PHYSICAL REVIEW D **92,** 034514 (2015)

Renormalization of gauge theories on general anisotropic lattices and high-energy scattering in QCD

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(Received 22 June 2015; published 26 August 2015)

We study the renormalization of $SU(N_c)$ gauge theories on general anisotropic lattices, to one-loop order in perturbation theory, employing the background field method. The results are then applied in the context of two different approaches to hadronic high-energy scattering. In the context of the Euclidean nonperturbative approach to soft high-energy scattering based on Wilson loops, we refine the nonperturbative justification of the analytic continuation relations of the relevant Wilson-loop correlators, required to obtain physical results. In the context of longitudinally rescaled actions, we study the consequences of one-loop corrections on the relation between the $SU(N_c)$ gauge theory and its effective description in terms of two-dimensional principal chiral models.

DOI: 10.1103/PhysRevD.92.034514 PACS numbers: 11.15.Ha, 11.10.Gh, 12.38.Bx, 13.85.-t

I. INTRODUCTION

Anisotropic lattices are a standard tool in modern lattice calculations, and have been used in the study of a large variety of problems, ranging from glueball [1] and light-hadron [2] spectroscopy to properties of QCD at finite temperature [3,4]. Numerical calculations in four dimensions usually employ lattices with $3+1$ anisotropy, i.e., only one of the lattice spacings is different from the others, while more general anisotropy classes have received much less attention [5], due to the increasing difficulty in the scale setting procedure. Indeed, for anisotropy classes other than $3+1$, one needs to appropriately tune the action in order to recover Lorentz invariance in the continuum, already at the pure-gauge theory level. A better understanding of these more general anisotropy classes would be useful, since they provide a more flexible setting for varying length scales independently in different directions. This would allow, for example, to enlarge the range of momenta accessible to lattice calculation at a reasonable computational cost, by improving the resolution only in a single spatial direction [5].

Anisotropic lattices provide, quite obviously, the natural setting for the nonperturbative study of anisotropic systems, also beyond numerical applications. An interesting case is that of longitudinally rescaled actions, which in recent years have been considered in the context of high-energy scattering in QCD [6–12]. The basic idea of Refs. [6–9] is to perform a rescaling of the longitudinal directions, which appear highly Lorentz-contracted in a high-energy scattering process, in order to derive an effective action starting from QCD. In Refs. [6–9] only the classically rescaled action was considered, while the important effect of quantum corrections was studied later in Refs. [10–12], in the framework of Wilsonian anisotropic renormalization in the continuum. In this context, the use of a gauge-invariant, anisotropic lattice regularization could lead to more insight in the structure of quantum corrections. The relevant anisotropy class here is $2+2$, with different lattice spacings in the longitudinal and in the transverse plane. This is also the case considered in Ref. [5], although for different purposes.

In Ref. [13] a classical anisotropic rescaling of the functional integral has been used to justify, on nonperturbative grounds, the analytic continuation from Euclidean to Minkowski space, required to obtain physical results in the Euclidean formulation [14–19] of the nonperturbative approach to soft high-energy scattering [20–27]. This approach has been recently used in Ref. [28] to obtain a theoretical estimate of the leading energy dependence of hadronic total cross sections, resulting in fair agreement with experiments. As the analytic continuation plays a key role in this approach, it is important to establish its correctness going beyond the formal argument of Ref. [13], which, as we have said above, is based only on a classical rescaling of the QCD action. To this end, quantum corrections to the effective action must be included to prove that the necessary analyticity requirements are actually fulfilled. The relevant anisotropy class in this case is $2+1+1$, with different lattice spacings in the transverse plane and in the two longitudinal directions.

The purpose of this paper is to perform the renormalization of a $SU(N_c)$ gauge theory regularized on a general anisotropic lattice, and to apply the results in the study of hadronic high-energy scattering through the approaches mentioned above. To avoid the complications related to the introduction of fermions on the lattice, we work here in the *quenched* approximation, i.e., pure gauge theory.

The plan of the paper is the following. In Sec. II we study renormalization for a general anisotropic lattice

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regularization, using the background field method on the lattice [29–37]. In Sec. III we use the results in the context of the nonperturbative approach to soft high-energy scattering of Refs. [20–27], refining the argument of Ref. [13] on the possibility of performing analytic continuation to Euclidean space. In Sec. IV we discuss the longitudinally rescaled actions of Refs. [6–12], focusing on the representation of the $SU(N_c)$ gauge theory as a set of coupled two-dimensional principal chiral models. Finally, Sec. V contains our conclusions and prospects. Some technical details are discussed in the Appendixes.

II. ANISOTROPIC RENORMALIZATION

Our aim is to renormalize the Euclidean $SU(N_c)$ gauge theory regularized on a 4D orthogonal anisotropic lattice. More precisely, lattice points are located at $x = \sum_{\mu=1}^{4} x_\mu \hat{\mu}$, where $\hat{\mu}$ are four orthogonal unit vectors, and the physical coordinates $x_\mu = x_\mu(n)$ in Euclidean space are $x_\mu(n) = a_\mu n_\mu$, $n_\mu \in \mathbb{Z}$. Here $a_\mu = a/\lambda_\mu$ is the lattice spacing in direction μ, with the dimensionless *anisotropy parameters* $\lambda_\mu \in \mathbb{R}^+$ being the inverse ratios of a_μ to a reference length scale a.[1] Consider the following Wilson-like action,

$$S_{\rm lat}^{\rm tree} = \beta \sum_n \sum_{\mu<\nu} C_{\mu\nu} \left(1 - \frac{1}{N_c} \mathrm{Re\,tr} U_{\mu\nu}(n)\right) = \beta \sum_n \sum_{\mu<\nu} C_{\mu\nu} \mathcal{P}_{\mu\nu}(n), \quad (1)$$

where $U_{\mu\nu}(n)$ are the usual plaquette variables built up with the link variables $U_\mu(n) \in SU(N_c)$,

$$U_{\mu\nu}(n) = U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu^\dagger(n+\hat{\nu}) U_\nu^\dagger(n), \quad (2)$$

$\beta = 2N_c/g^2$ with g the coupling constant, and $C_{\mu\nu}$, $\mu \neq \nu$, are the *plaquette coefficients*,[2]

$$C_{\mu\nu} = C_{\nu\mu} = C_{\mu\nu}(\lambda) = (\lambda_\mu \lambda_\nu)^2 \mathcal{J}, \qquad \mathcal{J}^{-1} \equiv \prod_{\alpha=1}^{4} \lambda_\alpha. \quad (3)$$

It is straightforward to show that Eq. (1) yields the correct naïve continuum limit upon identification of the continuum, physical gauge fields $A_\mu(x)$ through

$$U_\mu(n) = e^{iga_\mu A_\mu(x(n))}, \quad (4)$$

as appropriate for an anisotropic lattice. The choice of plaquette coefficients $C_{\mu\nu}$ is easily understood by noticing that $\mathcal{J}a^4$ is just the volume of an elementary cell, so that $\mathcal{J}$ is the Jacobian for the change of variables from isotropic to anisotropic coordinates, while $a_\mu a_\nu = (\lambda_\mu \lambda_\nu)^{-1} a^2$ is the area of the faces of an elementary cell lying in the $\mu\nu$ plane.

As is well known, divergencies appear in the continuum limit when taking into account quantum corrections. These divergencies need to be subtracted through a suitable renormalization of the couplings in order to obtain a finite continuum theory. On the isotropic lattice, the symmetry under the unbroken hypercubic subgroup of $O(4)$ guarantees that all the plaquette terms in the action need to be renormalized in the same way, so that a single redefinition of g is sufficient to reabsorb the divergencies. The form of the action is therefore unchanged, and one recovers the full $O(4)$ invariance in the continuum limit.

On a general anisotropic lattice this residual symmetry is broken, except for reflections through lattice hyperplanes, and so in general different terms will require a different renormalization. Since there are six different plaquette terms and only four lattice spacings, it will not be possible in the general case to reabsorb completely the quantum corrections into a redefinition of λ_μ, keeping at the same time the same form of the tree-level action [5]. In turn, this implies that the continuum limit of Eq. (1) cannot be made into an $O(4)$-invariant theory by an appropriate, simple rescaling of the lattice spacings, since in the general case one will still find different coefficients for the six continuum field-strength terms. To recover $O(4)$ invariance one must ensure that these coefficients are equal, and this requires that we take the action to be of the more general form

$$S_{\rm lat} = \sum_n \sum_{\mu<\nu} \beta_{\mu\nu} C_{\mu\nu} \left(1 - \frac{1}{N_c} \mathrm{Re\,tr} U_{\mu\nu}(n)\right) = \sum_n \sum_{\mu<\nu} \beta_{\mu\nu} C_{\mu\nu} \mathcal{P}_{\mu\nu}(n), \quad (5)$$

where the couplings $\beta_{\mu\nu} = \beta_{\nu\mu} = \beta_{\mu\nu}(\lambda)$ have to be properly tuned to yield a finite, $O(4)$-invariant theory in the continuum limit.

The need for tuning comes, as we have said, from the fact that there are in general more couplings than anisotropy parameters. It is however easy to show that one has to tune at most only two combinations of the couplings to achieve restoration of $O(4)$ invariance in the continuum limit, while the other four independent combinations can be interpreted as the coupling fixing the overall lattice scale, and renormalizations of the anisotropies $a_\nu/a_\mu = \lambda_\mu/\lambda_\nu$. To see this, let us remove the redundancy in the set $\{\lambda_\mu\}$ by imposing the symmetric condition $\prod_\mu \lambda_\mu = 1$, thus defining a in terms of the volume of an elementary cell. Any other equivalent choice (i.e., giving the same a_μ) is obtained by a simple global rescaling of $\{\lambda_\mu\}$ and of a. The six plaquette terms can be grouped in pairs of "complementary" $\mu\nu$ and

[1] The five parameters a and $\{\lambda_\mu\}$ are obviously redundant, and some condition has to be imposed on $\{\lambda_\mu\}$ to remove this redundancy. This notation is however convenient, as we treat all the directions on the same footing.

[2] For definiteness, we define also $C_{\mu\mu} = 0$.

$\bar{\mu}\bar{\nu}$ plaquettes, i.e., (U_{12}, U_{34}), etc., which we denote as $(\mu\nu|\bar{\mu}\bar{\nu}) = (12|34), (13|24), (14|23)$. It is also easily noticed that $C_{\mu\nu} = \frac{\lambda_\mu\lambda_\nu}{\lambda_{\bar{\mu}}\lambda_{\bar{\nu}}}$, so that $C_{\bar{\mu}\bar{\nu}} = C_{\mu\nu}^{-1}$. This suggests to parametrize $\beta_{\mu\nu}$ as follows,

$$\beta_{\mu\nu} = \beta Z_{(\mu\nu|\bar{\mu}\bar{\nu})} \frac{z_\mu z_\nu}{z_{\bar{\mu}} z_{\bar{\nu}}}. \tag{6}$$

As there are two redundant parameters, we choose to fix $\prod_\mu z_\mu = 1$, so that our condition on $\{\lambda_\mu\}$ is not renormalized,[3] and $\prod_{(\mu\nu|\bar{\mu}\bar{\nu})} Z_{(\mu\nu|\bar{\mu}\bar{\nu})} = 1$. In this way β, z_μ and $Z_{(\mu\nu|\bar{\mu}\bar{\nu})}$ are unambiguously defined and can be obtained from $\beta_{\mu\nu}$ as follows,

$$\beta = \left(\prod_{\mu<\nu} \beta_{\mu\nu}\right)^{\frac{1}{6}}, \qquad z_\mu = \left(\prod_{\nu\neq\mu} \frac{\beta_{\mu\nu}}{\beta_{\bar{\mu}\bar{\nu}}}\right)^{\frac{1}{8}},$$
$$Z_{(\mu\nu|\bar{\mu}\bar{\nu})} = \left[\frac{\beta_{\mu\nu}\beta_{\bar{\mu}\bar{\nu}}}{(\beta_{\mu\bar{\nu}}\beta_{\bar{\mu}\nu}\beta_{\mu\bar{\mu}}\beta_{\nu\bar{\nu}})^{\frac{1}{2}}}\right]^{\frac{1}{3}}. \tag{7}$$

This makes it explicit that the restoration or not of $O(4)$ invariance in the continuum depends only on the values of the ratios of the couplings $\beta_{\mu\nu}$. Defining now the bare anisotropy parameters $\lambda_\mu^B \equiv z_\mu\lambda_\mu$, and the bare plaquette coefficients $C_{\mu\nu}^B \equiv C_{\mu\nu}(\lambda^B)$, one can rewrite Eq. (5) as

$$S_{\rm lat} = \beta \sum_{n,(\mu\nu|\bar{\mu}\bar{\nu})} Z_{(\mu\nu|\bar{\mu}\bar{\nu})}[C_{\mu\nu}^B P_{\mu\nu}(n) + C_{\bar{\mu}\bar{\nu}}^B P_{\bar{\mu}\bar{\nu}}(n)]. \tag{8}$$

This equation shows that to obtain an $O(4)$-invariant theory in the continuum limit, one can choose freely λ_μ^B (up to a constraint to remove the redundancy), and then tune only the two independent ratios of $Z_{(\mu\nu|\bar{\mu}\bar{\nu})}$ to the appropriate values. The physical anisotropy parameters λ_μ are related to the bare ones through the renormalization $\lambda_\mu = z_\mu^{-1}\lambda_\mu^B$, and can be measured *ex post*.

Using the parametrization Eq. (7), it is possible to set up a rather simple nonperturbative scheme to achieve restoration of $O(4)$ invariance in the continuum, for an arbitrary choice of bare anisotropy parameters. The basic idea is to impose that the string tension, determined from the asymptotic behavior of large rectangular $T \times R$ on-axis Wilson loops $W_{\alpha\beta} \sim \exp\{-\hat{\sigma}_{\alpha\beta}TR\}$, is the same for all pairs of directions α, β. Denoting with σ the physical (dimensionful) string tension, this amounts to impose $\lambda_\alpha\lambda_\beta\hat{\sigma}_{\alpha\beta} = a^2\sigma$, for all pairs of different α, β. Multiplying the relations for $\hat{\sigma}_{\alpha\beta}$ and its complementary $\hat{\sigma}_{\bar{\alpha}\bar{\beta}}$, one obtains the following consistency conditions,

$$\hat{\sigma}_{12}\hat{\sigma}_{34} = \hat{\sigma}_{13}\hat{\sigma}_{24} = \hat{\sigma}_{14}\hat{\sigma}_{23}, \tag{9}$$

which have to be imposed to recover $O(4)$ invariance. This can be done without any prior knowledge of the physical λ_α, and requires only to properly tune two of the coefficients $Z_{(\mu\nu|\bar{\mu}\bar{\nu})}$ (the third one being constrained by our choice $\prod_{(\mu\nu|\bar{\mu}\bar{\nu})} Z_{(\mu\nu|\bar{\mu}\bar{\nu})} = 1$). Having done this, the anisotropies can then be obtained from the ratio $\lambda_\mu/\lambda_\nu = \hat{\sigma}_{\nu\alpha}/\hat{\sigma}_{\mu\alpha}$ for any $\alpha \neq \mu, \nu$. Imposing $\prod_\mu \lambda_\mu = 1$ one can explicitly determine all λ_μ's, and set the lattice scale a from the relation $a^4\sigma^2 = \hat{\sigma}_{12}\hat{\sigma}_{34}$. While the string tension is known not to be the best observable for setting the physical scale, nevertheless it could be useful for the tuning, as it can be determined to high precision by means of multilevel algorithms [38]. It is worth mentioning that the tuning of two parameters is only required when all the lattice spacings are different: if at least a pair of lattice spacings are equal, one easily sees that only one parameter has to be tuned.[4]

A. Background field method

From the discussion above, we see that our task is to find the relations among the couplings $\beta_{\mu\nu}$ that will lead to an $O(4)$-invariant theory in the continuum limit. We will study this problem to lowest order in perturbation theory, making use of the background field method [29–32] on the lattice [33–37]. The advantage of this method is that it allows us to keep an exact gauge invariance on the lattice after gauge fixing, which greatly simplifies the calculations. A full account on the background field method can be found elsewhere [39,40]. Here we briefly recall the main points of the method to fix the notation.

The first step is to introduce a background field $U_\mu^{(c)}$ in the gauge action as follows,

$$S_{\rm BF}[U^{(c)}, V] \equiv S_{\rm lat}[VU^{(c)}], \tag{10}$$

where we now denote with V the gauge links, to be integrated over with the usual Haar measure. As a consequence of the gauge invariance of $S_{\rm lat}$, the action $S_{\rm BF}$ is invariant under the *background gauge transformation*

$$U_\mu^{(c)G}(n) = G(n)U_\mu^{(c)}(n)G^\dagger(n+\hat{\mu}),$$
$$V_\mu^G(n) = G(n)V_\mu(n)G^\dagger(n), \tag{11}$$

[3] Any other choice is of course allowed. If, for example, the scale a is defined to be one of the lattice spacings by choosing $\lambda_\mu = 1$ for some μ, then it is convenient to choose $z_\mu = 1$. The new values of z_ν are obtained from those corresponding to the symmetric condition by replacing $z_\nu \to z_\nu/z_\mu$, while $Z_{(\mu\nu|\bar{\mu}\bar{\nu})}$ and β are unaffected.

[4] In the $3+1$ case, where a single lattice spacing differs from the others, there are only two kinds of plaquette terms and so only two independent lattice string tensions. In this case there is thus no consistency condition to be satisfied and no tuning is needed, as is well known.

with $G(n) \in SU(N_c)$, as well as under the following gauge transformation of V alone,

$$V_\mu(n) \to G(n) V_\mu(n) U_\mu^{(c)}(n) G^\dagger(n+\hat{\mu}) U_\mu^{(c)\dagger}(n). \quad (12)$$

The integration measure is also obviously invariant under the transformations Eqs. (11) and (12). One then proceeds to set up perturbation theory in the usual way, setting

$$V_\mu(n) = e^{i\frac{g}{\lambda_\mu} q_\mu(n)}, \qquad U_\mu^{(c)}(n) = e^{i a_\mu B_\mu(n)}, \quad (13)$$

where[5] $q_\mu(n) = q_\mu^a(n) t^a$ and $B_\mu(n) = B_\mu^a(n) t^a$, with t^a the generators of $SU(N_c)$ in the fundamental representation, $a = 1, \ldots, N_c^2 - 1$, and $\mathrm{tr}\{t^a t^b\} = \frac{1}{2}\delta^{ab}$. One then changes variables of integration to q, expressing the Jacobian as a contribution $S_{\mathrm{meas}}[q]$ to the action. Notice that powers of g and a are chosen so that q is dimensionless, while B has dimensions of mass. This distinction is convenient for bookkeeping purposes [36].

