

Inverse approach to solutions of the Dirac equation for space-time dependent fields

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Exact solutions of the Dirac equation in external electromagnetic background fields are very helpful for understanding nonperturbative phenomena in quantum electrodynamics. However, for the limited set of known solutions, the field often depends on one coordinate only, which could be the time t , a spatial coordinate such as x or r , or a light-cone coordinate such as $ct - x$. By swapping the roles of known and unknown quantities in the Dirac equation, we are able to generate families of solutions of the Dirac equation in the presence of genuinely space-time dependent electromagnetic fields in $1 + 1$ and $2 + 1$ dimensions.

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I. INTRODUCTION

Quantum electrodynamics (QED) as the theory of charged particles interacting with electromagnetic fields is well understood in the context of standard perturbation theory and can describe several intriguing phenomena of nature. However, QED contains other fascinating effects that cannot be explained using perturbative methods. Such nonperturbative effects can arise when the electromagnetic field is so strong that it cannot be treated as a perturbation. In order to understand these phenomena, it is often useful to study the behavior of exact solutions in those external background fields.

Unfortunately, although the Dirac equation was first formulated more than eighty years ago [1], the set of known exact solutions is still quite limited (see, e.g., [2] for a review). Apart from the Coulomb field $\propto 1/r^2$ [3,4], exact solutions are known for a constant electric field and a Sauter profile in space $\propto 1/\cosh^2(kx)$ or time $\propto 1/\cosh^2(\omega t)$, for example. The latter are relevant for the nonperturbative Sauter-Schwinger effect [5–7] corresponding to electron-positron pair creation from vacuum via tunneling. In contrast to electron-positron pair creation in the perturbative (multiphoton) regime which has been observed at SLAC [8], this nonperturbative prediction of quantum field theory has not been conclusively experimentally verified yet. However, there are several experimental initiatives which might be able to eventually reach the ultrastrong field regime necessary for observing this striking effect [9].

Furthermore, exact solutions are known for a constant magnetic field (relativistic Landau levels, see [10,11]) and plane waves, where the fields depend on one of the light-cone coordinates such as $ct - x$ (Volkov solutions, see e.g. [12–15]). These (transverse) fields do not induce pair creation from vacuum.

Nevertheless, in all these cases, the fields depend on one coordinate only (such as r , x , t , or $ct - x$). As a result of this high degree of symmetry, the set of partial differential equations can be reduced to an ordinary differential equation, which greatly simplifies the analysis. An analogous limitation applies to our theoretical understanding of the Sauter-Schwinger effect. Even though there are many results for fields which depend on one coordinate only, we are just beginning to understand the impact of the interplay between spatial and temporal dependencies, see, e.g., [16–21].

In the following, we develop a method which allows us to obtain solutions of the Dirac equation for genuinely space-time dependent fields. To this end, we pursue a different approach by assuming that we already know a solution to the Dirac equation. We then calculate the vector potential A_μ corresponding to the given solution from the Dirac equation. This is feasible as the Dirac equation does not contain any derivatives of the vector potential. More generally speaking, we write down a solution to a partial differential equation and then try to find a physical problem associated with the solution—a concept also well known in the field of fluid dynamics, see, for example, [22].

II. LIGHT-CONE COORDINATES

Let us start with the most simple and yet nontrivial case—the Dirac equation in $1 + 1$ dimensions. For the following derivation, it is convenient to transform to light-cone coordinates x_\pm defined as ($\hbar = c = 1$)

$$x_+ = \frac{t + x}{\sqrt{2}}, \quad x_- = \frac{t - x}{\sqrt{2}}. \quad (1)$$

Since perhaps not all readers will be familiar with the form of the subsequent expressions in light-cone coordinates, let us insert a brief reminder. The Jacobian matrix of the coordinate transformation between Cartesian and light-cone coordinates,

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$$J^\mu{}_\nu = \frac{\partial(x_+, x_-)}{\partial(t, x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (2)$$

yields the transformation laws for tensors such as the partial derivatives

$$\partial'_\mu = (J^{-1})^\nu{}_\mu \partial_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} \partial_t + \partial_x \\ \partial_t - \partial_x \end{pmatrix} = \begin{pmatrix} \partial_{x_+} \\ \partial_{x_-} \end{pmatrix} =: \begin{pmatrix} \partial_+ \\ \partial_- \end{pmatrix}. \quad (3)$$

