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Wilson fermion doubling phenomenon on an irregular lattice: Similarity and difference with the case of a regular lattice

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It is shown that the Wilson fermion doubling phenomenon on irregular lattices (simplicial complexes) does exist. This means that the irregular (not smooth) zero or soft modes exist in the case when the "naive fermions" are introduced. The statement is proved on a four-dimensional lattice by means of the Atiyah-Singer index theorem, and then it is extended easily into the cases $D < 4$. But there is a fundamental difference between doubled quanta on regular and irregular lattices: in the latter case, the propagator decreases exponentially. This means that the doubled quanta on irregular lattices are "bad" quasiparticles.

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I. INTRODUCTION

The Wilson fermion doubling phenomenon on the regular periodic lattices was discovered long ago in Ref. [1]. The phenomenon and its influence on physics was studied in a number of works (for example, see Refs. [2–4]). It was proved in Refs. [5,6] that the fermion doubling phenomenon indeed takes place on any periodic lattice with local fermion action transforming to the usual Dirac action in the long-wavelength region. But the question about the existence of the Wilson fermion doubling on irregular lattices is open at present. This means that the problem is unsolved in the case of lattice quantum gravity theory [7] (see Refs. [8,9]).

In this paper, I show that the Wilson fermion doubling phenomenon on irregular lattices (simplicial complexes) with $D \leq 4$ does exist [10]. The statement is proved on a four-dimensional lattice by means of the Atiyah-Singer index theorem, then it is extended easily into the cases $D < 4$. However, there exists a fundamental difference between the propagation of doubling modes on regular and irregular lattices. In the first case, the propagator of the irregular modes is the same as the propagator of the regular modes from the spectrum origin, i.e., power behaved. On the contrary, the propagation of irregular modes on an irregular lattice is similar to the Markov process of a random walk. So it turns out that the propagator of irregular modes on an irregular lattice decreases very quickly (exponentially); the doubled irregular modes are "bad" quasiparticles.

From here, the motivation of the work follows. The subsequent considerations in this section have a speculative character.

Let us suppose that the space-time is discrete on the microscopic level; the corresponding lattice is an irregular and "breathing" one. This means that the dynamics of the variables $\hat{e}_{\mathcal{W}ij}$ (see Sec. II) describing the metrics is governed by a wave function [see also Sec. V and (5.9)]. Suppose also that there are nonzero densities of the irregular quanta of the three known neutrinos.

Nonzero densities ($n^{\mathcal{I}} \neq 0$) of the irregular quanta do not contradict the fundamental notions of astrophysics, since the irregular quanta energy can be arbitrarily small (see the end of Sec. IV).

The following consequences might have resulted from these suppositions:

(1) *The problem of dark matter in cosmology.*

Do nonzero densities of the neutrino irregular quanta form dark matter in cosmology? It seems that this hypothesis does not contradict the main properties of dark matter: (i) the irregular quanta are "bad" quasiparticles, so such dark matter can be localized; and (ii) the irregular quanta interact very slightly with all normal quanta.

But nonzero densities of the neutrino irregular quanta give a contribution to the energy-momentum tensor, and therefore to the gravitational potential in the vicinity of a metagalaxy.

(2) *The problem of neutrino oscillation*.

The neutrino oscillations, i.e., the mutual oscillating transitions of the neutrinos of different generations, have been observed for a long time now. The common explanation of the phenomenon is based on the assumption that the neutrino mass matrix is nondiagonal. Moreover, in order to match all the experimental evidence, extra neutrino fields are introduced which are sterile regarding all interactions (naturally, except for the gravitational one). These sterile neutrinos cannot be observed directly; they are coupled to the three known neutrino generations only by means of a common mass matrix, and this is the way they give a

