ABJM membrane instanton from a pole cancellation mechanism

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The coefficients of the membrane instantons in the ABJM theory are known to be quadratic polynomials of the chemical potential. For better insight into this nonconstantness, we consider more general superconformal Chern–Simons theories labelled by two parameters (q, p). In these theories, we show that the membrane instantons split into three types of nonperturbative effects, one more type compared with the previous observation. We also determine their explicit coefficients which are independent of the chemical potential. We find that, although these constants contain poles at certain values of q and p including the ABJM case, all of the poles cancel among themselves, and the finite quadratic polynomial coefficients are reproduced at these values. This is similar to what happens between the membrane instantons and the worldsheet instantons in the ABJM theory.

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I. INTRODUCTION AND SUMMARY

A. ABJM membrane instantons revisited

Recently there has been much progress in understanding the world volume theory of multiple M2-branes. It was found in [1] that the world volume theory of N coincident M2-branes on a geometry $\mathbb{C}^4/\mathbb{Z}_k$ is described by the $\mathcal{N}=6$ $U(N)_k \times U(N)_{-k}$ superconformal Chern–Simons theory where the subscripts k and $-k$ denote the Chern–Simons levels associated to each $U(N)$ factor.[1] We shall call this theory Aharony-Bergman-Jafferis-Maldacena (ABJM) theory hereafter. After applying the localization theorem [6,7], the infinite-dimensional path integral in defining the partition function of the ABJM theory on S^3 is reduced to a finite-dimensional matrix integral.

One of the most remarkable results in the study of this matrix model is the determination of the coefficient of the membrane instantons. It was found that the nonperturbative effects in $1/N$ consist of two types of instantons and their bound states. One is called the worldsheet instanton. If we define the grand potential $J_{\mathrm{ABJM}}(\mu)$ dual to the partition function

$$e^{J_{\mathrm{ABJM}}(\mu)} = \sum_{N=0}^{\infty} Z_{\mathrm{ABJM}}(N) e^{\mu N}, \tag{1.1}$$

by introducing the chemical potential μ dual to N, the effects of the worldsheet instantons can be written as

$$J^{\mathrm{WS}}_{\mathrm{ABJM}}(\mu) = \sum_{n=1}^{\infty} d_m(k) e^{-\frac{4m\mu}{k}}. \tag{1.2}$$

In the IIA picture, the exponent $e^{-\frac{4\mu}{k}}$ is interpreted as a fundamental string wrapping $\mathbb{CP}^1$ [8,9] in $\mathbb{C}^4/\mathbb{Z}_k$.

The other is the membrane instanton. Though the membrane instanton was first introduced in Ref. [10], quantitative studies of it in general situations remain difficult. In the case of the ABJM theory where the membrane instanton corresponds to a D2-brane wrapping $\mathbb{RP}^3$ [11] in $\mathbb{C}^4/\mathbb{Z}_k$, however, the complete determination of the membrane instanton effects was finally achieved,

$$J^{\mathrm{MB}}_{\mathrm{ABJM}}(\mu) = \sum_{\ell=1}^{\infty} (a_\ell(k)\mu^2 + b_\ell(k)\mu + c_\ell(k)) e^{-2\ell\mu}, \tag{1.3}$$

where $a_\ell(k)$, $b_\ell(k)$, and $c_\ell(k)$ are μ-independent constants given in Refs. [12,13]. For example, the explicit form of the coefficients of the first membrane instanton is given with [14,15]

$$a_1(k) = -\frac{4\cos\frac{\pi k}{2}}{\pi^2 k}, \qquad b_1(k) = \frac{2\cos^2\frac{\pi k}{2}}{\pi \sin\frac{\pi k}{2}},$$
$$c_1(k) = \frac{\pi}{6}\left(1 + \frac{k^2}{8}\right) a_1(k) - \frac{k^2}{2}\frac{\partial}{\partial k}\left(\frac{b_1(k)}{k}\right). \tag{1.4}$$

Compared with the worldsheet instanton, it is perplexing to find that the coefficients of the membrane instanton (1.3) are quadratic polynomials of the chemical potential due to the following reasons:

(i) In the standard situations, an instanton coefficient is usually a constant independent of the instanton exponent, which is related to the volume of the instanton moduli space.

(ii) From the M-theoretical viewpoint, both the worldsheet instantons and the membrane instantons stem

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[1]See Refs. [2–4] for earlier works on superconformal Chern-Simons theories and Ref. [5] for their dual geometries.

from the same M2-branes. It is reasonable to expect both of them to appear in the same manner in the grand potential.

The fact of the coefficients being polynomials may suggest that the membrane instanton contains some further structures to be clarified. In this paper, after identifying the structures, we can resolve the polynomial coefficients into constants.

B. Clues

Some clues to this unfamiliar situation were already found in the developments so far. The first clue is the so-called pole cancellation mechanism [14] used to determine the expression of (1.3). Let us first recapitulate it. Following many interesting aspects of the ABJM matrix model [9,16–19], it was discovered [20] that we can regard the partition function as that of a noninteracting ideal Fermi gas system with N particles which are governed by a nontrivial one-particle Hamiltonian,

$$e^{-\hat{H}_{\mathrm{ABJM}}} = \frac{1}{2\cosh\frac{\hat{Q}}{2}} \frac{1}{2\cosh\frac{\hat{P}}{2}}, \tag{1.5}$$

with the Planck constant in the canonical commutation relation $[\hat{Q}, \hat{P}] = i\hbar$ given by $\hbar = 2\pi k$. In terms of this Hamiltonian, the grand potential is given by

$$J_{\mathrm{ABJM}}(\mu) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1} e^{\ell\mu}}{\ell} \operatorname{tr} e^{-\ell \hat{H}_{\mathrm{ABJM}}}. \tag{1.6}$$

This Fermi gas formalism is not only suitable for the systematic Wentzel-Kramers-Brilouin (WKB) $\hbar$ expansion [15,20] but also applicable to the study of the exact values of the partition function [21,22] which lead directly to the numerical results of the grand potential [12,14]. Combining with the results from the 't Hooft genus expansion [9,18,23] and the dual description through the topological string theory on local $\mathbb{P}^1 \times \mathbb{P}^1$ [16], finally the whole large μ expansion of the grand potential including the nonperturbative terms were written down explicitly [13]. The worldsheet instanton (1.2) were found to be described by the free energy of the topological string theory on local $\mathbb{P}^1 \times \mathbb{P}^1$ [9,14,16]. For the membrane instantons, after the whole studies of the partition function of the ABJM theory, we finally discovered that they are described by the free energy of the refined topological string theory in the Nekrasov–Shatashvili limit [24] on the same background [13].[2]

In the determination of these nonperturbative effects, the so-called pole cancellation mechanism [14] played a crucial role. It was found [14] that the coefficients of the worldsheet instanton (1.2) contain poles at certain values of k. Since the matrix model itself takes finite values, these poles must be cancelled by those from other nonperturbative contributions. If we assume that the coefficients of the membrane instantons also have the poles thus required, we finally obtain the exact expressions of the coefficients of the membrane instantons, which are consistent with the WKB $\hbar$ expansion [15,20] and reproduce the numerical results of [12,14] after the pole cancellation. Furthermore, if we adopt the free energy of the refined topological strings in the Nekrasov–Shatashvili limit for the membrane instantons, we can see [13] that all of the poles from the free energy of the topological strings describing the worldsheet instantons are cancelled. In this sense, we can say that the whole membrane instantons in the ABJM theory are determined by the pole cancellation mechanism.