In 1 + 1 dimensions, the electromagnetic field strength tensor contains only one independent component, the electric field $E(t, x)$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \quad (4)$$

which thus reads in light-cone coordinates

$$F'_{\mu\nu} = (J^{-1})^\lambda{}_\mu (J^{-1})^\rho{}_\nu F_{\lambda\rho} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}. \quad (5)$$

Transforming the Cartesian Minkowski metric tensor $\eta_{\mu\nu} = \text{diag}(+1, -1)$ to light-cone coordinates as well gives

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (6)$$

A possible choice of light-cone gamma matrices satisfying the Clifford algebra's anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad (7)$$

therefore is

$$\gamma^+ = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad \gamma^- = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}. \quad (8)$$

Note that in 1 + 1 and 2 + 1 dimensions, the Clifford algebra can be satisfied with 2×2 -matrices.

III. INVERSE APPROACH

The Dirac equation, minimally coupled to the electromagnetic potential A_μ via the charge q

$$(i\gamma^\mu[\partial_\mu + iqA_\mu] - m)\psi = 0, \quad (9)$$

assumes the following simple form in terms of the light-cone gamma matrices (8):

$$\begin{pmatrix} -m & i\sqrt{2}[\partial_+ + iqA_+] \\ i\sqrt{2}[\partial_- + iqA_-] & -m \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \quad (10)$$

Traditionally, the Dirac equation is treated as a partial differential equation. A solution ψ for a specific potential

A_μ is typically calculated by reducing the Dirac equation to an ordinary differential equation. In our approach, we assume that we know a specific spinor $\psi = (\psi_1, \psi_2)^\top$ which is a solution to the Dirac equation and calculate the corresponding potential. Thus, we solve (10) for the components of A_μ

$$\begin{aligned} qA_+ &= i \frac{\partial_+ \psi_2}{\psi_2} - \frac{m}{\sqrt{2}} \frac{\psi_1}{\psi_2}, \\ qA_- &= i \frac{\partial_- \psi_1}{\psi_1} - \frac{m}{\sqrt{2}} \frac{\psi_2}{\psi_1}. \end{aligned} \quad (11)$$

For arbitrary ψ , these expressions are not necessarily real. Therefore, we require the imaginary parts of qA_+ and qA_- to vanish, giving two conditions which we use to eliminate 2 real degrees of freedom of the spinor ψ . Using the polar representation for the spinor components $\psi_k = r_k e^{i\varphi_k}$, these conditions can be written as

$$\begin{aligned} r_2 \partial_+ r_2 - \frac{m}{\sqrt{2}} r_1 r_2 \sin(\varphi_1 - \varphi_2) &= 0, \\ r_1 \partial_- r_1 + \frac{m}{\sqrt{2}} r_1 r_2 \sin(\varphi_1 - \varphi_2) &= 0. \end{aligned} \quad (12)$$

Adding the two equations gives

$$\partial_- r_1^2 = -\partial_+ r_2^2, \quad (13a)$$

$$\partial_- r_1 = \frac{m}{\sqrt{2}} r_2 \sin(\varphi_2 - \varphi_1). \quad (13b)$$

The first equation (13a) can be solved for r_2 by integrating with respect to x_+

$$r_2 = \sqrt{c(x_-) - \int \partial_- r_1^2 dx_+}, \quad (14)$$

where $c(x_-)$ is an integration constant that may still depend on x_- . The remaining equation (13b) determines the phase difference $\varphi_2 - \varphi_1$

$$\varphi_2 - \varphi_1 = \arcsin\left(\frac{\sqrt{2} \partial_- r_1}{m r_2}\right), \quad (15)$$

where we could also use other branches of the arcsin function such as $\Delta\varphi = \varphi_2 - \varphi_1 \rightarrow \pi - \Delta\varphi$, leading to different solutions in general—see the remark after Eq. (17). Using the abbreviation

$$s = \sqrt{c - \int \partial_- r^2 dx_+ - \frac{2}{m^2} (\partial_- r)^2}, \quad (16)$$

and Eqs. (14) and (15), we can calculate the form of the spinor ψ

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = e^{i\varphi} \begin{pmatrix} r \\ \pm s + i\frac{\sqrt{2}}{m}\partial_- r \end{pmatrix}, \quad (17)$$

where we have set $r = r_1$ and $\varphi = \varphi_1$. Note that we find two different solutions with $\pm s$ corresponding to the different branches of the arcsin or square-root functions in Eqs. (15) and (16), respectively.