Under the transformation Eq. (11), the background field B transforms as a gauge field, while the "quantum" field q transforms as a matter field in the adjoint representation. The symmetry under the gauge transformation Eq. (12) requires to impose a gauge condition on q to define the corresponding propagator. This is done *à la* Faddeev–Popov, adding a gauge-fixing term to the action, together with the corresponding ghost term. The key point is that there is an appropriate choice of gauge, called the *background field gauge*, for which the gauge-fixing and the ghost terms are invariant under the background gauge transformation Eq. (11). This gauge-fixing term is [29,30,37]

$$S_{\mathrm{g.f.}}[B, q] = \mathcal{J} \sum_n \mathrm{tr}\left(\sum_\mu D_\mu^- q_\mu \right)^2, \quad (14)$$

where $D_\mu^\pm$ are the lattice background covariant differences,

$$\begin{aligned} D_\mu^+ f(n) &\equiv \lambda_\mu [U_\mu^{(c)}(n) f(n+\hat{\mu}) U_\mu^{(c)\dagger}(n) - f(n)], \\ D_\mu^- f(n) &\equiv \lambda_\mu [U_\mu^{(c)\dagger}(n-\hat{\mu}) f(n-\hat{\mu}) U_\mu^{(c)}(n-\hat{\mu}) - f(n)], \end{aligned} \quad (15)$$

in which a factor λ_μ is also included for convenience. The usual lattice differences $\Delta_\mu^\pm$ are obtained setting $U_\mu^{(c)} = \mathbf{1}$ in the expressions above, where $\mathbf{1}$ denotes the unit matrix. The corresponding ghost term is

$$S_{\mathrm{ghost}}[B, q, c, \bar{c}] = 2\mathcal{J} \sum_{n,\mu} \mathrm{tr}\left\{ [D_\mu^+ \bar{c}(n)] \left[M^{-1}\left(\frac{g}{\lambda_\mu} q_\mu(n) \right) D_\mu^+ + i\mathrm{Ad}\left(\frac{g}{\lambda_\mu} q_\mu(n) \right) \right] c(n) \right\}, \quad (16)$$

where $c = c^a t^a$, $\bar{c} = \bar{c}^a t^a$, with $c^a, \bar{c}^a$ independent Grassmann variables, and where

$$M(X) \equiv \frac{1 - e^{-i\mathrm{Ad}(X)}}{i\mathrm{Ad}(X)}, \qquad \mathrm{Ad}(X) Y \equiv [X, Y]. \quad (17)$$

It is straightforward to prove invariance of these two terms under the background gauge transformation, Eq. (11), supplemented by the transformation laws for the ghost fields,

$$\begin{aligned} c^G(n) &= G(n) c(n) G^\dagger(n), \\ \bar{c}^G(n) &= G(n) \bar{c}(n) G^\dagger(n). \end{aligned} \quad (18)$$

Expanding Eq. (16) up to $\mathcal{O}(g^0)$, one finds

$$\begin{aligned} S_{\mathrm{ghost}}[B, q, c, \bar{c}] &= 2\mathcal{J} \sum_{n,\mu} \mathrm{tr}\{ [D_\mu^+ \bar{c}(n)][D_\mu^+ c(n)] \} + \mathcal{O}(g) \\ &= 2\mathcal{J} \sum_{n,\mu} \mathrm{tr}\{ \bar{c}(n) D_\mu^- D_\mu^+ c(n) \} + \mathcal{O}(g) \\ &\equiv S^0_{\mathrm{ghost}}[B, c, \bar{c}] + \mathcal{O}(g), \end{aligned} \quad (19)$$

where we have used "integration by parts" on a lattice (infinite or with periodic boundary conditions),

$$\sum_n \mathrm{tr}\{ [D_\mu^+ f(n)] g(n) \} = \sum_n \mathrm{tr}\{ f(n) [D_\mu^- g(n)] \}. \quad (20)$$

The starting point for the perturbative analysis is the generating functional

$$\begin{aligned} Z[B, J, \bar{\eta}, \eta] &= \int \mathcal{D}q \mathcal{D}c \mathcal{D}\bar{c}\, e^{-S_{\mathrm{tot}}[B,q,c,\bar{c}] + J\cdot q + \bar{\eta}\cdot c + \eta\cdot\bar{c}} \\ &= e^{W[B,J,\bar{\eta},\eta]}, \\ S_{\mathrm{tot}}[B, q, c, \bar{c}] &= S_{\mathrm{BF}}[B, q] + S_{\mathrm{meas}}[q] + S_{\mathrm{g.f.}}[B, q] \\ &\quad + S_{\mathrm{ghost}}[B, q, c, \bar{c}], \end{aligned} \quad (21)$$

where with a small abuse of notation we have written $S_{\mathrm{BF}}[B, q] = S_{\mathrm{BF}}[U_c, V]$, and we have added source terms for the various fields. Here $J = J(n) = J_\mu^a(n) t^a$ and $J \cdot q \equiv \sum_{n,\mu} J_\mu^a(n) q_\mu^a(n)$, and similarly for the other terms. A Legendre transform gives the effective action (generating functional for 1PI graphs),

$$\Gamma[B, Q, C, \bar{C}] = -W[B, J, \bar{\eta}, \eta] + J \cdot Q + \bar{\eta} \cdot C + \bar{C} \cdot \eta, \quad (22)$$

[5] Here and in the following, the sum over repeated color indices is understood.

where the classical fields Q, C and $\bar{C}$ are defined as

$$Q^a_\mu(n) = \frac{\partial W[B,J,\bar{\eta},\eta]}{\partial J^a_\mu(n)},$$
$$C^a(n) = \frac{\partial W[B,J,\bar{\eta},\eta]}{\partial \bar{\eta}^a(n)},$$
$$\bar{C}^a(n) = \frac{\partial W[B,J,\bar{\eta},\eta]}{\partial \eta^a(n)}, \tag{23}$$

i.e., they are the expectation values of the quantum fields for prescribed values of B and of the sources.

Defining a background gauge transformation for the classical fields, imposing that they transform as the corresponding quantum fields, Eqs. (11) and (18), leads finally to the identity

$$\Gamma[B^G, Q^G, C^G, \bar{C}^G] = \Gamma[B, Q, C, \bar{C}] \tag{24}$$

for the effective action. This is the key relation that allows us to simplify the calculations. Indeed, setting $S_{\rm eff}[B] \equiv \Gamma[B,0,0,0] - \Gamma[0,0,0,0]$, as a consequence of the background gauge invariance, of the discrete symmetries of the action (translations and reflections[6]), and of the locality of divergencies, to one-loop accuracy and to lowest order in perturbation theory we are guaranteed to find in the continuum limit

$$\lim_{a\to 0} S_{\rm eff}[B] = \frac{1}{2}\sum_{\mu,\nu}\int d^4x \left[\frac{\beta_{\mu\nu}}{2N_c} - K_{\mu\nu}\right] {\rm tr}\mathcal{F}^2_{\mu\nu}(x) + (\text{nonlocal finite terms}) + \mathcal{O}(g^2), \tag{26}$$

where $K_{\mu\nu} = K_{\nu\mu} = K_{\mu\nu}(a,\lambda)$ is $\mathcal{O}(g^0)$, and where $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu + i[\mathcal{B}_\mu, \mathcal{B}_\nu]$ is the field strength for the continuum background field $\mathcal{B}_\mu(x)$, $\mathcal{B}_\mu(x(n)) = B(n)$. For our purposes it is therefore sufficient to compute the two-point function of the background field to have enough information to renormalize the theory and impose $O(4)$ invariance. To one-loop accuracy it is enough to set

$$\frac{\beta_{\mu\nu}}{2N_c} - K_{\mu\nu} = \frac{1}{g^2} + \frac{\delta\beta_{\mu\nu}}{2N_c} - K_{\mu\nu} = \frac{1}{g_r^2}, \tag{27}$$

where g_r is the renormalized, λ-independent coupling.

[6]Reflections Π_α act as follows on the coordinates, $\Pi_\alpha n_\mu = n_\mu$ for $\mu \neq \alpha$, $\Pi_\alpha n_\alpha = -n_\alpha$. The corresponding transformation laws for B and q are the following,

$$B^{\Pi_\alpha}_\mu(n) = \begin{cases} B_\mu(\Pi_\alpha n) & \mu \neq \alpha, \\ -B_\alpha(\Pi_\alpha n - \hat{\alpha}), \end{cases}$$
$$q^{\Pi_\alpha}_\mu(n) = \begin{cases} q_\mu(\Pi_\alpha n) & \mu \neq \alpha, \\ -U^{(c)}_\alpha(\Pi_\alpha n - \hat{\alpha}) q_\alpha(\Pi_\alpha n - \hat{\alpha}) U^{(c)}_\alpha(\Pi_\alpha n - \hat{\alpha}). \end{cases} \tag{25}$$

We notice that Eq. (5), with the couplings chosen according to Eq. (27), can be interpreted in two ways. Under the identification $U_\mu(n) = e^{iga_\mu A_\mu(x(n))}$ with $x_\alpha = a_\alpha n_\alpha$, it leads in the continuum to the renormalized, *isotropic* action for the gauge fields $A_\mu(x)$, for which it provides an appropriate lattice discretization. On the other hand, identifying $U_\mu(n) = e^{iga\phi_\mu(y(n))}$ with $y_\alpha = an_\alpha$, in the continuum limit one obtains the following renormalized *anisotropic* action,

$$S \to \frac{1}{2g_r^2}\sum_{\mu,\nu} C_{\mu\nu}\int d^4y\, {\rm tr}\Phi^2_{\mu\nu}(y), \tag{28}$$

with $\Phi_{\mu\nu}$ the usual field-strength tensor for ϕ_μ, for which Eq. (5) provides therefore a lattice discretization. This is the form of the action obtained by classically rescaling coordinates and fields in the Yang-Mills action, discussed in Refs. [6–9,13].

B. One-loop calculation

To compute $K_{\mu\nu}$ it is enough to expand the action to order $\mathcal{O}(g^0)$, which in turn means expanding the gauge action up to second order in q. Contributions from $S_{\rm meas}$ are at least $\mathcal{O}(g^2)$ and can be ignored. Let us expand the action $S_{\rm BF} + S_{\rm g.f.}$ in powers of q,

$$S_{\rm BF}[B,q] + S_{\rm g.f.}[B,q] = S_c[B] + S_{\rm g1}[B,q] + S_{\rm g2}[B,q] + \cdots, \tag{29}$$

where $S_c[B] = S_{\rm BF}[B,0]$ is the classical action, $S_{\rm g1}[B,q]$ is linear in q, $S_{\rm g2}[B,q]$ is quadratic and so on, and set

$$S_2[B,q,c,\bar{c}] = S_{\rm g2}[B,q] + S^0_{\rm ghost}[B,c,\bar{c}] = \sum_{n,m,\mu,\nu}\frac{1}{2}q^a_\mu(n)(\Pi[B])^{ab}_{nm;\mu\nu}q^b_\nu(m) + \sum_{n,m}\bar{c}^a(n)(\hat{\Pi}[B])^{ab}_{nm}c^b(m). \tag{30}$$

A straightforward calculation then shows that

$$S_{\rm eff}[B]|_{\mathcal{O}(g^0)} = S_{\rm BF}[B,0] + \frac{1}{2}\log\frac{\det\Pi[B]}{\det\Pi[0]} - \log\frac{\det\hat{\Pi}[B]}{\det\hat{\Pi}[0]}. \tag{31}$$

Terms linear in q play no role and can be ignored.[7] Equation (31) can be conveniently written as

[7]These terms are usually discarded by requiring B to satisfy the equations of motion, but this is actually not necessary.

$$e^{-S_{\rm eff}[B]}|_{\mathcal{O}(g^0)} = e^{-S_c[B]}\langle e^{-(S_2-S^{\rm free})}\rangle_0, \tag{32}$$

where $S^{\rm free}[q,c,\bar{c}] = S_2[0,q,c,\bar{c}]$ is the free action with no background field, and $\langle\ldots\rangle_0$ denotes the corresponding expectation value,

$$\langle\mathcal{O}[B,q,c,\bar{c}]\rangle_0 = Z_{\rm free}^{-1}\int \mathcal{D}q\mathcal{D}c\mathcal{D}\bar{c}e^{-S^{\rm free}[q,c,\bar{c}]}\mathcal{O}[B,q,c,\bar{c}],$$
$$Z_{\rm free} = \int \mathcal{D}q\mathcal{D}c\mathcal{D}\bar{c}e^{-S^{\rm free}[q,c,\bar{c}]}. \tag{33}$$

For future utility, we define the connected correlation function $\langle\mathcal{O}_1\mathcal{O}_2\rangle_{0c} \equiv \langle\mathcal{O}_1\mathcal{O}_2\rangle_0 - \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle_0$. Since we are interested only in the two-point function for B, only terms up to $\mathcal{O}(B^2)$ will be kept in S_2.

1. The quadratic action

The gauge action in a background field can be conveniently written as follows,

$$S_{\rm BF}[B,q] = S_{\rm lat}[VU^{(c)}] = \sum_{n,\mu<\nu}\beta_{\mu\nu}C_{\mu\nu}\left(1-\frac{1}{N_c}{\rm Re\,tr}\{V_{\mu\nu}(n)U^{(c)}_{\mu\nu}(n)\}\right), \tag{34}$$

where the "quantum" and the "background" plaquette are given respectively by

$$V_{\mu\nu}(n) \equiv e^{-ig\frac{1}{\lambda_\mu}(\frac{1}{\lambda_\nu}D^+_\nu q_\mu(n)+q_\mu(n))}e^{-ig\frac{1}{\lambda_\nu}q_\nu(n)}e^{ig\frac{1}{\lambda_\mu}q_\mu(n)} \times e^{ig\frac{1}{\lambda_\nu}(\frac{1}{\lambda_\mu}D^+_\mu q_\nu(n)+q_\nu(n))},$$
$$U^{(c)}_{\mu\nu}(n) \equiv U^{(c)}_\mu(n)U^{(c)}_\nu(n+\hat{\mu})U^{(c)\dagger}_\mu(n+\hat{\nu})U^{(c)\dagger}_\nu(n). \tag{35}$$

A standard application of the Baker-Campbell-Hausdorff formula gives

$$U^{(c)}_{\mu\nu}(n) = \exp\left\{i\frac{a^2}{\lambda_\mu\lambda_\nu}f_{\mu\nu}(n) + \mathcal{O}(a^3B\partial B, a^4(\partial B)^2) + \mathcal{O}(a^3B^3)\right\}, \tag{36}$$

with[8]

$$f_{\mu\nu} = a^{-1}(\Delta^+_\mu B_\nu - \Delta^+_\nu B_\mu) + i[B_\mu, B_\nu], \tag{37}$$

which in the continuum limit reduces to the usual field strength tensor for the background field.[9] For $V_{\mu\nu}$ we have instead

$$V_{\mu\nu}(n) = \exp\left\{ig\frac{1}{\lambda_\mu\lambda_\nu}[F_{\mu\nu}(n) + gR_{\mu\nu}(n)] + \mathcal{O}(g^3)\right\}, \tag{38}$$

where

$$F_{\mu\nu} = D^+_\mu q_\nu - D^+_\nu q_\mu,$$
$$R^{(1)}_{\mu\nu} = \frac{i}{2\lambda_\mu\lambda_\nu}[D^+_\mu q_\nu, D^+_\nu q_\mu] + i[q_\mu, q_\nu],$$
$$R^{(2)}_{\mu\nu} = \frac{i}{2}\left(\frac{1}{\lambda_\mu}[q_\mu, D^+_\nu q_\mu] - \frac{1}{\lambda_\nu}[q_\nu, D^+_\mu q_\nu]\right), \tag{39}$$

and $R_{\mu\nu} = R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$. Expanding up to quadratic terms in B and q we find

$$S_{\rm BF}[B,q] = S_c[B] + S_q[B,q] + (\text{linear in } q) + \mathcal{O}(q^3), \tag{40}$$

where S_c is the classical action, already defined above,

$$S_c = \mathcal{J}a^4\sum_{n,\mu,\nu}\frac{\beta_{\mu\nu}}{2N_c}\frac{1}{2}{\rm tr}f^2_{\mu\nu}(n) \underset{a\to 0}{\to} \frac{1}{2}\int d^4x\sum_{\mu,\nu}\frac{\beta_{\mu\nu}}{2N_c}{\rm tr}\mathcal{F}^2_{\mu\nu}(x), \tag{41}$$

while the "quantum" piece S_q is given by

$$S_q[B,q] = \mathcal{J}\sum_{n,\mu,\nu}\frac{1}{2}{\rm tr}\{F^2_{\mu\nu}(n) + 2a^2R_{\mu\nu}(n)f_{\mu\nu}(n)\} - \frac{1}{4}\frac{a^4}{(\lambda_\mu\lambda_\nu)^2}{\rm tr}\{F^2_{\mu\nu}(n)f^2_{\mu\nu}(n)\}. \tag{42}$$

The gauge-fixing term is quadratic in q, and can be conveniently rearranged as follows,

$$S_{\rm g.f.} = S^{(1)}_{\rm g.f.} + S^{(2)}_{\rm g.f.} + S_{\rm T'}, \tag{43}$$

where

$$S^{(1)}_{\rm g.f.} = \mathcal{J}\sum_{n,\mu,\nu}{\rm tr}\{D^+_\nu q_\mu(n)D^+_\mu q_\nu(n)\},$$
$$S^{(2)}_{\rm g.f.} = \mathcal{J}a^2\sum_{n,\mu,\nu}{\rm tr}\{\bar{R}^{(1)}_{\mu\nu}(n)f_{\mu\nu}(n)\},$$
$$\bar{R}^{(1)}_{\mu\nu} = i\left([q_\mu,q_\nu] + \frac{1}{\lambda_\mu}[q_\mu, D^+_\mu q_\nu] - \frac{1}{\lambda_\nu}[q_\nu, D^+_\nu q_\mu] + \frac{1}{\lambda_\mu\lambda_\nu}[D^+_\nu q_\mu, D^+_\mu q_\nu]\right),$$
$$S_{\rm T'} = \mathcal{J}a^4\sum_{n,\mu,\nu}{\rm tr}\bar{R}^{(2)}_{\mu\nu}(n)$$
$$\bar{R}^{(2)}_{\mu\nu} = \frac{i}{2\lambda_\mu\lambda_\nu}\left[f_{\mu\nu}, \frac{1}{\lambda_\mu}D^+_\mu q_\nu + q_\nu\right]\left[f_{\mu\nu}, \frac{1}{\lambda_\nu}D^+_\nu q_\mu + q_\mu\right]. \tag{44}$$

[8] In the following equations we will sometimes drop the dependence on the lattice site n to make the expressions more readable.

[9] In principle, also the higher-order terms of order $\mathcal{O}(a^3B\partial B, a^4(\partial B)^2)$ appearing in Eq. (36) could contribute to the two-point function in the continuum. This however is not the case, as we will see below (see footnotes 10 and 11).

Finally, the ghost term is independent of q to $\mathcal{O}(g^0)$. Putting all the terms together, one obtains for the quadratic lattice action

$$S_2 = S_c + S^{\text{free}} + S^{\text{int}}_{\text{gluon}} + S^{\text{int}}_{\text{ghost}} + S_{\text{A}} + S_{\text{B}} + S_{\text{T}} + S_{\text{T}'}, \tag{45}$$

where the terms have been grouped so that each quantity in the equation above is separately invariant under a background gauge transformation [36]. Here $S^{\text{free}} = S^{\text{free}}_{\text{gluon}} + S^{\text{free}}_{\text{ghost}}$, with

$$\begin{aligned} S^{\text{free}}_{\text{gluon}} &= \mathcal{J}\sum_{n,\mu,\nu} \text{tr}[\Delta^+_\mu q_\nu(n)]^2, \\ S^{\text{free}}_{\text{ghost}} &= 2\mathcal{J}\sum_{n,\mu,\nu} \text{tr}\{\bar{c}(n)\Delta^-_\mu \Delta^+_\mu c(n)\}, \end{aligned} \tag{46}$$

being the free actions for gluons and ghosts, respectively, in terms of which the propagators are defined, while the interaction terms are given by

$$\begin{aligned} S^{\text{int}}_{\text{gluon}} &= S_q + S^{(1)}_{\text{g.f.}} - S^{\text{free}}_{\text{gluon}} \\ &= \mathcal{J}\sum_{n,\mu,\nu} \text{tr}\{[D^+_\mu q_\nu(n)]^2 - [\Delta^+_\mu q_\nu(n)]^2\}, \\ S^{\text{int}}_{\text{ghost}} &= S^0_{\text{ghost}} - S^{\text{free}}_{\text{ghost}} \\ &= 2\mathcal{J}\sum_{n,\mu} \text{tr}\{\bar{c}(n)[D^-_\mu D^+_\mu - \Delta^-_\mu \Delta^+_\mu]c(n)\}. \end{aligned} \tag{47}$$

Moreover, extra vertices come from the terms

$$\begin{aligned} S_{\text{A}} &= \mathcal{J}a^2\sum_{n,\mu,\nu} \text{tr}\{R^{(1)}_{\mu\nu}(n) f_{\mu\nu}(n)\} + S^{(2)}_{\text{g.f.}} \\ &= \mathcal{J}a^2\sum_{n,\mu,\nu} \text{tr}\{[R^{(1)}_{\mu\nu}(n) + \bar{R}^{(1)}_{\mu\nu}(n)] f_{\mu\nu}(n)\}, \\ S_{\text{B}} &= \mathcal{J}a^2\sum_{n,\mu,\nu} \text{tr}\{R^{(2)}_{\mu\nu}(n) f_{\mu\nu}(n)\}, \\ S_{\text{T}} &= -\mathcal{J}a^4\sum_{n,\mu,\nu} \frac{1}{(2\lambda_\mu\lambda_\nu)^2} \text{tr}\{F^2_{\mu\nu}(n) f^2_{\mu\nu}(n)\}. \end{aligned} \tag{48}$$

Explicitly, we have for S_A and S_B the expressions[10]

$$\begin{aligned} S_{\text{A}} &= \mathcal{J}a^2\sum_{n,\mu,\nu} \frac{1}{2}\text{tr}\{A_{\mu\nu}(n) f_{\mu\nu}(n)\}, \\ A_{\mu\nu} &= 2i\Bigg(2[q_\mu, q_\nu] + \frac{1}{\lambda_\mu}[q_\mu, D^+_\mu q_\nu] - \frac{1}{\lambda_\nu}[q_\nu, D^+_\nu q_\mu] \\ &\quad - \frac{1}{2\lambda_\mu\lambda_\nu}[D^+_\mu q_\nu, D^+_\nu q_\mu]\Bigg), \\ S_{\text{B}} &= \mathcal{J}a^2\sum_{n,\mu,\nu} \frac{1}{2}\text{tr}\{B_{\mu\nu}(n) f_{\mu\nu}(n)\}, \\ B_{\mu\nu} &= i\left(\frac{1}{\lambda_\mu}[q_\mu, D^+_\nu q_\mu] - \frac{1}{\lambda_\nu}[q_\nu, D^+_\mu q_\nu]\right). \end{aligned} \tag{49}$$

Notice that the terms S_A and $S_{T'}$ are odd in a given component q_μ of the gluon field, while the other terms are even. Since the propagator is diagonal, this implies [33,35,36] that[11] $\langle S_{\text{A}}\rangle_0 = \langle S_{\text{T}'}\rangle_0 = 0$, and also that $\langle S^{\text{int}}_{\text{gluon}} S_{\text{A}}\rangle_{0c} = \langle S_{\text{A}} S_{\text{B}}\rangle_{0c} = 0$.

