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Isospin breaking decay $\eta(1405) \to f_0(980)\pi^0 \to 3\pi$

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There are attempts in the literature to theoretically explain the large breaking of isotopic invariance in the decay $\eta(1405) \to f_0(980)\pi^0 \to 3\pi$ by the mechanism containing the logarithmic (triangle) singularity, i.e., as being due to the transition $\eta(1405) \to (K^*\bar{K} + \bar{K}^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to 3\pi$. The corresponding calculations were fulfilled for a hypothetic case of the stable K^* meson. Here, we show that the account of the finite width of the K^* ($\Gamma_{K^*\to K\pi} \approx 50$ MeV) smoothes the logarithmic singularities in the amplitude and results in the suppression of the calculated decay width $\eta(1405) \to f_0(980)\pi^0 \to 3\pi$ by the factor of 6–8 as compared with the case of $\Gamma_{K^*\to K\pi} = 0$. We also analyze the difficulties related with the assumption of the dominance of the $\eta(1405) \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi$ decay mechanism and discuss the possible dynamics of the decay $\eta(1405) \to \eta\pi\pi$. The decisive improvement of the experimental data on the $K\bar{K}$, $K\pi$, $\eta\pi$, and $\pi\pi$ mass spectra in the decay of the resonance structure $\eta(1405/1475)$ to $K\bar{K}\pi$ and $\eta\pi\pi$, and on the shape of the resonance peaks themselves in the $K\bar{K}\pi$ and $\eta\pi\pi$ decay channels is necessary for further establishing the $\eta(1405) \to 3\pi$ decay mechanism.

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I. INTRODUCTION

In the seventies, a threshold phenomenon known as the mixing of $a_0^0(980)$ and $f_0(980)$ resonances that breaks the isotopic invariance was theoretically discovered in Ref. [1]; see also Ref. [2]. Recently, the interest in the $a_0^0(980) - f_0(980)$ mixing has been renewed. New proposals for searching it [3–24] have appeared, and the results of the first experiments reporting its discovery with the help of detectors VES [25,26] and BESIII [27,28] have been presented. The VES Collaboration observed for the first time the isospin breaking decay $f_1(1285) \to \pi^+\pi^-\pi^0$ [25,26]; the proposal for searching it was put in Ref. [1,2]. The BESIII Collaboration has obtained the indications on manifestation of the $a_0^0(980) - f_0(980)$ mixing in the decays $J/\psi \to \phi f_0(980) \to \phi a_0(980) \to \phi\eta\pi$ and $\chi_{c1} \to a_0(980)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ [27], suggested for studies in Ref. [21,22]. In another experiment, the BESIII Collaboration has measured the decays $J/\psi \to \gamma\pi^+\pi^-\pi^0$ and $J/\psi \to \gamma\pi^0\pi^0\pi^0$ and observed the resonance structure in the three pion mass spectra in the vicinity of 1.4 GeV with the width of about 50 MeV [28]. At the same time, the corresponding $\pi^+\pi^-$- and $\pi^0\pi^0$ mass spectra in the vicinity of 990 MeV (i.e., in the K^+K^- and $K^0\bar{K}^0$ threshold domain) possess the narrow structure with the width of about 10 MeV [28]. So, in this experiment, the isospin breaking decay $J/\psi \to \gamma\eta(1405) \to \gamma f_0(980)\pi^0$ followed by the transition $f_0(980) \to \pi^+\pi^-(\pi^0\pi^0)$ was observed for the first time [28] with the statistical significance exceeding 10σ. In the same experiment, the decay $f_1(1285)/\eta(1295) \to \pi^+\pi^-\pi^0$ [28] was also observed, with the branching ratio by a factor of two lower than that reported by VES [26].

The narrow resonancelike structure observed in the $\pi^+\pi^-$ and $\pi^0\pi^0$ mass spectra in the decays $\eta(1405) \to \pi^+\pi^-\pi^0$, $\pi^0\pi^0\pi^0$ in the K^+K^- and $K^0\bar{K}^0$ threshold domain looks like the structure expected to originate from the isospin breaking $a_0^0(980) - f_0(980)$ mixing [1], i.e., due to the transition $a_0^0(980) \to (K^+K^- + K^0\bar{K}^0) \to f_0(980) \to \pi\pi$ caused by the mass difference of the K^+K^- and $K^0\bar{K}^0$ intermediate states. It should be recalled that the corresponding S wave amplitude responsible for the breaking of isotopic invariance in the region between $K\bar{K}$ thresholds (the width of this region is about 8 MeV) turns out to be of the order of $\sqrt{(m_{K^0} - m_{K^+})/m_{K^0}}$ [1,29], but not $(m_{K^0} - m_{K^+})/m_{K^0}$, i.e., by the order of magnitude greater than could be expected from the naive considerations. It is natural to expect the relative magnitude of the isospin violation to be suppressed outside the $K\bar{K}$ threshold region, i.e., at the level of $(m_{K^0} - m_{K^+})/m_{K^0}$. To the first approximation, one can neglect this and the similar not really calculable contributions.

The mechanism of the breaking of isotopic invariance in the decay $\eta(1405) \to f_0(980)\pi^0 \to 3\pi$ is similar to the mechanism of the $a_0^0(980) - f_0(980)$ mixing in that it is caused by the transition $\eta(1405) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to 3\pi$. Its amplitude does not vanish due to the nonvanishing mass difference of K^+ and K^0 mesons, and turns out to be appreciable in

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the narrow region between the K^+K^- and $K^0\bar{K}^0$ thresholds.

The aim of the present work is the elucidation of the possible mechanism of the decay $\eta(1405) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$. There are attempts in the literature to theoretically explain this decay as being due to the mechanism that includes the logarithmic (triangle) singularities [30–32], i.e., due to the transition $\eta(1405) \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$. We pay attention to the fact that in the cited works the vector $K^*(892)$ meson in the intermediate state was considered to be stable, and show that the account of the finite width of K^*, $\Gamma_{K^*} \approx \Gamma_{K^* \to K\pi} \approx 50$ MeV, smoothes the logarithmic singularities in the amplitude resulting in the suppression of the calculated width of the decay $\eta(1405) \to f_0(980)\pi^0 \to 3\pi$ by the factor of 6–8 in comparison with the case of $\Gamma_{K^*} = 0$. We also analyze the difficulties related to the assumption of the dominance of the decay $\eta(1405) \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi$ and discuss the possible dynamics of the decay $\eta(1405) \to \eta\pi\pi$. The decisive improvement of the experimental data on the $K\bar{K}$, $K\pi$, $\eta\pi$, and $\pi\pi$ mass spectra in the decays of the resonance structure $\eta(1405/1475)$ [33] to $K\bar{K}\pi$ and $\eta\pi\pi$, and on the shape of the resonance peaks themselves in the $K\bar{K}\pi$ and $\eta\pi\pi$ decay channels is necessary for further establishing the $\eta(1405) \to 3\pi$ decay mechanism.