Local gauge invariance allows us to eliminate the phase $e^{i\varphi}$ by applying a gauge transformation $\psi \mapsto \psi' = e^{-i\varphi}\psi$, which adds a term $\partial_\mu\varphi$ to qA_μ . The components of A_μ using the spinor given in (17) finally are

$$\begin{aligned} qA_+ &= \mp \frac{m}{\sqrt{2}} \frac{r}{s} \mp \frac{\sqrt{2}}{m} \frac{\partial_+ \partial_- r}{s}, \\ qA_- &= \mp \frac{m}{\sqrt{2}} \frac{s}{r}. \end{aligned} \quad (18)$$

These are obviously real as long as r and s are real, too. The electric field corresponding to this potential according to (5) is

$$E = \partial_- A_+ - \partial_+ A_-. \quad (19)$$

In summary, by choosing a real generating function $r(x_+, x_-)$ and a real supplementary boundary value function $c(x_-)$, we can generate arbitrary space-time dependent solutions $\psi(x_+, x_-)$ of the Dirac equation in the presence of an electromagnetic background A_μ , which can also depend on space and time.

Obviously, the associated electromagnetic field strength tensor $F^{\mu\nu}$ in Eq. (5) automatically satisfies the homogeneous Maxwell equations as it has been derived from a vector potential A_μ . If we demand that it also obeys the inhomogeneous Maxwell equations ($\mu_0 = 1$)

$$\partial_\nu F^{\mu\nu} = j^\mu, \quad (20)$$

we have to specify the sources j^μ accordingly. In $1+1$ dimensions we find

$$\begin{aligned} \rho &= -\partial_x E = \frac{1}{\sqrt{2}}(\partial_- E - \partial_+ E), \\ j &= \partial_t E = \frac{1}{\sqrt{2}}(\partial_- E + \partial_+ E), \end{aligned} \quad (21)$$

where ρ is the charge density and j is the current density. For nontrivial field profiles $E(t, x)$, they will be nonzero in general. However, this is no surprise because the only vacuum solution of the Maxwell equations in $1+1$ dimensions is a constant electric field $E = \text{const}$.

IV. SOLUTIONS

In order to illustrate the approach presented in the previous section, let us discuss some exemplary solutions

that can be found using this method, starting with the most simple ones. The expressions for the spinor and the potential components are significantly simplified if r is independent of x_- .

A. Plane waves

Choosing r and $s = \pm\sqrt{c}$ to be constant,

$$\psi = \begin{pmatrix} r \\ s \end{pmatrix} = \text{const}, \quad (22)$$

leads to a constant electromagnetic vector potential

$$\begin{aligned} qA_+ &= -\frac{m}{\sqrt{2}} \frac{r}{s} = \text{const}, \\ qA_- &= -\frac{m}{\sqrt{2}} \frac{s}{r} = \text{const}. \end{aligned} \quad (23)$$

Thus, a gauge transformation $\psi \mapsto \psi' = e^{-ip_\mu x^\mu} \psi$ with

$$p_\mu := \begin{pmatrix} p_+ \\ p_- \end{pmatrix} = \frac{m}{\sqrt{2}} \begin{pmatrix} r/s \\ s/r \end{pmatrix} \quad (24)$$

can be used to set the potential components to zero and reveals that these solutions are plane wave solutions to the free Dirac equation of either positive or negative energy. Transforming the light-cone momenta p_\pm back to the usual Cartesian representation $p_0 = (p_+ + p_-)/\sqrt{2}$, we find that the energy is given by $p_0 = m(r/s + s/r)/2$. Thus a solution where both r and s are positive (or both negative) corresponds to a positive energy whereas different signs of r and s yield a negative energy.