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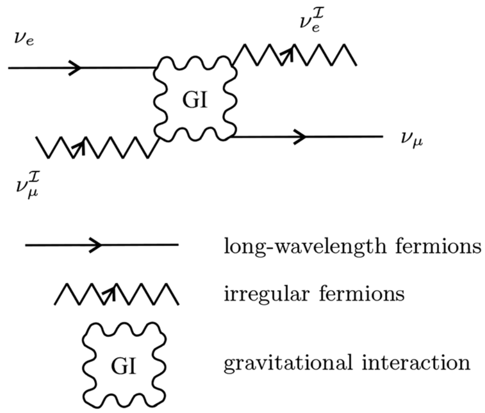

FIG. 1. The process of the electron neutrino transition to the muon one.

contribution to the neutrino oscillations. The introduction of sterile neutrinos does not exhaust all difficulties of the theory: possibly, the most confounding factor of the theory consists in the fact that the electroweak interaction becomes nonrenormalizable.

A detailed description of the neutrino oscillation experiments and theory can be found, for example, in Refs. [11,12], and in numerous references there.

Now, let us consider the possibility of another physics which may provide the neutrino oscillations. The basis for this physics is the Wilson fermion doubling phenomenon on irregular lattices discussed above.

Let us consider the scattering of the usual normal long-wavelength electron neutrino quantum with the momentum k_e, $|k_e| \ll l_P^{-1}$, by the condensate of muon irregular quanta. Suppose the interaction is mediated by the gravitational field [13]. This scattering process is pictured in Fig. 1. Obviously, the time-mean value of the irregular quantum momentum is equal to zero, and the corresponding necessary minimal averaging time is $\tau \sim l_P$. This means that the irregular quantum has zero momentum in the interaction process of the long-wavelength neutrino quantum with neutrino irregular excitation. Suppose also that vacuum is translation invariant. Then the scattering process in Fig. 1 conserves the momentum of the long-wavelength neutrino: $k_\mu = k_e$. The same process as in Fig. 1 takes place with ν_e and ν_μ interchanging. Finally, we conclude that the neutrino oscillations should be observed, since there are mutual transitions of the electron and muon neutrinos with fixed and equal momenta.

II. FERMIONS ON AN IRREGULAR LATTICE

First of all, one must outline shortly the Dirac system on irregular lattices. This goal is solved more gracefully in the frame of the problem of discrete gravity theory on simplicial complexes (see Refs. [8,9]). The definition and necessary properties of the simplicial complexes can also be found in Refs. [8,9]. Here the orientable four-dimensional simplicial complexes are interesting. Below, only the necessary designations concerning simplicial complexes are introduced.

The vertices of the complex $\mathfrak{K}$ are denoted as $a_{\mathcal{V}}$; the index $\mathcal{V} = 1, 2, \ldots, \mathfrak{N} \to \infty$ enumerates the vertices. Let the index $\mathcal{W}$ enumerate 4-simplices. It is necessary to use the local enumeration of the vertices $a_{\mathcal{V}}$ attached to a given 4-simplex: all five vertices of a 4-simplex with index $\mathcal{W}$ are enumerated as $a_{\mathcal{W}i}$, $i = 1, 2, 3, 4, 5$. It must be kept in mind that the same vertex, 1-simplex, etc., can belong to another adjacent 4-simplex. The later notations with the extra index $\mathcal{W}$ indicate that the corresponding quantities belong to the 4-simplex with index $\mathcal{W}$. The Levi-Cività symbol within pairs different indices $\varepsilon_{\mathcal{W}ijklm} = \pm 1$ depending on whether the order of vertices $s^4_{\mathcal{W}} = a_{\mathcal{W}i} a_{\mathcal{W}j} a_{\mathcal{W}k} a_{\mathcal{W}l} a_{\mathcal{W}m}$ defines the positive or negative orientation of the 4-simplex $s^4_{\mathcal{W}}$. We introduce the following notation for oriented 1-simplices in the case where the vertices a_i and a_j belong to the 4-simplex with index $\mathcal{W}$:

$$X^{\mathcal{W}}_{ij} = a_i a_j = -X^{\mathcal{W}}_{ji}. \qquad (2.1)$$

Let

$$s^4_{\mathcal{W}} = a_{\mathcal{W}i_0} a_{\mathcal{W}i_1} a_{\mathcal{W}i_2} a_{\mathcal{W}i_3} a_{\mathcal{W}i_4} \qquad (2.2)$$

be a positively oriented 4-simplex. An oriented frame of a simplex (2.2) at a vertex a_{i_0} is the ordered set of four oriented 1-simplices (2.1): by definition, the frame

$$\mathcal{R}^{\mathcal{W}i_0} = (X^{\mathcal{W}}_{i_0 i_1}, X^{\mathcal{W}}_{i_0 i_2}, X^{\mathcal{W}}_{i_0 i_3}, X^{\mathcal{W}}_{i_0 i_4}) \qquad (2.3)$$

is oriented positively, and each permutation of these 1-simplices conserves or changes the orientation of the frame depending on the permutation parity.

Let $\gamma^a, a, b, c, \ldots = 1, 2, 3, 4$ be 4×4 Dirac matrices with Euclidean signature. Thus, all Dirac matrices as well as the matrix

$$\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4, \qquad \mathrm{tr} \gamma^5 \gamma^a \gamma^b \gamma^c \gamma^d = 4\varepsilon^{abcd} \qquad (2.4)$$

are Hermitian. The Dirac spinors $\psi_{\mathcal{V}}$ and $\psi^\dagger_{\mathcal{V}}$, each of whose components assumes values in a complex Grassman algebra, are assigned to each vertex $a_{\mathcal{V}}$. In the case of Euclidean signature, the spinors $\psi_{\mathcal{V}}$ and $\psi^\dagger_{\mathcal{V}}$ are independent variables and are interchanged under the Hermitian conjugation.

Let us assign to each oriented edge $a_{\mathcal{W}i} a_{\mathcal{W}j}$ an element of the group Spin(4):

$$\Omega_{\mathcal{W}ij} = \Omega^{-1}_{\mathcal{W}ji} = \exp(\omega_{\mathcal{W}ij}),$$
$$\omega_{\mathcal{W}ij} \equiv \frac{1}{2} \sigma^{ab} \omega^{ab}_{\mathcal{W}ij}, \qquad \sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]. \qquad (2.5)$$

Holonomy element $\Omega_{\mathcal{W}ij}$ of the gravitational field executes a parallel transformation of spinor $\psi_{\mathcal{W}j}$ from vertex $a_{\mathcal{W}j}$ of edge $a_{\mathcal{W}i}a_{\mathcal{W}j}$ to neighboring vertex $a_{\mathcal{W}i}$. Let each oriented edge $a_{\mathcal{W}i}a_{\mathcal{W}j}$ be put in correspondence with element $\hat{e}_{\mathcal{W}ij} \equiv e^a_{\mathcal{W}ij}\gamma^a$, such that

$$\hat{e}_{\mathcal{W}ij} \equiv -\Omega_{\mathcal{W}ij}\hat{e}_{\mathcal{W}ji}\Omega^{-1}_{\mathcal{W}ij}. \tag{2.6}$$

The quantities assigned to each oriented edge $a_{\mathcal{W}i}a_{\mathcal{W}j}$ and satisfying Eq. (2.6) are called 1-forms.

We define the orientation of the complex by defining the orientation of each 4-simplex. In this case, if two 4-simplices have a common tetrahedron, the two orientations of the tetrahedron, which are defined by the orientations of these two 4-simplices, are opposite. In our case, the complex obviously has only two orientations.