II. EXPERIMENTAL DATA

According to BESIII [28], the mass and width of the $\eta(1405)$ peak in the $\pi^+\pi^-\pi^0$ channel are 1409.0 ± 1.7 MeV and 48.3 ± 5.2 MeV, respectively, while the branching ratio is

$$\mathrm{BR}(J/\psi \to \gamma\eta(1405) \to \gamma f_0(980)\pi^0 \to \gamma\pi^+\pi^-\pi^0) = (1.50 \pm 0.11 \pm 0.11) \times 10^{-5}. \quad (1)$$

Comparing the above with the result of Particle Data Group (PDG) [33],

$$\mathrm{BR}(J/\psi \to \gamma\eta(1405/1475) \to \gamma K\bar{K}\pi) = (2.8 \pm 0.6) \times 10^{-3}, \quad (2)$$

one gets

$$\frac{\mathrm{BR}(J/\psi \to \gamma\eta(1405) \to \gamma f_0(980)\pi^0 \to \gamma\pi^+\pi^-\pi^0)}{\mathrm{BR}(J/\psi \to \gamma\eta(1405/1475) \to \gamma K\bar{K}\pi)} = (0.53 \pm 0.13)\%. \quad (3)$$

The magnitude of this ratio tells us about the very large breaking of the isotopic invariance in the decay $\eta(1405) \to f_0(980)\pi^0$. Guided by naive considerations, this ratio is expected to be at the level of $[(m_{K^0} - m_{K^+})/m_{K^0}]^2 \lesssim 10^{-4}$. Notice that, in Eq. (3), the magnitude of the forbidden by isotopic invariance decay $\eta(1405) \to f_0(980)\pi^0$ is compared to the magnitude of the main allowed decay $\eta(1405/1475) \to K\bar{K}\pi$ [33–36].

To illustrate the observed breaking of isotopic invariance, the BESIII Collaboration [28] gives the ratio

$$\frac{\mathrm{BR}(\eta(1405) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0)}{\mathrm{BR}(\eta(1405) \to a_0^0(980)\pi^0 \to \eta\pi^0\pi^0)} = (17.9 \pm 4.2)\%. \quad (4)$$

However, it is large in comparison with Eq. (3) due only to the fact that the isospin-allowed transition $\eta(1405/1475) \to a_0^0(980)\pi \to \eta\pi^0\pi^0$ is small. Really, using the PDG branching ratio $J/\psi \to \gamma\eta(1405/1475) \to \gamma\eta\pi^+\pi^-)$ [37] and the largest PDG value of $\Gamma(\eta(1405) \to a_0(980)\pi)/\Gamma(\eta(1405) \to \eta\pi\pi)$ [38], the BESIII Collaboration [28] estimated $\mathrm{BR}(J/\psi \to \gamma\eta(1405) \to \gamma a_0^0(980)\pi^0 \to \gamma\eta\pi^0\pi^0) = (8.40 \pm 1.75) \times 10^{-5}$. So the ratio Eq. (4) is an unreliable characteristic of the isospin violation.

In what follows we also use the notation $\iota \equiv \eta(1405)$ for brevity. Since the decay $\iota \to f_0(980)\pi^0$ is measured in the radiative decay of the J/ψ meson, then, when analyzing the situation, it is natural to base the treatment on the information about the decays $J/\psi \to \gamma\iota \to \gamma K\bar{K}\pi, \gamma\eta\pi\pi$ [33–36,39–48]. However, this information is rather scarce. The matters are further complicated by the fact that the data [39–48] refer to the decays $J/\psi \to \gamma\eta(1405/1475) \to \gamma K\bar{K}\pi, \gamma\eta\pi\pi$, in which the resonance structure $\eta(1405/1475)$ [33–36] may correspond to some mixture of the overlapping states $\eta(1405)$ and $\eta(1475)$ [it is called sometimes $\eta(1440)$ in current literature]. In the meantime, there is no single established opinion concerning the reality of two pseudoscalars and the dynamics of the decays $\eta(1405/1475) \to K\bar{K}\pi$ and $\eta\pi\pi$ [30–36].

III. THE DECAY $\iota \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi^0 \to \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$

If the ι decays to $(K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi$ (see Fig. 1), then, due to the final state interaction among K and $\bar{K}$ mesons, i.e., due to the transitions $K^+K^- \to f_0(980) \to \pi^+\pi^-$ and $K^0\bar{K}^0 \to f_0(980) \to \pi^+\pi^-$, the isospin breaking decay $\iota \to (K^*\bar{K} + \bar{K}^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ is induced (see Fig. 2). It should be mentioned that here we consider the effect of the isospin

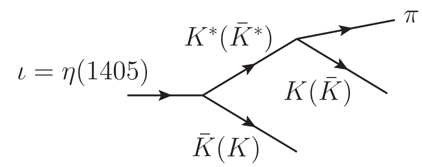

FIG. 1. The diagram of the decay $\iota \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi$.

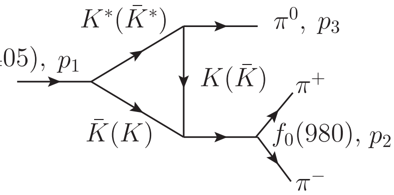

FIG. 2. The diagram of the decay $\iota \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ via the $K^*\bar{K} + \bar{K}^*K$ intermediate states; p_1, p_2, p_3 stand for the 4-momenta of particles participating in the reaction, $p_1^2 = s_1$ being the invariant mass squared of the ι resonance or of the final $\pi^+\pi^-\pi^0$ system; $p_2^2 = s_2 = m^2_{\pi^+\pi^-}$ is the invariant mass squared of the $f_0(980)$ or of the final $\pi^+\pi^-$ system, $p_3^2 = m^2_{\pi^0}$.

violation in the decay $\iota \to \pi^+\pi^-\pi^0$ as being due solely to the mass difference of the stable charged and neutral K mesons. The contributions from the production of the K^+K^- and $K^0\bar{K}^0$ pairs are not compensated completely. The smallest compensation among them should naturally take place at the invariant mass of the $\pi^+\pi^-$ system, $\sqrt{s_2}$, in the region between the K^+K^- and $K^0\bar{K}^0$ thresholds. However, there is some complexity in the present case. The fact is that just in the region of the ι resonance all intermediate particles in the loop of triangle diagram in Fig. 2, at the definite values of the kinematic variables $\sqrt{s_1}$ and $\sqrt{s_2}$, can lie on their mass shells. This means that in the hypothetic case of the stable K^* meson the logarithmic singularity appears in the imaginary part of the triangle diagram [49–51]. Figure 3 shows the location of the logarithmic singularities for the contributions of the $K^{*+}K^-$ and $K^{*0}\bar{K}^0$ intermediate states. As is seen, in the ι resonance region, they are located very close to the $K\bar{K}$ thresholds. For example, at $\sqrt{s_1} = 1.420$ GeV, the singularities from the $K^{*+}K^-$ and $K^{*0}\bar{K}^0$ intermediate state contributions in the $\pi^+\pi^-$ mass spectrum take place at $\sqrt{s_2} = 0.989$ and 0.998 GeV, respectively (see Fig. 3). Since the singularities located at different positions from the charged and neutral intermediate states do not compensate each other, the considered mechanism may seem to result in a catastrophic violation of isotopic symmetry in the decay $\iota \to \pi^+\pi^-\pi^0$. However, the accounting of the finite width of the K^* resonance, i.e., the averaging of the amplitude over the resonance Breit-Wigner distribution in accord with the spectral Källén-Lehmann representation for the propagator of the unstable K^* meson [49–51], smoothes the logarithmic singularities of the amplitude and hence makes the compensation of the contributions of the $K^{*+}K^- + K^{*-}K^+$ and $K^{*0}\bar{K}^0 + \bar{K}^{*0}K^0$ intermediate states more strong [52]. This results in both the diminishing of the calculated width of the decay $\iota \to \pi^+\pi^-\pi^0$ by a number of times in comparison with the case of $\Gamma_{K^*\to K\pi} = 0$, and in the concentration of the main effect of the isospin breaking in the domain of the $\pi^+\pi^-$ invariant mass between the $K\bar{K}$ thresholds. Figures 4–5 show the influence of allowing for the instability of K^* on the energy dependent width