B. Single pulses

In this subsection, we find solutions for arbitrary light-cone fields $E(x_+)$ and $E(x_-)$, i.e., pulses moving along the light lines. Such solutions were found before using traditional methods as well [14,15].

1. x_+ -dependent pulse

Let us assume that the function r depends on x_+ only while $s = \pm\sqrt{c} = \text{const}$

$$\psi = \begin{pmatrix} r(x_+) \\ s \end{pmatrix}. \quad (25)$$

In this case, neither the spinor nor the vector potential depends on x_- which simplifies the expression for the electric field

$$qE = \underbrace{q\partial_- A_+}_{=0} - q\partial_+ A_- = \frac{m}{\sqrt{2}} s \partial_+ \frac{1}{r(x_+)}. \quad (26)$$

This is a first-order ordinary differential equation for $r(x_+)$ which can be integrated easily

$$r(x_+) = r_{\text{in}} \left[1 + \frac{\sqrt{2} r_{\text{in}}}{m s} q \int_{-\infty}^{x_+} E(\tilde{x}_+) d\tilde{x}_+ \right]^{-1}, \quad (27)$$

with $r_{\text{in}} = r(x_+ \rightarrow -\infty)$. Comparison with Sec. IV A reveals that the prefactor $\sqrt{2}r_{\text{in}}/(ms)$ in front of the above x_+ -integral over qE is just the inverse initial momentum $1/p_-^{\text{in}}$. As already discussed in [14], the term in the square bracket in Eq. (27) vanishes and thus r diverges when this x_+ -integral over qE becomes large enough to compensate p_-^{in} . Note that the light-cone dispersion relation $p_+p_- = m^2/2$ shows that p_+ must diverge when p_- vanishes and vice versa.

Now, let us recall that the phenomenon of electron-positron pair creation (such as in the Sauter-Schwinger effect) can be described by the situation where an initial solution with positive energy transforms into a final solution which contains contributions with negative energies (or vice versa). Assuming that r becomes constant initially and finally, we find that pair creation can only occur if $r(x_+)$ changes its sign somewhere, i.e., if $r(x_+)$ vanishes or diverges at some point. If $r(x_+)$ crosses zero, the electric field (26) diverges—whereas a diverging $r(x_+)$ precisely corresponds to the case discussed above, see also [14]. Thus, we find that we cannot describe particle creation in this case without introducing some singularity (see the Appendix).

2. x_- -dependent pulse

In a similar way, we can derive solutions for electric fields only depending on x_- by setting $r = \text{const}$ and letting $s(x_-) = \pm\sqrt{c(x_-)}$ depend on x_-

$$\psi = \begin{pmatrix} r \\ s(x_-) \end{pmatrix}. \quad (28)$$

Thus, the electric field can be calculated as follows:

$$qE = q\partial_- A_+ - \underbrace{q\partial_+ A_-}_{=0} = -\frac{m}{\sqrt{2}} r \partial_- \frac{1}{s(x_-)}, \quad (29)$$

which is again a first-order ordinary differential equation for $s(x_-)$. The solution is given by

$$s(x_-) = s_{\text{in}} \left[1 - \frac{\sqrt{2} s_{\text{in}}}{m r} q \int_{-\infty}^{x_-} E(\tilde{x}_-) d\tilde{x}_- \right]^{-1}, \quad (30)$$

with $s_{\text{in}} = s(x_- \rightarrow -\infty)$. In complete analogy, the same arguments as for an x_+ -dependent pulse apply in this case.

