetic turbulence [57], according to which the description of decaying turbulence lasting for τ_T can be replaced by the description of stationary turbulence with a time duration of $\tau_T/2$. Below we briefly discuss our approach.

Turbulence during PTs generated through magnetic helicity can be described through two major stages [36]. During the first stage the main process is determined by the magnetic energy direct cascade that last a few largest eddy turnover times $\tau_0 = 2\pi/(k_0 v_0)$, where $v_0 < 1$ is the turbulent eddy velocity ($v_0 \simeq M$ for kinetic turbulence where M is the Mach number and $v_0 \simeq v_A$ for magnetic turbulence) determined by the PT and magnetogenesis model parameters (see Refs. [31,97]), i.e. $\tau_\mathrm{T} = s_0 \tau_0$ (with $s_0 = 3 - 5$). The magnetic field induces vorticity fluctuations, and at the end of the first (semi)equipartition between kinetic and magnetic energies is reached, this results in doubling the value of the source for GWs. The magnetic energy density power spectra are then determined by the proper dissipation rate per unit enthalpy ε_M as $S_0 = \pi^2 C_K \varepsilon^{2/3}$ where C_K is a constant order of unity, and $\varepsilon = k_0 v_0^3$. Note that the autocorrelation function $\eta(k) = \varepsilon^{1/3} k^{2/3}/\sqrt{2\pi}$ [98]. Although the Kolmogoroff model is valid *only* for nonrelativistic turbulence, while during PTs we might deal with $v_0 \simeq 1$ (relativistic turbulence), our estimates for the amplitude and polarization degrees of GW signals are qualitatively justified; see Ref. [29]. The second stage consists in helicity transfer (inverse cascade). The scaling laws for this stage are still under debate. Based on our previous consideration [36], we assume that (i) the details of the scaling laws during this stage will not substantially affect our estimates; (ii) instead of considering decay turbulence we will again consider the stationary turbulence with a scale-dependent duration time. Then we obtain for the helical Kolmogoroff model, $A_0 = \pi^2 C_K \sigma/(k_0 \varepsilon^{1/3})$, where σ is the magnetic helicity dissipation rate per unit enthalpy, leading to $A_0/S_0 = \sigma/(\varepsilon k_0)$. The helical Kolmogoroff model is mainly relevant for weakly helical fields, $|A(k)| \ll S(k)$, which is a case of magnetic fields generated during PTs.[7] According to results of recent numerical simulations (see Ref. [56]), the weakly helical turbulence even accounting for the free decay of turbulence, shows an establishment of the spectra in good agreement with the helical Kolmogoroff model, as well as equipartition between magnetic and kinetic energy densities. Thus we adopt $n_S = -11/3$ and $n_H = -14/3$ for the inertial range, with $S_0 = \pi^2 k_0^{2/3} v_0^2$ and $A_0 = hS_0$ where h determines the fraction of helicity dissipation, $h \equiv \sigma/(\varepsilon k_0)$. The turbulence fluctuation velocity $v_0 = v_A$ is determined by the magnetogenesis mechanism and for the model of our interest is given by $v_0 \simeq 0.2$ ($B_{\rm eff,in} \simeq 5\times 10^{-7}$ G) [38] for the EWPT model of Refs. [73,74] and $v_0 \simeq 0.01$ ($B_{\rm eff,in} \simeq 2\times 10^{-8}$ G) [54] for the QCDPT model of Refs. [60,75].

IV. GRAVITATIONAL-WAVE SIGNAL AMPLITUDE AND POLARIZATION

In this section we compute the GW signal (stain amplitude) and polarization degree from the hydro and MHD turbulence generated during the first-order cosmological PTs. To determine the amplitude of GWs we proceed as described in Ref. [36]. We derive the energy density spectrum of the GWs at the end of the PT (in our approximation the end of turbulence). The energy density of GWs is given through the ensemble average as

$$\rho_{GW}(\mathbf{x},t) = \frac{1}{32\pi G}\langle \partial_t h_{ij}(\mathbf{x},t)\partial_t h_{ij}(\mathbf{x},t)\rangle. \tag{12}$$

As we noted the rescaling of the GW amplitude and frequency given through Eq. (7) is irrelevant when computing the polarization degree, while it is crucial for the estimation of the GW energy density.