2. The effective action

Expanding now Eq. (32) up to terms quadratic in the background field, we obtain the following expression for $S_{\text{eff}}[B]$,

$$\begin{aligned} S_{\text{eff}}|_{\mathcal{O}(B^2),\mathcal{O}(g^0)} &= S_c + \frac{1}{2}\left(\langle S^{\text{int}}_{\text{gluon}}\rangle_0 - \frac{1}{2}\langle (S^{\text{int}}_{\text{gluon}})^2\rangle_{0c}\right) \\ &\quad - \frac{1}{2}\langle (S_{\text{A}})^2\rangle_{0c} - \frac{1}{2}\langle (S_{\text{B}})^2\rangle_{0c} + \langle S_{\text{T}}\rangle_0 \\ &\quad + (\langle S_{\text{B}}\rangle_0 - \langle S^{\text{int}}_{\text{gluon}} S_{\text{B}}\rangle_{0c}) \\ &\equiv S_c + \Delta S_{\text{g}} + \Delta S_{\text{A}} + \Delta S_{\text{B}} + \Delta S_{\text{T}} + \Delta S_{\text{gB}}. \end{aligned} \tag{50}$$

Here we have taken into account the remarks after Eq. (49), and the fact that in four dimensions the ghost contribution exactly cancels half of the gluon contribution from $S^{\text{int}}_{\text{gluon}}$ [33–36]. Terms have been grouped so that each contribution is separately gauge-invariant [36].

The evaluation of the various terms is performed generalizing the techniques developed in [36] to the anisotropic case. Since such a generalization is straightforward, here we simply list the results, giving in Table I the contribution $\Delta K_{\mu\nu}$ of each term to the quantity $K_{\mu\nu}$ [see Eqs. (26) and (27)] in front of $1/2\int d^4x\, \text{tr}\mathcal{F}^2_{\mu\nu}(x)$. The relevant technical

[10] In these quantities one should in principle include also the higher-order terms mentioned above in footnote 9, by properly redefining $f_{\mu\nu}$.

[11] This clearly remains true also if higher-order terms neglected in Eq. (36) are included in the definition of $f_{\mu\nu}$, see footnotes 9 and 10. Since $S^{\text{int}}_{\text{gluon}}$ is $\mathcal{O}(B)$, the only contribution of a higher-order term which should still be considered is that of the $\mathcal{O}(a^3 B\partial B)$ term in Eq. (36) to $\langle S_{\text{B}}\rangle_0 \propto \sum \text{tr}\{\langle B_{\mu\nu}\rangle_0 t^a\} f^a_{\mu\nu}$; however, (global) background gauge invariance implies that $\text{tr}\{\langle B_{\mu\nu}\rangle_0 t^a\} \propto B^a$, and so higher-order terms can be safely ignored.

TABLE I. Contribution of the various terms in Eq. (50) to $K_{\mu\nu}$ in the effective action, Eq. (26).

Term	$\Delta K_{\mu\nu}$
$\Delta S_{\rm g}$	$-\frac{N_c}{3(4\pi)^2}\left[-\gamma+\log\frac{1}{(aM)^2}+\frac{8}{3}\right]$
	$-\frac{N_c}{3}[\mathcal{G}_{\mu\nu}(\lambda)-\mathcal{G}_\mu(\lambda)-\mathcal{G}_\mu(\lambda)+\mathcal{G}(\lambda)]$
$\Delta S_{\rm A}$	$\frac{4N_c}{(4\pi)^2}\left[-\gamma+\log\frac{1}{(aM)^2}+2\right]$
	$+N_c[\mathcal{G}_{\mu\nu}(\lambda)-2\mathcal{G}_\mu(\lambda)-2\mathcal{G}_\nu(\lambda)+4\mathcal{G}(\lambda)]$
$\Delta S_{\rm B}$	$\frac{N_c}{4}\left[\mathcal{Z}(\lambda)\left(\frac{1}{\lambda_\mu^2}+\frac{1}{\lambda_\nu^2}\right)-\frac{\mathcal{Z}_\mu(\lambda)}{\lambda_\nu^2}-\frac{\mathcal{Z}_\nu(\lambda)}{\lambda_\mu^2}\right]$
$\Delta S_{\rm T}$	$\frac{N_c^2-1}{2N_c}\left[\frac{\mathcal{Z}_\mu(\lambda)}{\lambda_\nu^2}+\frac{\mathcal{Z}_\nu(\lambda)}{\lambda_\mu^2}\right]$
$\Delta S_{\rm gB}$	0

details can be found in the appendix of Ref. [36]. Summing up, one obtains

$$K_{\mu\nu}(a,\lambda)=K^{\rm div}(a)+\mathcal{K}_{\mu\nu}(\lambda)=\beta_0\log\frac{1}{(aM)^2}+\mathcal{K}_{\mu\nu}(\lambda), \tag{51}$$

with M a mass scale which sets the renormalization point, β_0 the first coefficient of the Yang-Mills β-function [41–43],

$$\beta_0=\frac{11}{3}\frac{N_c}{(4\pi)^2}, \tag{52}$$

and with $\mathcal{K}_{\mu\nu}$ finite, a-independent coefficients,

$$\begin{aligned}\mathcal{K}_{\mu\nu}(\lambda)=&\frac{11}{3}\frac{N_c}{(4\pi)^2}\left[-\gamma+\frac{64}{33}\right]\\&+N_c\left[\frac{2}{3}\mathcal{G}_{\mu\nu}(\lambda)-\frac{5}{3}(\mathcal{G}_\mu(\lambda)+\mathcal{G}_\nu)(\lambda)+\frac{11}{3}\mathcal{G}(\lambda)\right]\\&+\frac{N_c}{4}\left[\mathcal{Z}(\lambda)\left(\frac{1}{\lambda_\nu^2}+\frac{1}{\lambda_\mu^2}\right)-\frac{\mathcal{Z}_\mu(\lambda)}{\lambda_\nu^2}-\frac{\mathcal{Z}_\nu(\lambda)}{\lambda_\mu^2}\right]\\&+\frac{N_c^2-1}{2N_c}\left[\frac{\mathcal{Z}_\mu(\lambda)}{\lambda_\nu^2}+\frac{\mathcal{Z}_\nu(\lambda)}{\lambda_\mu^2}\right],\end{aligned} \tag{53}$$

where $\gamma\simeq 0.5772$ is the Euler–Mascheroni constant, and $\mathcal{G}_{\mu\nu}$, $\mathcal{G}_\mu$, $\mathcal{G}$, $\mathcal{Z}_\mu$ and $\mathcal{Z}$ are functions of $\{\lambda_\mu\}$ defined in terms of integrals involving the modified Bessel functions of the first kind. Their precise form is not needed for the analysis of the present section, and can be found in Appendix A, Eqs. (A1) and (A3).

To renormalize the theory and recover $O(4)$ invariance in the continuum limit it is enough to set

$$\begin{aligned}&\frac{1}{g^2}=\frac{1}{g_r^2(M)}+\beta_0\log\frac{1}{(aM)^2}=\beta_0\log\frac{1}{(a\Lambda)^2},\\&\frac{\delta\beta_{\mu\nu}}{2N_c}=\mathcal{K}_{\mu\nu}.\end{aligned} \tag{54}$$

Here $\Lambda=M\exp\{-1/(2\beta_0 g_r^2(M))\}$ is a renormalization-group-invariant mass scale, whose value can be determined by comparing lattice results with experiments. Since a shift $\delta\beta_{\mu\nu}\to\delta\beta_{\mu\nu}+\tilde\beta$ can always be reabsorbed in a redefinition of g, any choice satisfying the set of conditions $\delta\beta_{\mu\nu}-\delta\beta_{\rho\sigma}=\mathcal{K}_{\mu\nu}-\mathcal{K}_{\rho\sigma}$ will actually lead to restoration of $O(4)$ invariance at one-loop accuracy.[12] As we show in Appendix A, under a global rescaling $\lambda_\mu\to\zeta\lambda_\mu$, $\mathcal{Z}$ and $\mathcal{Z}_\mu$ get a factor ζ^2, $\mathcal{G}_{\mu\nu}$ and $\mathcal{G}_\mu$ are unchanged, and $\mathcal{G}\to\mathcal{G}+\frac{1}{(4\pi)^2}\log\zeta^2$, so that overall $\mathcal{K}_{\mu\nu}\to\mathcal{K}_{\mu\nu}+\beta_0\log\zeta^2$. Since the additive term can be canceled by $a\to\zeta a$, this means that the couplings $\beta_{\mu\nu}$ depend on a and λ_μ only through the combinations provided by the lattice spacings a_μ, as they should. As we have already remarked, to avoid redundancy one has to impose a condition on the λ_μ's, like, e.g., setting $\lambda_\alpha=1$ for some α, so using one of the lattice spacings as reference length, or imposing the symmetric condition $\prod_\alpha\lambda_\alpha=1$, thus using the volume of the elementary cell to define a.

We have compared our results with the ones available in the literature for the isotropic [33,34,36], $3+1$ [5,35,44,45] and $2+2$ [5] anisotropic cases.[13] In particular, we have successfully checked that in the isotropic case we recover the result of [36], and we have compared the differences of $\delta\beta_{\mu\nu}$ with the ones reported in Ref. [5] for the $3+1$ and $2+2$ cases. While there is full agreement for the $3+1$ case, we found a discrepancy in the analytic expression of one of the two independent differences in the $2+2$ case.[14] On the other hand, the numerical values also reported in Ref. [5] agree with ours. It has to be noted that the analytic result reported in Ref. [5] for that difference does not vanish when there is no anisotropy, as it should, so most likely it contains some misprint.

For future utility, we report the lowest-order approximation for the expectation value $\langle\mathcal{P}_{\mu\nu}\rangle$ of the plaquette terms. Setting $U_\mu(n)=e^{i\frac{g}{\lambda_\mu}q_\mu}$ and expanding in g, one finds

$$\langle\mathcal{P}_{\mu\nu}\rangle=\frac{g^2}{2N_c}\frac{1}{\lambda_\mu^2\lambda_\nu^2}\langle{\rm tr}F_{\mu\nu}^2\rangle_0+\mathcal{O}(g^3), \tag{55}$$

where $F_{\mu\nu}=\Delta_\mu^+q_\nu-\Delta_\nu^+q_\mu$ [see Eq. (39)], and $\langle\ldots\rangle_0$ has been defined in Eq. (33). A straightforward calculation yields

[12] More generally, it is the ratios $\beta_{\mu\nu}/\beta_{\rho\sigma}$ that will be constrained by the request of restoration of $O(4)$ invariance, see the discussion in Sec. II.

[13] In the $3+1$ anisotropy class one lattice spacing differs from the other three, e.g., $\lambda_4\neq\lambda_1=\lambda_2=\lambda_3$, while in the $2+2$ class the lattice spacings are equal pairwise, e.g., $\lambda_4=\lambda_1\neq\lambda_2=\lambda_3$.

[14] In the notation of Ref. [5], the discrepancy is in $\eta_{ff}^{(1)}-\eta_{cf}^{(1)}$, in particular in the coefficients of the quantities $\mathcal{B}_\xi^c(2,1)$ and $\mathcal{B}_\xi^f(2,1,1)$, for which we find respectively $\frac{N_c}{2}(\frac{1}{\xi^2}+\frac{5}{3\xi^4})$ and $\frac{N_c}{6}(\frac{1}{2}+\frac{1}{\xi^2})$.

$$\langle \mathcal{P}_{\mu\nu} \rangle = g^2 \frac{N_c^2 - 1}{2N_c} \left[\frac{\mathcal{Z}_\mu(\lambda)}{\lambda_\nu^2} + \frac{\mathcal{Z}_\nu(\lambda)}{\lambda_\mu^2} \right] + \mathcal{O}(g^3). \quad (56)$$

III. ANALYTIC CONTINUATION IN THE NONPERTURBATIVE APPROACH TO SOFT HIGH-ENERGY SCATTERING

In this section we use the results of Sec. II in the context of the nonperturbative approach to soft high-energy scattering. After a brief review of this approach (the interested reader can confer Refs. [20–27] for a more detailed discussion), we discuss its formulation on a Euclidean anisotropic lattice, and we refine the arguments of Ref. [13] on the analytic continuation back to Minkowski spacetime.

A. Euclidean approach to soft high-energy scattering

Soft high-energy scattering is characterized by small transferred momentum squared t, $|t| \lesssim 1$ GeV^2, and very large total center-of-mass energy squared s, $s \gg 1$ GeV^2. In the approach of Ref. [20], hadronic scattering amplitudes in the soft high-energy regime can be obtained from partonic scattering amplitudes after folding with appropriate hadronic wave functions. In particular, for meson-meson scattering the basic quantity is the scattering amplitude of two colorless transverse dipoles, which in the soft high-energy regime is given in impact-parameter space by the correlation function of two rectangular Minkowskian Wilson loops [21,22]. These Wilson loops are computed on the paths described by the classical trajectories of the dipoles, so forming a large hyperbolic angle χ in the longitudinal plane, and are cut at proper times $\pm T$ for infrared regularization purposes [6]. In turn, their (Minkowskian) correlation function is obtained after analytic continuation in the angular variable and in the length of the loops from the correlation function of two Euclidean Wilson loops of length $2T$ forming an angle θ in the longitudinal Euclidean plane [13–19]. This approach can be generalized to describe scattering processes involving baryons [20–26,46]. As the constructions and the arguments of this section are easily adapted to this case, we restrict the discussion to meson-meson (dipole-dipole) scattering for simplicity.

The relevant Euclidean correlator is given by[15]

$$\mathcal{G}_E(\theta, T; \vec{z}_\perp; \vec{R}_{1\perp}, f_1; \vec{R}_{2\perp}, f_2) = \frac{\langle \mathcal{W}_1^{(T)} \mathcal{W}_2^{(T)} \rangle_E}{\langle \mathcal{W}_1^{(T)} \rangle_E \langle \mathcal{W}_2^{(T)} \rangle_E} - 1, \quad (57)$$

where $\langle \ldots \rangle_E$ denotes the expectation value in the sense of the Euclidean functional integral, $\vec{z}_\perp$ is the impact-parameter distance between the dipoles, and $\vec{R}_{i\perp}$ and f_i are the transverse size of the dipoles and the longitudinal momentum fraction of the quarks in the two mesons, respectively ("dipole variables"). The Wilson loops $\mathcal{W}_{1,2}^{(T)}$ are computed on the following paths (see Fig. 1),

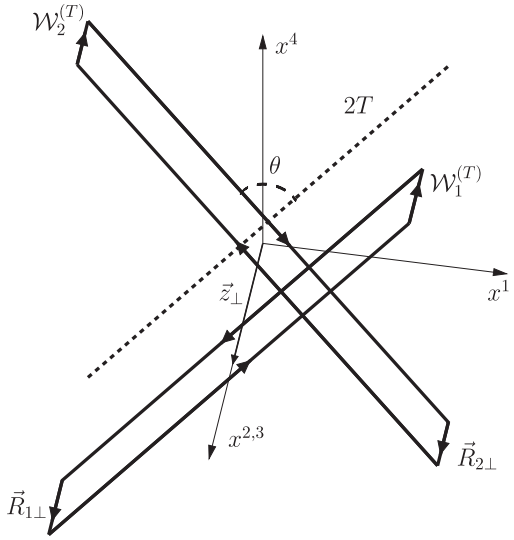

FIG. 1. The Euclidean Wilson loops $\mathcal{W}_1^T$ and $\mathcal{W}_2^T$, defined in Eq. (58).

$$\mathcal{C}_1^{(T)}: X_{E1}^\pm(\tau) = \pm u_1 \tau + z + f_1^\pm R_1 = \pm u_1 \tau + d_1^\pm,$$
$$\mathcal{C}_2^{(T)}: X_{E2}^\pm(\tau) = \pm u_2 \tau + f_2^\pm R_2 = \pm u_2 \tau + d_2^\pm, \quad (58)$$

with $\tau \in [-T, T]$, and closed by straight-line paths in the transverse plane at $\tau = \pm T$. The four-vectors $u_{1,2}$ are chosen to be $u_{1,2} = (\pm \sin\frac{\theta}{2}, \vec{0}_\perp, \cos\frac{\theta}{2})$, θ being the angle formed by the two trajectories, i.e., $u_1 \cdot u_2 = \cos\theta$. Moreover, $R_i = (0, \vec{R}_{i\perp}, 0)$, $z = (0, \vec{z}_\perp, 0)$ and $f_i^+ \equiv 1 - f_i$, $f_i^- \equiv -f_i$, with $f_i \in [0, 1]$. The Minkowskian correlation function is obtained from Eq. (57) by means of analytic continuation as follows [13,17],

$$\mathcal{G}_M(\chi, T; \vec{z}_\perp; \vec{R}_{1\perp}, f_1; \vec{R}_{2\perp}, f_2) = \mathcal{G}_E(-i\chi, iT; \vec{z}_\perp; \vec{R}_{1\perp}, f_1; \vec{R}_{2\perp}, f_2). \quad (59)$$

Physical amplitudes are finally obtained from $\mathcal{G}_M$ in the limit $T \to \infty$, and for asymptotically large $\chi \sim \log s$. It is worth mentioning that combining Eq. (59) with the $O(4)$ symmetry of the Euclidean theory one obtains the following crossing-symmetry relations [18,19],

$$\mathcal{G}_M(\chi, T; \vec{z}_\perp; \vec{R}_{1\perp}, f_1; -\vec{R}_{2\perp}, 1 - f_2) = \mathcal{G}_M(i\pi - \chi, T; \vec{z}_\perp; \vec{R}_{1\perp}, f_1; \vec{R}_{2\perp}, f_2), \quad (60)$$

which allow us to relate the scattering amplitudes in the direct (meson-meson) and crossed (meson-antimeson) channels.

[15]Here and in the following we denote by $\vec{v}_\perp$ a two-dimensional vector in the transverse plane.

The analytic continuation relation, Eq. (59), has allowed studies of the correlators through nonperturbative Euclidean techniques [28,47–56]. For a brief review of the older results and a comparison to lattice data cf. [53–55].

B. Anisotropic lattice formalism

It is well known that the functional integral needs to be regularized to become a well-defined mathematical object. Furthermore, the analytic continuation relation Eq. (59) is meaningful only if a sufficiently wide analyticity domain exists. The first issue can be dealt with by discretizing the theory on a lattice, so that the relevant Wilson loop correlator can then be computed nonperturbatively, for example by means of numerical simulations, using off-axis operators to approximate the continuum Wilson loops. Numerical simulations using an isotropic lattice have been reported in Refs. [53–55]. Unfortunately, only a discrete set of angles is accessible in this case; furthermore, for each angle one has to use a different off-axis Wilson loop, which makes the angular dependence even less analytically controllable. Since our purpose here is to study the analytic dependence on θ and T, it is more convenient to use an appropriate anisotropic lattice keeping fixed the Wilson-loop operator, which allows us to expose the dependence on the relevant variables in the action. In this way we make the functional integral a well-defined object, and at the same time we can study the analyticity domain of the correlator.

To avoid complications related to the well-known difficulties in treating fermions on the lattice, in this study we consider the *quenched* approximation of QCD, i.e., the pure-gauge theory case. We hope to return in a future paper on the inclusion of fermionic effects, which may be more important than usually expected for soft high-energy processes (see Refs. [28,56]).