The second clue is the appearance of two types of membrane instantons in the generalizations of the ABJM theory. It is interesting to ask how general it is that the pole cancellation mechanism can determine the nonperturbative expansions.[3] In our previous work [38], we proceeded to more general $\mathcal{N} = 4$ superconformal Chern–Simons theories of the circular quiver type [39] with the levels given by[4]

$$k_a = \frac{k}{2}(s_a - s_{a-1}), \quad s_a = \pm 1. \tag{1.7}$$

The Fermi gas formalism is also applicable to this class of theories, and the Hamiltonian is given by

$$e^{-\hat{H}} = \frac{1}{(2\cosh\frac{\hat{Q}}{2})^{q_1}} \frac{1}{(2\cosh\frac{\hat{P}}{2})^{p_1}} \frac{1}{(2\cosh\frac{\hat{Q}}{2})^{q_2}} \frac{1}{(2\cosh\frac{\hat{P}}{2})^{p_2}} \cdots \frac{1}{(2\cosh\frac{\hat{Q}}{2})^{q_m}} \frac{1}{(2\cosh\frac{\hat{P}}{2})^{p_m}}, \tag{1.8}$$

for $\{s_a\}_{a=1}^{M} = \{(+1)^{q_1}, (-1)^{p_1}, (+1)^{q_2}, (-1)^{p_2}, \ldots, (+1)^{q_m}, (-1)^{p_m}\}$. Here this expression denotes a sequence consisting of q_1 elements of $+1$, p_1 elements of -1 and so on in this ordering. For the perturbative part and the membrane instanton part, we fully utilized the WKB $\hbar$ expansion for this Fermi gas system

[2] Some further studies such as the spectral problem, the perturbation series, and the special supersymmetry enhancements can be found in Refs. [25–30].

[3] For a generalization to the case of two different ranks $U(N_1)_k \times U(N_2)_{-k}$ [31,32], see Refs. [33–37].

[4] A special case of the $\mathcal{N} = 4$ theories called orbifold ABJM theory [40–42] was studied in Ref. [43]. Also, a similar analysis on a closely related model [44] in a slightly different language, which corresponds to the $\{s_a\}_{a=1}^{M} = \{(+1)^{N_f}, (-1)\}$ case in our language, can be found in Ref. [45], which appeared almost simultaneously as Ref. [38].

$$J^{\mathrm{pert+MB}}(\mu) = \sum_{n=1}^{\infty} \hbar^{n-1} J_n(\mu). \tag{1.9}$$

In Ref. [38] we analyzed the first few terms in the $\hbar$ expansion in the models with

$$q = \sum_{a=1}^{m} q_a, \qquad p = \sum_{a=1}^{m} p_a, \tag{1.10}$$

for several pairs of values $(q, p) \in \mathbb{N}^2$. As a result, we observed two types of nonperturbative effects with the exponents $e^{-\frac{2\mu}{q}}$ and $e^{-\frac{2\mu}{p}}$, though the explicit form of the coefficients was obscure. As the exponents are independent of k, we expected that they can be interpreted as generalizations of the membrane instantons.

C. Resolutions

Now let us come back to the original question, the quadratic polynomial coefficients in the membrane instantons. From the above two clues, if we introduce two deformation parameters (q, p), it is natural to expect that the ABJM membrane instanton (1.3) splits into two or more fundamental nonperturbative effects with constant coefficients containing poles at certain values of (q, p) and that the polynomial coefficients in (1.3) appear after cancelling these poles. In fact, in this paper we shall see that this is the case.

Our setup is as follows. We study the "minimal" generalization $\{s_a\}_{a=1}^{M} = \{(+1)^q, (-1)^p\}$ with general values of (q, p), which reduces to the ABJM case for $(q, p) = (1, 1)$.[5] We consider only the WKB expansion of the membrane instanton around $k = 0$. The grand potential in this case was found to be [38]

$$\begin{aligned} J_0(\mu) &= \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1} e^{\ell\mu}}{\ell} \int \frac{dQdP}{2\pi} \frac{1}{(2\cosh\frac{Q}{2})^{q\ell}} \frac{1}{(2\cosh\frac{P}{2})^{p\ell}} \\ &= \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1} e^{\ell\mu}}{2\pi\ell} \frac{\Gamma(\frac{q\ell}{2})^2}{\Gamma(q\ell)} \frac{\Gamma(\frac{p\ell}{2})^2}{\Gamma(p\ell)}, \end{aligned} \tag{1.11}$$

without much change from the ABJM case [20].

Let us summarize our main results. Although the original definition (1.11) is given in the small e^{μ} expansion, if we generalize (q, p) to irrational numbers, we can rewrite it into the large e^{μ} expansion. Aside from the perturbative part

$$J_0^{\mathrm{pert}}(\mu) = \frac{4}{3\pi qp}\mu^3 + \frac{\pi(4 - q^2 - p^2)}{3qp}\mu + \frac{2(q^3 + p^3)}{\pi qp}\zeta(3), \tag{1.12}$$

we obtain the explicit expression of the instanton coefficients for $e^{-\frac{2\mu}{q}}$ and $e^{-\frac{2\mu}{p}}$,

$$\begin{aligned} J_0^{(q)}(\mu) &= \sum_{m=1}^{\infty} \binom{2m}{m} \frac{1}{m \sin\frac{2\pi m}{q}} \frac{\Gamma(-\frac{pm}{q})^2}{\Gamma(-\frac{2pm}{q})} e^{-\frac{2m\mu}{q}}, \\ J_0^{(p)}(\mu) &= \sum_{n=1}^{\infty} \binom{2n}{n} \frac{1}{n \sin\frac{2\pi n}{p}} \frac{\Gamma(-\frac{qn}{p})^2}{\Gamma(-\frac{2qn}{p})} e^{-\frac{2n\mu}{p}}. \end{aligned} \tag{1.13}$$

Moreover, we discover the third kind of instantons

$$J_0^{(2)}(\mu) = \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{2\pi l} \frac{\Gamma(-\frac{ql}{2})^2}{\Gamma(-ql)} \frac{\Gamma(-\frac{pl}{2})^2}{\Gamma(-pl)} e^{-l\mu}. \tag{1.14}$$

Note that the coefficients of the nonperturbative effects are not quadratic polynomials anymore but constants independent of the chemical potential. If we take the deformation parameters (q, p) back to (1, 1) for the ABJM theory, we encounter various poles. After cancelling all the poles, we come back to the original quadratic polynomials of the ABJM theory. This indicates that we have decomposed the original membrane instanton of the ABJM theory into more fundamental ones.

Before closing the Introduction, at this point let us stress some bonus of the resolution:

(i) Technically, the resolution is helpful in disentangling the complexity and clarifying the mathematical structure of the nonperturbative effects. In fact, in the ABJM theory, it was not until we split the nonperturbative effects into the worldsheet instantons and the membrane instantons that we were able to describe them in terms of the (refined) topological string.

(ii) We have encountered a new type of instanton effects (1.14). Interestingly, if q, p are integers, these instantons never have distinct exponents from the other two. These instantons become detectable only after we deform (q, p) to irrational numbers. For these reasons we shall call them "ghost instantons."