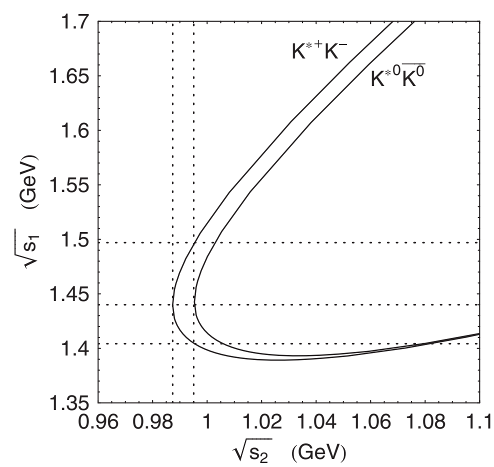

FIG. 3. Solid curves on the plane $(\sqrt{s_2}, \sqrt{s_1})$ show the location of the logarithmic singularity of the imaginary part of the triangle diagram shown in Fig. 2, in the case of the $K^{*+}K^-$ and $K^{*0}\bar{K}^0$ intermediate states. The dashed vertical lines show the K^+K^- and $K^0\bar{K}^0$ thresholds in the variable $\sqrt{s_2}$ (i.e., its values equal to $2m_{K^+} = 0.987354$ and $2m_{K^0} = 0.995344$ GeV). The dashed horizontal lines correspond to the values of the variable $\sqrt{s_1}$ equal to 1.404, 1.440, and 1.497 GeV. At 1.404 GeV $< \sqrt{s_1} <$ 1.497 GeV the logarithmic singularity, in the case of the $K^{*+}K^-$ intermediate state, is located at the values of $\sqrt{s_2}$ between the K^+K^- and $K^0\bar{K}^0$ thresholds, while in the case of the $K^{*0}\bar{K}^0$ intermediate state it does not go away from the $K^0\bar{K}^0$ threshold by farther than 6 MeV. At approximately $\sqrt{s_1} = 1.440$ GeV, the singularities reach the $K\bar{K}$ thresholds.

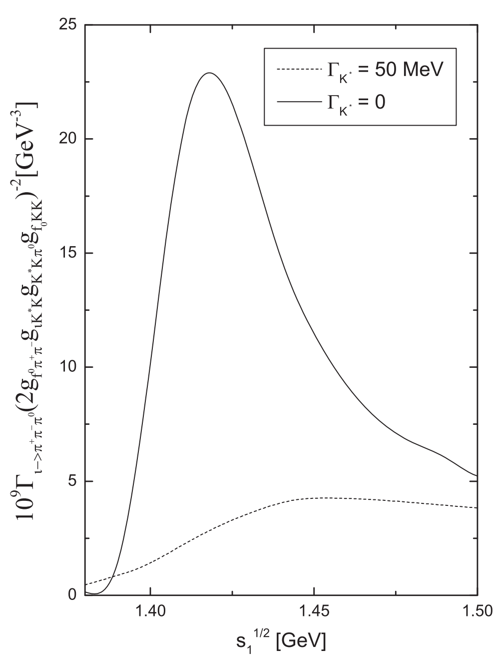

FIG. 4. The illustration of the influence of instability of the intermediate K^* meson on the calculated width of the decay $\iota \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$.

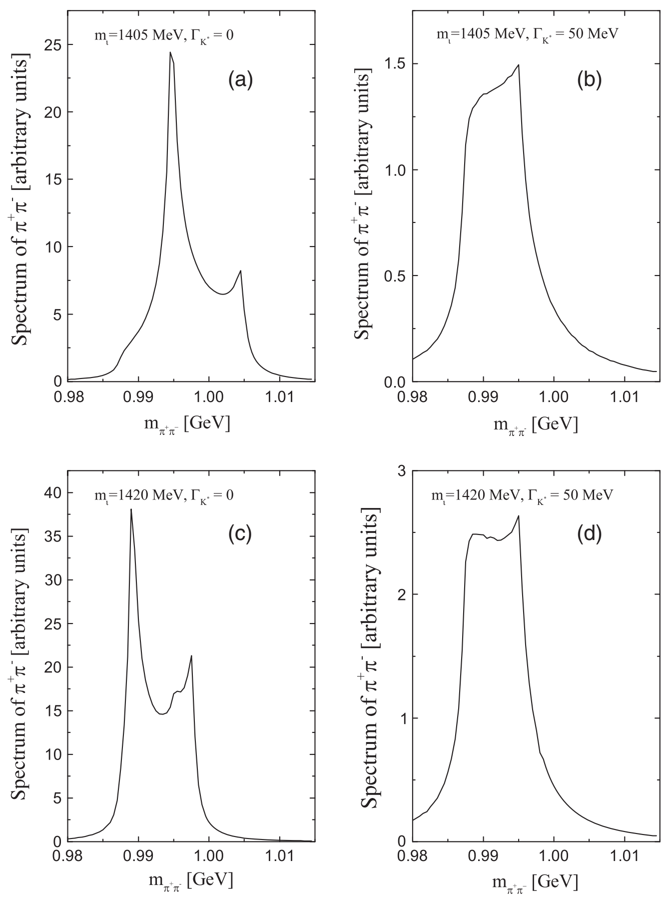

FIG. 5. The illustration of the influence of instability of the intermediate K^* meson on the $\pi^+\pi^-$ mass spectra in the decay $\iota \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$. The units are arbitrary but nevertheless the same for all $\pi^+\pi^-$ mass spectra in (a)–(d).

$\Gamma_{\iota\to\pi^+\pi^-\pi^0}(s_1)$ and on the mass spectra of the $\pi^+\pi^-$ system, $d\Gamma_{\iota\to\pi^+\pi^-\pi^0}(s_1, s_2)/d\sqrt{s_2}$, $\sqrt{s_2} = m_{\pi^+\pi^-}$. Figure 4 shows that in the region $1.400\ \text{GeV} < \sqrt{s_1} < 1.425\ \text{GeV}$ the calculated width of the decay $\iota \to \pi^0\pi^+\pi^-$ is lowered by the factor of 6–8. The $\pi^+\pi^-$ mass spectra, see Fig. 5, are distorted strongly. Notice that the nonzero experimental resolution in the $\pi^+\pi^-$ mass (in the BESIII experiment [28] it was about 2 MeV) would smooth the peaks in the domain of singularity in Figs. 5(a) and 5(c), but the area under the curves would remain practically the same.