A good choice is to use the anisotropic action discussed previously, Eq. (5), taking the anisotropy parameters to be such that the long sides of the Wilson loops lie in a lattice plane at 45° from two of the lattice axes, and are of fixed length. This amounts to set

$$\lambda_4(\theta,\bar{T}) = \frac{1}{\sqrt{2}\bar{T}\cos\frac{\theta}{2}}, \qquad \lambda_1(\theta,\bar{T}) = \frac{1}{\sqrt{2}\bar{T}\sin\frac{\theta}{2}},$$
$$\lambda_2(\theta,\bar{T}) = \lambda_3(\theta,\bar{T}) = 1, \tag{61}$$

where $\bar{T} \equiv T/T_0$ with T_0 some fixed length, and θ is restricted to $\theta \in (0,\pi)$ without loss of generality [18]. This yields for the plaquette coefficients

$$C_{41}(\theta,\bar{T}) = \frac{1}{C_{23}(\theta,\bar{T})} = \frac{1}{\bar{T}^2\sin\theta},$$
$$C_{42}(\theta,\bar{T}) = C_{43}(\theta,\bar{T}) = \frac{1}{C_{12}(\theta,\bar{T})} = \frac{1}{C_{13}(\theta,\bar{T})} = \tan\frac{\theta}{2}. \tag{62}$$

Notice that the following relations hold,

$$\lambda_4^2(\theta,\bar{T}) = \frac{C_{42}(\theta,\bar{T})}{C_{23}(\theta,\bar{T})}, \qquad \lambda_1^2(\theta,\bar{T}) = \frac{C_{12}(\theta,\bar{T})}{C_{23}(\theta,\bar{T})},$$
$$\mathcal{J}(\theta,\bar{T}) = \bar{T}^2\sin\theta = C_{23}(\theta,\bar{T}). \tag{63}$$

The action defined by Eq. (5), with anisotropy parameters Eq. (61), will be denoted by $S[U;\theta,\bar{T}]$, and the corresponding expectation value will be denoted by $\langle\ldots\rangle_{\theta,\bar{T}}$.

The lattice Wilson loops are defined as

$$\mathcal{W}_{Li}^{(T_0)} = \frac{1}{N_c}\mathrm{tr}\{W_i^+ H_i^+ W_i^{-\dagger} H_i^{-\dagger}\}, \tag{64}$$

where the "tilted" Wilson lines $W_i^\pm$ are defined as (see Fig. 2)

$$W_i^\pm = \prod_{j=-t_0}^{t_0-1} \mathcal{U}^{(i)}(jv_{1,2} + d_{Li}^\pm), \tag{65}$$

where $v_{1,2} = (\pm 1,0,0,1)$, $t_0 = \frac{T_0}{a\sqrt{2}}$ with $t_0 \in \mathbb{N}$, and $d_{Li}^\pm = d_i^\pm/a$ denotes the transverse position in lattice units, see Eq. (58), while $H_i^\pm$ are the appropriate Wilson lines made of the usual link variables in the transverse plane, closing the loops.[16] It is clear that $\sqrt{2}t_0 = \frac{T_0}{a}$ is the distance in lattice units between the center of a long side of the loop and its endpoints, i.e., loosely speaking, the half-length in lattice units of the Wilson loops. The tilted links $\mathcal{U}^{(i)}(n)$ are appropriate functionals $\mathcal{U}^{(i)}[U;n]$ of the lattice links, which in the continuum limit have to satisfy[17] [see Eq. (4)]

$$\mathcal{U}^{(1)}(n) = \mathbf{1} + ia\bar{T}\sqrt{2}\left[\cos\frac{\theta}{2}A_4(x(n)) + \sin\frac{\theta}{2}A_1(x(n))\right] + \mathcal{O}(a^2),$$
$$\mathcal{U}^{(2)}(n) = \mathbf{1} + ia\bar{T}\sqrt{2}\left[\cos\frac{\theta}{2}A_4(x(n)) - \sin\frac{\theta}{2}A_1(x(n))\right] + \mathcal{O}(a^2), \tag{66}$$

and which under a gauge transformation behave as

[16] One can properly choose $\lambda_{2,3}$ and use tilted links also in the transverse plane. This would however leave the discussion and the conclusions of this section unchanged.

[17] The factor in front of the square brackets takes into account that the diagonal of a plaquette in the longitudinal plane has length $\sqrt{a_4^2 + a_1^2} = \sqrt{2}a\bar{T}$. Notice that we are using path-ordered Wilson loops, as it is customary on the lattice, rather than the time-ordered Wilson loops appearing in the formulas for the scattering amplitudes (see, e.g., Refs. [27,53]). This has no consequence on the results, as the theory is invariant under charge conjugation, and so under reversing the loop orientation.

$$\mathcal{U}^{(1)}(n) \to G(n)\mathcal{U}^{(1)}(n)G^\dagger(n+\hat{4}+\hat{1}),$$
$$\mathcal{U}^{(2)}(n) \to G(n)\mathcal{U}^{(2)}(n)G^\dagger(n+\hat{4}-\hat{1}). \quad (67)$$

The simplest possibility in building $\mathcal{U}^{(1,2)}$ is obviously to use a combination of the gauge transporters along the two shortest paths connecting opposite corners of an elementary plaquette, namely

$$U_1^{(1)}(n) = U_4(n)U_1(n+\hat{4}), \quad U_2^{(1)}(n) = U_1(n)U_4(n+\hat{1}),$$
$$U_1^{(2)}(n) = U_4(n)U_1^\dagger(n+\hat{4}-\hat{1}),$$
$$U_2^{(2)}(n) = U_1^\dagger(n-\hat{1})U_4(n-\hat{1}). \quad (68)$$

It is convenient to adopt a definition of $\mathcal{U}^{(j)}$ which is symmetric under the exchange $U_1^{(j)}(n) \leftrightarrow U_2^{(j)}(n)$. A viable choice is (see Fig. 3)

$$\mathcal{U}^{(j)}(n) = \mathrm{Proj}_{SU(N_c)}[U_1^{(j)}(n) + U_2^{(j)}(n)], \quad (69)$$

with $\mathrm{Proj}_{SU(N_c)}$ denoting the projection on $SU(N_c)$. This symmetry requirement comes out naturally if we want that the Wilson loop correlator satisfies on the lattice the same "crossing property" [18,19] that it satisfies in the continuum. It is easy to show that in the continuum the correlation function of the two Wilson loops $\mathcal{W}_{1,2}^{(T)}$, defined in Eq. (58), at angle $\pi-\theta$ is equal to the correlation function of $\mathcal{W}_{1,2}^{(T)}$ at angle θ but with the orientation of one of the loops being reversed. In formulas,

$$\langle \mathcal{W}_1^{(T)}\mathcal{W}_2^{(T)}\rangle_E|_{\theta=\pi-\vartheta} = \langle \mathcal{W}_1^{(T)}\mathcal{W}_2^{(T)*}\rangle_E|_{\theta=\vartheta}$$
$$= \langle \mathcal{W}_1^{(T)*}\mathcal{W}_2^{(T)}\rangle_E|_{\theta=\vartheta}. \quad (70)$$

In order to impose this symmetry on the lattice, let us first notice that the anisotropic lattice action defined by Eqs. (5) and (61) is invariant under the transformation $U = \Xi U^\Xi$ acting on the links, defined by

$$U_4(n) = U_1^\Xi(n^\Xi), \quad U_1(n) = U_4^\Xi(n^\Xi), \quad U_{2,3}(n) = U_{2,3}^\Xi(n^\Xi),$$
$$n_4^\Xi = n_1, \quad n_1^\Xi = n_4, \quad n_{2,3}^\Xi = n_{2,3}, \quad (71)$$

if at the same time one also sends $\theta \to \pi-\theta$. Indeed, it suffices to verify that $C_{42}(\pi-\theta,\bar{T}) = C_{12}(\theta,\bar{T})$ and $C_{41}(\pi-\theta,\bar{T}) = C_{41}(\theta,\bar{T})$ [see Eq. (62)]. Consequently, the one-loop corrections $\mathcal{K}_{\mu\nu}$ will transform in the same way as $C_{\mu\nu}$, as can be also verified explicitly. We have then $S[\Xi U;\theta,\bar{T}] = S[U;\pi-\theta,\bar{T}]$, and since the integration measure is clearly invariant, the expectation value of some observable $\mathcal{O}[U]$ satisfies $\langle\mathcal{O}[U]\rangle_{\pi-\theta,\bar{T}} = \langle\mathcal{O}[\Xi U]\rangle_{\theta,\bar{T}}$. In order to maintain the crossing property also on the lattice, the tilted links must therefore transform as

$$\mathcal{U}^{(1)}[\Xi U; n^\Xi] = \mathcal{U}^{(1)}[U;n],$$
$$\mathcal{U}^{(2)}[\Xi U; n^\Xi] = \mathcal{U}^{(2)\dagger}[U; n-\hat{4}+\hat{1}]. \quad (72)$$

One can then readily show that the definition Eq. (69) satisfies the properties Eq. (66), Eq. (67) and Eq. (72). In Appendix B we show that using Eq. (69) in the case of the compact $U(1)$ gauge theory one correctly recovers the continuum result of Ref. [17] in the weak-coupling limit.

One can then define the relevant Euclidean correlator as the continuum limit of the appropriate lattice correlator,

$$\mathcal{G}_E(\theta, T = T_0\bar{T}) = \lim_{a\to 0, V\to\infty} \mathcal{G}_L(\theta, T_0, \bar{T}; a, V),$$
$$\mathcal{G}_L(\theta, T_0, \bar{T}; a, V) \equiv \frac{\langle \mathcal{W}_{L1}^{(T_0)}\mathcal{W}_{L2}^{(T_0)}\rangle_{\theta,\bar{T}}}{\langle \mathcal{W}_{L1}^{(T_0)}\rangle_{\theta,\bar{T}}\langle \mathcal{W}_{L2}^{(T_0)}\rangle_{\theta,\bar{T}}} - 1, \quad (73)$$

where V is the lattice volume, and we have dropped the dependence on the impact parameter and on the dipole variables, since they play no role in the following.

C. Analytic continuation

We now argue that $\mathcal{G}_E(w,T)$ is analytic in a complex domain $\mathcal{D}$ which makes the analytic continuation relations Eq. (59) meaningful. Here w and T are now *complex* variables, which we parametrize as $w = \theta - i\chi$, with real θ,χ, and $T = T_0\bar{T} = T_0|\bar{T}|e^{i\frac{\varphi}{2}}$, with $\varphi \in (-2\pi, 2\pi]$. Since one has to take two possibly dangerous limits, i.e., the infinite-volume limit and the continuum limit, which currently are not under full theoretical control, our argument is not rigourous. Nevertheless, a few reasonable technical assumptions are sufficient to complete the proof.

The first thing to check is that the couplings, $\beta_{\mu\nu}(w,\bar{T})$, and the plaquette coefficients, $C_{\mu\nu}(w,\bar{T})$, are analytic functions of w and $\bar{T} = |\bar{T}|e^{i\frac{\varphi}{2}}$. This is obvious at tree level, since $\beta_{\mu\nu} = 2N_c/g^2$ and the only singular points of $C_{\mu\nu}$ are $w = n\pi$ with $n \in \mathbb{Z}$, and $\bar{T} = 0$. Analyticity of the one-loop corrections $\mathcal{K}_{\mu\nu}(w,\bar{T})$, and so of $\beta_{\mu\nu}(w,\bar{T})$ at the one-loop level, is studied in Appendix A.

The next step is to require that the theory has the desired continuum limit. This requires positivity of the real part of the action to guarantee convergence. The tree-level convergence conditions have been discussed in Ref. [13], and read

$$\mathrm{Re}C_{\mu\nu}(w,\bar{T}) > 0 \quad \forall\ \mu,\nu. \quad (74)$$

These conditions define a complex domain $\mathcal{D}$ which has been fully worked out in Ref. [13]. Although its detailed form will not be used here, it is worth mentioning that $\mathcal{D}$ is defined only in terms of the complex angle w and of φ, i.e., $|\bar{T}|$ is not restricted (except for asking $|\bar{T}| \neq 0$). The Euclidean region corresponds to $\theta \in (0,\pi)$, $\chi = 0$, $\varphi = 0$. The Minkowskian region $\theta = 0$, $\chi > 0$, $\varphi = \pi$ lies at the

boundary of $\mathcal{D}$, and so does also the "crossed" Minkowskian region $\theta = \pi$, $\chi < 0$, $\varphi = \pi$; we will refer to these as the "physical" boundaries of $\mathcal{D}$. Notice that both in the Euclidean and in the Minkowskian regions the restrictions on the angular variables do not lead to any loss of information [18]. As it is shown in details in Appendix A, the one-loop corrections $\mathcal{K}_{\mu\nu}(w, \bar{T})$ are analytic in $\mathcal{D}$. For small enough lattice spacing, the one-loop corrections will therefore not spoil the positivity of the real part of the action enforced at tree level, for any choice of parameters in a compact subdomain of $\mathcal{D}$.

At finite volume and finite lattice spacing, and at one-loop accuracy, we have therefore proved that the relevant correlators are analytic functions in a domain $\mathcal{D}$, within which positivity of the real part of the action is guaranteed. This domain of analyticity will survive the infinite volume limit if the convergence is uniform. Proving this is currently out of reach. However, if a lattice system has short-range interactions, then correlation functions of operators localized in some finite region $\mathcal{R}$ of spacetime will become insensitive to the lattice size when this exceeds the size of $\mathcal{R}$ by a few correlation lengths. Notice that T_0 is fixed, so that our operators are indeed localized. If interactions remain short-ranged throughout $\mathcal{D}$, then it is enough to take the lattice size required by the largest correlation length (within some compact subdomain of $\mathcal{D}$) to make finite-size corrections uniformly negligible. This essentially amounts to assuming that the theory remains confining as one moves in $\mathcal{D}$. Although we cannot prove this, we find it plausible: for example, it is easy to see that it is true at strong coupling by means of a character expansion.

At this point one has to take the continuum limit. This limit is expected to exist and be finite within $\mathcal{D}$ (again, a rigorous proof is out of question). In particular, Wilson-loop operators renormalize multiplicatively [57,58], so that the normalized correlation function appearing in Eq. (73) does not require any further renormalization on top of the renormalization of the couplings in the action, discussed in the previous section. A rigorous proof of uniform convergence is currently out of reach; however, deviations from the continuum limit are expected to be of order $\mathcal{O}(a)$, independently of w and $\bar{T}$, and in this case it is possible to make them uniformly negligible.

The conclusion, within the present accuracy, is that G_E is analytic in the complex domain $\mathcal{D}$, which, as shown in Ref. [13], is sufficiently wide to make the analytic continuation relation Eq. (59) and the crossing-symmetry relations Eq. (60) fully meaningful.

As it was implicit in the discussion above, singularities in the correlator may develop at the boundaries of $\mathcal{D}$. As the analytic continuation Eq. (59) is formally equivalent to the usual definition of the Minkowskian correlator making use of the "$-i\varepsilon$" prescription [18], no singularities are expected at the physical boundaries of the domain. The anisotropic action itself is singular at $\theta = 0, \pi$, $\chi = 0$ and $\theta = 0, \pi$, $\chi = \infty$, but as this is an artifact of the construction it is not clear if true singularities are present there. At finite T (i.e., for Wilson loops of finite physical length), no singularity is expected in the Euclidean correlator ($\chi = 0$) also at $\theta = 0, \pi$; however, as $T \to \infty$, a true singularity is expected to appear there, which has its physical origin in the relation between the correlator Eq. (57) at $\theta = 0, \pi$ and the static dipole-dipole potential [59–62]. This is also supported by numerical results [53,54]. On the other hand, the points $\theta = 0, \chi = \infty$ and $\theta = \pi, \chi = -\infty$, in the limit $T \to \infty$, are the ones actually relevant to soft scattering at asymptotically high energy, where the approach initiated by Ref. [20] applies. A better understanding of the correlator near these points would help in the study of the asymptotic high-energy behavior of scattering amplitudes and total cross sections. In particular, in order to establish that the expressions for the scattering amplitudes derived in this approach satisfy unitarity, it is crucial to show that for vanishing θ and large χ the correlator is a properly bounded function of the impact parameter and of the dipole variables. Furthermore, the existence (or not) of the strict $\chi \to \infty$ limit at fixed impact parameter, and the properties of the correlator in this limit, are closely connected to the issue of universality of hadronic total cross sections observed in experiments (see, e.g., Refs. [63,64] and references therein). For more details on these problems, we invite the interested reader to confer Ref. [28].

Other singularities could appear when $|T| \to 0$ or $|T| \to \infty$. Working at fixed T_0, this corresponds to $|\bar{T}| \to 0$ or $|\bar{T}| \to \infty$, which are again singular points of the anisotropic action. However, since the analytic continuation to the physical boundaries requires only the phase of $\bar{T}$ to be changed, one can take as well $T_0 = |T|$ and $\bar{T} = \exp\{i\frac{\varphi}{2}\}$, and study the two limits above by changing the length of the tilted Wilson loops.[18] The above limits therefore correspond to the limit of tilted Wilson loops of vanishing or infinite length. In the first case no singularity is expected; in any case this limit is irrelevant for our purposes. On the other hand, the limit of infinite length is the one entering the physical scattering amplitudes. In this case, the short-ranged nature of strong interactions (which is assumed to remain unchanged throughout the analyticity domain) implies that distant parts of the two Wilson loops do not "feel" each other, i.e., those parts of the loops that lie beyond a certain distance from the centers interact mutually only very weakly, and essentially contribute only to the self-interaction of the loops. These contributions are canceled by the normalization factors, so that the correlator becomes basically insensitive to the length of the loops beyond some critical value, and a finite limit $|T| \to \infty$ is therefore expected. In the Euclidean case, this has already been checked on the lattice, although in an isotropic setting

[18] In the continuum limit the choice of T_0 should be irrelevant, as long as it is compensated by the appropriate choice of $\bar{T}$.

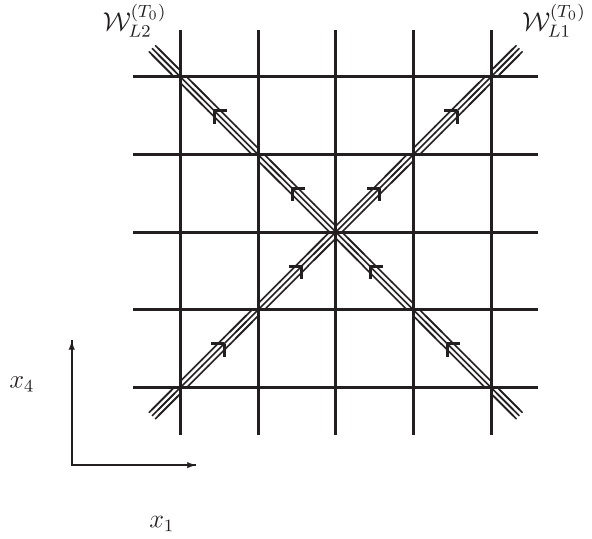

FIG. 2. The "tilted" lattice Wilson loops $\mathcal{W}_{L1}^{(T_0)}$ and $\mathcal{W}_{L2}^{(T_0)}$, Eqs. (64) and (65), projected on the longitudinal plane.

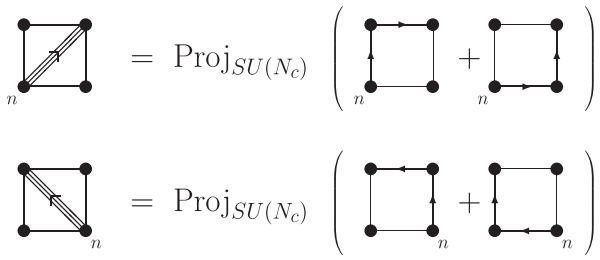

FIG. 3. The tilted links of Eq. (69), built from the shortest paths connecting opposite corners of a plaquette.

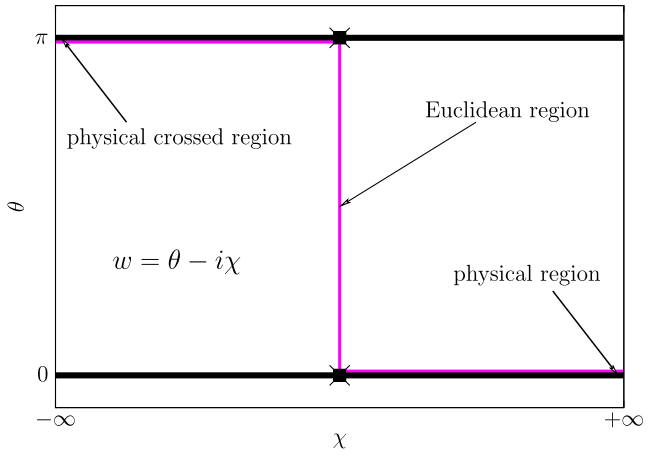

FIG. 4 (color online). Analyticity domain of the Wilson-loop correlator with the infrared cutoff removed, $\mathcal{C}_E$ [see Eq. (75)]. The solid black lines indicate the boundaries of the domain, and crosses signal the singularities.