The remaining part of this paper is organized as follows. In Sec. II, we shall rewrite the small e^{μ} expansion of the grand potential (1.11) into the large e^{μ} expansion, where we find three types of nonperturbative effects (1.13) and (1.14). Although the coefficients contain poles at various values of (q, p), all of the poles cancel among themselves to reproduce the quadratic polynomials, as we shall see in Sec. III. In Sec. IV, we apply our large μ expansion to the subsequent orders in the WKB $\hbar$ expansion. We conclude in Sec. V with discussions, emphasizing the above bonus.

[5] The explicit expansion of $J_0(\mu)$ is valid also for general $\mathcal{N} = 4$ theories with (1.10), since the ordering of operators is irrelevant in the strictly classical limit. Our argument on the pole cancellation among the membrane instantons can be straightforwardly extended also for the higher-order corrections. This is because the arguments of the hypergeometric series (1.11) and (4.1) depend on $\{q_a, p_a\}$ only through (q, p) (1.10), as observed in Ref. [38].

II. FROM SMALL e^μ TO LARGE e^μ

The grand potential in the classical limit $\hbar\to 0$, $J_0(\mu)$, is obtained as a power series in e^μ (1.11), which is appropriate at $\mu\to-\infty$. In this section, we shall rewrite this series into a large μ expansion to derive the perturbative part (1.12) and the nonperturbative corrections (1.13), (1.14) in $J_0(\mu)$. Below we generalize q and p to be irrational numbers, to avoid any divergences which possibly appear.

We first introduce numerical constants γ_m defined by

$$\frac{\Gamma(x)^2}{\Gamma(2x)}=\frac{2}{2^{2x}}\sum_{m=0}^{\infty}\frac{\gamma_m}{m+x},\qquad \gamma_m=\frac{1}{2^{2m}}\binom{2m}{m}. \tag{2.1}$$

Using these constants, the power series expansion of $J_0(\mu)$ with respect to e^μ (1.11) is rewritten into

$$J_0(\mu)=-\frac{8}{\pi qp}\sum_{\ell=1}^{\infty}(-e^{\mu'})^\ell\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\gamma_m\gamma_n}{\ell(\ell+\frac{2m}{q})(\ell+\frac{2n}{p})}. \tag{2.2}$$

Here we have introduced

$$\mu'=\mu-(q+p)\log 2 \tag{2.3}$$

for abbreviation. Using the partial fraction decomposition, we find that the coefficient in the summand is written as

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\gamma_m\gamma_n}{\ell(\ell+\frac{2m}{q})(\ell+\frac{2n}{p})}=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{q^2\gamma_m\gamma_n}{4m^2(1-\frac{nq}{mp})}\frac{1}{\ell+\frac{2m}{q}}+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{p^2\gamma_m\gamma_n}{4n^2(1-\frac{mp}{nq})}\frac{1}{\ell+\frac{2n}{p}}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{qp\gamma_m\gamma_n}{4mn}\frac{1}{\ell}$$
$$+\sum_{m=1}^{\infty}\gamma_m\left(\frac{q}{2m}\frac{1}{\ell^2}-\frac{q^2}{4m^2}\frac{1}{\ell}\right)+\sum_{n=1}^{\infty}\gamma_n\left(\frac{p}{2n}\frac{1}{\ell^2}-\frac{p^2}{4n^2}\frac{1}{\ell}\right)+\frac{1}{\ell^3}, \tag{2.4}$$

where we have used $\gamma_0=1$. Now let us perform the summation over ℓ in (2.2). To obtain the large μ expansion, we use the formulas

$$\sum_{\ell=1}^{\infty}\frac{(-e^\mu)^\ell}{\ell+\alpha}=-\frac{1}{\alpha}+\frac{\pi}{\sin\pi\alpha}e^{-\alpha\mu}-\sum_{\ell=1}^{\infty}\frac{(-e^\mu)^{-\ell}}{-\ell+\alpha} \tag{2.5}$$

and

$$\begin{aligned}\mathrm{Li}_1(-e^\mu)&=-\mu+\mathrm{Li}_1(-e^{-\mu}),\\ \mathrm{Li}_2(-e^\mu)&=-\frac{\mu^2}{2}-\frac{\pi^2}{6}-\mathrm{Li}_2(-e^{-\mu}),\\ \mathrm{Li}_3(-e^\mu)&=-\frac{\mu^3}{6}-\frac{\pi^2\mu}{6}+\mathrm{Li}_3(-e^{-\mu}),\end{aligned} \tag{2.6}$$

for the polylogarithm function

$$\mathrm{Li}_s(z)=\sum_{\ell=1}^{\infty}\frac{z^\ell}{\ell^s}. \tag{2.7}$$

Here all of these formulas (2.5) and (2.6) can be derived from[6]

$$\sum_{\ell=-\infty}^{\infty}\frac{(-e^\mu)^\ell}{\ell+\alpha}=\frac{\pi}{\sin\pi\alpha}e^{-\alpha\mu}. \tag{2.8}$$

With the help of these formulas, we divide $J_0(\mu)$ into four parts: the perturbative terms and the nonperturbative terms of $e^{-\frac{2\mu}{q}}$, $e^{-\frac{2\mu}{p}}$, $e^{-\mu}$.

First let us consider the nonperturbative terms of $e^{-\frac{2\mu}{q}}$ and $e^{-\frac{2\mu}{p}}$, which are collected as

$$\begin{aligned}J_0^{(q)}(\mu)&=2\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{\gamma_m\gamma_n}{m(n-\frac{mp}{q})\sin\frac{2\pi m}{q}}e^{-\frac{2m\mu'}{q}},\\ J_0^{(p)}(\mu)&=2\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{\gamma_m\gamma_n}{n(m-\frac{nq}{p})\sin\frac{2\pi n}{p}}e^{-\frac{2n\mu'}{p}}.\end{aligned} \tag{2.9}$$

In these expressions, we can perform the summation over n in $J_0^{(q)}(\mu)$ [or over m in $J_0^{(p)}(\mu)$] just by the definition (2.1), and we finally obtain (1.13).

Next we consider the nonperturbative terms of $e^{-\mu}$, which are

$$J_0^{(2)}(\mu)=\frac{8}{\pi qp}\sum_{\ell=1}^{\infty}(-e^{\mu'})^{-\ell}\Bigg[\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{q^2\gamma_m\gamma_n}{4m^2(1-\frac{nq}{mp})}\frac{1}{-\ell+\frac{2m}{q}}+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{p^2\gamma_m\gamma_n}{4n^2(1-\frac{mp}{nq})}\frac{1}{-\ell+\frac{2n}{p}}$$
$$-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{qp\gamma_m\gamma_n}{4mn}\frac{1}{\ell}+\sum_{m=1}^{\infty}\gamma_m\left(\frac{q}{2m}\frac{1}{\ell^2}+\frac{q^2}{4m^2}\frac{1}{\ell}\right)+\sum_{n=1}^{\infty}\gamma_n\left(\frac{p}{2n}\frac{1}{\ell^2}+\frac{p^2}{4n^2}\frac{1}{\ell}\right)-\frac{1}{\ell^3}\Bigg]. \tag{2.10}$$

[6] It is interesting to note that the same formula with μ purely imaginary was used in the light-cone string field theory [46,47] to prove the unitarity [48,49] of the overlapping matrices.