We can now write the Euclidean action in the model in question:

$$\mathfrak{A} = \frac{1}{5\times 24}\sum_{\mathcal{W}}\sum_{i,j,k,l,m}\varepsilon_{\mathcal{W}ijklm}\mathrm{tr}\gamma^5 \times \left\{-\frac{1}{2l_P^2}\Omega_{\mathcal{W}mi}\Omega_{\mathcal{W}ij}\Omega_{\mathcal{W}jm}\hat{e}_{\mathcal{W}mk}\hat{e}_{\mathcal{W}ml} - \frac{1}{24}\hat{\Theta}_{\mathcal{W}mi}\hat{e}_{\mathcal{W}mj}\hat{e}_{\mathcal{W}mk}\hat{e}_{\mathcal{W}ml}\right\}, \tag{2.7}$$

$$\hat{\Theta}_{\mathcal{W}ij} = \frac{i}{2}\gamma^a(\psi^\dagger_{\mathcal{W}i}\gamma^a\Omega_{\mathcal{W}ij}\psi_{\mathcal{W}j} - \psi^\dagger_{\mathcal{W}j}\Omega_{\mathcal{W}ji}\gamma^a\psi_{\mathcal{W}i}) \equiv \Theta^a_{\mathcal{W}ij}\gamma^a. \tag{2.8}$$

The quantity $\hat{\Theta}_{\mathcal{W}ij}$, as well as the whole action (2.7), represents a Hermitian operator. One can easily verify that the 1-form (2.8), just as the 1-form $\hat{e}_{ij}$, satisfies the relation (2.6). This fact is established by the repeated application of the formula

$$S^{-1}\gamma^a S = S^a_b\gamma^b, \tag{2.9}$$

where

$$S \equiv \exp\frac{1}{2}\varepsilon_{ab}\sigma^{ab}, \qquad \varepsilon_{ab} = -\varepsilon_{ba} = \varepsilon^a_b, S^a_b \equiv (\exp\varepsilon)^a_b = \delta^a_b + \varepsilon^a_b + \frac{1}{2}\varepsilon^a_c\varepsilon^c_b + \cdots. \tag{2.10}$$

The dynamic variables are quantities $\Omega_{\mathcal{W}ij}$ and $\hat{e}_{\mathcal{W}ij}$, which describe the gravitational degrees of freedom, and fields $\psi^\dagger_{\mathcal{W}i}$ and $\psi_{\mathcal{W}i}$, which are material fermion fields.

In the space of fields, there acts a gauge group according to the following rule: To each vertex $a_{\mathcal{W}i}$, let us assign an element of the group $S_{\mathcal{W}i} \in \mathrm{Spin}(4)$. According to the principle of gauge invariance, the fields Ω, e, ψ, and the transformed fields

$$\begin{aligned}
\tilde{\Omega}_{\mathcal{W}ij} &= S_{\mathcal{W}i}\Omega_{\mathcal{W}ij}S^{-1}_{\mathcal{W}j},\\
\tilde{e}_{\mathcal{W}ij} &= S_{\mathcal{W}i}e_{\mathcal{W}ij}S^{-1}_{\mathcal{W}i},\\
\tilde{\psi}_{\mathcal{W}i} &= S_{\mathcal{W}i}\psi_{Ai}, \qquad \tilde{\psi^\dagger}_{\mathcal{W}i} = \psi^\dagger_{\mathcal{W}i}S^{-1}_{\mathcal{W}i}
\end{aligned} \tag{2.11}$$

are physically equivalent. This means that the action (2.7) is invariant under the transformations (2.11). Under the gauge transformations (2.11), the 1-form Θ is transformed in the same way as the form e:

$$\tilde{\hat{\Theta}}_{\mathcal{W}ij} = S_{\mathcal{W}i}\hat{\Theta}_{\mathcal{W}ij}S^{-1}_{\mathcal{W}i}. \tag{2.12}$$

The last formula is verified with the help of Eqs. (2.9), (2.10), and (2.11). Gauge invariance of the action (2.7) is established by using Eqs. (2.11) and (2.12).

It is natural to interpret the quantity

$$l^2_{\mathcal{W}ij} \equiv \frac{1}{4}\mathrm{tr}(\hat{e}_{\mathcal{W}ij})^2 = \sum_{a=1}^{4}(e^a_{\mathcal{W}ij})^2 \tag{2.13}$$

as the square of the length of the edge $a_{\mathcal{W}i}a_{\mathcal{W}j}$. Thus, the geometric properties of a simplicial complex prove to be completely defined.