[53]. As discussed in Ref. [13], the boundedness and the analyticity properties of the correlator as a function of T imply through the Phragmén-Lindelöf theorem (see, e.g., Ref. [65]) that the analytic continuation to Minkowski spacetime and the infinite-length limit commute. Setting $\mathcal{C}_{E,M} = \lim_{T\to\infty}\mathcal{G}_{E,M}$, this means that one can obtain the physical correlator by means of an analytic continuation in the angular variable only, i.e.,

$$\mathcal{C}_M(\chi;\vec{z}_\perp;\vec{R}_{1\perp},f_1;\vec{R}_{2\perp},f_2) = \mathcal{C}_E(-i\chi;\vec{z}_\perp;\vec{R}_{1\perp},f_1;\vec{R}_{2\perp},f_2). \tag{75}$$

The analyticity domain for $\mathcal{C}_E(w=\theta-i\chi)$, already discussed in Ref. [13], is clearly not changed by one-loop corrections, and it is simply the strip $\theta\in(0,\pi)$, $\chi\in\mathbb{R}$, shown in Fig. 4.

IV. LONGITUDINALLY RESCALED ACTION

The results of Sec. II can be used to obtain some insight in the approach to high-energy scattering based on longitudinally rescaled actions [6–12]. The physical idea behind this approach is that in high-energy scattering processes the longitudinal directions appear highly Lorentz-contracted, so that it should be possible to achieve an effective description through an appropriately rescaled action. While initially only a classical rescaling was considered [6–9], in recent years the effects of quantum corrections have been computed by means of anisotropic renormalization in the continuum theory [10–12]. Here we will consider the same problem in the lattice approach, which will allow us to clarify, to some extent, the structure of the action in the limit of large anisotropy. Notice that the anisotropy class $(2+2)$ is the same considered in Ref. [5].

On the lattice, the tree-level anisotropic action is given by Eq. (1) with the following anisotropy parameters,

$$\lambda_4^{(LR)}(\xi) = \lambda_1^{(LR)}(\xi) = \xi, \quad \lambda_2^{(LR)}(\xi) = \lambda_3^{(LR)}(\xi) = 1. \tag{76}$$

In the following, the superscript LR is used to specify that this particular choice has been made. We will refer to directions 4 and 1 as longitudinal, and directions 2 and 3 as transverse, and use the notation $n_\parallel=(n_4,n_1)$, $n_\perp=(n_2,n_3)$, $a_\parallel = a_4 = a_1 = a/\xi$, $a_\perp = a_2 = a_3 = a$. The plaquette coefficients $C_{\mu\nu}^{(LR)}$ in the anisotropic action read

$$\begin{aligned} &C_{23}^{(LR)}(\xi) = \frac{1}{\xi^2}, \\ &C_{24}^{(LR)}(\xi) = C_{21}^{(LR)}(\xi) = C_{34}^{(LR)}(\xi) = C_{31}^{(LR)}(\xi) = 1, \\ &C_{41}^{(LR)}(\xi) = \xi^2. \end{aligned} \tag{77}$$

The interesting case is that of large ξ. Taking naïvely the limit $\xi\to\infty$ in the tree-level action, the transverse-transverse plaquette term drops from the action, while the

longitudinal-longitudinal term yields essentially a "delta function" forcing the longitudinal links to be trivial. The resulting effective action would read

$$S^{\rm tree}_{\rm lat} \underset{\xi\to\infty}{\to} S^{(2D)} = \sum_{n_\perp} S^{(2)}_\chi(n_\perp) + S^{(3)}_\chi(n_\perp),$$
$$S^{(\mu)}_\chi(n_\perp) = \frac{\beta}{2N_c}\sum_{n_\parallel}\sum_{\alpha=4,1}{\rm tr}\{[\Delta^+_\alpha U_\mu(n)][\Delta^+_\alpha U_\mu(n)]^\dagger\}, \tag{78}$$

which describes a set of independent 2D principal chiral models involving the transverse link variables, each one living in the longitudinal plane at a given point $n_\perp$ in the transverse plane. Here Δ^+_α has been redefined by omitting the λ_α factor [see Eq. (15)]. Taking into account quantum corrections, however, a different coupling has to be used for each of the three different kinds of plaquette terms, namely $\beta^{(LR)}_{\parallel\parallel} = \beta^{(LR)}_{41}$ for the longitudinal-longitudinal term, $\beta^{(LR)}_{\perp\perp} = \beta^{(LR)}_{23}$ for the transverse-transverse term and $\beta^{(LR)}_{\parallel\perp} = \beta^{(LR)}_{42} = \beta^{(LR)}_{43} = \beta^{(LR)}_{12} = \beta^{(LR)}_{13}$ for the longitudinal-transverse terms. Recall that the quantum corrections are of the form

$$\mathcal{K}^{(LR)}_{\mu\nu}(\xi) = \frac{11}{3}\frac{N_c}{(4\pi)^2}\left[-\gamma+\frac{64}{33}\right] + \Delta\mathcal{G}^{(LR)}_{\mu\nu}(\xi) + \Delta\mathcal{Z}^{(LR)}_{\mu\nu}(\xi)$$
$$= -\beta_0\log c^2 + \Delta\mathcal{G}^{(LR)}_{\mu\nu}(\xi) + \Delta\mathcal{Z}^{(LR)}_{\mu\nu}(\xi), \tag{79}$$

with $\Delta\mathcal{G}^{(LR)}_{\mu\nu}$ and $\Delta\mathcal{Z}^{(LR)}_{\mu\nu}$ containing respectively the contributions of the $\mathcal{G}$- and $\mathcal{Z}$-integrals, Eqs. (A1) and (A3). Obviously, $\Delta\mathcal{G}^{(LR)}_{42} = \Delta\mathcal{G}^{(LR)}_{43} = \Delta\mathcal{G}^{(LR)}_{12} = \Delta\mathcal{G}^{(LR)}_{13}$, and similarly for $\Delta\mathcal{Z}^{(LR)}_{\mu\nu}$. Using the large-$\xi$ behavior of these integrals, derived in Appendix A, one gets

$$\frac{\beta^{(LR)}_{\perp\perp}(a,\xi)}{2N_c} = \beta_0\log\frac{1}{(a\Lambda c)^2} + 2\frac{N_c^2-1}{2N_c}\frac{1}{4\pi}\log\xi^2 + \Delta\mathcal{G}^{(LR),fin}_{\perp\perp}(\xi) + \Delta\mathcal{Z}^{(LR),fin}_{\perp\perp}(\xi),$$
$$\frac{\beta^{(LR)}_{\parallel\parallel}(a,\xi)}{2N_c} = \beta_0\log\frac{1}{(a\Lambda c)^2} + \Delta\mathcal{G}^{(LR),fin}_{\parallel\parallel}(\xi) + \Delta\mathcal{Z}^{(LR),fin}_{\parallel\parallel}(\xi),$$
$$\frac{\beta^{(LR)}_{\parallel\perp}(a,\xi)}{2N_c} = \beta_0\log\frac{1}{(a\Lambda c)^2} + \frac{N_c}{4}\frac{1}{4\pi}\log\xi^2 + \Delta\mathcal{G}^{(LR),fin}_{\parallel\perp}(\xi) + \Delta\mathcal{Z}^{(LR),fin}_{\parallel\perp}(\xi), \tag{80}$$

where the superscript fin on a quantity indicates that it is finite in the limit $\xi\to\infty$. If we keep the transverse spacing $a_\perp = a$ fixed, then taking $\xi\to\infty$ means taking $a_\parallel = a_\perp/\xi$ to zero, i.e., taking the continuum limit in the longitudinal plane only. One then sees that in general it is not allowed to discard the transverse-transverse plaquette term, since $\sum_{n_\parallel}\frac{\beta_{\perp\perp}}{2N_c}\xi^{-2}\mathcal{P}_{23} = \sum_{n_\parallel}\frac{\beta_{\perp\perp}}{2N_c}(\frac{a_\parallel}{a_\perp})^2\mathcal{P}_{23}$ contains the right power of $a_\parallel$ to become the two-dimensional integral over the longitudinal plane in the limit $a_\parallel\to 0$.

The action can now be recast in a form appropriate for a set of coupled two-dimensional principal chiral models. To this end, it is convenient to introduce the following couplings,

$$\beta_{(2D)}(a_\parallel,a_\perp) = \left.\frac{\beta^{(LR)}_{\parallel\perp}(a,\xi)}{2N_c}\right|_{\mathcal{O}(\xi^0)} = \frac{N_c}{2}\frac{1}{4\pi}\log\frac{1}{a_\parallel\Lambda^{(2D)}(a_\perp)},$$
$$\tilde\beta_{(2D)}(a_\parallel,a_\perp) = \left.\frac{\beta^{(LR)}_{\perp\perp}(a,\xi)}{2N_c}\right|_{\mathcal{O}(\xi^0)} = \frac{N_c^2-1}{2N_c}\frac{1}{\pi}\log\frac{1}{a_\parallel\tilde\Lambda^{(2D)}(a_\perp)},$$
$$\hat\beta_{(2D)}(a_\perp) = \left.\frac{\beta^{(LR)}_{\parallel\parallel}(a,\xi)}{2N_c}\right|_{\mathcal{O}(\xi^0)} = \beta_0\log\frac{1}{(a_\perp\Lambda c)^2}, \tag{81}$$

where the $a_\perp$-dependent scales $\Lambda^{(2D)}$ and $\tilde\Lambda^{(2D)}$ are given in terms of the original Λ-scale as follows,

$$\Lambda^{(2D)}(a_\perp) = \Lambda c(a_\perp\Lambda c)^{\frac{16\pi\beta_0}{N_c}-1}e^{-\frac{8\pi}{N_c}[\Delta\mathcal{G}^{(LR),fin}_{\parallel\perp}(\infty)+\Delta\mathcal{Z}^{(LR),fin}_{\parallel\perp}(\infty)]},$$
$$\tilde\Lambda^{(2D)}(a_\perp) = \Lambda c(a_\perp\Lambda c)^{\frac{4\pi\beta_0 N_c}{N_c^2-1}-1}e^{-\frac{2\pi N_c}{N_c^2-1}[\Delta\mathcal{G}^{(LR),fin}_{\perp\perp}(\infty)+\Delta\mathcal{Z}^{(LR),fin}_{\perp\perp}(\infty)]}. \tag{82}$$

The action can be equivalently written as follows,

$$S_{\rm lat} = \sum_{n_\perp} S^{(2)}_\chi(n_\perp) + S^{(3)}_\chi(n_\perp) + S_{\rm int1}(n_\perp) + S_{\rm int2}(n_\perp), \tag{83}$$

where $S^{(\mu)}_\chi$ correspond to principal chiral models,

$$S^{(\mu)}_\chi(n_\perp) = \beta_{(2D)}(a_\parallel,a_\perp) \times\sum_{n_\parallel}\sum_{\alpha=4,1}{\rm tr}\{[\Delta^+_\alpha U_\mu(n)][\Delta^+_\alpha U_\mu(n)]^\dagger\}, \tag{84}$$

and the interaction terms read

$$S_{\rm int1}(n_\perp) = \tilde\beta_{(2D)}(a_\parallel,a_\perp)\sum_{n_\parallel}\frac{a_\parallel^2}{a_\perp^2}\mathcal{P}_{23}(n),$$
$$S_{\rm int2}(n_\perp) = \beta_{(2D)}(a_\parallel,a_\perp)\sum_{\mu=2,3}\sum_{n_\parallel}\sum_{\alpha=4,1}[2N_c\mathcal{P}_{\mu\alpha}(n) - {\rm tr}\{[\Delta^+_\alpha U_\mu(n)][\Delta^+_\alpha U_\mu(n)]^\dagger\}] + \hat\beta_{(2D)}(a_\perp)\sum_{n_\parallel}2N_c\frac{a_\perp^2}{a_\parallel^2}\mathcal{P}_{41}(n). \tag{85}$$

The only approximation made here is to discard $o(\xi^0)$ terms in the couplings, so that this is just a rewriting of the original action in the limit of large ξ. Nevertheless, this expression displays a remarkable feature: the coupling $\beta_{(2D)}$ is precisely the one appropriate for a 2D principal

chiral model with lattice spacing $a_\parallel$, to one-loop accuracy (see, e.g., Ref. [66]). The principal chiral models are clearly not independent, with the precise form of the interaction dictated by the full 4D action. Notice that identifying the longitudinal links with $U_\mu = \exp\{ia_\parallel q_\mu\}$ and expanding in powers of $a_\parallel$, the summands in the interaction term $S_{\rm int2}$ are of order $\mathcal{O}(a_\parallel^2)$, as appropriate to obtain an integral over the longitudinal plane in the naïve $a_\parallel \to 0$ limit, so there is no reason to discard these contributions.[19] It is not surprising that the interaction terms cannot be neglected *a priori*: after all, no matter how anisotropic the lattice is made, by construction the action has to describe QCD in the continuum limit. The possibility or not to neglect the interaction terms will depend on the properties of the specific observables relevant to the study of high-energy processes.

The expectation values of the different plaquette terms can be used to estimate the range of applicability of the expressions above. Using Eq. (56) one gets to lowest order [see Eq. (A37)]

$$\begin{aligned}
\langle \mathcal{P}_{41}\rangle &= g^2\frac{N_c^2-1}{N_c}\frac{\mathcal{Z}_\parallel^{(LR)}(\xi)}{\xi^2} \simeq g^2\frac{N_c^2-1}{N_c}\frac{z_{10}}{\xi^2},\\
\langle \mathcal{P}_{42}\rangle &= g^2\frac{N_c^2-1}{2N_c}\left(\mathcal{Z}_\parallel^{(LR)}(\xi)+\frac{\mathcal{Z}_\perp^{(LR)}(\xi)}{\xi^2}\right) \simeq g^2\frac{N_c^2-1}{2N_c}z_{10},\\
\langle \mathcal{P}_{23}\rangle &= g^2\frac{N_c^2-1}{N_c}\mathcal{Z}_\perp^{(LR)}(\xi) \simeq g^2\frac{N_c^2-1}{N_c}\frac{1}{4\pi}\log\xi^2,
\end{aligned} \tag{86}$$

with z_{10} a constant defined in Eq. (A21), so that in order to have small fluctuations one needs $g^2\log\xi \ll 1$. Together with the basic assumption $g^2 \ll 1$, and the fact that we work here at $\xi \gg 1$, the requirement $\langle \mathcal{P}_{23}\rangle \ll 1$ defines the range of applicability of perturbation theory, which in terms of the lattice spacings reads

$$1 \gg a_\perp\Lambda \gg a_\parallel\Lambda \gg (a_\perp\Lambda)^{1+\frac{4\pi\beta_0 N_c}{N_c^2-1}} \geq (a_\perp\Lambda)^2. \tag{87}$$

The important fact is that Eq. (87) does not allow us to strictly take the continuum limit in the longitudinal plane before taking $a_\perp$ to zero. This was already suggested in Ref. [10], although there it is claimed that perturbation theory makes sense only for ξ slightly larger than 1; according to our results, a much larger region seems to be accessible.

A comparison of our results with those of Refs. [10–12] is not straightforward. First of all, since we use a different regularization, we expect different finite contributions to the renormalization of the couplings in the limit $a \to 0$ (at fixed ξ); ultraviolet divergences, on the other hand, have to be the same. Indeed, to account for a change in the cutoff, Orland and collaborators integrate over an anisotropic ellipsoidal shell in momentum space, while on the lattice a change in the cutoff requires us to integrate over an anisotropic parallelepipedal shell. It would be interesting to compare the divergent terms in the limit $\xi \to \infty$, but in Refs. [10–12] only the case $\xi \gtrsim 1$ is studied.

We conclude by noticing that a similar recasting of the action can be done also in the case discussed in Sec. III, considering the limit of large $\bar{T}$. The results are briefly discussed in Appendix C.

V. CONCLUSIONS

In this paper we have performed the renormalization of $SU(N_c)$ gauge theories on a general four-dimensional anisotropic lattice, with different lattice spacings in the four directions, using perturbation theory to one-loop order and the background field method on the lattice (Sec. II). To avoid the complications related to the introduction of fermions on the lattice, we have discussed here the pure-gauge case only. For general anisotropy, the various couplings in the gauge action need to be properly tuned in order to recover $O(4)$ invariance in the continuum limit, as already observed in Ref. [5]. In practice, however, only two parameters need to be tuned for this purpose, which reduce to one if there is at least a pair of equal lattice spacings (and to none in the $3+1$ case). A simple nonperturbative scheme for this tuning, based on the string tensions obtained in different lattice planes, has also been proposed.

In Sec. III, the possibility to vary continuously the anisotropy parameters has been exploited in the context of the nonperturbative approach to soft high-energy hadron-hadron scattering based on Wilson loops [20–27], in order to refine the arguments of Ref. [13] on the analyticity properties of the relevant Wilson-loop correlators. The results reported here give further support to the possibility of performing the desired analytic continuation between Euclidean and Minkowski space, and thus on the very possibility of using Euclidean techniques to study soft high-energy processes. This is particularly important in the light of recent progress on the problem of hadronic total cross sections [28,56], which is based on the possibility of recovering the physical amplitudes starting from Euclidean space.

In Sec. IV we have applied our results to the longitudinally rescaled actions considered in Refs. [6–12] to study high-energy scattering in QCD. At the classical level, in the limit of large anisotropy the action reduces to that of a set of coupled two-dimensional principal chiral models, living in the longitudinal plane at each point of the transverse plane. Our main result in this context is that this interpretation holds also at the one-loop level, as the bare coupling resulting in the free part of each principal chiral model

[19] We notice that working in the axial gauge $U_1 = 1$ and expanding $S_{\rm int2}$ to $\mathcal{O}(a_\parallel^2)$, the resulting expression is quadratic in q_4 and the corresponding integration can be carried out. This leads to the appearance of complicated, nonlocal interaction terms involving the transverse link variables.

behaves appropriately as a function of the longitudinal lattice spacing. The precise form of the interactions among the principal chiral models is dictated by the full gauge action. However, the limit of large anisotropy cannot be taken independently of the continuum limit, at least in the perturbative approach. Indeed, the requirement of small gauge field fluctuations defines a range of validity of the form $1 \gg a_\perp \Lambda \gg a_\parallel \Lambda \gg (a_\perp \Lambda)^{1+\gamma}$ for the longitudinal and transverse lattice spacings $a_\parallel$ and $a_\perp$, where Λ is the QCD scale and $\gamma > 0$. Nevertheless, our findings suggest that there may be a deeper relation between gauge theories and principal chiral models than just at the classical level.

There are several open directions for future studies. An obvious possibility is the inclusion of fermions in the analysis. This is particularly relevant to the nonperturbative approach to soft high-energy scattering, since the presence or not of dynamical fermions seems to have large effects on total cross sections [28,56]. It would be interesting to extend the perturbative analysis to nonorthogonal lattices, which would allow us to use on-axis Wilson loops in the relevant lattice correlator. However, in this case more terms appear in the action, so leading to a more intricate calculation.

ACKNOWLEDGMENTS

I want to thank E. Meggiolaro and T. G. Kovács for many useful remarks. I also acknowledge interesting discussions and correspondence with G. Burgio, E. Follana, S. Katz, and P. Orland. This work is supported by the Hungarian Academy of Sciences under "Lendület" Grant No. LP2011-011.