A. Gravitational-wave signal

Assuming the homogeneous and isotropic turbulent source lasting for τ_T, and using the far-field approximation (see Ref. [34]), the total energy density of GWs at a given spatial point and a given time can be obtained by integrating over all sources within a spherical shell centered at that observer, with a shell thickness corresponding to the duration of the turbulent source (in our case the duration of the PT), and a radius equal to the proper distance along any light-like path from the observer to the source (causality requirement), and then

$$\rho_{GW}(\omega) = \frac{d\rho_{GW}}{d\ln\omega} = 16\pi^3\omega^3 G \mathrm{w}^2 \tau_T H_{ijij}(\omega,\omega), \tag{13}$$

where ω is the angular frequency measured at the moment of generation of GWs, and $H_{ijij}(\omega,\omega)$ is a complicated function of ω (which is computed using the aeroacoustic approximation and Millionshchikov quasinormality [94]), given as

$$H_{ijij}(\mathbf{k},\omega) \simeq H_{ijij}(0,\omega) = \frac{7C_K^2\varepsilon}{6\pi^{3/2}}\int_{k_0}^{k_D}\frac{dk}{k^6} \times \exp\left(-\frac{\omega^2}{\varepsilon^{2/3}k^{4/3}}\right)\mathrm{erfc}\left(-\frac{\omega}{\varepsilon^{1/3}k^{2/3}}\right). \tag{14}$$

Here, $\mathrm{erfc}(x)$ is the complementary error function defined as $\mathrm{erfc}(x) = 1-\mathrm{erf}(x)$, where $\mathrm{erf}(x) = \int_0^x dy \exp(-y^2)$ is the error function [99]. The integral in Eq. (14) is dominated by the large-scale ($k \simeq k_0$) contribution so, for direct-cascade turbulence during the first stage (direct cascade; see Sec. III), the peak frequency is

$$\omega_{\rm max}^{(I)} \simeq k_0 M \tag{15}$$

where M is the Mach number. To compute the GW signal arising from the inverse-cascade stage we have to consider two models separately: Model A assumes that the correlation length during the inverse cascade scales as $\xi_M \propto t^{1/2}$ and Model B corresponds to the correlation length scaling as $\xi_M \propto t^{2/3}$. We obtain that in both models the peak frequencies during the second stage are equal and are determined by the Hubble frequency as [36]

$$\omega_{\rm max}^{(II)} \simeq 2\pi H_\star \tag{16}$$

while the GW amplitudes are slightly different in Models A and B

$$H_{ijij}^{(\mathrm{A})}(\mathbf{k},\omega) \simeq H_{ijij}(0,\omega) = \frac{7C_1^2M^3\zeta_\star^{3/2}}{12\pi^{3/2}k_0}\int_{k_S}^{k_0}\frac{dk}{k^4} \times \exp\left(-\frac{\omega^2k_0^2}{\zeta_\star M^2k^4}\right)\mathrm{erfc}\left(-\frac{\omega k_0}{\zeta_\star^{1/2}Mk^2}\right), \tag{17}$$

and

$$H_{ijij}(\mathbf{k},\omega)^{(\mathrm{A})} \simeq H_{ijij}(0,\omega) = \frac{7C_1^2M^3\zeta_\star^{3/2}}{6\pi^{3/2}k_0^{3/2}}\int_{k_S}^{k_0}\frac{dk}{k^{7/2}} \times \exp\left(-\frac{\omega^2k_0}{\zeta_\star M^2k^3}\right)\mathrm{erfc}\left(-\frac{\omega k_0^{1/2}}{\zeta_\star^{1/2}Mk^{3/2}}\right). \tag{18}$$

Here $\zeta_\star$ determines the amount of initial magnetic helicity and is equal to $\zeta_\star = \langle \mathbf{a}(\mathbf{x})\cdot\mathbf{b}(\mathbf{x})\rangle/(\xi_M\mathcal{E}_M)$ [where $\mathbf{a}(\mathbf{x}) = \mathbf{A}(\mathbf{x})/\mathrm{w}$ is the normalized vector potential], and

[7]Note that the substantially helical case is usually determined by the helicity transfer (inverse cascade), $S_0 = C_S\sigma^{2/3}$ and $A_0 = C_A\sigma^{2/3}$ [95].