C. Two pulses

As a nontrivial extension of these two cases, we can combine the previous two solutions into a single spinor

$$\psi = \begin{pmatrix} r(x_+) \\ s(x_-) \end{pmatrix}, \quad (31)$$

where the two components are given by

$$\begin{aligned} r(x_+) &= r_{\text{in}} \left[1 + \frac{\sqrt{2} r_{\text{in}}}{m s_{\text{in}}} q \int_{-\infty}^{x_+} E_+(\tilde{x}_+) d\tilde{x}_+ \right]^{-1}, \\ s(x_-) &= s_{\text{in}} \left[1 - \frac{\sqrt{2} s_{\text{in}}}{m r_{\text{in}}} q \int_{-\infty}^{x_-} E_-(\tilde{x}_-) d\tilde{x}_- \right]^{-1}. \end{aligned} \quad (32)$$

We may calculate the electric field using (18) and (19)

$$E(x_+, x_-) = \frac{r(x_+)}{r_{\text{in}}} E_-(x_-) + \frac{s(x_-)}{s_{\text{in}}} E_+(x_+). \quad (33)$$

Initially, we have $E(x_+, x_-) = E_-(x_-) + E_+(x_+)$ which corresponds to two independent pulses approaching each other from different directions. When these two pulses meet, however, this is no longer true—which shows that the mapping from ψ (i.e., r and s) to A_μ is not linear. For late times, these pulses propagate again independently, but with modified amplitudes in general.

D. Emerging pulses

Another solution where the corresponding electric field consists of two pulses can be generated by setting

$$r(x_+, x_-) = r_{\text{in}} + \frac{\xi}{1 + e^{-\gamma x_+} + e^{-\gamma x_-}}. \quad (34)$$

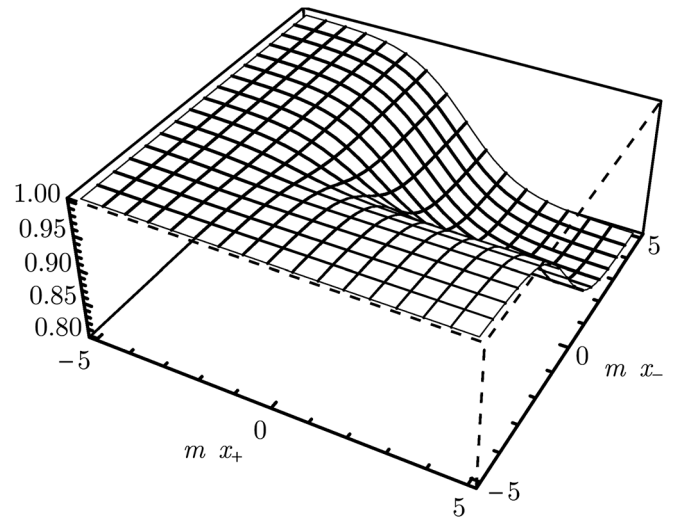


FIG. 1. Plot of $r(x_+, x_-)$ as given in (34) with $r_{\text{in}} = 1$, $\xi = 0.2$, and $\gamma = 1.2/m$.

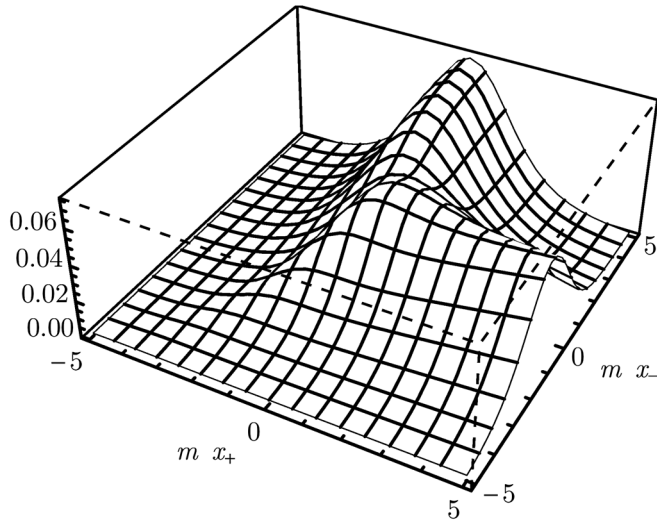


FIG. 2. Plot of the electric field qE corresponding to the solution generated by $r(x_+, x_-)$ given in (34) with $r_{in} = 1$, $\xi = 0.2$, and $\gamma = 1.2/m$.

For nonvanishing ξ and $\gamma > 0$, the chosen $r(x_+, x_-)$ will be constant almost everywhere except in the vicinity of the forward light cone (see Fig. 1).