This expression of $J_0^{(2)}(\mu)$ seems lengthy. However, we can compute it without much effort. First we notice that the expression (2.10) is obtained by using (2.5) and (2.6). The formula (2.5) converts the e^{μ} terms into the $e^{-\mu}$ terms just by replacing ℓ with $-\ell$ and simultaneously changing the overall signs. This is the case also for (2.6) if we substitute the power series expression of the polylogarithm function (2.7). This observation means that $J_0^{(2)}(\mu)$ can be computed by using the formula (2.4) inversely, with the same flips of signs

$$J_0^{(2)}(\mu)=\frac{8}{\pi qp}\sum_{\ell=1}^{\infty}(-e^{\mu'})^{-\ell}\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\gamma_m\gamma_n}{-\ell(-\ell+\frac{2m}{q})(-\ell+\frac{2n}{p})}. \tag{2.11}$$

Summing over m and n by (2.1), one ends up with (1.14).

Finally we consider the perturbative terms,

$$J_0^{\rm pert}(\mu)=-\frac{q}{\pi p}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{\gamma_m\gamma_n}{m^2(\frac{n}{p}-\frac{m}{q})}-\frac{p}{\pi q}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{\gamma_m\gamma_n}{n^2(\frac{m}{q}-\frac{n}{p})}+\left(\frac{2\bar\gamma_1^2}{\pi}-\frac{2\bar\gamma_2}{\pi}\left(\frac{q}{p}+\frac{p}{q}\right)\right)\mu'$$
$$+\frac{4\bar\gamma_1}{\pi}\left(\frac{1}{p}+\frac{1}{q}\right)\left(\frac{\mu'^2}{2}+\frac{\pi^2}{6}\right)+\frac{8}{\pi qp}\left(\frac{\mu'^3}{6}+\frac{\pi^2\mu'}{6}\right). \tag{2.12}$$

Here we have introduced other numerical constants

$$\bar\gamma_s=\sum_{m=1}^{\infty}\frac{\gamma_m}{m^s}, \tag{2.13}$$

the explicit values of which are

$$\bar\gamma_1=2\log 2,\qquad \bar\gamma_2=\frac{\pi^2}{6}-2(\log 2)^2,$$
$$\bar\gamma_3=-\frac{\pi^2\log 2}{3}+\frac{4(\log 2)^3}{3}+2\zeta(3). \tag{2.14}$$

To calculate this expression, note that the first two terms sum up to

$$\frac{\bar\gamma_3}{\pi}\left(\frac{q^2}{p}+\frac{p^2}{q}\right)-\frac{\bar\gamma_1\bar\gamma_2(q+p)}{\pi}. \tag{2.15}$$

Plugging this in, with the explicit values of $\bar\gamma_s$ (2.14), we obtain the result (1.12).

III. POLE CANCELLATION MECHANISM

In the previous section, we have seen the large μ expansion of the classical limit of the grand potential $J_0(\mu)$. We have found that the large μ expansion contains three types of nonperturbative contributions $e^{-\frac{2\mu}{q}}$, $e^{-\frac{2\mu}{p}}$, and $e^{-\mu}$ respectively in $J_0^{(q)}(\mu)$, $J_0^{(p)}(\mu)$, and $J_0^{(2)}(\mu)$ with coefficients being constant independent of the chemical potential μ. There we have extrapolated (q,p) into general irrational numbers to obtain the results (1.13) and (1.14). These resulting expressions indicate that, in the case of integral (q,p), which is our original interest, the coefficient of each sector contains divergent contributions.

In this section, we shall see that these divergences completely cancel among themselves. The cancellation is indeed consistent, since the grand potential $J_0(\mu)$ (1.11) itself is well defined for arbitrary positive (q,p). Remarkably, the coefficients in the nonperturbative effects remaining after these pole cancellations are generally polynomials in μ.

In the following, we first rewrite the results into a symmetric expression which is suitable for seeing how the pole cancellation occurs. Then, restricting ourselves to the cases where all the three sectors contribute to the cancellation (which is the only possibility for the ABJM theory), we explicitly write down the general form of the remaining coefficients. We obtain quadratic polynomials in these cases, which exactly coincide with the previously obtained ones for the ABJM theory [15,20] and the $\mathcal{N}=4$ theories [38]. Finally we see an implication of the form of these quadratic polynomials.

A. Symmetric expression

To simplify the discussion of the pole cancellation, let us first rewrite the three sectors of nonperturbative contributions, $J_0^{(q)}(\mu)$, $J_0^{(p)}(\mu)$, and $J_0^{(2)}(\mu)$, into an expression symmetric under the exchange of q, p, and 2. We find that they can be expressed as[7]

$$J_0^{(z_i)}(\mu)=\sum_{\ell_i=1}^{\infty}\frac{F(\frac{\ell_i}{z_i};\mu)}{\ell_i}\prod_{j=1(\neq i)}^{3}\cot\frac{\pi z_j\ell_i}{z_i}, \tag{3.1}$$

where we have introduced $z_i=(q,p,2)$, $\ell_i=(m,n,l)$, and

$$F(r;\mu)=-\frac{2\pi}{\cos 2\pi r}\frac{\Gamma(2qr+1)}{\Gamma(qr+1)^2}\frac{\Gamma(2pr+1)}{\Gamma(pr+1)^2}e^{-2r\mu}. \tag{3.2}$$

[7] In the discovery of this expression, we are partially stimulated by some previous works. In Ref. [50], the n-ple sine function is decomposed into n sectors symmetric under the exchange of the n parameters, each of which takes the form of the series expansion. In Ref. [51], the partition function on S^5 is expressed similarly. Also in a note by Kazumi Okuyama, he was trying to formulate the cancellation mechanism between the membrane instantons and the worldsheet instantons in the analogy of these works.

Indeed it is not difficult to find that each sector in (3.1) reduces to (1.13) and (1.14) after the substitution $(z_1, z_2, z_3) = (q, p, 2)$. In the derivation, we need to flip the signs in the arguments of the Gamma functions using

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{3.3}$$

In the expression (3.1) all of the Gamma functions in the coefficients are free from divergence, while the cotangent factors imply that each sector contains the nonperturbative effects with divergent coefficients. Explicitly speaking, the divergence appears at $m \in \frac{q}{\gcd(q,p)}\mathbb{N} \cup \frac{q}{\gcd(q,2)}\mathbb{N}$ in $J_0^{(q)}(\mu)$, at $n \in \frac{p}{\gcd(p,2)}\mathbb{N} \cup \frac{p}{\gcd(p,q)}\mathbb{N}$ in $J_0^{(p)}(\mu)$ and at $l \in \frac{2}{\gcd(2,q)}\mathbb{N} \cup \frac{2}{\gcd(2,p)}\mathbb{N}$ in $J_0^{(2)}(\mu)$. However, as $F(r;\mu)$ from different sectors share the same instanton exponent at these points, we expect that the divergences are cancelled among those terms with the same exponent. By replacing (q, p) with $(q(1+\epsilon_1), p(1+\epsilon_2))$ to regularize the divergences and taking the limit $\epsilon_1, \epsilon_2 \to 0$ after summing all the contributions, we find that our expectation is indeed correct. In the next subsection, as an example, we demonstrate this in detail for the cancellation among the three sectors and determine the finite coefficients remaining after the cancellation.