Shown in Fig. 6 is the behavior of the $\iota \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ and $\iota \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi$ decay widths against the invariant mass $\sqrt{s_1}$ of the ι resonance calculated at $\Gamma_{K^*} = 50$ MeV. Both widths demonstrate the strong dependence on $\sqrt{s_1}$. The ratio of these widths is an important characteristic of the violation of the isotopic invariance in the considered model. It does not depend on the magnitude of the ι coupling with $K^*\bar{K}$ ($g_{\iota K^*\bar{K}}$), and its order of magnitude is controlled by the factor $[(m_{K^0} - m_{K^+})/m_{K^0}] \times (g^2_{f_0K^+K^-}/g^2_{f_0\pi^+\pi^-})$ and decay

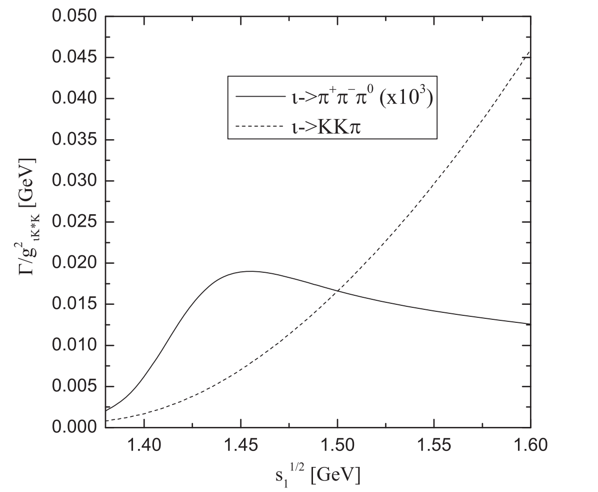

FIG. 6. The dependence of the $\iota \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ and $\iota \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi$ decay widths on the invariant mass of the ι resonance $\sqrt{s_1}$ ($\Gamma_{K^*} = 50$ MeV).

kinematics. For the ratio of the widths in Fig. 6 averaged over the region $1.400\ \text{GeV} < \sqrt{s_1} < 1.425\ \text{GeV}$, one has

$$R = \frac{\bar{\Gamma}_{\iota\to\pi^+\pi^-\pi^0}}{\bar{\Gamma}_{\iota\to K\bar{K}\pi}} \equiv \frac{\langle\Gamma_{\iota\to\pi^+\pi^-\pi^0}(s_1)\rangle}{\langle\Gamma_{\iota\to K\bar{K}\pi}(s_1)\rangle} \approx 4\times 10^{-3}. \qquad (5)$$

Now, using Eqs. (2) and (5) for evaluation of $\mathrm{BR}(J/\psi \to \gamma\iota \to \gamma f_0(980)\pi^0 \to \gamma\pi^+\pi^-\pi^0)$, one obtains

$$\begin{aligned} &\mathrm{BR}(J/\psi \to \gamma\iota \to \gamma f_0(980)\pi^0 \to \gamma\pi^+\pi^-\pi^0) \\ &\quad\approx R \times \mathrm{BR}(J/\psi \to \gamma\eta(1405/1475) \to \gamma K\bar{K}\pi) \\ &\quad\approx 1.12\times 10^{-5}, \end{aligned} \qquad (6)$$

in agreement with the data of BESIII [28] given in Eq. (1).

The estimate Eq. (6) includes the assumption of dominance of the $\eta(1405/1475) \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi$ mechanism in the decay $\eta(1405/1475) \to K\bar{K}\pi$ to be discussed below. Moreover, in view of the absence of the detailed data, one forcedly assumes that ι ($\eta(1405)$), $\eta(1440)$, and the resonance complex $\eta(1405/1475)$ constitute the single object looking differently in various channels. Hence, the magnitude of $\mathrm{BR}(J/\psi \to \gamma\iota \to \gamma f_0(980)\pi^0 \to \gamma\pi^+\pi^-\pi^0)$ given by Eq. (6) should be considered in the present model as the upper estimate. See also remarks in Ref. [53].

IV. THE DECAY $\iota \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi$

Guided by the data about the resonance complex $\eta(1405/1475)$ produced in the radiative decays of the J/ψ meson one can conclude that it decays to $K\bar{K}\pi$ with the probability of about 80–90% [33–36,39–48]. The information about the contribution of the $K^*\bar{K}+\bar{K}^*K$, $a_0(980)\pi$, $\kappa(800)\bar{K}+\bar{\kappa}(800)K$ intermediate states to the decay $\eta(1405/1475)\to K\bar{K}\pi$ is contained in two-particle mass spectra of the states $K\bar{K}$, $K\pi$, and $\bar{K}\pi$. Available statistics of the $K\bar{K}\pi$ events are not sufficient [39–45], so the quality of the data does not permit one to reliably isolate the possible contributions. To a very rough approximation it is assumed [33–36] that the decay $\eta(1405/1475)\to K\bar{K}\pi$ in the vicinity of 1475 MeV proceeds mainly via the $K^*\bar{K}+\bar{K}^*K$ state. As for the region of 1405 MeV, it is considered that it can proceed via the $a_0(980)\pi$ state [33–36], though the admixture of the $K^*\bar{K}+\bar{K}^*K$ channel and even its dominance are discussed too [33–36,43]. If, nevertheless, one admits dominance of the $a_0(980)\pi$ channel, then it would be natural to expect a rather sizeable signal from the decay $\iota\to a_0(980)\pi\to\eta\pi\pi$ [$a_0(980)$ resonance is located near the $K\bar{K}$ threshold and decays more intensively into $\eta\pi$ than into $K\bar{K}$]. In experiments, the decay $J/\psi\to\gamma\iota\to\gamma\eta\pi\pi$ is seen [33–36,41,43,46–48], but it is small. See the next section concerning this fact. One can definitely state that the pointlike mechanism of the decay $\iota\to K\bar{K}\pi$ does not describe the data. So the assumption of the dominance of the decay $\iota\to(K^*\bar{K}+\bar{K}^*K)\to K\bar{K}\pi$ cannot be rejected as yet. The high statistics experimental studies of the basic decay channels $\iota\to K\bar{K}\pi$ and $\iota\to\eta\pi\pi$ are necessary for elucidation of the situation.

In connection with the $\iota\to(K^*\bar{K}+\bar{K}^*K)\to K\bar{K}\pi$ decay dominance we also want to pay attention to the difficulty of using the simplest Breit-Wigner expressions for the description of the ι resonance. For example, let us take the recent BES data [45] on the $K\bar{K}\pi$ spectrum in the decay $J/\psi\to\gamma\eta(1440)\to\gamma K\bar{K}\pi$, see Fig. 7, and fit them with the help of the standard expression

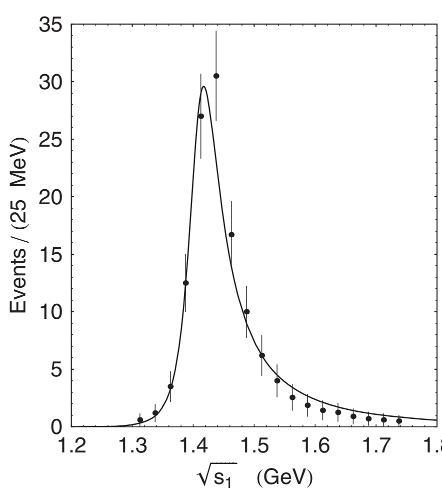

FIG. 7. The $K\bar{K}\pi$ mass spectrum in the decay $J/\psi\to\gamma\iota\to\gamma K\bar{K}\pi$ as the function of the invariant mass of the ι resonance $\sqrt{s_1}$. Points with error bars are the BES data [45]. The curve is obtained in the $\iota\to(K^*\bar{K}+\bar{K}^*K)\to K\bar{K}\pi$ decay model. See the main text for more detail.