In this case, the expression for s according to (16) is not as simple as before because r is not independent of x_- . Nevertheless, s can be calculated analytically, although the resulting expressions for s and the electric field qE are quite lengthy. Thus, we will only give a plot of the resulting electric field which shows the two pulses emerging from the origin and moving along the forward light lines (see Fig. 2).

V. EXTENSION TO 2 + 1 DIMENSIONS

The approach presented here can be extended to 2 + 1 dimensional space-times as well. We use the Cartesian coordinate y in addition to the light-cone coordinates x_+ and x_- . Thus, the metric tensor becomes

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (35)$$

In order to complete our set of gamma matrices from (8), we choose the third γ -matrix according to

$$\gamma^2 = i\sigma_z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (36)$$

Thus the Dirac equation in 2 + 1 dimensions is given by

$$\begin{pmatrix} -m - [\partial_y + iqA_y] & i\sqrt{2}[\partial_+ + iqA_+] \\ i\sqrt{2}[\partial_- + iqA_-] & -m + [\partial_y + iqA_y] \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \quad (37)$$

In complete analogy to Sec. III, we solve the Dirac equation for qA_+ and qA_- and reduce the spinor's number of degrees of freedom by requiring the imaginary parts of the electromagnetic potential's components to vanish. After some calculation, we are able to write the spinor and the electromagnetic potential in terms of three real functions $r_1(x_+, x_-, y)$, $r_2(x_+, x_-, y)$ and $c(x_+, x_-)$. Explicitly, a spinor of the form

$$\psi = \begin{pmatrix} r_1 \\ s - iu \end{pmatrix}, \quad (38)$$

with

$$s = \pm \sqrt{r_2^2 - u^2} \quad (39)$$

and

$$u = \frac{1}{\sqrt{2}r_1} \left[c(x_+, x_-) + \int (\partial_- r_1^2 + \partial_+ r_2^2) dy \right] \quad (40)$$

is a solution of the Dirac equation with the potential components

$$\begin{aligned} qA_+ &= -\frac{m}{\sqrt{2}} \frac{r_1}{s} - \frac{1}{\sqrt{2}} \frac{\partial_y r_1}{s} + \frac{\partial_+ u}{s}, \\ qA_- &= -\frac{m}{\sqrt{2}} \frac{r_2^2}{r_1 s} + \frac{1}{\sqrt{2}} \frac{r_2 \partial_y r_2}{r_1 s} - \frac{u \partial_- r_1}{r_1 s}, \\ qA_y &= -m \frac{u}{s} - \frac{u \partial_y r_1}{r_1 s} + \frac{1}{\sqrt{2}} \frac{\partial_+ r_2^2}{r_1 s}. \end{aligned} \quad (41)$$

In contrast to 1 + 1 dimensions, the electromagnetic field strength tensor contains three independent components, for example the two electric fields $E_{x,y}$ in x and y directions plus the perpendicular magnetic field B_z . These components of the electromagnetic field can be calculated as follows:

$$\begin{aligned} E_x &= \partial_- A_+ - \partial_+ A_-, \\ E_y &= \frac{1}{\sqrt{2}} (\partial_- A_y - \partial_y A_- + \partial_+ A_y - \partial_y A_+), \\ B_z &= \frac{1}{\sqrt{2}} (\partial_- A_y - \partial_y A_- - \partial_+ A_y + \partial_y A_+). \end{aligned} \quad (42)$$

We see that these expressions simplify significantly if r_1 and r_2 are independent of y . In that case, the electromagnetic field does only depend on the light-cone coordinates as before and similar solutions as in the 1 + 1 dimensional

case can be found, e.g. one and two wave fronts. In fact, the solutions given in Sec. IV are solutions to the 2 + 1 dimensional Dirac equation as well but can be extended to also include a transverse electric and magnetic field component.