B. Cancellation among three sectors

When the instanton numbers of the three sectors (m, n, l) satisfy

$$\frac{m}{q} = \frac{n}{p} = \frac{l}{2} (=: r), \tag{3.4}$$

where $r \in \mathbb{N}/\gcd(q, p, 2)$, all the three sectors contribute to the nonperturbative effect of $e^{-2r\mu}$.

Let us see how the pole cancellation works. For this purpose, we substitute $z_i(1+\epsilon_i)$ for z_i and send $\epsilon_i \to 0$. Note that we do not have to introduce ϵ_3 to shift $z_3 = 2$ in discussing the cancellation. The cancellation becomes clearer, however, by introducing ϵ_3 and treating three z_i on the equal footing. For simplicity, we introduce the notation

$$F_\epsilon(r;\mu) = F(r;\mu)|_{q\to q', p\to p'} \tag{3.5}$$

with $q' = q(1+\epsilon_1)$, $p' = p(1+\epsilon_2)$ and leave q' and p' in $F_\epsilon(r;\mu)$ untouched while expanding other factors around $\epsilon_i \to 0$. Then we find that the term in $J_0^{(z_i)}(\mu)$ contributing to $e^{-2r\mu}$ is

$$\begin{aligned} F_\epsilon\left(\frac{\ell_i}{z_i(1+\epsilon_i)};\mu\right)\frac{1}{\ell_i}\cot\frac{\pi z_j \ell_i}{z_i}\frac{1+\epsilon_j}{1+\epsilon_i}\cot\frac{\pi z_k \ell_i}{z_i}\frac{1+\epsilon_k}{1+\epsilon_i} \\ = \left(F_\epsilon(r;\mu) - \frac{\epsilon_i r}{1+\epsilon_i}\partial F_\epsilon(r;\mu) + \frac{\epsilon_i^2 r^2}{2(1+\epsilon_i)^2}\partial^2 F_\epsilon(r;\mu) + \mathcal{O}(\epsilon^3)\right) \\ \times \frac{1}{z_i r}\left(\frac{1+\epsilon_i}{\pi z_j r \epsilon_{ji}} - \frac{1}{3}\frac{\pi z_j r \epsilon_{ji}}{1+\epsilon_i} + \mathcal{O}(\epsilon^3)\right)\left(\frac{1+\epsilon_i}{\pi z_k r \epsilon_{ki}} - \frac{1}{3}\frac{\pi z_k r \epsilon_{ki}}{1+\epsilon_i} + \mathcal{O}(\epsilon^3)\right), \end{aligned} \tag{3.6}$$

where j, k denote the two indices[8] other than i and we have introduced the shorthand notation $\epsilon_{ji} = \epsilon_j - \epsilon_i$. Collecting the terms which formally scale in the nonpositive powers in ϵ, we find

$$\frac{F_\epsilon(r;\mu)}{\pi^2 r^3 z_1 z_2 z_3}\frac{(1+\epsilon_i)^2}{\epsilon_{ji}\epsilon_{ki}} - \frac{\partial F_\epsilon(r;\mu)}{\pi^2 r^2 z_1 z_2 z_3}\frac{\epsilon_i(1+\epsilon_i)}{\epsilon_{ji}\epsilon_{ki}} + \frac{\partial^2 F_\epsilon(r;\mu)}{2\pi^2 r z_1 z_2 z_3}\frac{\epsilon_i^2}{\epsilon_{ji}\epsilon_{ki}} - \frac{F_\epsilon(r;\mu)}{3 r z_1 z_2 z_3}\left(\frac{z_k^2 \epsilon_{ki}}{\epsilon_{ji}} + \frac{z_j^2 \epsilon_{ji}}{\epsilon_{ki}}\right). \tag{3.7}$$

With the help of the identities

$$\sum_{i=1}^{3}\frac{1}{\epsilon_{ji}\epsilon_{ki}} = 0, \qquad \sum_{i=1}^{3}\frac{\epsilon_i}{\epsilon_{ji}\epsilon_{ki}} = 0, \qquad \sum_{i=1}^{3}\frac{\epsilon_i^2}{\epsilon_{ji}\epsilon_{ki}} = 1, \tag{3.8}$$

we can show that the terms in the formally negative power of ϵ vanish after summed over all the three sectors.[9] Because of it, we can safely change $F_\epsilon(r;\mu)$ back to $F(r;\mu)$. Finally, the finite part is given by

[8] The readers should not confuse the index k appearing only in this subsection with the Chern–Simons level $k = \hbar/2\pi$.

[9] Note that the terms of formally positive power in ϵ simplify into a homogeneous polynomial of that degree. For example, the terms proportional to $(\epsilon_{ji}\epsilon_{ki})^{-1}$ sum up to the Schur polynomial $(n > 2)$

$$\sum_{i=1}^{3}\frac{\epsilon_i^n}{\epsilon_{ji}\epsilon_{ki}} = \chi_{(n-2)}(\epsilon_1, \epsilon_2, \epsilon_3). \tag{3.9}$$

This fact guarantees that these contributions vanish in the limit of $\epsilon_i \to 0$, regardless of the direction of the limit.

$$\frac{F(r;\mu)-r\partial_r F(r;\mu)+\frac{1}{2}r^2\partial_r^2 F(r;\mu)}{\pi^2 z_1 z_2 z_3 r^3}-\frac{(z_1^2+z_2^2+z_3^2)F(r;\mu)}{3z_1 z_2 z_3 r}. \tag{3.10}$$

Calculating $F(r;\mu)$ and its derivatives, with the explicit form of $F(r;\mu)$ in (3.2), we finally obtain the following contribution of the nonperturbative effects $e^{-2r\mu}$:

$$\frac{F(r;\mu)}{2\pi^2 qpr^3}\left[2r^2\mu^2+(2r-4r^2H_1(r))\mu+1-2rH_1(r)+r^2(2H_1(r)^2-H_2(r))+\frac{\pi^2 r^2(4-q^2-p^2)}{6}\right]. \tag{3.11}$$

Here $H_s(r)$ is defined with the harmonic numbers

$$h_s(m)=\sum_{\ell=1}^{m}\frac{1}{\ell^s}, \tag{3.12}$$

as

$$\begin{aligned}H_s(r)&=q^s(2^{s-1}h_s(2qr)-h_s(qr))\\&\quad+p^s(2^{s-1}h_s(2pr)-h_s(pr)).\end{aligned} \tag{3.13}$$

These $H_s(r)$ result from the derivatives of the Gamma functions in $F(r;\mu)$, using the formula

$$\begin{aligned}\psi^{(0)}(m)&=-\gamma+h_1(m-1),\\ \psi^{(1)}(m)&=\frac{\pi^2}{6}-h_2(m-1),\end{aligned} \tag{3.14}$$

where γ is the Euler–Mascheroni constant and the polygamma functions are defined as

$$\psi^{(s-1)}(x)=\left(\frac{d}{dx}\right)^s\log\Gamma(x). \tag{3.15}$$

As we have expected in section I, quadratic polynomial coefficients have appeared in (3.11) as a result of the pole cancellation.