$$\frac{dN}{dm}=A(1-m^2/m^2_{J/\psi})^3\mathrm{BR}(\iota\to K\bar{K}\pi;m),\qquad(7)$$

where $m\equiv\sqrt{s_1}$, and

$$\mathrm{BR}(\iota\to K\bar{K}\pi;m)=\frac{2m}{\pi}\frac{m\Gamma_{\iota\to K\bar{K}\pi}(m)}{|m_\iota^2-m^2-im\Gamma_\iota^{\mathrm{tot}}(m)|^2}.\qquad(8)$$

In the case of the total dominance of the $K^*\bar{K}+\bar{K}^*K$ channel, i.e., when

$$\Gamma_\iota^{\mathrm{tot}}(m)=\Gamma_{\iota\to K\bar{K}\pi}(m)=\Gamma_{\iota\to(K^*\bar{K}+\bar{K}^*K)\to K\bar{K}\pi}(m),\quad(9)$$

the fit, shown in Fig. 7 with the solid line, gives $\chi^2/\mathrm{n.d.f.}=10/15$, $A=20$, $m_\iota=1.465$ GeV, and $g_{\iota K^{*+}K^-}=6.91$ [hence $\Gamma_{\iota\to(K^*\bar{K}+\bar{K}^*K)\to K\bar{K}\pi}(m_\iota)=448$ MeV, but the visible width of the peak is essentially lower]. Our normalization is such that in the case of the stable K^* meson the coupling constant $g_{\iota K^{*+}K^-}$ is related with the $\iota\to K^*\bar{K}+\bar{K}^*K$ decay width in accord with the expression

$$\Gamma_{\iota\to K^*\bar{K}+\bar{K}^*K}=\frac{g^2_{\iota K^{*+}K^-}}{4\pi}\frac{8p_K^3}{m^2_{K^*}},\qquad(10)$$

where p_K stands for the momentum of the K meson in the ι rest frame. If one evaluates the total $\iota\to K\bar{K}\pi$ decay probability than instead of the expected value close to 1 one would get

$$\mathrm{BR}(\iota\to K\bar{K}\pi)=\int_{1.3\ \mathrm{GeV}}^{3\ \mathrm{GeV}}\mathrm{BR}(\iota\to K\bar{K}\pi;m)dm\approx0.34.\qquad(11)$$

The reason for this violation of the normalization is the sharp P wave growth of $\Gamma_{\iota\to(K^*\bar{K}+\bar{K}^*K)\to K\bar{K}\pi}(m)$ with increasing m (see Fig. 6).

Recall that, in the case of the scalar mesons $\sigma(600)$, $a_0(980)$, $f_0(980)$, their propagators obtained upon taking into account the finite width corrections, satisfying the Källén-Lehmann representation and, due to this fact, preserve the total decay probability normalization to unity [54,55]; see also Ref. [56]. Unfortunately, we have not yet succeeded in constructing the propagator for the ι resonance, providing the desired normalization to unity, as in the case of scalar mesons.

So one can conclude that the fittings of the data on the ι resonance and the results of the determination of its parameters from seemingly natural expressions should be considered as tentative guesses.

V. THE DECAY $\iota\to a_0^0(980)\pi^0\to f_0(980)\pi^0\to\to\pi^+\pi^-\pi^0$

The decay $\iota\to\pi^+\pi^-\pi^0$ can also proceed due to the $a_0^0(980)-f_0(980)$ mixing [1]: $\iota\to a_0^0(980)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$. As a result, the $\pi^+\pi^-$ mass spectrum is sharply enhanced in the region between the K^+K^- and $K^0\bar{K}^0$ thresholds and looks very similar to the spectra shown in Figs. 5(b) and 5(d). However, it is difficult, with the help of this mechanism, to obtain the magnitude of $\mathrm{BR}(J/\psi \to \gamma\iota \to \gamma\pi^+\pi^-\pi^0)$ close to the experimental value Eq. (1).

Let us take the data about the $a_0^0(980) - f_0(980)$ mixing obtained by BESIII [27],

$$\xi_{af} = \frac{\Gamma_{a_0^0 \to f_0 \to \pi^+\pi^-}}{\Gamma_{a_0^0 \to \eta\pi^0}} = (0.31 \pm 0.16 \pm 0.143)\%. \quad (12)$$

Notice that the upper limit on ξ_{af} is 1.0% at 90% confidence level [27]. Let us also base the consideration on the magnitude

$$\begin{aligned}&\mathrm{BR}(J/\psi \to \gamma\iota \to \gamma\eta\pi^+\pi^-)\\ &\quad = \mathrm{BR}(J/\psi \to \gamma\eta(1405/1475) \to \gamma\eta\pi^+\pi^-)\\ &\quad = (3.0 \pm 0.5) \times 10^{-4} \end{aligned} \quad (13)$$

[33], and let us consider the decay $\iota \to \eta\pi^+\pi^-$ as proceeding via the $(a_0^+(980)\pi^- + a_0^-(980)\pi^+)$ intermediate states. Then one obtains for $\mathrm{BR}(J/\psi \to \gamma\iota \to \gamma\pi^+\pi^-\pi^0)$:

$$\begin{aligned}&\mathrm{BR}(J/\psi \to \gamma\iota \to \gamma\pi^+\pi^-\pi^0)\\ &\quad = \frac{\xi_{af}}{2}\mathrm{BR}(J/\psi \to \gamma\iota \to \gamma\eta\pi^+\pi^-)\\ &\quad \approx (4.5 \pm 3.3) \times 10^{-7}. \end{aligned} \quad (14)$$

The central value in Eq. (14) is by approximately 30 times lower than the central value given by Eq. (1). However, the experimental uncertainties of the data on ξ_{af} are large, and one needs additional measurements to make definite conclusions.

In general, the suppression of the decay $J/\psi \to \gamma\iota \to \gamma\eta\pi^+\pi^-$ as compared with the $J/\psi \to \gamma\iota \to \gamma K\bar{K}\pi$ one [33] is not directly related with the smallness of the $\iota \to a_0(908)\pi$ decay probability. Hence, the branching ratio $\mathrm{BR}(J/\psi \to \gamma\iota \to \gamma\pi^+\pi^-\pi^0)$, caused by the $a_0^0(980) - f_0(980)$ mixing mechanism, can be few times greater than that given in Eq. (14). The fact is that the $a_0(980)\pi$ intermediate state in the $\iota \to \eta\pi\pi$ decay channel can be hidden due to the destructive interference with other contributions. As our estimates show, the interference between $a_0(980)\pi$ and $\sigma(600)\eta$ intermediate states can reduce the probability of the decay $\iota \to \eta\pi\pi$ by the factor of about 1.5; see also Ref. [57]. Besides, the S wave $\pi\pi$ final state interaction in the decay $\iota \to a_0(980)\pi \to \eta\pi\pi$ is capable of suppressing its width by the factor of approximately two. The possible influence of this interaction on the $\pi\pi$ mass spectrum in the decay $\iota \to \eta\pi\pi$ is shown in Fig. 8. So the estimate Eq. (14) can be enhanced by the factor of approximately three. If such a possibility is realized, it would mean that the contribution of the $a_0^0(980) - f_0(980)$ mixing mechanism can provide up to 30% of the $\iota \to \pi^+\pi^-\pi^0$ decay amplitude.