To verify that our method reproduces known solutions, we insert the lowest Landau level solution

$$\psi = \mathcal{N} \exp\left(-\frac{1}{2}qB\left[x - \frac{k_y}{qB}\right]^2\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (43)$$

into our formalism, i.e. we set

$$\begin{aligned} r_1(x_+, x_-) &= \mathcal{N} \exp\left(-\frac{1}{2}qB\left[\frac{x_+ - x_-}{\sqrt{2}} - \frac{k_y}{qB}\right]^2\right), \\ r_2(x_+, x_-) &= r_1(x_+, x_-), \quad c = 0, \end{aligned} \quad (44)$$

where \mathcal{N} is a normalization constant. Calculating the potential components gives

$$qA_+ = qA_- = -\frac{m}{\sqrt{2}}, \quad qA_y = -qB\frac{x_+ - x_-}{\sqrt{2}} + k_y, \quad (45)$$

so that the electromagnetic field is

$$E_x = E_y = 0, \quad B_z = B, \quad (46)$$

which is the expected result.

VI. CONCLUSIONS AND OUTLOOK

We have developed an inverse approach for generating families of exact solutions of the Dirac equation in the presence of space-time dependent electromagnetic fields in 1 + 1 and 2 + 1 dimensions. Somewhat similar to optimal control theory, we start with a suitable ansatz for the spinor ψ and then derive the appropriate background field A_μ which supports this solution. In 1 + 1 dimensions, we may choose a real generating function $r(x_+, x_-)$ and a suitable real supplementary boundary value function $c(x_-)$ such that the radicand in Eq. (14) stays positive. In 2 + 1 dimensions, we may choose two real generating functions $r_1(x_+, x_-, y)$ and $r_2(x_+, x_-, y)$ as well as one real boundary value function $c(x_+, x_-)$.

The solutions generated in this way may depend on space and time in a complicated manner—a situation which is quite difficult to treat with traditional methods. As one possible application, our method could be used to solve steering problems such as: given an initial wave packet ψ_{in} , which electromagnetic field A_μ induces an evolution to a prescribed final wave packet ψ_{out} ? As another application, these exact solutions could be used as touchstones for already existing exact or approximate nonperturbative derivation techniques (e.g. the worldline instanton method [23]) or as starting point for new approximative methods,

such as WKB [24] or linearization around a given background solution (see the Appendix).

The structure of the Dirac equation suggests that this general strategy can also be applied to 3 + 1 dimensions, where both the potential A_μ and the Dirac bispinor ψ have four components. Thus, for a given ψ , we get four equations for the four components A_μ , which can be solved (except in singular cases). However, the four constraints $\mathfrak{S}(A_\mu) = 0$ assume a form which is far more complicated than in 1 + 1 and 2 + 1 dimensions. This renders the identification of real generating functions which correspond to the remaining degrees of freedom rather cumbersome. The analysis could be simplified by restricting the space-time dependence to 1 + 1 and 2 + 1 dimensions, which should be the subject of further investigations.

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APPENDIX: PERTURBED SOLUTION

To find solutions for electric fields that create electron-positron pairs (see, e.g., [26–29]), we use the ansatz

$$r = \alpha + \beta \sin(m\gamma), \quad (A1)$$

where the Bogoliubov coefficients α and β as well as the eikonal function γ are slowly varying functions of the light cone coordinates. (We consider 1 + 1 dimensions for simplicity.) The main idea here is that α is an exact solution and β is used to slowly turn on an oscillating perturbation. The value of β then is related to the pair creation rate.

However, the calculation of s and qE is rather complicated for arbitrary functions α , β , and γ because s depends nonlinearly on r . Hence, as the perturbation should be small, we calculate the electric field only up to linear order in β

$$E = E^{(\alpha)} + E^{(\beta)} + \mathcal{O}(\beta^2), \quad (A2)$$

where $qE^{(\alpha)}$ is the unperturbed force of order β^0 and $qE^{(\beta)}$ is the first-order perturbation of order β^1 . Apart from this linearization, we assume that the mass m represents the largest energy scale in the problem and thus we employ a large- m expansion on top of the approximation in Eq. (A2). Expanding $qE^{(\beta)}$ into powers of m and keeping only the highest-order term gives