This explicit form indeed reproduces the previous results in the $\mathcal{N}=4$ theories of $\{s_a\}=\{(+1)^q,(-1)^p\}$ [38] which were obtained by expressing the grand potential $J_0(\mu)$ with the generalized hypergeometric function ${}_{q+p+2}F_{q+p+1}(e^{2\mu'})$ where q,p were the numbers of the parameters and should be integers throughout the analysis. Especially, with $q=p=1$, the membrane instanton coefficients in the limit $k\to 0$ in the ABJM theory [15,20] are reproduced.

At the poles where only two of the three sectors contribute, on the other hand, we obtain linear polynomials in μ as the remaining finite parts. These are again consistent with the results obtained in Ref. [38].

C. Effective chemical potential

As a byproduct, in this subsection we shall discuss an implication of the expressions (3.11) for general $\mathcal{N}=4$ theories. Let us express the results for the WKB expansion (1.9) schematically as

$$\begin{aligned}J^{\mathrm{pert+MB}}(\mu)&=\frac{C}{3}\mu^3+B\mu+A+J_a(\mu)\mu^2\\&\quad+J_b(\mu)\mu+J_c(\mu).\end{aligned} \tag{3.16}$$

Here A, B, and C are perturbative coefficients. The explicit form of C [52] and B [38] is

$$C=\frac{4}{\pi\hbar qp},\qquad B=\frac{1}{\pi}\left(\frac{\hbar qp}{48}+\pi^2\frac{4-q^2-p^2}{3\hbar qp}\right), \tag{3.17}$$

while the explicit form of A is not used below. On the other hand, the nonperturbative contributions $J_a(\mu)$, $J_b(\mu)$, and $J_c(\mu)$ are given by $(r\in\mathbb{N}/\gcd(q,p,2))$

$$\begin{aligned}J_a(\mu)&=\frac{1}{\pi\hbar}\sum_r a_r e^{-2r\mu}+\mathcal{O}(\hbar),\qquad J_b(\mu)=\frac{1}{\pi\hbar}\sum_r b_r e^{-2r\mu}+\cdots+\mathcal{O}(\hbar),\\ J_c(\mu)&=\frac{1}{\pi\hbar}\sum_r(c_r+\pi^2c'_r)e^{-2r\mu}+\cdots+\mathcal{O}(\hbar),\end{aligned} \tag{3.18}$$

where all of the coefficients a_r, b_r, c_r, and c'_r are rational numbers of which the explicit forms are given in (3.11). Note that there are also nonperturbative contributions with different exponents in $J_b(\mu)$ and $J_c(\mu)$, though they do not affect the argument in this subsection. In the case of the ABJM theory, it was found [12] that the large μ expansion simplifies extensively if we redefine the chemical potential μ into

$$\mu_{\mathrm{eff}} = \mu + \frac{J_a(\mu)}{C}. \tag{3.19}$$

Indeed, the worldsheet instanton part takes care of all the bound states of the worldsheet instanton and the membrane instanton; the quadratic part of the instanton coefficients is completely absorbed into the perturbative part; the c_r' terms are also absorbed, and the c_r terms are the derivatives of b_r. In this subsection we shall find that in the redefinition $\mu_{\mathrm{eff}}(\mu)$ in a general (q,p) model one of the simplifications, the cancellation of the c_r' terms, still takes place.

In fact, in terms of μ_{eff}, it is not difficult to find that the linear part and the constant part are shifted as

$$\tilde{J}_b(\mu_{\mathrm{eff}}) = J_b(\mu) - \frac{J_a(\mu)^2}{C},$$
$$\tilde{J}_c(\mu_{\mathrm{eff}}) = J_c(\mu) - \frac{J_a(\mu)J_b(\mu)}{C} - \frac{BJ_a(\mu)}{C} + \frac{2J_a(\mu)^3}{3C^2}. \tag{3.20}$$

Now we find that not only the coefficients in $\tilde{J}_b(\mu_{\mathrm{eff}})$ but also those in $\tilde{J}_c(\mu_{\mathrm{eff}})$ are rational numbers except the overall factor $1/\pi$. Indeed the terms in $\pi\tilde{J}_c(\mu_{\mathrm{eff}})$ proportional to π^2, coming only from $J_c(\mu)$ and $-BJ_a(\mu)/C$, completely cancel as

$$c_r' - \frac{B}{C}\cdot a_r = \frac{r^2(4-q^2-p^2)}{6} - \frac{\frac{4-q^2-p^2}{3\hbar qp}}{\frac{4}{\hbar qp}}\cdot 2r^2 = 0. \tag{3.21}$$

Remarkably, this cancellation of irrationality is also true for the higher $\hbar$ corrections, as we explain at the end of the next section.

In the ABJM theory, the introduction of the effective chemical potential μ_{eff} was important as we have explained above. This nontrivial rationality in the coefficients of nonperturbative contributions might imply that the effective chemical potential also plays an important role in the $\mathcal{N}=4$ theories.

IV. HIGHER-ORDER CORRECTIONS

So far we have considered the grand potential $J_0(\mu)$ in the leading order of the classical limit $\hbar\to 0$. In this section, we shall consider the higher-order correction in $\hbar$ to the grand potential. We shall see that our results for $J_0(\mu)$ obtained in the previous sections are straightforwardly generalized to these corrections.

In Ref. [38], we found that, introducing a generalization of the power series (1.11),

$$\mathcal{F}(\alpha,\beta;\mu) = \sum_{\ell=1}^{\infty}\frac{(-1)^{\ell-1}e^{\ell\mu}}{\ell}\int\frac{dQdP}{2\pi}\frac{1}{(2\cosh\frac{Q}{2})^{q\ell+\alpha}}\frac{1}{(2\cosh\frac{P}{2})^{p\ell+\beta}}$$
$$= \sum_{\ell=1}^{\infty}\frac{(-1)^{\ell-1}e^{\ell\mu}}{2\pi\ell}\frac{\Gamma(\frac{q\ell+\alpha}{2})^2}{\Gamma(q\ell+\alpha)}\frac{\Gamma(\frac{p\ell+\beta}{2})^2}{\Gamma(p\ell+\beta)}, \tag{4.1}$$

with α and β being non-negative even integers, then, as well as the leading-order $J_0(\mu)=\mathcal{F}(0,0;\mu)$, the $\hbar$ corrections $J_2(\mu)$ and $J_4(\mu)$ to the grand potential (1.9) are also expressed in terms of $\mathcal{F}(\alpha,\beta;\mu)$ as

$$J_2(\mu) = \frac{qp}{24}(1-\partial_\mu^2)\mathcal{F}(2,2;\mu),$$
$$J_4(\mu) = \frac{(qp)^2}{5760}[-(1-\partial_\mu^2)(9-\partial_\mu^2)f_{41} + (1-\partial_\mu^2)(4-\partial_\mu^2)f_{42}], \tag{4.2}$$

with

$$f_{41} = \mathcal{F}(4,4;\mu) + \frac{1}{2}\mathcal{F}(2,4;\mu) + \frac{1}{2}\mathcal{F}(4,2;\mu) + \frac{1}{4}\mathcal{F}(2,2;\mu), \qquad f_{42} = \mathcal{F}(2,2;\mu). \tag{4.3}$$