$k_S = 2\pi/l_S$ is the typical scale at which the inverse cascade stops—either because the cascade time $\tau_{\rm cas}$ reaches the expansion time scale $H_\star^{-1}$ or because the characteristic length scale $l_S \simeq \xi_M$ reaches the Hubble radius $H_\star^{-1}$. The value of k_S can be found by using the above conditions, being equal to $k_S = k_0 \zeta_\star^{-1/4}(\gamma/M)^{1/2}$. Note that the integrals in Eqs. (17)–(18) are dominated by the large scale $k \simeq k_S$ contributions and are maximal at $\omega_{\rm max}^{(II)}$.

B. Gravitational-wave polarization

To compute the polarization degree of GWs we need to estimate the tensor perturbations source two-point correlation function $\mathcal{F}_{ijlm}(k,\tau) \equiv \langle \Pi_{ij}^{(T)\star}(\mathbf{k},t)\Pi_{lm}^{(T)}(\mathbf{k}',t'+\tau)\rangle$, which can be expressed through the forms $\mathcal{M}_{ijlm}$ and $\mathcal{A}_{ijlm}$ [which are defined below Eq. (5)], as

$$\mathcal{F}_{ijlm}(k,\tau) = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \times [\mathcal{M}_{ijlm}\mathcal{S}(k,\tau) + i\mathcal{A}_{ijlm}\mathcal{Q}(k,\tau)]. \quad (19)$$

As we discussed above for the nonstationary turbulence $S(k,\tau)$ and $A(k,\tau)$ are complex functions of τ and k [see Eqs. (10)–(11) for the time decorrelation function and the magnetic field spectrum]. Following Ref. [32] we split the spatial and temporal dependence as $S(k,\tau) = S(k)D_S(\tau)$ and $A(k,\tau) = A(k)D_A(\tau)$ which is a valid approximation for $k \simeq k_0$ (the range which mostly contributes to the GW signal). Generalizing the stationary case [69] by accounting for the time-dependent functions $D_S(\tau)$ and $D_A(\tau)$ the forms for $\mathcal{S}(k,\tau)$ and $\mathcal{Q}(k,\tau)$ are given as,

$$\mathcal{S}(k,\tau) = \frac{\mathrm{w}^2}{(2\pi)^6}\int d^3p_1 \int d^3p_2 \delta^{(3)}(\mathbf{k}-\mathbf{p}_1-\mathbf{p}_2) \times [(1+\alpha^2)(1+\beta^2)D_S^2(\tau)S(p_1)S(p_2) + 4\alpha\beta D_A^2(\tau)A(p_1)A(p_2)], \quad (20)$$

$$\mathcal{Q}(k,\tau) = \frac{\mathrm{w}^2 D_S(\tau)D_A(\tau)}{128\pi^6}\int d^3p_1 \int d^3p_2 \times \delta^{(3)}(\mathbf{k}-\mathbf{p}_1-\mathbf{p}_2)[(1+\alpha^2)\beta S(p_1)S(p_2) + (1+\beta^2)\alpha A(p_1)A(p_2)], \quad (21)$$

where $\alpha = \hat{\mathbf{k}}\cdot\hat{\mathbf{p}}_1$ and $\beta = \hat{\mathbf{k}}\cdot\hat{\mathbf{p}}_2$ with $\hat{\mathbf{p}}_1 = \mathbf{p}_1/p_1$ $(p_1 = |\mathbf{p}_1|)$ and $\hat{\mathbf{p}}_2 = \mathbf{p}_2/p_2$ $(p_1 = |\mathbf{p}_1|)$. The helical source term $\mathcal{Q}(k,\tau)$ vanishes for turbulence without helicity. Since in the helical Kolmogoroff model the time decorrelation is mostly determined by the energy density dissipation, in the first-order approximation we can assume that $D_S(\tau) \simeq D_A(\tau)$ (both functions are monotonically decreasing functions). Next we should connect $\mathcal{S}$ and $\mathcal{Q}$ with the $H(k,t)$ and $\mathcal{H}(k,t)$ functions, which determine the GW polarization degree; see Ref. [32] for details.

Magnetic helicity generated via bubble wall collisions during the first-order PTs is determined by the corresponding energy scales $\Lambda_{\rm PT}$ ($\Lambda_{\rm PT} \simeq 100$ and 0.15 GeV for EWPT and QCDPT respectively) and the PT bubble lengths (l_b). In addition, the bubble wall velocity substantially affects the development of turbulent motion [82].