$$qE^{(\beta)} = \sqrt{2}\beta \frac{\cos(m\gamma)}{s_\alpha \partial_+ \gamma} \left[m^2 (\partial_+ \gamma)^2 (\partial_- \gamma)^2 - \left(\underbrace{\frac{m}{\sqrt{2}} \frac{\alpha}{s_\alpha} \partial_- \gamma}_{=-qA_+^{(\alpha)}} + \underbrace{\frac{m}{\sqrt{2}} \frac{s_\alpha}{\alpha} \partial_+ \gamma}_{=-qA_-^{(\alpha)}} \right)^2 \right] + \mathcal{O}(m^1), \quad (\text{A3})$$

with the abbreviation

$$s_\alpha = \sqrt{c - \int \partial_- \alpha^2 dx_+}. \quad (\text{A4})$$

[$qA_+^{(\alpha)}$ and $qA_-^{(\alpha)}$ are the leading-order contributions to the vector potential.] Since α , β , and γ are supposed to be slowly varying, the leading contribution (A3) would be rapidly oscillating due to the prefactor $\cos(m\gamma)$ unless the

phase function γ has a stationary point (see below). Of course, such a rapidly oscillating force with a frequency of order m could well create pairs, but this process would be typically in the perturbative (multiphoton) regime. Here, we are interested in nonperturbative phenomena such as the Sauter-Schwinger effect and thus we demand that these rapidly oscillating contributions are absent—at least to leading order. Thus, we require the term of order m^2 in $qE^{(\beta)}$ to vanish. This is the case if $S = m\gamma$ solves the eikonal equation

$$\frac{m^2}{2} = (\partial_+ S + qA_+^{(\alpha)})(\partial_- S + qA_-^{(\alpha)}). \quad (\text{A5})$$

Therefore, this condition can be used to fix γ for a given α . Then, the leading order of $qE^{(\beta)}$ is of order m^1

$$qE^{(\beta)} = \frac{m}{\sqrt{2}} \frac{1}{s_\alpha} \sin(m\gamma) \left\{ 2(\partial_+ \beta) \left[\frac{\partial_- \gamma}{\partial_+ \gamma} + \left(\frac{\alpha}{s_\alpha} \frac{\partial_- \gamma}{\partial_+ \gamma} \right)^2 \right] + 2(\partial_- \beta) \left[1 + \left(\frac{s_\alpha}{\alpha} \right)^2 \frac{\partial_+ \gamma}{\partial_- \gamma} \right] + \beta \left[\frac{\partial_+ \alpha}{\alpha} \left(\frac{\partial_- \gamma}{\partial_+ \gamma} + \left(\frac{\alpha}{s_\alpha} \frac{\partial_- \gamma}{\partial_+ \gamma} \right)^2 + 2 \left(\frac{s_\alpha}{\alpha} \right)^2 \right) + \frac{\partial_- \alpha}{\alpha} \left(2 + \left(\frac{s_\alpha}{\alpha} \right)^2 \frac{\partial_+ \gamma}{\partial_- \gamma} + \left(\frac{\alpha}{s_\alpha} \right)^2 \left(2(\partial_- \gamma)^2 - \frac{\partial_- \gamma}{\partial_+ \gamma} \right) \right) + \partial_+ (\partial_- \gamma)^2 + \left(\frac{\alpha}{s_\alpha} \right)^2 \frac{\partial_- \gamma}{\partial_+ \gamma} \partial_+ \left(\frac{\partial_- \gamma}{\partial_+ \gamma} \right) \right] \right\} + \mathcal{O}(m^0), \quad (\text{A6})$$

where we have used the eikonal equation (A5) to simplify some expressions. If we require this rapidly oscillating term to vanish as well, we get a linear first-order partial differential equation for β . However, this linear equation does not have any source term. Therefore, a solution where β vanishes initially will not generate any pairs unless the coefficients of $\partial_+ \beta$ and $\partial_- \beta$ vanish at some point. As those are proportional to

$$\frac{m}{\sqrt{2}} \frac{\alpha}{s_\alpha} \partial_- \gamma + \frac{m}{\sqrt{2}} \frac{s_\alpha}{\alpha} \partial_+ \gamma, \quad (\text{A7})$$

we only obtain pair creation at this level of description if (A7) vanishes somewhere. According to the eikonal equation (A3), this in turn implies that $\partial_+ \gamma$ or $\partial_- \gamma$ has to vanish somewhere, i.e., that the phase function becomes stationary.

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