APPENDIX A: THE $\mathcal{G}$- AND $\mathcal{Z}$-INTEGRALS

In the expression for the one-loop contributions $\mathcal{K}_{\mu\nu}$, Eq. (53), there appear a few integrals involving the modified Bessel functions of the first kind $I_n(z)$, which are special cases of the integrals

$$\mathcal{G}_n(\lambda) = \begin{cases} \displaystyle\int_0^\infty d\rho\rho \left[\prod_{\alpha=1}^4 \lambda_\alpha \tilde{I}_0^{(n_\alpha)}(2\lambda_\alpha^2\rho)\right], & n \neq (0,0,0,0), \\ \displaystyle\int_0^\infty d\rho\rho \left\{\left[\prod_{\alpha=1}^4 \lambda_\alpha \tilde{I}_0(2\lambda_\alpha^2\rho)\right] - \Theta(\rho-1)\frac{1}{(4\pi\rho)^2}\right\}, & n = (0,0,0,0), \end{cases}$$
$$\mathcal{Z}_n(\lambda) = \int_0^\infty d\rho \left[\prod_{\alpha=1}^4 \lambda_\alpha \tilde{I}_0^{(n_\alpha)}(2\lambda_\alpha^2\rho)\right], \tag{A1}$$

defined for a general four-vector of integers n, where

$$\tilde{I}_n(z) \equiv e^{-z} I_n(z), \qquad \tilde{I}_n^{(m)}(z) \equiv (-\partial/\partial z)^m \tilde{I}_n(z) \tag{A2}$$

and $\Theta(z)$ is the step function. In particular, in Eq. (53) we have denoted as follows the relevant cases,

$$\mathcal{G}_{\mu\nu} = \mathcal{G}_n|_{n_\alpha=\delta_{\alpha\mu}+\delta_{\alpha\nu}}, \qquad \mathcal{G}_\mu = \mathcal{G}_n|_{n_\alpha=\delta_{\alpha\mu}}, \qquad \mathcal{G} = \mathcal{G}_n|_{n_\alpha=0},$$
$$\mathcal{Z}_\mu = \mathcal{Z}_n|_{n_\alpha=\delta_{\alpha\mu}}, \qquad \mathcal{Z} = \mathcal{Z}_n|_{n_\alpha=0}. \tag{A3}$$

These integrals are not all independent; in particular, the following sum rules hold,

$$\sum_{\mu=1}^4 2\lambda_\mu^2 \mathcal{Z}_\mu = \prod_{\mu=1}^4 \lambda_\mu = \mathcal{J}^{-1}, \qquad \sum_{\mu=1}^4 2\lambda_\mu^2 \mathcal{G}_\mu = \mathcal{Z},$$
$$2\lambda_\mu^2 \mathcal{G}_\mu + \sum_{\nu\neq\mu} 2\lambda_\nu^2 \mathcal{G}_{\mu\nu} = \sum_{\nu\neq\mu} 2\lambda_\nu^2 \mathcal{G}_\nu. \tag{A4}$$

A global rescaling $\lambda_\alpha \to \zeta\lambda_\alpha$ ($\zeta > 0$) of the anisotropy parameters can be essentially reabsorbed in $\mathcal{Z}_n$ and in $\mathcal{G}_n$ by changing variables to $\rho' = \zeta^2\rho$, which brings about a multiplicative factor for $\mathcal{Z}_n$, and an additive contribution proportional to $\log\zeta$ to $\mathcal{G}_0$. More precisely, we find

$$\mathcal{Z}_n(\zeta\lambda) = \zeta^2 \mathcal{Z}_n(\lambda),$$
$$\mathcal{G}_n(\zeta\lambda) = \begin{cases} \mathcal{G}_n(\lambda), & n \neq 0, \\ \mathcal{G}_0(\lambda) + \frac{1}{(4\pi)^2}\log\zeta^2, & n = 0. \end{cases} \tag{A5}$$

1. Analyticity properties

We discuss now the analyticity properties of $\mathcal{K}_{\mu\nu}$. It is clear that for $\lambda_\alpha \neq 0\ \forall\,\alpha$ these depend only on the analyticity properties of the $\mathcal{G}$- and $\mathcal{Z}$-integrals defined in Eq. (A4). Since these are integrals of analytic functions of ρ and $\{\lambda_\alpha\}$, it suffices to show that they converge uniformly in $\{\lambda_\alpha\}$ within some complex domain. In turn, a sufficient condition for this is that we can bound the modulus of the integrand uniformly in $\{\lambda_\alpha\}$ by some function f, whose integral is also convergent. To do this, we need the following inequalities,

$$|\tilde{I}_0(z)| \leq \tilde{I}_0(\mathrm{Re}\, z), \qquad |\tilde{I}_1(z)| \leq \tilde{I}_0(\mathrm{Re}\, z),$$
$$|\tilde{I}_0(z)| \leq 1 \quad \text{if } \mathrm{Re}\, z \geq 0, \tag{A6}$$

which are easily proved using the integral representation for $I_n(z)$. We also need the monotonicity property

$$\frac{\partial}{\partial x}\tilde{I}_0(x) \le 0, \quad \forall\, x \in \mathbb{R}, \tag{A7}$$

and the asymptotic behavior of $\tilde{I}_0(z)$,

$$\tilde{I}_0(z) \sim \frac{1}{\sqrt{2\pi z}}\left(1 - \frac{1}{8z} + \mathcal{O}(z^{-2})\right), \tag{A8}$$

valid for $|\arg z| < \pi$ (see, e.g., Ref. [67]).

(a) The quantities $\mathcal{Z}$ and $\mathcal{Z}_\mu$ are given by the product of the analytic factor $\mathcal{J}^{-1} = \prod_\alpha \lambda_\alpha$ and an integral of the product of functions $\tilde{I}_0$ and, in the case of $\mathcal{Z}_\mu$, also $\tilde{I}_0 - \tilde{I}_1$, so that we may write

$$\begin{aligned}\mathcal{Z}(\{\lambda_\alpha\}) &= \mathcal{J}^{-1}\tilde{\mathcal{Z}}(\{\lambda_\alpha^2\}),\\ \mathcal{Z}_\mu(\{\lambda_\alpha\}) &= \mathcal{J}^{-1}\tilde{\mathcal{Z}}_\mu(\{\lambda_\alpha^2\}).\end{aligned} \tag{A9}$$

For $\{\lambda_\alpha\}$ such that for every α one has $\mathrm{Re}\lambda_\alpha^2 \in [u_\alpha, v_\alpha]$, with $u_\alpha, v_\alpha \in \mathbb{R}$, $0 < u_\alpha < v_\alpha < \infty$, the first two inequalities in Eq. (A6) and the monotonicity property Eq. (A7) tell us that a possible choice for $f(\rho)$ to bound the modulus of the integrands both in $\tilde{\mathcal{Z}}$ and $\tilde{\mathcal{Z}}_\mu$ is $f(\rho) = 2\prod_\alpha f_\alpha(\rho)$, $f_\alpha(\rho) = \tilde{I}_0(2u_\alpha\rho)$. In particular, this shows that $\tilde{\mathcal{Z}}$ and $\tilde{\mathcal{Z}}_\mu$ are analytic functions of $\{\lambda_\alpha^2\}$.

(b) To study $\mathcal{G}$ we split the integral into two parts, $\int_0^\infty = \int_0^1 + \int_1^\infty$. For the first piece, the third inequality in Eq. (A6) indicates that we can take $f(\rho) = \rho$. The integrand of the second piece is conveniently written as

$$\rho\left(\prod_{\alpha=1}^{4}\lambda_\alpha\tilde{I}_0(2\lambda_\alpha^2\rho) - \frac{1}{(4\pi\rho)^2}\right) = \frac{1}{(4\pi\rho)^2}\tilde{f}(\{\lambda_\alpha\},\rho), \tag{A10}$$

where $\tilde{f}$ is analytic $\forall\lambda$ and $\rho \neq 0$, and furthermore it is certainly bounded for $\mathrm{Re}\,\lambda_\alpha^2 \in [u_\alpha, v_\alpha]$ and $\rho \in [0,\infty)$, since it has a finite limit as $\rho \to \infty$, see Eq. (A8). In this case we can then take $f(\rho) = M/(4\pi\rho)^2$ for a properly chosen constant M.

(c) Finally, analyticity properties of $\mathcal{G}_\mu$ and $\mathcal{G}_{\mu\nu}$ are inherited from $\mathcal{Z}$ and $\mathcal{Z}_\mu$. Indeed, since one can bring derivatives under the sign of integral due to uniform convergence, one shows immediately that

$$\begin{aligned}\lambda_\mu\frac{\partial}{\partial\lambda_\mu}\mathcal{Z}(\lambda) &= \mathcal{Z}(\lambda) - 4\lambda_\mu^2\mathcal{G}_\mu(\lambda),\\ \lambda_\nu\frac{\partial}{\partial\lambda_\nu}\mathcal{Z}_\mu(\lambda) &= \mathcal{Z}_\mu(\lambda) - 4\lambda_\nu^2\mathcal{G}_{\mu\nu}(\lambda) \quad (\nu \neq \mu).\end{aligned} \tag{A11}$$

Notice that $\mathcal{G}_\mu$ and $\mathcal{G}_{\mu\nu}$ are of the form

$$\begin{aligned}\mathcal{G}_\mu(\{\lambda_\alpha\}) &= \mathcal{J}^{-1}\tilde{\mathcal{G}}_\mu(\{\lambda_\alpha^2\}),\\ \mathcal{G}_{\mu\nu}(\{\lambda_\alpha\}) &= \mathcal{J}^{-1}\tilde{\mathcal{G}}_{\mu\nu}(\{\lambda_\alpha^2\}),\end{aligned} \tag{A12}$$

with $\tilde{\mathcal{G}}_\mu$ and $\tilde{\mathcal{G}}_{\mu\nu}$ analytic in $\{\lambda_\alpha^2\}$.

In conclusion, $\mathcal{K}_{\mu\nu}$ are analytic in any compact domain with $\mathrm{Re}\lambda_\alpha^2 > 0$, $\forall\alpha$. For our purposes, it is convenient to extend further the domain of analyticity. To this end, notice that for real positive λ_α, one can rewrite $\mathcal{Z}$, $\mathcal{Z}_\mu$, $\mathcal{G}_\mu$ and $\mathcal{G}_{\mu\nu}$ as follows by exploiting their behavior under global rescaling, Eq. (A5),

$$\begin{aligned}&\mathcal{Z}(\{\lambda_\alpha\}) = \tilde{\mathcal{Z}}(\{\mathcal{J}\lambda_\alpha^2\}), \quad \mathcal{Z}_\mu(\{\lambda_\alpha\}) = \tilde{\mathcal{Z}}_\mu(\{\mathcal{J}\lambda_\alpha^2\}),\\ &\mathcal{G}_\mu(\{\lambda_\alpha\}) = \mathcal{J}\tilde{\mathcal{G}}_\mu(\{\mathcal{J}\lambda_\alpha^2\}), \quad \mathcal{G}_{\mu\nu}(\{\lambda_\alpha\}) = \mathcal{J}\tilde{\mathcal{G}}_{\mu\nu}(\{\mathcal{J}\lambda_\alpha^2\}),\end{aligned} \tag{A13}$$

where Eqs. (A9) and (A12) have been used. The domain of analyticity of these quantities can thus be straightforwardly extended to $\mathrm{Re}(\mathcal{J}\lambda_\alpha^2) > 0$. Furthermore, for real positive λ_α, one has

$$\begin{aligned}\mathcal{G} = \int d\rho\rho\left[\mathcal{J}\left(\prod_\alpha \tilde{I}_0(2\mathcal{J}\lambda_\alpha^2\rho)\right) - \Theta(\rho-1)\frac{1}{(4\pi\rho)^2}\right]\\ - \frac{1}{(4\pi)^2}\log\mathcal{J},\end{aligned} \tag{A14}$$

where we have used Eq. (A5) again. By the same token used above in point (b), the first term in Eq. (A14) is analytic for $\mathrm{Re}\mathcal{J}\lambda_\alpha^2 > 0$. The logarithmic term is an analytic function in the cut complex plane for $|\arg\mathcal{J}| < \pi$, so we conclude that $\mathcal{K}_{\mu\nu}$ are analytic also in the domain defined by $\mathrm{Re}(\mathcal{J}\lambda_\alpha^2) > 0$, $|\arg\mathcal{J}| < \pi$.

We now analyze the specific case discussed in Sec. III, corresponding to the following choice of anisotropy parameters,

$$\begin{aligned}&\lambda_4(\theta,\bar{T}) = \frac{1}{\sqrt{2}\bar{T}\cos\frac{\theta}{2}}, \qquad \lambda_1(\theta,\bar{T}) = \frac{1}{\sqrt{2}\bar{T}\sin\frac{\theta}{2}},\\ &\lambda_2(\theta,\bar{T}) = \lambda_3(\theta,\bar{T}) = 1,\end{aligned} \tag{A15}$$

which, in the light of the extension of the analyticity domain discussed above, can be recast more conveniently as follows,

$$\mathcal{J}(\theta,\bar{T})\lambda_4^2(\theta,\bar{T}) = C_{42}(\theta,\bar{T}),$$
$$\mathcal{J}(\theta,\bar{T})\lambda_1^2(\theta,\bar{T}) = C_{12}(\theta,\bar{T}),$$
$$\mathcal{J}(\theta,\bar{T}) = \mathcal{J}(\theta,\bar{T})\lambda_2^2(\theta,\bar{T}) = \mathcal{J}(\theta,\bar{T})\lambda_3^2(\theta,\bar{T}) = C_{23}(\theta,\bar{T}). \quad \text{(A16)}$$

As functions of complex angle and length, $C_{\mu\nu}(w,\bar{T})$ are analytic everywhere, except at $w = n\pi$ with $n \in \mathbb{Z}$, and $|\bar{T}| = 0$. Since the domain $\mathcal{D}$ considered in Sec. III is defined by $\mathrm{Re}C_{\mu\nu}(w,\bar{T}) > 0$, in $\mathcal{D}$ one has that $|\arg \mathcal{J}(w,\bar{T})| < \frac{\pi}{2}$ and $\mathrm{Re}(\mathcal{J}(w,\bar{T})\lambda_\alpha^2(w,\bar{T})) > 0$, so that the $\mathcal{G}$- and $\mathcal{Z}$-integrals are analytic there, and in conclusion the one-loop corrections $\mathcal{K}_{\mu\nu}(w,\bar{T})$ are analytic in $\mathcal{D}$.

2. Large-$\bar{T}$ behavior

We now determine, for real $\bar{T}$, the large-$\bar{T}$ behavior of the $\mathcal{Z}$- and $\mathcal{G}$-integrals for the choice of anisotropy parameters of Eqs. (61) and (A15). To this end, it is convenient to define the following auxiliary quantities,

$$B_n(\theta,\bar{T}) = \int_0^\infty d\rho \left[\prod_\alpha \lambda_\alpha \tilde{I}_{n_\alpha}(2\lambda_\alpha^2\rho)\right],$$
$$D_n(\theta,\bar{T}) = \int_0^\infty d\rho\rho \left[\prod_\alpha \lambda_\alpha \tilde{I}_{n_\alpha}(2\lambda_\alpha^2\rho) - \Theta(\rho-1)\frac{1}{(4\pi\rho)^2}\right], \quad \text{(A17)}$$

where $\{\lambda_\alpha\}$ are chosen according to Eq. (61). It is straightforward to show that

$$\mathcal{Z} = B_n|_{n_\alpha=0}, \qquad \mathcal{Z}_\mu = B_n|_{n_\alpha=0} - B_n|_{n_\alpha=\delta_{\alpha\mu}},$$
$$\mathcal{G} = D_n|_{n_\alpha=0}, \qquad \mathcal{G}_\mu = D_n|_{n_\alpha=0} - D_n|_{n_\alpha=\delta_{\alpha\mu}},$$
$$\mathcal{G}_{\mu\nu} = D_n|_{n_\alpha=\delta_{\alpha\mu}+\delta_{\alpha\nu}} + D_n|_{n_\alpha=0} - D_n|_{n_\alpha=\delta_{\alpha\mu}} - D_n|_{n_\alpha=\delta_{\alpha\nu}}. \quad \text{(A18)}$$

A rather simple calculation shows that at large $\bar{T}$

$$B_n(\theta,\bar{T}) = \frac{1}{\bar{T}^2\sin\theta}\left\{\frac{1}{4\pi}\tilde{I}_{n_4}(0)\tilde{I}_{n_1}(0)\log\bar{T}^2 + b_n(\theta) + o(\bar{T}^0)\right\},$$
$$b_n(\theta) = \tilde{I}_{n_4}(0)\tilde{I}_{n_1}(0)\int_0^1 d\rho\tilde{I}_{n_2}(2\rho)\tilde{I}_{n_3}(2\rho) + \frac{1}{4\pi}\int_0^\infty\frac{d\rho}{\rho}\left[\tilde{I}_{n_4}\left(\frac{\rho}{\cos^2\frac{\theta}{2}}\right)\tilde{I}_{n_1}\left(\frac{\rho}{\sin^2\frac{\theta}{2}}\right) - \Theta(1-\rho)\tilde{I}_{n_4}(0)\tilde{I}_{n_1}(0)\right],$$
$$D_n(\theta,\bar{T}) = -\frac{1}{(4\pi)^2}\log\bar{T}^2 + d_n(\theta) + o(\bar{T}^0),$$
$$d_n(\theta) = \frac{1}{4\pi}\int_0^\infty d\rho\left[\frac{1}{\sin\theta}\tilde{I}_{n_4}\left(\frac{\rho}{\cos^2\frac{\theta}{2}}\right)\tilde{I}_{n_1}\left(\frac{\rho}{\sin^2\frac{\theta}{2}}\right) - \Theta(\rho-1)\frac{1}{4\pi\rho}\right]. \quad \text{(A19)}$$

It is now straightforward to obtain the large-$\bar{T}$ behavior of the relevant quantities. For the $\mathcal{Z}$-integrals we have

$$\mathcal{Z}(\theta,\bar{T}) = \frac{1}{\bar{T}^2\sin\theta}\left\{\frac{1}{4\pi}\log\bar{T}^2 + z_{00} + \tilde{z}_{00}(\theta) + o(\bar{T}^0)\right\},$$
$$\mathcal{Z}_4(\theta,\bar{T}) = \frac{1}{\bar{T}^2\sin\theta}\left\{\frac{1}{4\pi}\log\bar{T}^2 + z_{00} + \tilde{z}_{10}(\theta) + o(\bar{T}^0)\right\},$$
$$\mathcal{Z}_1(\theta,\bar{T}) = \frac{1}{\bar{T}^2\sin\theta}\left\{\frac{1}{4\pi}\log\bar{T}^2 + z_{00} + \tilde{z}_{01}(\theta) + o(\bar{T}^0)\right\},$$
$$\mathcal{Z}_2(\theta,\bar{T}) = \mathcal{Z}_3(\theta,\bar{T}) = \frac{1}{\bar{T}^2\sin\theta}\{z_{10} + o(\bar{T}^0)\}, \quad \text{(A20)}$$

where we have introduced the following quantities,

$$z_{nm} = \int_0^1 d\rho\tilde{I}_0^{(n)}(2\rho)\tilde{I}_0^{(m)}(2\rho),$$
$$\tilde{z}_{nm}(\theta) = \frac{1}{4\pi}\int_0^\infty\frac{d\rho}{\rho}\left\{\tilde{I}_0^{(n)}\left(\frac{\rho}{\cos^2\frac{\theta}{2}}\right)\tilde{I}_0^{(m)}\left(\frac{\rho}{\sin^2\frac{\theta}{2}}\right) - \tilde{I}_0^{(n)}(0)\tilde{I}_0^{(m)}(0)\Theta(1-\rho)\right\}. \quad \text{(A21)}$$