In the present paper we focus on magnetic helicity generation mechanisms following Refs. [60,75]. In the framework of these magnetogenesis scenarios magnetic helicity during PTs with the magnetic wall in the $x-y$ plane is given by

$$\mathcal{H}_M = A_z B_z \quad \text{or} \quad \mathcal{H}_M = \frac{(B_z)^2}{\Lambda_{\rm PT}}. \quad (22)$$

We note that the EWPT energy (mass) scale is approximately equal to the Higgs mass, $\Lambda_{\rm EWPT} \simeq M_{\rm Higgs}$. The model parameters such as bubble and wall sizes and wall velocity depend on PT modeling, and were determined in Refs. [60,73,74]. We quote also the physical values of magnetic field amplitudes as $B_\star^{(\rm EW)} \simeq 6.45 \times 10^4\ \mathrm{GeV}^2$ [73,74] and $B_\star^{(\rm QCD)} \simeq 1.5\times 10^{-3}\ \mathrm{GeV}^2$ [60].

The fraction of initial magnetic helicity ($\zeta_\star$) can be expressed in terms of the magnetic field correlation length (which can be taken to be equal to l_b) and the maximal allowed length scale $H_\star^{-1}$, as $\zeta_\star \simeq l_b/H_\star^{-1}$. Following Refs. [73,74] the normalized magnetic field generated during the first-order EWPT (at the time moment $\simeq 10^{-11}$ sec) is equal to $v_A \simeq 0.2$, and assuming around 100 bubbles within the Hubble length scale, the fractional helicity is $\zeta_\star^{(\rm EW)} \simeq 0.01$. The comoving magnetic helicity itself is expressed as

$$\mathcal{H}_{M,\star}^{(\rm EW)} \simeq \frac{(B^{(\rm EW)})^2}{125\ \mathrm{GeV}}. \quad (23)$$

Assuming that the magnetic field (with $v_A \simeq 0.01$) is correlated over the wall thickness (the QCD momentum is 0.15 GeV) [60], results in extremely small magnetic helicity generated during the first-order QCDPT (at the time moment $\simeq 10^{-5}$ sec),

$$\mathcal{H}_M^{(\rm QCD)}(t_\star) \simeq \frac{(B^{(\rm QCD)})^2}{0.15\ \mathrm{GeV}}, \quad (24)$$

which corresponds to $\zeta_\star^{(\rm QCD)} \ll 1$. On the other hand, making the field correlated over the bubble length scale, leads to the fractional helicity $\zeta_\star^{(\rm QCD)} \simeq 0.2$.

Using the approximation given above, the polarization degree of GWs, $\mathcal{P}^{\rm GW}(k)$, for the Kolmogoroff helical turbulence model can be estimated through [in our simplified description the time dependence is canceled because of $D_S(\tau) \simeq D_A(\tau)$]

$$\mathcal{P}^{\rm GW}(k) = \frac{\mathcal{H}(k)}{H(k)} = \frac{\mathcal{I}_A(K)}{\mathcal{I}_S(K)} \quad (25)$$

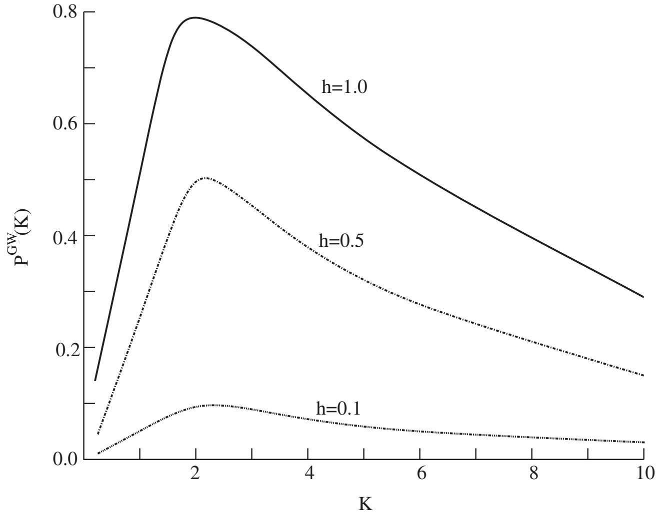

FIG. 1. The GW polarization degree $\mathcal{P}^{\rm GW}(K)$ ($K = k/k_0$) in terms of the model parameter $h = 1.0, 0.5, 0.1$.