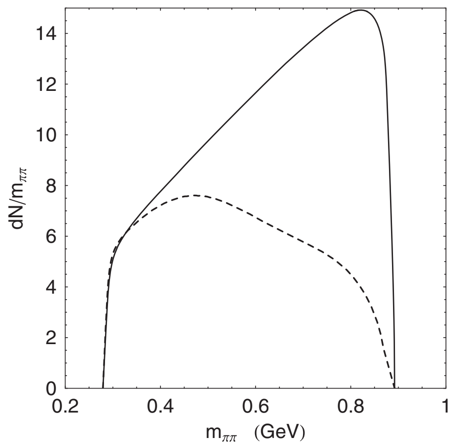

FIG. 8. The $\pi\pi$ mass spectrum in the decay $\iota \to \eta\pi\pi$. The solid curve corresponds to the $\iota \to a_0(980)\pi \to \eta\pi\pi$ decay mechanism. The dashed curve corresponds to the decay via the $a_0^{\pm,0}(980)\pi^{\mp,0}$ intermediate states upon taking into account the S wave $\pi\pi$ final state interaction.

The high statistics experimental investigations on both the form of the mass spectrum of the ι resonance in the $\eta\pi\pi$ decay channel and the $\eta\pi$ and $\pi\pi$ subsystem mass spectra in the region of ι peak could elucidate considerably the production dynamics and the role of the $a_0(980)\pi$ intermediate state.

VI. DETAILS OF CALCULATIONS

To estimate the effect, we use the following expression for the propagator of stable K^* meson:

$$\frac{g_{\mu\nu} - k_\mu k_\nu/k^2}{m_{K^*}^2 - k^2 - i\varepsilon}. \quad (15)$$

It preserves the conservation of the unit spin in the presence of interaction and the convergence of the triangle diagram in Fig. 2 for the intermediate states with the specific charge. It should be stressed that the convergence or divergence of the triangle diagram as well as of the $K\bar{K}$ loops in the case of the $a_0^0(980) \to (K^+K^- + K^0\bar{K}^0) \to f_0(980)$ transition is not related with the effect under discussion. The sum of the subtraction constants for the contributions of the charged and neutral intermediate states in the dispersion representation for the isospin breaking amplitude should have the natural order of smallness $\sim(m_{K^0} - m_{K^+})$, and it cannot be responsible for the enhancement of the symmetry violation in the vicinity of the K^+K^- and $K^0\bar{K}^0$ thresholds.

The contribution of the triangle diagram in Fig. 2, divided by the product of coupling constants $g_{\iota K^* K} g_{K^* K\pi^0} g_{f_0 KK}$ is

$$T \equiv T(s_1, s_2, m^2, m_K^2) = \int \frac{d^4 p_K}{(2\pi)^4} \cdot \frac{(p_1 + p_{\bar{K}})(p_\pi - p_K) - (s_1 - p_{\bar{K}}^2)(m_\pi^2 - p_K^2)/p_{K^*}^2}{(m^2 - p_{K^*}^2 - i\varepsilon)(m_K^2 - p_K^2 - i\varepsilon)(m_K^2 - p_{\bar{K}}^2 - i\varepsilon)}, \tag{16}$$

where s_1, s_2, m^2 are, respectively, the invariant masses squared of ι, f_0, and K^*. The numerator of the integrand contains polynomials $(m^2 - p_{K^*}^2)$, $(m_K^2 - p_K^2)$, and $(m_K^2 - p_{\bar{K}}^2)$, which cancel some poles in the denominator. Hence the expression for T reduces to the sum of terms with three and two propagators each treated using the Feynman parametrization:

$$\frac{1}{a_1 a_2 a_3} = 2\int_0^1 dx_1 \int_0^{x_1} \frac{dx_2}{[a_1 x_2 + a_2(x_1 - x_2) + a_3(1 - x_1)]^3},$$
$$\frac{1}{a_1 a_2} = \int_0^1 \frac{dx}{[a_1 x + a_2(1 - x)]^2}.$$

After integration over p_K the logarithmic divergences in the two-propagator contributions cancel, and the resulting expression can be represented in the following form:

$$\begin{aligned} T = \frac{1}{16\pi^2}\Bigg\{ & \left[s_1 + m_\pi^2 - m^2 + 2(m_K^2 - s_2) + \frac{(s_1 - m_K^2)(m_K^2 - m_\pi^2)}{m^2}\right] C_3(s_1, s_2, m^2, m_K^2) \\ & - \frac{(s_1 - m_K^2)(m_K^2 - m_\pi^2)}{m^2} \cdot C_3(s_1, s_2, 0, m_K^2) + \frac{s_1 - m_K^2}{m^2}[C_2(s_1, m^2, m_K^2) - C_2(s_1, 0, m_K^2)] \\ & + \frac{m_\pi^2 - m_K^2}{m^2}[C_2(m_\pi^2, m^2, m_K^2) - C_2(m_\pi^2, 0, m_K^2)] - C_2(s_1, m^2, m_K^2) - C_2(m_\pi^2, m^2, m_K^2) \\ & + C_1(s_2, m_K^2) + \ln m^2 - 1 \Bigg\}. \end{aligned} \tag{17}$$

Here,

$$\begin{aligned} C_1(s_2, m_K^2) &= \int_0^1 dx \ln[m_K^2 - s_2 x(1 - x) - i\varepsilon], \\ C_2(s_1, m^2, m_K^2) &= \int_0^1 dx \ln[(1 - x)(m_K^2 - s_1 x) + m^2 x - i\varepsilon], \end{aligned} \tag{18}$$

$$C_3(s_1, s_2, m^2, m_K^2) = \int_0^1 dx_1 \int_o^{x_1} \frac{dx_2}{m_K^2 + x_2(m^2 - m_K^2) - (x_1 - x_2)[s_2(1 - x_1) + m_\pi^2 x_2] - s_1 x_2(1 - x_2) - i\varepsilon}. \tag{19}$$

We use the analytical expression for C_1 and C_2, while C_3 is evaluated numerically. Note that in the kinematical region of our interest the net contribution from the two-propagator terms $C_{1,2}$ is negligible in comparison with the pure triangle contribution $\propto C_3$, where all three poles are essential. The knowledge of the explicit imaginary parts of the amplitude (the discontinuities on the $K^*\bar{K}$, $\bar{K}^*K$, and $K\bar{K}$ cuts) permits one to control the result of numerical evaluations. In the case of one of the four charge modes they look like

$$\begin{aligned} \mathrm{Im} g_{\iota f_0 \pi}^{(K^{*+}K^-)}(m^2) = \frac{1}{2i}\mathrm{Disc}_{K^{*+}K^-}(m^2) = \frac{g_{\iota K^{*+}K^-} g_{K^{*+}K^+\pi^0} g_{f_0 K^+ K^-}}{32\pi\sqrt{s_1}|p_\pi|} & \left\{-4|p_\pi||p_K|\left(1 + \frac{s_1 - m_{K^-}^2}{m^2}\right)\right. \\ & \left. + \left[s_1 + m_\pi^2 + 2m_{K^-}^2 - m^2 - 2s_2 + \frac{(s_1 - m_{K^-}^2)(m_{K^-}^2 - m_\pi^2)}{m^2}\right] \ln\frac{a_{K^{*+}K^-} + 1 + i\varepsilon}{a_{K^{*+}K^-} - 1 + i\varepsilon}\right\}, \end{aligned} \tag{20}$$