If we continue q and p to irrational numbers, we can obtain the large μ expansion of the function $\mathcal{F}(\alpha,\beta;\mu)$ by the same method used in Sec. II. In the current case, instead of (2.1), the expansion of the ratio of the Gamma functions reads

$$\frac{\Gamma(x+\frac{\alpha}{2})^2}{\Gamma(2x+\alpha)} = \frac{2}{2^{2x+\alpha}}\sum_{m=\frac{\alpha}{2}}^{\infty}\frac{\gamma_{m-\frac{\alpha}{2}}}{m+x}, \tag{4.4}$$

and, instead of (2.4), for $\alpha,\beta\geq 2$ the partial fraction decomposition is simply

$$\frac{1}{\ell(\ell+\frac{2m}{q})(\ell+\frac{2n}{p})}=\frac{qp}{4mn}\frac{1}{\ell}-\frac{q}{2m(\frac{2n}{p}-\frac{2m}{q})}\frac{1}{\ell+\frac{2m}{q}}-\frac{p}{2n(\frac{2m}{q}-\frac{2n}{p})}\frac{1}{\ell+\frac{2n}{p}}. \tag{4.5}$$

We finally obtain the large μ expansion of $\mathcal{F}(\alpha,\beta;\mu)$ which consists of, other than the perturbative parts,

$$\mathcal{F}^{\mathrm{pert}}(\alpha,\beta;\mu)=\frac{1}{2\pi}\frac{\Gamma(\frac{\alpha}{2})^2}{\Gamma(\alpha)}\frac{\Gamma(\frac{\beta}{2})^2}{\Gamma(\beta)}[\mu-q(\psi^{(0)}(\alpha)-\psi^{(0)}(\alpha/2))-p(\psi^{(0)}(\beta)-\psi^{(0)}(\beta/2))], \tag{4.6}$$

the three nonperturbative parts

$$\mathcal{F}^{(z_i)}(\alpha,\beta;\mu)=\sum_{\ell_i=\lambda_i}^{\infty}\frac{F_{(\alpha,\beta)}(\frac{\ell_i}{z_i};\mu)}{\ell_i}\prod_{j=1(\neq i)}^{3}\cot\frac{\pi z_j\ell_i}{z_i}. \tag{4.7}$$

Here we have defined $(\lambda_1,\lambda_2,\lambda_3)=(\frac{\alpha}{2},\frac{\beta}{2},1)$ and

$$F_{(\alpha,\beta)}(r;\mu)=-\frac{2\pi}{\cos 2\pi r}\frac{\Gamma(2qr-\alpha+1)}{\Gamma(qr-\frac{\alpha}{2}+1)^2}\frac{\Gamma(2pr-\beta+1)}{\Gamma(pr-\frac{\beta}{2}+1)^2}e^{-2r\mu}. \tag{4.8}$$

In the derivation, we have used (3.3) to change the arguments of the Gamma functions as previously.

Roughly speaking, the pole cancellation works in the same way as in the case of $\alpha=\beta=0$ discussed in Sec. III: terms from different sectors share the same instanton exponent at the point where the cotangent factors diverge. The main difference is that the pole cancellation among the three sectors happens at $(m,n,l)=(qr,pr,2r)$ with $r\in\mathbb{N}/\gcd(q,p,2)$, only when the instanton number is large enough to satisfy $m\geq\frac{\alpha}{2}$ and $n\geq\frac{\beta}{2}$. Finally, the finite part remaining after the cancellation is given by

$$\frac{F_{(\alpha,\beta)}(r;\mu)-r\partial_r F_{(\alpha,\beta)}(r;\mu)+\frac{1}{2}r^2\partial_r^2 F_{(\alpha,\beta)}(r;\mu)}{\pi^2 z_1z_2z_3r^3}-\frac{(z_1^2+z_2^2+z_3^2)F_{(\alpha,\beta)}(r;\mu)}{3z_1z_2z_3r}, \tag{4.9}$$

or explicitly, as a quadratic polynomial in μ,

$$\frac{F_{(\alpha,\beta)}(r;\mu)}{2\pi^2 qpr^3}\left[2r^2\mu^2+(2r-4r^2H_{1(\alpha,\beta)}(r))\mu+1-2rH_{1(\alpha,\beta)}(r)+r^2(2H_{1(\alpha,\beta)}(r)^2-H_{2(\alpha,\beta)}(r))+\frac{\pi^2r^2(4-q^2-p^2)}{6}\right]. \tag{4.10}$$

Here we define the generalization of $H_s(r)$ in (3.13), $H_{s(\alpha,\beta)}(r)$ as

$$H_{s(\alpha,\beta)}(r)=q^s\left(2^{s-1}h_s(2qr-\alpha)-h_s\left(qr-\frac{\alpha}{2}\right)\right)+p^s\left(2^{s-1}h_s(2pr-\beta)-h_s\left(pr-\frac{\beta}{2}\right)\right), \tag{4.11}$$

which again comes from the derivatives of the Gamma functions in $F_{(\alpha,\beta)}(r;\mu)$.

For the small instanton number, we have to be careful, since the corresponding contribution from $\mathcal{F}^{(q)}(\alpha,\beta;\mu)$ or from $\mathcal{F}^{(p)}(\alpha,\beta;\mu)$ sometimes do not exist due to the lower bounds on the instanton number, $m\geq\frac{\alpha}{2}$ and $n\geq\frac{\beta}{2}$. At first sight it might seem that we have too many divergent cotangent factors to obtain the finite result. In these cases, however, the ratio of the Gamma functions becomes zero, which reduces the power of divergences. This can also be seen from the expression before the rewriting using (3.3).

In Sec. III C, we have discussed the simplification of the nonperturbative effects of $e^{-2r\mu}$ with $r\in\mathbb{N}/\gcd(q,p,2)$ associated to the redefinition of the chemical potential (3.19). In the discussion there, the following properties of the coefficient (3.11) are essential: the rationality of a_r, b_r, c_r, c'_r in (3.18) and the r-independence of the ratio of a_r and c'_r (3.21). As we have claimed in Sec. III C, the same simplification occurs also in the higher $\hbar$ corrections.

Here we shall see it explicitly by showing these properties. Since these properties are preserved under the differential operations in (4.2) which convert $\mathcal{F}(\alpha,\beta;\mu)$ to $J_n(\mu)$, we have only to care about the coefficients of the nonperturbative effects in $\mathcal{F}(\alpha,\beta;\mu)$ themselves. For the case of the large instanton number, $m \geq \frac{\alpha}{2}$ and $n \geq \frac{\beta}{2}$, the coefficients in $\mathcal{F}(\alpha,\beta;\mu)$ are given by (4.10), and these properties can be explicitly checked as for $J_0(\mu)$ in Sec. III C. For the case where one of these two conditions is not satisfied, the result (4.10) is no longer valid. However, we can see $a_r = c'_r = 0$. First, since the divergence is at most $\mathcal{O}(\varepsilon^{-1})$, as argued in the paragraph below (4.11), the second derivative of $F_{(\alpha,\beta)}(r;\mu)$ does not appear, and thus $a_r = 0$. Secondly, the relative π^2 factor would only appear in the second derivative of $F_{(\alpha,\beta)}(r;\mu)$ or in the cross terms of $\mathcal{O}(\varepsilon^{-1})$ and $\mathcal{O}(\varepsilon)$ between two cotangent factors. Since both of these terms are absent in this case, c'_r is also zero. Moreover, the explicit calculation shows the rationality of the other two, b_r and c_r. Therefore, the required properties hold also in this case.