where $K \equiv k/k_0$ is a normalized wave number, and

$$\mathcal{I}_S(K) \simeq \int dP_1 P_1 \int dP_2 P_2 \bar{\Theta}[(1+\alpha_p^2)(1+\beta_p^2) P_1^{n_S} P_2^{n_S} + 4h^2 \alpha_p \beta_p P_1^{n_A} P_2^{n_A}], \tag{26}$$

$$\mathcal{I}_A(K) \simeq 2h \int dP_1 P_1 \int dP_2 P_2 \bar{\Theta}[(1+\alpha_p^2)\beta_p P_1^{n_S} P_2^{n_A} + (1+\beta_p^2)\alpha_p P_1^{n_A} P_2^{n_S}]. \tag{27}$$

Here h is the model parameter (which is related to the helicity fraction; see below), and we assume it to be equal to 1, 0.5, 0.1. $P_1 = p_1/k_0$, $P_2 = p_2/k_0$, $\alpha_p = (K^2 + P_1^2 - P_2^2)/(2KP_1)$, $\beta_p = (P^2 + P_2^2 - P_1^2)/(2KP_2)$, $\bar{\Theta} \equiv \theta(P_1 + P_2 - K)\theta(P_1 + K - P_2)\theta(P_2 + K - P_1)$, and θ is the Heaviside step function which is zero (unity) for negative (positive) argument. The integration limit ranges from 1 (we discard the source existence for the wave numbers below k_0) to k_D/k_0.

We emphasize that the fractional helicity parameter $\zeta_\star$ discussed above is defined through the normalized magnetic helicity (the integral quantity), while the parameter $h \equiv \sigma/(k_0 \varepsilon)$ is defined through the normalized magnetic energy density and normalized magnetic helicity (e.g. it is determined by the power spectra for the magnetic energy density and helicity at small length scales). Under the model adopted here (the Kolmogoroff helical turbulence with $n_S = -11/3$ and $n_A = -14//3$) both of these quantities coincide $\zeta_\star \simeq h$.

To keep our description as general as possible we present our results for the GW polarization degree $\mathcal{P}^{\rm GW}(k)$ in terms of the normalized wave number K. As we discussed above the typical wave number k_0 is determined by the turbulent eddy length scale l_0 ($k_0 = 2\pi/l_0$) and is significantly different for EWPT and QCDPT. The model parameter h, which determines the helicity fraction, varies depending on the magnetogenesis model. The results for $\mathcal{P}^{\rm GW}(k)$ with $h = 0.1$, 0.5, and 1 are shown in Fig. 1.

V. CONCLUSIONS

We computed the GW signal produced during first-order cosmological PTs through hydro and MHD turbulence. We also derived the polarization degree of GWs assuming the validity of the helical Kolmogoroff model, shown in Fig. 1. The GW polarization is present at the background level, and for maximally helical sources the polarization degree approaches unity at its maximum, around $k \sim 2k_0$, and decreases fast at small scales $k \gg k_0$. The formalism presented in this paper might be used to estimate the polarization degree of GWs from helical hydro and MHD turbulence in differential rotating neutron stars [100] or stellar convection [101]. Note from Fig. 1 that the detectability of the polarization degree is determined by the helicity fraction parameter h. We plan in our future research to make estimates of h values depending on magnetogenesis models during cosmological PTs.

Previously we have estimated the GW amplitude, $h_C(f)$, from the first-order EWPT and QCDPT [38], through assumptions of nonhelical magnetic fields [60,73,74]. We have shown that EWPT-generated GWs are potentially detectable through LISA-like missions [48] in the case of strong enough EWPT [44] (for QCDPT-generated GW detection prospects see Ref. [46]). In the present paper we expanded our previous results by considering GWs from helical magnetic and hydro turbulence. Probing the circular polarization of the GW background is a challenging task [66], and it is quite difficult at the monopole mode. To detect the circular polarization at the dipole and/or the octupole mode requires at least a system of two unaligned detectors, and LISA was designed ideally to provide detection of these anisotropic components whose magnitudes are as small as 1% of the detector noise [61–63]. Similar detection prospects are expected from eLISA data. The planned eLISA mission [49] was originally designed to detect unpolarized GW backgrounds (including GWs from cosmological PTs) [102] with the sensitivity given in Fig. 1 of Ref. [51] (see also the comparison with LISA's sensitivity). Although GW polarization detection is beyond the currently discussed eLISA science, our study should help further developments.

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