For the $\mathcal{G}$-integrals we find

$$\mathcal{G}(\theta,\bar{T}) = -\frac{1}{(4\pi)^2}\log\bar{T}^2 + \tilde{g}_{00}(\theta) + o(\bar{T}^0), \qquad \mathcal{G}_{41}(\theta,\bar{T}) = \tilde{g}_{11}(\theta) + o(\bar{T}^0),$$
$$\mathcal{G}_4(\theta,\bar{T}) = \tilde{g}_{10}(\theta) + o(\bar{T}^0), \qquad \mathcal{G}_1(\theta,\bar{T}) = \tilde{g}_{01}(\theta) + o(\bar{T}^0), \tag{A22}$$

where we have introduced the following quantities,

$$\tilde{g}_{nm}(\theta) = \frac{1}{4\pi}\int_0^1 d\rho \frac{1}{\sin\theta}\tilde{I}_0^{(n)}\left(\frac{\rho}{\cos^2\frac{\theta}{2}}\right)\tilde{I}_0^{(m)}\left(\frac{\rho}{\sin^2\frac{\theta}{2}}\right), \qquad (n,m)\neq(0,0),$$
$$\tilde{g}_{00}(\theta) = \frac{1}{4\pi}\int_0^\infty d\rho\left\{\frac{1}{\sin\theta}\tilde{I}_0\left(\frac{\rho}{\cos^2\frac{\theta}{2}}\right)\tilde{I}_0\left(\frac{\rho}{\sin^2\frac{\theta}{2}}\right) - \Theta(\rho-1)\frac{1}{4\pi\rho}\right\}, \tag{A23}$$

while the remaining integrals are all $o(\bar{T}^0)$. One can now easily determine the contributions of both kinds of terms to $\mathcal{K}_{\mu\nu}$, namely

$$\Delta\mathcal{G}_{\mu\nu}(\theta,\bar{T}) = N_c\left[\frac{2}{3}\mathcal{G}_{\mu\nu}(\theta,\bar{T}) - \frac{5}{3}(\mathcal{G}_\mu(\theta,\bar{T}) + \mathcal{G}_\nu(\theta,\bar{T})) + \frac{11}{3}\mathcal{G}(\theta,\bar{T})\right]$$
$$= \beta_0\log\frac{1}{\bar{T}^2} + \Delta\mathcal{G}_{\mu\nu}^{fin}(\theta,\bar{T}), \tag{A24}$$

where β_0 is defined in Eq. (52), and

$$\Delta\mathcal{G}_{41}^{fin}(\theta,\bar{T}) = N_c\left[\frac{11}{3}\tilde{g}_{00}(\theta) - \frac{5}{3}(\tilde{g}_{10}(\theta)+\tilde{g}_{01}(\theta)) + \frac{2}{3}\tilde{g}_{11}(\theta)\right] + o(\bar{T}^0),$$
$$\Delta\mathcal{G}_{42}^{fin}(\theta,\bar{T}) = \Delta\mathcal{G}_{43}^{fin}(\theta,\bar{T}) = N_c\left[\frac{11}{3}\tilde{g}_{00}(\theta) - \frac{5}{3}\tilde{g}_{10}(\theta)\right] + o(\bar{T}^0),$$
$$\Delta\mathcal{G}_{12}^{fin}(\theta,\bar{T}) = \Delta\mathcal{G}_{13}^{fin}(\theta,\bar{T}) = N_c\left[\frac{11}{3}\tilde{g}_{00}(\theta) - \frac{5}{3}\tilde{g}_{01}(\theta)\right] + o(\bar{T}^0),$$
$$\Delta\mathcal{G}_{23}^{fin}(\theta,\bar{T}) = o(\bar{T}^0), \tag{A25}$$

and

$$\Delta\mathcal{Z}_{\mu\nu}(\theta,\bar{T}) = \frac{N_c}{4}\left[\mathcal{Z}(\theta,\bar{T})\left(\frac{1}{\lambda_\nu^2(\theta,\bar{T})}+\frac{1}{\lambda_\mu^2(\theta,\bar{T})}\right) - \frac{\mathcal{Z}_\mu(\theta,\bar{T})}{\lambda_\nu^2(\theta,\bar{T})} - \frac{\mathcal{Z}_\nu(\theta,\bar{T})}{\lambda_\mu^2(\theta,\bar{T})}\right] + \frac{N_c^2-1}{2N_c}\left[\frac{\mathcal{Z}_\mu(\theta,\bar{T})}{\lambda_\nu^2(\theta,\bar{T})} + \frac{\mathcal{Z}_\nu(\theta,\bar{T})}{\lambda_\mu^2(\theta,\bar{T})}\right], \tag{A26}$$

where $\Delta\mathcal{Z}_{\mu\nu}$ can be split into a divergent and a finite part,

$$\Delta\mathcal{Z}_{41}(\theta,\bar{T}) = \Delta\mathcal{Z}_{41}^{div}(\theta,\bar{T}) + \Delta\mathcal{Z}_{41}^{fin}(\theta,\bar{T}),$$
$$\Delta\mathcal{Z}_{42}(\theta,\bar{T}) = \Delta\mathcal{Z}_{43}(\theta,\bar{T}) = \Delta\mathcal{Z}_{4\perp}^{div}(\theta,\bar{T}) + \Delta\mathcal{Z}_{4\perp}^{fin}(\theta,\bar{T}),$$
$$\Delta\mathcal{Z}_{12}(\theta,\bar{T}) = \Delta\mathcal{Z}_{13}(\theta,\bar{T}) = \Delta\mathcal{Z}_{1\perp}^{div}(\theta,\bar{T}) + \Delta\mathcal{Z}_{1\perp}^{fin}(\theta,\bar{T}),$$
$$\Delta\mathcal{Z}_{23}(\theta,\bar{T}) = \Delta\mathcal{Z}_{23}^{fin}(\theta,\bar{T}) = o(\bar{T}^0), \tag{A27}$$

with divergent parts given by

$$\Delta\mathcal{Z}_{41}^{div}(\theta,\bar{T}) = \frac{N_c^2-1}{2N_c}\frac{2}{\sin\theta}\frac{1}{4\pi}\log\bar{T}^2,$$
$$\Delta\mathcal{Z}_{4\perp}^{div}(\theta,\bar{T}) = \cot\frac{\theta}{2}\frac{N_c}{4}\frac{1}{4\pi}\log\bar{T}^2, \qquad \Delta\mathcal{Z}_{1\perp}^{div}(\theta,\bar{T}) = \tan\frac{\theta}{2}\frac{N_c}{4}\frac{1}{4\pi}\log\bar{T}^2, \tag{A28}$$

and finite parts given by

$$\Delta\mathcal{Z}_{41}^{fin}(\theta,\bar{T}) = \frac{N_c^2-1}{2N_c}\left[\frac{2}{\sin\theta}z_{00} + \cot\frac{\theta}{2}\tilde{z}_{01}(\theta) + \tan\frac{\theta}{2}\tilde{z}_{10}(\theta)\right] + \frac{N_c}{4}\left[\frac{2}{\sin\theta}\tilde{z}_{00}(\theta) - \cot\frac{\theta}{2}\tilde{z}_{01}(\theta) - \tan\frac{\theta}{2}\tilde{z}_{10}(\theta)\right] + o(\bar{T}^0),$$
$$\Delta\mathcal{Z}_{4\perp}^{fin}(\theta,\bar{T}) = \cot\frac{\theta}{2}\left[\frac{N_c}{4}(z_{00}+\tilde{z}_{00}(\theta)) + \frac{N_c^2-2}{4N_c}z_{10}\right] + o(\bar{T}^0),$$
$$\Delta\mathcal{Z}_{1\perp}^{fin}(\theta,\bar{T}) = \tan\frac{\theta}{2}\left[\frac{N_c}{4}(z_{00}+\tilde{z}_{00}(\theta)) + \frac{N_c^2-2}{4N_c}z_{10}\right] + o(\bar{T}^0), \tag{A29}$$

from which one can easily reconstruct the behavior of $\mathcal{K}_{\mu\nu}(\theta,\bar{T})$ up to $o(\bar{T}^0)$.

The results above allow us to easily derive the large-ξ behavior of the couplings when the anisotropy parameters are chosen appropriately for the longitudinally rescaled action of Sec. IV, i.e., $\lambda_4^{(LR)}=\lambda_1^{(LR)}=\xi$ and $\lambda_2^{(LR)}=\lambda_3^{(LR)}=1$, see Eq. (76). This is accomplished through the following steps. First of all, notice that $\lambda_\mu^{(LR)}$ are just a particular case of $\lambda_\mu(\theta,\bar{T})$, namely $\lambda_\mu^{(LR)}(\xi)=\lambda_\mu(\frac{\pi}{2},\frac{1}{\xi})$. Next, it is straightforward to show that

$$B_n\left(\frac{\pi}{2},\frac{1}{\xi}\right) = \xi^2 B_{\tilde{n}}\left(\frac{\pi}{2},\xi\right),$$
$$D_n\left(\frac{\pi}{2},\frac{1}{\xi}\right) = \frac{1}{(4\pi)^2}\log\xi^2 + D_{\tilde{n}}\left(\frac{\pi}{2},\xi\right), \tag{A30}$$

where $\tilde{n}_\mu = n_{\tilde{\mu}}$ with $\{\tilde{\mu}\}=\{\tilde{1},\tilde{2},\tilde{3},\tilde{4}\}=\{3,4,1,2\}$. Finally, one easily shows that

$$\frac{\xi^2}{[\lambda_\mu(\frac{\pi}{2},\frac{1}{\xi})]^2} = \frac{1}{[\lambda_{\tilde{\mu}}(\frac{\pi}{2},\xi)]^2}. \tag{A31}$$

Putting these results together one finds that

$$\Delta\mathcal{Z}_{\mu\nu}^{(LR)}(\xi) = \Delta\mathcal{Z}_{\mu\nu}\left(\frac{\pi}{2},\frac{1}{\xi}\right) = \Delta\mathcal{Z}_{\tilde{\mu}\tilde{\nu}}\left(\frac{\pi}{2},\xi\right),$$
$$\Delta\mathcal{G}_{\mu\nu}^{(LR)}(\xi) = \Delta\mathcal{G}_{\mu\nu}\left(\frac{\pi}{2},\frac{1}{\xi}\right) + \frac{1}{(4\pi)^2}\log\xi^2 = \Delta\mathcal{G}_{\tilde{\mu}\tilde{\nu}}^{fin}\left(\frac{\pi}{2},\xi\right). \tag{A32}$$

Explicitly,

$$\Delta\mathcal{G}_{23}^{(LR)}(\xi) = \Delta\mathcal{G}_{\perp\perp}^{(LR)}(\xi) = N_c\left[\frac{11}{3}\tilde{g}_{00}\left(\frac{\pi}{2}\right) - \frac{10}{3}\tilde{g}_{10}\left(\frac{\pi}{2}\right) + \frac{2}{3}\tilde{g}_{11}\left(\frac{\pi}{2}\right)\right] + o(\xi^0),$$
$$\Delta\mathcal{G}_{42}^{(LR)}(\xi) = \Delta\mathcal{G}_{43}^{(LR)}(\xi) = \Delta\mathcal{G}_{12}^{(LR)}(\xi) = \Delta\mathcal{G}_{13}^{(LR)}(\xi) = \Delta\mathcal{G}_{\parallel\perp}^{(LR)}(\xi) = N_c\left[\frac{11}{3}\tilde{g}_{00}\left(\frac{\pi}{2}\right) - \frac{5}{3}\tilde{g}_{10}\left(\frac{\pi}{2}\right)\right] + o(\xi^0),$$
$$\Delta\mathcal{G}_{41}^{(LR)}(\xi) = \Delta\mathcal{G}_{\parallel\parallel}^{(LR)}(\xi) = o(\xi^0), \tag{A33}$$

for the contributions $\Delta\mathcal{G}_{\mu\nu}^{(LR)}$, and

$$\Delta\mathcal{Z}_{23}^{(LR)}(\xi) = \Delta\mathcal{Z}_{\perp\perp}^{(LR),div}(\xi) + \Delta\mathcal{Z}_{\perp\perp}^{(LR),fin}(\xi),$$
$$\Delta\mathcal{Z}_{42}^{(LR)}(\xi) = \Delta\mathcal{Z}_{43}^{(LR)}(\xi) = \Delta\mathcal{Z}_{12}^{(LR)}(\xi) = \Delta\mathcal{Z}_{13}^{(LR)}(\xi) = \Delta\mathcal{Z}_{\parallel\perp}^{(LR),div}(\xi) + \Delta\mathcal{Z}_{\parallel\perp}^{(LR),fin}(\xi),$$
$$\Delta\mathcal{Z}_{41}^{(LR)}(\xi) = \Delta\mathcal{Z}_{\parallel\parallel}^{(LR)}(\xi) = o(\xi^0), \tag{A34}$$

for the contributions $\Delta\mathcal{Z}_{\mu\nu}^{(LR)}$, with divergent parts

$$\Delta\mathcal{Z}_{\perp\perp}^{(LR),div}(\xi) = \frac{N_c^2-1}{2N_c}\frac{1}{2\pi}\log\xi^2,$$
$$\Delta\mathcal{Z}_{\parallel\perp}^{(LR),div}(\xi) = \frac{N_c}{4}\frac{1}{4\pi}\log\xi^2, \tag{A35}$$

and finite parts

$$\Delta\mathcal{Z}_{\perp\perp}^{(LR),fin}(\xi) = \frac{N_c^2-1}{N_c}\left(z_{00}+\tilde{z}_{10}\left(\frac{\pi}{2}\right)\right) + \frac{N_c}{2}\left(\tilde{z}_{00}-\tilde{z}_{10}\left(\frac{\pi}{2}\right)\right) + o(\xi^0),$$
$$\Delta\mathcal{Z}_{\parallel\perp}^{(LR),fin}(\xi) = \frac{N_c}{4}\left(z_{00}+\tilde{z}_{00}\left(\frac{\pi}{2}\right)\right) + \frac{N_c^2-2}{4N_c}z_{10} + o(\xi^0). \tag{A36}$$

We also report the values of the $\mathcal{Z}$-integrals,

$$\mathcal{Z}_\parallel^{(LR)}(\xi) = \mathcal{Z}_4^{(LR)}(\xi) = \mathcal{Z}_1^{(LR)}(\xi) = z_{10} + o(\xi^0),$$
$$\mathcal{Z}_\perp^{(LR)}(\xi) = \mathcal{Z}_2^{(LR)}(\xi) = \mathcal{Z}_3^{(LR)}(\xi) = \frac{1}{4\pi}\log\xi^2 + z_{00} + \tilde{z}_{10}\left(\frac{\pi}{2}\right) + o(\xi^0). \tag{A37}$$

APPENDIX B: ABELIAN CASE

In this Appendix we compute the Wilson-loop correlator considered in Sec. III in the compact $U(1)$ lattice theory and in the weak-coupling limit. The starting point is the 4D anisotropic lattice formulation for the $U(1)$ gauge group,

$$S_{\rm lat}^{U(1)} = \frac{1}{e^2}\sum_{n,\mu<\nu} C_{\mu\nu}(1-{\rm Re}U_{\mu\nu}(n)). \tag{B1}$$

Here the plaquettes are built with the $U(1)$ links $U_\mu(n) = \exp\{i\phi_\mu(n)\}$, and can be written as ${\rm Re}U_{\mu\nu}(n) = \cos\Phi_{\mu\nu}(n)$, with $\Phi_{\mu\nu}(n) = \phi_\mu(n) + \phi_\nu(n+\hat{\mu}) - \phi_\mu(n+\hat{\nu}) - \phi_\nu(n)$. The Haar measure is simply $\int dU_\mu(n) = \int_{-\pi}^{+\pi}\frac{d\phi_\mu(n)}{2\pi}$. Setting

$$\begin{aligned}
\mathcal{U}_1^{(1)}(n) &= U_4(n)U_1(n+\hat{4}) = e^{i[\phi_4(n)+\phi_1(n+\hat{4})]} = e^{i\varphi_1^{(1)}(n)},\\
\mathcal{U}_2^{(1)}(n) &= U_1(n)U_4(n+\hat{1}) = e^{i[\phi_4(n+\hat{1})+\phi_1(n)]} = e^{i\varphi_2^{(1)}(n)},\\
\mathcal{U}_1^{(2)}(n) &= U_4(n)U_1^*(n+\hat{4}-\hat{1})\\
&= e^{i[\phi_4(n)-\phi_1(n+\hat{4}-\hat{1})]} = e^{i\varphi_1^{(2)}(n)},\\
\mathcal{U}_2^{(2)}(n) &= U_1^*(n-\hat{1})U_4(n-\hat{1})\\
&= e^{i[\phi_4(n-\hat{1})-\phi_1(n-\hat{1})]} = e^{i\varphi_2^{(2)}(n)},
\end{aligned} \tag{B2}$$

the analogue of Eq. (69) is here

$$\begin{aligned}
\mathcal{U}^{(j)}(n) &= \frac{e^{i\varphi_1^{(j)}(n)} + e^{i\varphi_2^{(j)}(n)}}{|e^{i\varphi_1^{(j)}(n)} + e^{i\varphi_2^{(j)}(n)}|}\\
&= \exp\left(\frac{i}{2}(\varphi_1^{(j)}(n) + \varphi_2^{(j)}(n))\right){\rm sign}\left(\cos\frac{\Phi^{(j)}(n)}{2}\right),
\end{aligned} \tag{B3}$$

where $\Phi^{(1)}(n) = \Phi_{41}(n)$ and $\Phi^{(2)}(n) = \Phi_{41}(n-\hat{1})$. The Wilson loops are written as $\mathcal{W}_{Lk} = e^{i\Omega_k}\sigma_k T_k$, $k = 1, 2$, with

$$\begin{aligned}
\Omega_k = \frac{1}{2}\sum_{j=-t_0}^{t_0-1} &(\varphi_1^{(k)}(jv_k + d_{k+}) + \varphi_2^{(k)}(jv_k + d_{k+})\\
&- \varphi_1^{(k)}(jv_k + d_{k-}) - \varphi_2^{(k)}(jv_k + d_{k-})),
\end{aligned} \tag{B4}$$

σ_k the product of the sign factors appearing in Eq. (B3), and T_k the contribution from the transverse links.

The calculation is greatly simplified if we take the limit $\bar{T}\to\infty$ first.[20] Discarding the longitudinal-longitudinal plaquette term, enforcing the triviality of the transverse links, and using $1-{\rm Re}U_{\mu\alpha} = \frac{1}{2}|\Delta_\alpha^+ U_\mu|^2$ for trivial U_α links, one ends up with

$$\begin{aligned}
&S_{\rm lat}^{U(1)} \underset{\bar{T}\to\infty}{\to} \frac{1}{2e^2}\sum_{\mu=4,1} c^{(\mu)} \sum_{n_\parallel, n_\perp}\sum_{\alpha=2,3} |\Delta_\alpha^+ U_\mu(n)|^2,\\
&c^{(4)} = \tan\frac{\theta}{2}, \qquad c^{(1)} = \cot\frac{\theta}{2},
\end{aligned} \tag{B5}$$

where $n_\parallel = (n_4, n_1)$ and $n_\perp = (n_2, n_3)$. The Wilson loops simplify to $\mathcal{W}_{Lk}\to e^{i\Omega_k}\sigma_k$. Since there is no interaction between link variables living at different sites of the longitudinal plane, and between U_4 and U_1 variables, one easily sees that the tilted links of Eq. (B3) interact with each other only if they are separated by at most one lattice spacing in the longitudinal plane, which leads to factorization of the Wilson-loop correlation function and expectation values.

It is convenient now to rescale the phases as $\phi_\mu(n) = e\bar{\phi}_\mu(n_\parallel, x)$ with $x = en_\perp$ (notice that x is dimensionless), in order to take the weak-coupling limit. One then obtains for the action

$$S_{\rm lat}^{U(1)} \underset{\bar{T}\to\infty, e\to 0}{\to} \sum_{\mu=4,1} c^{(\mu)} \sum_{n_\parallel}\int d^2x \sum_{\alpha=2,3}\frac{1}{2}[\partial_\alpha\bar{\phi}_\mu(n_\parallel; x)]^2, \tag{B6}$$

and the integration measure in the weak-coupling limit becomes

$$\int_{-\pi}^{+\pi}\frac{d\phi_\mu(n)}{2\pi} \to \int_{-\infty}^{+\infty} d\bar{\phi}_\mu(n_\parallel; x), \tag{B7}$$

where we have omitted a factor $e/(2\pi)$ since it gets canceled in expectation values. We passed to the continuum notation for simplicity: as the action is quadratic, the resulting continuum Gaussian integrals are fully under control. The propagator is readily obtained,

$$\begin{aligned}
D_{\mu\nu}(n_\parallel, m_\parallel; x, y) &\equiv \langle\bar{\phi}_\mu(n_\parallel; x)\bar{\phi}_\nu(m_\parallel; y)\rangle\\
&= \delta_{\mu\nu}\delta_{n_\parallel m_\parallel}\frac{1}{c^{(\mu)}}D(x-y),
\end{aligned} \tag{B8}$$

where $D(x)$ is the 2D scalar propagator,

$$D(x) = -\frac{1}{2\pi}\log|x|. \tag{B9}$$

From here on angular brackets without subscripts denote the expectation value with respect to the action Eq. (B6). In the weak-coupling limit, $\cos\Phi_{\mu\nu} = 1 + \mathcal{O}(e^2)$ and we can neglect the sign factors in the expression for the Wilson loops, i.e., $\mathcal{W}_{Lk}\to e^{i\Omega_k}$. Since the action is quadratic, one has for the relevant correlation function as $e\to 0$

$$\lim_{\bar{T}\to\infty}\frac{\langle\mathcal{W}_1\mathcal{W}_2\rangle_{\theta,\bar{T}}}{\langle\mathcal{W}_1\rangle_{\theta,\bar{T}}\langle\mathcal{W}_2\rangle_{\theta,\bar{T}}} = \frac{e^{-\frac{1}{2}\langle(\Omega_1+\Omega_2)^2\rangle}}{e^{-\frac{1}{2}\langle\Omega_1^2\rangle}e^{-\frac{1}{2}\langle\Omega_2^2\rangle}} = e^{-\langle\Omega_1\Omega_2\rangle}. \tag{B10}$$

Using now the explicit expression for Ω_k, see Eqs. (B2) and (B4), and exploiting the fact that the propagator is diagonal in the link directions and in the longitudinal coordinates, a straightforward calculation gives

[20] Since there is actually no continuum limit to be taken, in this case the complications of the non-Abelian case are absent.

$$\langle \Omega_1 \Omega_2 \rangle = \frac{e^2}{2\pi} \cot\theta \log \frac{|\vec{z}_\perp + \frac{\vec{R}_{1\perp}}{2} + \frac{\vec{R}_{2\perp}}{2}||\vec{z}_\perp - \frac{\vec{R}_{1\perp}}{2} - \frac{\vec{R}_{2\perp}}{2}|}{|\vec{z}_\perp + \frac{\vec{R}_{1\perp}}{2} - \frac{\vec{R}_{2\perp}}{2}||\vec{z}_\perp - \frac{\vec{R}_{1\perp}}{2} + \frac{\vec{R}_{2\perp}}{2}|}, \tag{B11}$$

which agrees with the known result for $\mathcal{C}_E$ in the 4D $U(1)$ pure gauge theory in the continuum limit [17]. Here we have set $f_1 = f_2 = \frac{1}{2}$ for convenience, without any loss of information [54].