where $a_{K^{*+}K^-} = (2E_{f_0}E_{K^-} - s_2)/(2|p_\pi||p_{K^-}|)$, $E_{f_0} = (s_1 + s_2 - m_\pi^2)/(2\sqrt{s_1})$, $E_{K^-} = (s_1 + m_{K^-}^2 - m^2)/(2\sqrt{s_1})$, $|p_\pi| = |p_{f_0}| = \sqrt{E_{f_0}^2 - s_2}$, $|p_{\bar{K}}| = \sqrt{E_{K^-}^2 - m_{K^-}^2}$ (here, the mass m of the K^* meson is not fixed to be m_{K^*});

$$\mathrm{Im}g^{(K^+K^-)}_{\iota f_0\pi}(m^2) = \frac{1}{2i}\mathrm{Disc}_{K^+K^-}(m^2) = \frac{g_{\iota K^{*+}K^-}g_{K^{*+}K^+\pi^0}g_{f_0K^+K^-}}{32\pi\sqrt{s_2}|p'_\pi|}\left\{4|p'_\pi||p'_K| + \left[s_1 + m^2_\pi + 2m^2_{K^-} - m^2 - 2s_2 + \frac{(s_1 - m^2_{K^-})(m^2_{K^-} - m^2_\pi)}{m^2}\right]\ln\frac{a_{K^+K^-} + 1 - i\varepsilon}{a_{K^+K^-} - 1 - i\varepsilon} - \frac{(s_1 - m^2_{K^-})(m^2_{K^-} - m^2_\pi)}{m^2}\ln\frac{a^{(0)}_{K^+K^-} + 1}{a^{(0)}_{K^+K^-} - 1}\right\}, \tag{21}$$

where $a_{K^+K^-} \equiv a_{K^+K^-}(m^2) = -(2E'_\pi E'_{K^-} + m^2_{K^-} + m^2_\pi - m^2)/(2|p'_\pi||p'_{K^-}|)$; $E'_\pi = (s_1 - s_2 - m^2_\pi)/(2\sqrt{s_2})$, $E'_{K^-} = \sqrt{s_2}/2$, $|p'_\pi| = \sqrt{E'^2_\pi - m^2_\pi}$, $|p'_{K^-}| = \sqrt{E'^2_{K^-} - m^2_{K^-}}$; $a^{(0)}_{K^+K^-} = a_{K^+K^-}(m^2 = 0)$.

The Lorenz transformation from the ι rest frame to f_0 gives the relation $\sqrt{s_1}|p_\pi| = \sqrt{s_2}|p'_\pi|$, so that the coefficients in front of two logarithms originating from the $K^*\bar{K}$, Eq. (20), and $K\bar{K}$, Eq. (21), cuts are coincident. Hence, in the kinematical region where imaginary parts of these logarithms appear, they cancel each other due to different signs in front of ε. The logarithm with $a^{(0)}_{K^+K_-}$ is explicitly real. So the imaginary part of the coupling constant

$$\mathrm{Im}g_{\iota f_0\pi}(m^2) = \frac{1}{2i}[\mathrm{Disc}_{K^{*+}K^-}(m^2) + \mathrm{Disc}_{K^+K^-}(m^2)] \tag{22}$$

is real. We have verified that the imaginary part of the numerically evaluated triangle diagram coincides with the evaluation of the analytically calculated one.

To account for the effect of the finite K^* width, we write the propagator of the unstable K^* meson in the form of the spectral Källén-Lehmann representation [49–51]

$$\frac{1}{m^2_{K^*} - p^2_{K^*} - im_{K^*}\Gamma_{K^*}} \to \int_{(m_K+m_\pi)^2}^{\infty} dm^2 \frac{\rho(m^2)}{m^2 - p^2_{K^*} - i\varepsilon} \tag{23}$$

and approximate $\rho(m^2)$ in the following way:

$$\rho(m^2) = \frac{1}{\pi}\frac{m_{K^*}\Gamma_{K^*}}{(m^2 - m^2_{K^*})^2 + (m_{K^*}\Gamma_{K^*})^2}. \tag{24}$$

Then, instead of amplitude $T \equiv T(s_1, s_2, m^2, m^2_K)$ from Eq. (16), we have the amplitude $\langle T\rangle$ weighted with the spectral density $\rho(m^2)$ [49–51],

$$\langle T\rangle = \int_{(m_K+m_\pi)^2}^{\infty} \rho(m^2)T(s_1, s_2, m^2, m^2_K)dm^2. \tag{25}$$

This integration eliminates the logarithmic infinities in the imaginary part of the triangle diagram. Notice that the contributions of the discontinuities on the $K^*\bar{K}$ and $\bar{K}^*K$ cuts in the s_1 channel are caused by the real three-body intermediate states $K\pi\bar{K}$ and $\bar{K}\pi K$, respectively. At the same time, the discontinuities of the triangle diagram in the s_2 channel correspond to the two-body intermediate states $K\bar{K}$.

The amplitude of the subprocess $K^+K^- \to f_0(980) \to \pi^+\pi^-$ (or $K^0\bar{K}^0 \to f_0(980) \to \pi^+\pi^-$), being a part of the amplitude of the diagram in Fig. 2, is taken in the form

$$f_S(s_2) = \frac{g_{f_0K^+K^-}g_{f_0\pi^+\pi^-}}{16\pi}\frac{1}{D_{f_0}(s_2)}e^{i\varphi(s_2)}, \tag{26}$$

where $g_{f_0K^+K^-}$ ($= g_{f_0K^0\bar{K}^0}$) and $g_{f_0\pi^+\pi^-}$ ($= \sqrt{2}g_{f_0\pi^0\pi^0}$) are the coupling constants of $f_0(980)$ with K^+K^- ($K^0\bar{K}^0$) and $\pi^+\pi^-$ ($\pi^0\pi^0$), the phase of the background is $\varphi(s_2) \approx \pi/2$, and $1/D_{f_0}(s_2)$ stands for the $f_0(980)$ propagator [54], the expression of which takes into account the couplings of $f_0(980)$ with the $\pi\pi$ and $K\bar{K}$ channels and the corresponding finite width corrections,

$$\frac{1}{D_{f_0}(s_2)} = \frac{1}{m^2_{f_0} - s_2 + \sum_{ab}[\mathrm{Re}\Pi^{ab}_{f_0}(m^2_{f_0}) - \Pi^{ab}_{f_0}(s_2)]}. \tag{27}$$

Here, $\Pi^{ab}_{f_0}(s_2)$ is the polarization operator for the $f_0(980)$, corresponding to the contribution of the ab intermediate state ($ab = \pi^+\pi^-, \pi^0\pi^0, K^+K^-, K^0\bar{K}^0$); $\mathrm{Im}\Pi^{ab}_{f_0}(s_2)/\sqrt{s_2} = \Gamma_{f_0\to ab}(s_2) = g^2_{f_0ab}\rho_{ab}(s_2)/(16\pi)$ is the width of the $f_0(980) \to ab$ decay; in this case $m_a = m_b$, and for $s_2 > 4m^2_a$

$$\Pi^{ab}_{f_0}(s) = \frac{g^2_{f_0ab}}{16\pi}\rho_{ab}(s)\left[i - \frac{1}{\pi}\ln\frac{1 + \rho_{ab}(s_2)}{1 - \rho_{ab}(s_2)}\right], \tag{28}$$

where $\rho_{ab}(s_2) = \sqrt{1 - 4m_a/s_2}$; for $0 < s_2 < 4m^2_a$, $\rho_{ab}(s_2)$ should be replaced by $i|\rho_{ab}(s_2)|$ and

$$\Pi^{ab}_{f_0}(s_2) = -\frac{g^2_{f_0ab}}{16\pi}|\rho_{ab}(s_2)|\left[1 - \frac{2}{\pi}\arctan|\rho_{ab}(s_2)|\right]. \tag{29}$$