There is still another way to obtain the large μ expansion of the function $\mathcal{F}(\alpha,\beta;\mu)$. From the power series definition (4.1), we find that the following differential relations are satisfied:

$$(q\partial_\mu + \alpha + 1)\mathcal{F}(\alpha+2,\beta;\mu) = \frac{1}{4}(q\partial_\mu + \alpha)\mathcal{F}(\alpha,\beta;\mu),$$
$$(p\partial_\mu + \beta + 1)\mathcal{F}(\alpha,\beta+2;\mu) = \frac{1}{4}(p\partial_\mu + \beta)\mathcal{F}(\alpha,\beta;\mu). \tag{4.12}$$

Decomposing these equations further into those for the terms with the same instanton exponents, we obtain the recursion relation between the coefficient in $\mathcal{F}(\alpha+2,\beta;\mu)$ [or in $\mathcal{F}(\alpha,\beta+2;\mu)$] and the corresponding one in $\mathcal{F}(\alpha,\beta;\mu)$. Regarding the constant coefficient in the nonperturbative sectors of $J_0(\mu) = \mathcal{F}(0,0;\mu)$ in (3.1) as the initial value for the recursion relation, we can reproduce the results for $\mathcal{F}(\alpha,\beta;\mu)$ in (4.7). In passing let us note that we can also use the relation (4.12) to obtain the perturbative part or the polynomial coefficients of the nonperturbative effects remaining after the pole cancellation.

To summarize our analysis for the higher-order corrections, we find that the total grand potential $J^{\text{pert+MB}}(\mu)$ in the WKB expansion obtained so far are given by

$$J^{\text{pert+MB}}(\mu) = \left(\frac{1}{\hbar}\mathcal{D}_0 + \hbar\mathcal{D}_2 + \hbar^3\mathcal{D}_4\right)J_0(\mu) + \mathcal{O}(\hbar^5), \tag{4.13}$$

with

$$\mathcal{D}_0 = 1, \qquad \mathcal{D}_2 = \frac{q^2p^2(1-\partial_\mu^2)\partial_\mu^2}{384(1+q\partial_\mu)(1+p\partial_\mu)},$$
$$\mathcal{D}_4 = \frac{q^3p^3(1-\partial_\mu^2)\partial_\mu^2}{92160(1+q\partial_\mu)(1+p\partial_\mu)} \times \left(-\frac{(9-\partial_\mu^2)(8+3q\partial_\mu)(8+3p\partial_\mu)}{16(3+q\partial_\mu)(3+p\partial_\mu)} + 4 - \partial_\mu^2\right). \tag{4.14}$$

Here we have used the recursion relation (4.12) to relate $\mathcal{F}(\alpha,\beta;\mu)$ to $\mathcal{F}(0,0;\mu)$. For the nonperturbative effects with constant coefficients, each ∂_μ is replaced with $-2m/q$, $-2n/p$ or $-l$. We hope that this expression is helpful in determining the coefficients of the membrane instantons at finite k.

V. CONCLUSION AND DISCUSSION

In this paper we have obtained a new understanding of the coefficients of the membrane instantons in the ABJM theory. First, the ABJM matrix model is generalized to include two parameters q and p. Due to these deformation parameters, the membrane instantons are subdivided into three instanton sectors $e^{-\frac{2\mu}{q}}$, $e^{-\frac{2\mu}{p}}$, and $e^{-\mu}$. The coefficients of these instantons are μ-independent constants, which are singular in the undeformed limit $(q,p) \to (1,1)$. The quadratic polynomial coefficients of the membrane instantons in the ABJM theory emerge as a result of the pole cancellation among these sectors.

The existence of the third kind of instanton $e^{-\mu}$, the exponent of which is completely independent of the deformation parameters (q,p), is rather surprising. Interestingly, if both q and p are odd integers, this instanton always vanishes for odd instanton numbers, while otherwise it cannot be distinguished from the other two kinds $e^{-\frac{2\mu}{q}}$ and $e^{-\frac{2\mu}{p}}$. This is why we could not detect it in our previous work [38] and decide to call it a ghost instanton. For this property, one would suspect that the third instantons are just artificial. However, since they form an infinite instanton series as the other two kinds of instantons and the poles appearing in the other two would never be cancelled without the third instantons, it is natural to regard them as physical.

Conceptually, we have to confess that we are still far from understanding the instanton effects clearly. Though the divergence of the instanton coefficients is essential in our argument, there is no intuitive description of the occurrence of the divergence itself. In view of the standard interpretation of the instanton coefficient as the volume of the instanton moduli space, we are tempted to give a similar interpretation to our results. From this viewpoint the divergence might denote the noncompactness of the instanton moduli space, while the cancellation implies the nonperturbative compactification of the moduli space.

Also, the role of the third instantons is unclear to us. Though their appearance is inevitable, they do not give any distinct exponents from the other two for integral (q, p). However, we do not have a concrete field theoretic picture of the instanton effects to answer these questions. Some hints may be provided from the gravity dual of the deformed theory.

The $\mathcal{N} = 4$ theory with s_a satisfying (1.10) is dual to the 11-dimensional supergravity on $\mathrm{AdS}_4 \times S^7/\Gamma$, where Γ is generated by three nonindependent operations $\mathbb{Z}_k$, $\mathbb{Z}_q$, and $\mathbb{Z}_p$, with the discrete torsion [41,53]. On the gravity side, the nonperturbative effects may be understood as M2-branes wrapping this complicated orbifold in various ways, as in Ref. [11]. An interesting point indicated by our explicit calculation is that, although there are bound states of the worldsheet instantons and the membrane instantons, there are no bound states among the three types of the membrane instantons without the worldsheet instantons. We hope that the study in the gravity side will shed new light on the instantons.

Technically, it is important to further analyze the partition function of the $\mathcal{N} = 4$ circular quiver superconformal Chern–Simons theories. Denoting the theory with $\{s_a\}_{a=1}^{M} = \{(+1)^q, (-1)^p\}$ as $(q, p)_k$, we would like to stress that our current work is the first one which succeeds in studying the model with the two-parameter deformation, $(q, p)_0$. This is in contrast to the recent studies of $(N_f, 1)_1$ in Ref. [45] and $(2, 1)_k$ in Ref. [38], both of which contain only one-parameter deformation. We believe that to study the deformations with as many parameters as possible along this line is important for the exact computation of the instanton effects. In fact, in the ABJM theory, it is only after we deformed the integral Chern–Simons level k into an irrational number that we were able to split the nonperturbative effects into the worldsheet instantons and the membrane instantons and describe them in terms of the refined topological strings. We expect that the irrationality of q, p will play a similar role in determining the membrane instantons in the $\mathcal{N} = 4$ theories.

Let us point out some possibilities that the nonperturbative effects have more abundant fine structures to be clarified by introducing additional deformations. After seeing that the combination $(q, p, 2)$ can be put on the equal footing in Sec. III B, we expect that we can introduce an additional deformation parameter which changes the exponent of the third instanton $e^{-\mu}$. Also, the results obtained in Refs. [43,45] suggest further deformations. The grand potential obtained in Ref. [43] for $r = 4$, $k = 2$ and the one obtained in [45] with $k = 3$, 6 contain polynomials of degree higher than 2 in instanton coefficients. Since the coefficients of the membrane instantons are at most quadratic, these results imply that the worldsheet instantons also have the nonconstant structure, which requires more deformations.

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