APPENDIX C: LARGE-$\bar{T}$ LIMIT OF THE $(\theta,\bar{T})$-DEPENDENT ACTION

For completeness, in this Appendix we report on the large-$\bar{T}$ limit of the anisotropic action with anisotropy parameters Eq. (61), discussed in Sec. III. The idea is that there could be some useful simplification if one takes $\bar{T} \to \infty$, corresponding to the limit of loops of infinite length, before taking the continuum limit. In full analogy with the discussion of Sec. IV, in this limit the action can be recast as that of a set of interacting principal chiral models, which however live now in the transverse plane at every site of the longitudinal plane. This is natural since the limit $\bar{T} \to \infty$ corresponds to taking the continuum limit in the transverse plane at fixed longitudinal spacing $a_\parallel \equiv \bar{T} a$, i.e., the same situation of Sec. IV but reversing the roles of the longitudinal and the transverse planes. In the large-$\bar{T}$ limit, the longitudinal-transverse couplings can be rewritten as follows,

$$\begin{aligned}
\beta^{(4)}_{2D}(a_\perp, a_\parallel, \theta) &= \frac{\beta_{42}}{2N_c} C_{42} = \frac{\beta_{43}}{2N_c} C_{43} \\
&= \frac{1}{2} \frac{N_c}{4\pi} \log \frac{1}{a_\perp \Lambda^{(4)}_{2D}(a_\parallel, \theta)}, \\
\Lambda^{(4)}_{2D}(a_\parallel, \theta) &= \Lambda c e^{-\frac{8\pi}{N_c} \tan\frac{\theta}{2} (\Delta \mathcal{G}^{fin}_{4\perp}(\infty) + \Delta \mathcal{Z}^{fin}_{4\perp}(\infty))} (a_\parallel \Lambda c)^{\frac{16\pi\beta_0}{N_c} \tan\frac{\theta}{2} - 1}, \\
\beta^{(1)}_{2D}(a_\perp, a_\parallel, \theta) &= \frac{\beta_{12}}{2N_c} C_{12} = \frac{\beta_{13}}{2N_c} C_{13} \\
&= \frac{1}{2} \frac{N_c}{4\pi} \log \frac{1}{a_\perp \Lambda^{(1)}_{2D}(a_\parallel, \theta)}, \\
\Lambda^{(1)}_{2D}(a_\parallel, \theta) &= \Lambda c e^{-\frac{8\pi}{N_c} \cot\frac{\theta}{2} (\Delta \mathcal{G}^{fin}_{1\perp}(\infty) + \Delta \mathcal{Z}^{fin}_{1\perp}(\infty))} (a_\parallel \Lambda c)^{\frac{16\pi\beta_0}{N_c} \cot\frac{\theta}{2} - 1},
\end{aligned} \tag{C1}$$

which is precisely the form of the bare coupling as a function of the lattice cutoff $a_\perp = a$ in the two-dimensional $SU(N_c)$ principal chiral model, to one-loop accuracy (see, e.g., Ref. [66]). Here we have neglected $o(T^0)$ terms. The remaining couplings read, in the same limit and in the same approximation,

$$\begin{aligned}
\frac{\beta_{41}}{2N_c} C_{41} &= \frac{1}{\bar{T}^2 \sin\theta} \frac{N_c^2 - 1}{2N_c} \frac{1}{\pi \sin\theta} \log \frac{1}{a_\perp \tilde{\Lambda}_{2D}(a_\parallel, \theta)} \\
&\equiv \frac{1}{\bar{T}^2} \tilde{\beta}_{2D}(a_\perp, a_\parallel, \theta), \\
\tilde{\Lambda}_{2D}(a_\parallel, \theta) &= \Lambda c (a_\parallel \Lambda c)^{\frac{4\pi\beta_0 N_c \sin\theta}{N_c^2 - 1} - 1} e^{-\frac{2\pi N_c \sin\theta}{N_c^2 - 1} (\Delta \mathcal{G}^{fin}_{41}(\infty) + \Delta \mathcal{Z}^{fin}_{41}(\infty))}, \\
\frac{\beta_{23}}{2N_c} C_{23} &= \bar{T}^2 \sin\theta \beta_0 \log \frac{1}{(a_\parallel \Lambda c)^2} \equiv \bar{T}^2 \hat{\beta}_{2D}(a_\parallel, \theta).
\end{aligned} \tag{C2}$$

The action can be recast as follows,

$$S^{(2D)} = \sum_{n_\parallel} S^{(4)}_\chi(n_\parallel) + S^{(1)}_\chi(n_\parallel) + S_{\text{int1}}(n_\parallel) + S_{\text{int2}}(n_\parallel), \tag{C3}$$

where $S^{(\mu)}_\chi$ correspond to principal chiral models,

$$\begin{aligned}
S^{(\mu)}_\chi(n_\parallel) &= \beta^{(\mu)}_{2D}(a_\parallel, a_\perp, \theta) \\
&\quad \times \sum_{n_\perp} \sum_{\alpha=2,3} \text{tr}\{[\Delta^+_\alpha U_\mu(n)][\Delta^+_\alpha U_\mu(n)]^\dagger\},
\end{aligned} \tag{C4}$$

and the mutual interactions are given by the remaining terms,

$$\begin{aligned}
S_{\text{int1}}(n_\parallel) &= \tilde{\beta}_{2D}(a_\parallel, a_\perp, \theta) \sum_{n_\perp} \frac{a_\perp^2}{a_\parallel^2} \mathcal{P}_{41}(n), \\
S_{\text{int2}}(n_\parallel) &= \sum_{\mu=4,1} \beta^{(\mu)}_{2D}(a_\parallel, a_\perp, \theta) \sum_{n_\perp} \sum_{\alpha=2,3} [2N_c \mathcal{P}_{\mu\alpha}(n) \\
&\quad - \text{tr}\{[\Delta^+_\alpha U_\mu(n)][\Delta^+_\alpha U_\mu(n)]^\dagger\}] \\
&\quad + \hat{\beta}_{2D}(a_\perp, \theta) \sum_{n_\perp} 2N_c \frac{a_\parallel^2}{a_\perp^2} \mathcal{P}_{23}(n).
\end{aligned} \tag{C5}$$

The two-dimensional scales $\Lambda^{(4,1)}_{2D}$ and $\tilde{\Lambda}_{2D}$ have prescribed values that depend on Λ, which is set in the 4D theory, and on $a_\parallel$, which has to be taken to zero at the end of the calculation. However, the average plaquette terms to lowest order and for large $\bar{T}$ read [see Eqs. (56) and (A20)]

$$\begin{aligned}
\langle \mathcal{P}_{41} \rangle &\simeq g^2 \frac{N_c^2 - 1}{N_c} \frac{\log \bar{T}^2}{4\pi \sin\theta}, \quad \langle \mathcal{P}_{23} \rangle \simeq g^2 \frac{N_c^2 - 1}{N_c} \frac{z_{10}}{\bar{T}^2 \sin\theta}, \\
\langle \mathcal{P}_{4\perp} \rangle &\simeq g^2 \frac{N_c^2 - 1}{2N_c} \cot\frac{\theta}{2} z_{10}, \quad \langle \mathcal{P}_{1\perp} \rangle \simeq g^2 \frac{N_c^2 - 1}{2N_c} \tan\frac{\theta}{2} z_{10},
\end{aligned} \tag{C6}$$

so that the range of applicability of perturbation theory is limited by $g^2 \log \bar{T} \ll 1$; more precisely, besides $a_\parallel \gg a_\perp$ one needs $a_\parallel \Lambda \ll (a_\perp \Lambda)^{1-\gamma}$ for some θ-dependent γ, which prevents from taking the continuum limit in the transverse plane independently from the longitudinal plane.

[1] C. J. Morningstar and M. J. Peardon, The glueball spectrum from an anisotropic lattice study, Phys. Rev. D **60**, 034509 (1999).

[2] H. W. Lin, S. D. Cohen, J. Dudek, R. G. Edwards, B. Joó, D. G. Richards, J. Bulava, J. Foley, C. Morningstar, E. Engelson, S. Wallace, K. J. Juge, N. Mathur, M. J. Peardon, and S. M. Ryan (Hadron Spectrum Collaboratio), First results from $2+1$ dynamical quark flavors on an anisotropic lattice: Light-hadron spectroscopy and setting the strange-quark mass, Phys. Rev. D **79**, 034502 (2009).

[3] Y. Namekawa, S. Aoki, R. Burkhalter, S. Ejiri, M. Fukugita, S. Hashimoto, N. Ishizuka, Y. Iwasaki, K. Kanaya, T. Kaneko, Y. Kuramashi, V. Lesk, M. Okamoto, M. Okawa, Y. Taniguchi, A. Ukawa, and T. Yoshie (CP-PACS Collaboration), Thermodynamics of SU(3) gauge theory on anisotropic lattices, Phys. Rev. D **64**, 074507 (2001).

[4] S. Borsányi, S. Dürr, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, S. Mages, D. Nógrádi, A. Pásztor, A. Schäfer, K. K. Szabó, B. C. Tóth, and N. Trombitás, Charmonium spectral functions from $2+1$ flavour lattice QCD, J. High Energy Phys. 04 (2014) 132.

[5] G. Burgio, A. Feo, M. J. Peardon, and S. M. Ryan, Gauge theories on a $2+2$ anisotropic lattice, Phys. Rev. D **67**, 114502 (2003).

[6] H. Verlinde and E. Verlinde, QCD at high energies and two-dimensional field theory, arXiv:hep-th/9302104.

[7] I. Ya. Aref'eva, Regge regime in QCD and asymmetric lattice gauge theory, Phys. Lett. B **325**, 171 (1994).

[8] I. Ya. Aref'eva, Large N QCD at high energies as two-dimensional field theory, Phys. Lett. B **328**, 411 (1994).

[9] P. Orland, Near-integrability and confinement for high-energy hadron-hadron collisions, Phys. Rev. D **77**, 056004 (2008).

[10] P. Orland and J. Xiao, Longitudinal rescaling and high-energy effective actions, Phys. Rev. D **80**, 016005 (2009).

[11] A. Cortés Cubero and P. Orland, Longitudinal rescaling of quantum electrodynamics, Phys. Rev. D **84**, 065034 (2011).

[12] A. Cortés Cubero and P. Orland, Longitudinal rescaling of quantum chromodynamics, arXiv:1203.5141.

[13] M. Giordano and E. Meggiolaro, A nonperturbative foundation of the Euclidean–Minkowskian duality of Wilson-loop correlation functions, Phys. Lett. B **675**, 123 (2009).

[14] E. Meggiolaro, The high-energy quark-quark scattering: from Minkowskian to Euclidean theory, Z. Phys. C **76**, 523 (1997).

[15] E. Meggiolaro, The high-energy quark-quark scattering: from Minkowskian to Euclidean theory, Eur. Phys. J. C **4**, 101 (1998).

[16] E. Meggiolaro, The analytic continuation of the high-energy parton–parton scattering amplitude with an IR cutoff, Nucl. Phys. **B625**, 312 (2002).

[17] E. Meggiolaro, On the loop–loop scattering amplitudes in Abelian and non-Abelian gauge theories, Nucl. Phys. **B707**, 199 (2005).

[18] M. Giordano and E. Meggiolaro, Analyticity and crossing symmetry of the eikonal amplitudes in gauge theories, Phys. Rev. D **74**, 016003 (2006).

[19] E. Meggiolaro, Can the pomeron be derived from a Euclidean–Minkowskian duality?, Phys. Lett. B **651**, 177 (2007).

[20] O. Nachtmann, Considerations concerning diffraction scattering in quantum chromodynamics, Ann. Phys. (N.Y.) **209**, 436 (1991).

[21] H. G. Dosch, E. Ferreira, and A. Krämer, Nonperturbative QCD treatment of high-energy hadron-hadron scattering, Phys. Rev. D **50**, 1992 (1994).

[22] O. Nachtmann, High energy collisions and nonperturbative QCD, Lectures on QCD **496**, 1 (1997).

[23] E. R. Berger and O. Nachtmann, Differential cross sections for high energy elastic hadron-hadron scattering in nonperturbative QCD, Eur. Phys. J. C **7**, 459 (1999).

[24] H. G. Dosch, Space-time picture of high energy scattering, in *At the Frontier of Particle Physics*, edited by M. Shifman (World Scientific, Singapore, 2001), Vol. 2, p. 1195.

[25] A. I. Shoshi, F. D. Steffen, and H. J. Pirner, S-matrix unitarity, impact parameter profiles, gluon saturation and high-energy scattering, Nucl. Phys. **A709**, 131 (2002).

[26] S. Donnachie, G. Dosch, P. Landshoff, and O. Nachtmann, *Pomeron Physics and QCD* (Cambridge University Press, Cambridge, England, 2002).

[27] M. Giordano, Wilson-loop formalism for Reggeon exchange in soft high-energy scattering, J. High Energy Phys. 07 (2012) 109; 01 (2013) 21(E).

[28] M. Giordano and E. Meggiolaro, Hadronic total cross sections at high energy and the QCD spectrum, J. High Energy Phys. 03 (2014) 002.

[29] B. S. DeWitt, Quantum theory of gravity. II. The manifestly covariant theory, Phys. Rev. **162**, 1195 (1967).

[30] B. S. DeWitt, Quantum theory of gravity. III. Applications of the covariant theory, Phys. Rev. **162**, 1239 (1967).

[31] L. F. Abbott, The background field method beyond one loop, Nucl. Phys. **B185**, 189 (1981).

[32] L. F. Abbott, M. T. Grisaru, and R. K. Schaefer, The background field method and the S matrix, Nucl. Phys. **B229**, 372 (1983).

[33] R. Dashen and D. J. Gross, Relationship between lattice and continuum definitions of the gauge-theory coupling, Phys. Rev. D **23**, 2340 (1981).

[34] A. Hasenfratz and P. Hasenfratz, The scales of euclidean and hamiltonian lattice QCD, Nucl. Phys. **B193**, 210 (1981).

[35] F. Karsch, SU(N) gauge theory couplings on asymmetric lattices, Nucl. Phys. **B205**, 285 (1982).

[36] A. Gonzalez-Arroyo and C. P. Korthals Altes, Asymptotic freedom scales for any single plaquette action, Nucl. Phys. **B205**, 46 (1982).

[37] M. Lüscher and P. Weisz, Background field technique and renormalization in lattice gauge theory, Nucl. Phys. **B452**, 213 (1995).

[38] M. Lüscher and P. Weisz, Locality and exponential error reduction in numerical lattice gauge theory, J. High Energy Phys. 09 (2001) 010.

[39] L. F. Abbott, Introduction to the background field method, Acta Phys. Pol. B **13**, 33 (1982).

[40] J. Zinn-Justin, *Quantum field theory and critical phenomena* (Oxford University Press, New York, 2002).

[41] G. 't Hooft, in *Proceedings of the Colloquium on Renormalization of Yang–Mills Fields and Applications to Particle Physics, Marseille, 1972*, edited by C. P. Korthals Altes (unpublished).

[42] D. J. Gross and F. Wilczek, Ultraviolet Behavior of Non-Abelian Gauge Theories, Phys. Rev. Lett. **30**, 1343 (1973).
[43] H. D. Politzer, Reliable Perturbative Results for Strong Interactions?, Phys. Rev. Lett. **30**, 1346 (1973).
[44] M. García Pérez and P. van Baal, One-loop anisotropy for improved actions, Phys. Lett. B **392**, 163 (1997).
[45] I. T. Drummond, A. Hart, R. R. Horgan, and L. C. Storoni, One loop calculation of the renormalized anisotropy for improved anisotropic gluon actions on a lattice, Phys. Rev. D **66**, 094509 (2002).
[46] M. Rueter and H. G. Dosch, Nucleon structure and high-energy scattering, Phys. Lett. B **380**, 177 (1996).
[47] A. I. Shoshi, F. D. Steffen, H. G. Dosch, and H. J. Pirner, Confining QCD strings, Casimir scaling, and a Euclidean approach to high-energy scattering, Phys. Rev. D **68**, 074004 (2003).
[48] E. Shuryak and I. Zahed, Instanton-induced effects in QCD high-energy scattering, Phys. Rev. D **62**, 085014 (2000).
[49] R. A. Janik and R. Peschanski, High energy scattering and the AdS/CFT correspondence, Nucl. Phys. **B565**, 193 (2000).
[50] R. A. Janik and R. Peschanski, Minimal surfaces and reggeization in the AdS/CFT correspondence, Nucl. Phys. **B586**, 163 (2000).
[51] R. A. Janik, String fluctuations, AdS/CFT and the soft pomeron intercept, Phys. Lett. B **500**, 118 (2001).
[52] M. Giordano and R. Peschanski, High energy bounds on soft $N = 4$ SYM amplitudes from AdS/CFT, J. High Energy Phys. 05 (2010) 037.
[53] M. Giordano and E. Meggiolaro, High-energy hadron-hadron (dipole-dipole) scattering from lattice QCD, Phys. Rev. D **78**, 074510 (2008).
[54] M. Giordano and E. Meggiolaro, Instanton effects on Wilson-loop correlators: A new comparison with numerical results from the lattice, Phys. Rev. D **81**, 074022 (2010).
[55] M. Giordano, E. Meggiolaro, and N. Moretti, Asymptotic energy dependence of hadronic total cross sections from lattice QCD, J. High Energy Phys. 09 (2012) 031.
[56] M. Giordano and E. Meggiolaro, Comments on high-energy total cross sections in QCD, Phys. Lett. B **744**, 263 (2015).
[57] V. S. Dotsenko and S. N. Vergeles, Renormalizability of phase factors in non-abelian gauge theory, Nucl. Phys. **B169**, 527 (1980).
[58] R. A. Brandt, F. Neri, and M. A. Sato, Renormalization of loop functions for all loops, Phys. Rev. D **24**, 879 (1981).
[59] T. Appelquist and W. Fischler, Some remarks on van der Waals forces in QCD, Phys. Lett. **77B**, 405 (1978).
[60] G. Bhanot, W. Fischler, and S. Rudaz, A multipole expansion and the Casimir-Polder effect in quantum chromodynamics, Nucl. Phys. **B155**, 208 (1979).
[61] M. E. Peskin, Short-distance analysis for heavy-quark systems: (I). Diagrammatics, Nucl. Phys. **B156**, 365 (1979).
[62] G. Bhanot and M. E. Peskin, Short-distance analysis for heavy-quark systems : (II). Applications, Nucl. Phys. **B156**, 391 (1979).
[63] M. Ishida and K. Igi, Universal rise of hadronic total cross sections based on forward πp and source $\bar{p}p(pp)$ scatterings, Phys. Lett. B **670**, 395 (2009).
[64] K. A. Olive *et al.* (Particle Data Group Collaboration), Review of particle physics, Chin. Phys. C **38**, 090001 (2014).
[65] E. C. Titchmarsh, *The Theory of Functions*, 2nd ed. (Cambridge University Press, Cambridge, England, 1939).
[66] A. M. Polyakov, *Gauge fields and strings* (Harwood Academic Publisher, Chur, Switzerland, 1987).
[67] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1980).