Our estimates are given for the following values: $m_{f_0} = 0.990$ GeV, $2g^2_{f_0K^+K^-}/(16\pi) = 0.4\ \mathrm{GeV}^2$, and $(3/2)g^2_{f_0\pi^+\pi^-}/(16\pi) = 0.1\ \mathrm{GeV}^2$. We have also tried different values of the $f_0(980)$ parameters, for instance, $m_{f_0} = 0.975$ GeV, $2g^2_{f_0K^+K^-}/(16\pi) = 0.5\ \mathrm{GeV}^2$, and $(3/2)g^2_{f_0\pi^+\pi^-}/(16\pi) = 0.1\ \mathrm{GeV}^2$ and verified that the results are not changed significantly.

For the example given in Fig. 8, the following values are used for the $a_0(980)$ resonance [54,58]: $m_{a_0} = 0.9847$ GeV, $2g^2_{a_0K^+K^-}/(16\pi) = 0.4\ \mathrm{GeV}^2$, and $g^2_{a_0\eta\pi} = g^2_{a_0\eta'\pi} = g^2_{a_0K^+K^-}$. To take into account the $\pi\pi$ final state interaction in the decay $\iota\to\eta\pi\pi$, the contribution of the amplitude $\iota\to a_0(908)\pi\to\eta(\pi\pi)_S$ is multiplied by the factor $[1+i\rho_{\pi\pi}(s)T_0^0(s)]=e^{i\delta_0^0(s)}\cos\delta_0^0(s)$, where $(\pi\pi)_S$ means the $\pi\pi$ system in S wave, $T_0^0(s)$ and $\delta_0^0(s)$ being, respectively, the amplitude and the phase of $\pi\pi$ scattering with the angular momentum $l=0$ and isospin $I=0$, s is the invariant mass squared of the $\pi\pi$ state. The data on $\delta_0^0(s)$ are approximated by the smooth curve [59,60].

The propagator of the $a_0(980)$ resonance with the invariant mass square s_2 is

$$\frac{1}{D_{a_0}(s_2)}=\frac{1}{m_{a_0}^2-s_2+\sum_{ab}[\mathrm{Re}\Pi_{a_0}^{ab}(m_{f_0}^2)-\Pi_{a_0}^{ab}(s_2)]},\tag{30}$$

where $ab=\pi\eta, K^+K^-, K^0\bar K^0, \pi\eta'$; $\mathrm{Im}\Pi_{a_0}^{ab}(s_2)/\sqrt{s_2}=\Gamma_{a_0\to ab}(s_2)=g_{a_0ab}^2\rho_{ab}(s_2)/(16\pi)$. For $s_2>m_{ab}^{(+)2}$ ($m_{ab}^{(\pm)}=m_b\pm m_a$, $m_b\ge m_a$), the polarization operator is given by [54,58]

$$\Pi_{a_0}^{ab}(s_2)=\frac{g_{a_0ab}^2}{16\pi}\left[\frac{m_{ab}^{(+)}m_{ab}^{(-)}}{\pi s_2}\ln\frac{m_a}{m_b}+\rho_{ab}(s_2)\times\left(i-\frac{1}{\pi}\ln\frac{\sqrt{s_2-m_{ab}^{(-)2}}+\sqrt{s_2-m_{ab}^{(+)2}}}{\sqrt{s_2-m_{ab}^{(-)2}}-\sqrt{s_2-m_{ab}^{(+)2}}}\right)\right],\tag{31}$$

where $\rho_{ab}(s_2)=\sqrt{s_2-m_{ab}^{(+)2}}\sqrt{s_2-m_{ab}^{(-)2}}\Big/s_2$, for $m_{ab}^{(-)2}<s_2<m_{ab}^{(+)2}$,

$$\Pi_{a_0}^{ab}(s_2)=\frac{g_{a_0ab}^2}{16\pi}\left[\frac{m_{ab}^{(+)}m_{ab}^{(-)}}{\pi s_2}\ln\frac{m_a}{m_b}-\rho_{ab}(s_2)\left(1-\frac{2}{\pi}\arctan\frac{\sqrt{m_{ab}^{(+)2}-s_2}}{\sqrt{s_2-m_{ab}^{(-)2}}}\right)\right],\tag{32}$$

where $\rho_{ab}(s_2)=\sqrt{m_{ab}^{(+)2}-s_2}\sqrt{s_2-m_{ab}^{(-)2}}/s_2$, and for $s_2\le m_{ab}^{(-)2}$

$$\Pi_{a_0}^{ab}(s_2)=\frac{g_{a_0ab}^2}{16\pi}\left[\frac{m_{ab}^{(+)}m_{ab}^{(-)}}{\pi s_2}\ln\frac{m_a}{m_b}-\rho_{ab}(s_2)\frac{1}{\pi}\ln\frac{\sqrt{m_{ab}^{(+)2}-s_2}+\sqrt{m_{ab}^{(-)2}-s_2}}{\sqrt{m_{ab}^{(+)2}-s_2}-\sqrt{m_{ab}^{(-)2}-s_2}}\right],\tag{33}$$

where $\rho_{ab}(s_2)=\sqrt{m_{ab}^{(+)2}-s_2}\sqrt{m_{ab}^{(-)2}-s_2}/s_2$.

VII. CONCLUSION

The phenomenon of the $a_0^0(980)-f_0(980)$ mixing [1] gave an impetus to conduct experiments of VES on the decay $f_1(1285)\to\pi^+\pi^-\pi^0$ [25,26] and BESIII on the decays $J/\psi\to\phi f_0(980)\to\phi a_0(980)\to\phi\eta\pi$, $\chi_{c1}\to a_0(980)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$ [27], and $J/\psi\to\gamma\eta(1405)\to\gamma f_0(980)\pi^0\to\gamma 3\pi$ [28]. We hope that the remarks presented here, on the mechanisms of the isospin breaking in the decay $\eta(1405)\to 3\pi$, will stimulate both the further studies of this decay and the principal improvement of the data about $K\bar K$, $K\pi$, $\eta\pi$, and $\pi\pi$ mass spectra in the decays of the resonance structure $\eta(1405/1475)$ into $K\bar K\pi$ and $\eta\pi\pi$, and about the shape of these resonance peaks in the $K\bar K\pi$ and $\eta\pi\pi$ channels.

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$$\left|\ln\left(\frac{\Gamma_{K^*}/2}{\sqrt{m^2_{K^0} - m^2_{K^+} + \Gamma^2_{K^*}/4}}\right)\right|.$$

This expression is not a result of calculations. It is constructed, so to speak, by hand to reflect the properties of this amplitude.

[53] As for the logarithmic singularity, higher-order corrections in the effective Lagrangian approach are reduced to renormalizations of the coupling constants and so are included in their physical values. The sum of the triangle diagrams with charged and neutral $K\bar{K}^* + \bar{K}K^*$ intermediate states gives the primary isospin violation contribution. Other isospin violation corrections are not essential since they are proportional to $m_{K^0} - m_{K^+}$.

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