Now, let us show in the limit of slowly varying fields that the action (2.7) reduces to the continuum action of gravity, minimally connected with a Dirac field, in a four-dimensional Euclidean space.

Consider a certain subset of vertices from the simplicial complex, and assign the coordinates (real numbers)

$$x^\mu_{\mathcal{W}i} = x^\mu_{\mathcal{V}} \equiv x^\mu(a_{\mathcal{W}i}) \equiv x^\mu(a_{\mathcal{V}}), \qquad \mu = 1,2,3,4 \tag{2.14}$$

to each vertex $a_{\mathcal{W}i}$ from this subset. We stress that these coordinates are defined only by their vertices rather than by the higher-dimensional simplices to which these vertices belong; moreover, the correspondence between the vertices from the considered subset and the coordinates (2.14) is one to one.

Suppose that

$$|x^\mu_{\mathcal{W}i} - x^\mu_{\mathcal{W}j}| \sim l_P, \tag{2.15}$$

where the parameter l_P is of the order of the lattice spacing. Estimate (2.15) can be valid only if the complex contains a very large number of simplices and its geometric realization is an almost smooth four-dimensional surface [14]. Suppose also that the four 4-vectors

$$\mathrm{d}x^\mu_{\mathcal{W}ji} \equiv x^\mu_{\mathcal{W}i} - x^\mu_{\mathcal{W}j}, \qquad i = 1, 2, 3, 4 \tag{2.16}$$

are linearly independent, and

$$\begin{vmatrix} \mathrm{d}x^1_{\mathcal{W}m1} & \mathrm{d}x^2_{\mathcal{W}m1} & \dots & \mathrm{d}x^4_{\mathcal{W}m1} \\ \dots & \dots & \dots & \dots \\ \mathrm{d}x^1_{\mathcal{W}m4} & \mathrm{d}x^2_{\mathcal{W}m4} & \dots & \mathrm{d}x^4_{\mathcal{W}m4} \end{vmatrix} \gtrless 0, \tag{2.17}$$

depending on whether the frame $(X^{\mathcal{W}}_{m1}, \dots, X^{\mathcal{W}}_{m4})$ is positively or negatively oriented. Here, the differentials of coordinates (2.16) correspond to one-dimensional simplices $a_{\mathcal{W}j}a_{\mathcal{W}i}$, so that if the vertex $a_{\mathcal{W}j}$ has the coordinates $x^\mu_{\mathcal{W}j}$, then the vertex $a_{\mathcal{W}i}$ has the coordinates $x^\mu_{\mathcal{W}j} + \mathrm{d}x^\mu_{\mathcal{W}ji}$.

In the continuum limit, the holonomy group elements (2.5) are close to the identity element, so that the quantities ω^{ab}_{ij} tend to zero, being of the order of $O(\mathrm{d}x^\mu)$. Thus, one can consider the following system of equations for $\omega_{\mathcal{W}m\mu}$:

$$\omega_{\mathcal{W}m\mu}\mathrm{d}x^\mu_{\mathcal{W}mi} = \omega_{\mathcal{W}mi}, \qquad i = 1, 2, 3, 4. \tag{2.18}$$

In this system of linear equations, the indices $\mathcal{W}$ and m are fixed, the summation is carried out over the index μ, and the index runs over all its values. Since the determinant (2.17) is nonzero, the quantities $\omega_{\mathcal{W}m\mu}$ are defined uniquely. Suppose that a one-dimensional simplex $X^{\mathcal{W}}_{mi}$ belongs to four-dimensional simplices with indices $\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_r$. Introduce the quantity

$$\omega_\mu\left(\frac{1}{2}(x_{\mathcal{W}m} + x_{\mathcal{W}i})\right) \equiv \frac{1}{r}\{\omega_{\mathcal{W}_1 m\mu} + \dots + \omega_{\mathcal{W}_r m\mu}\}, \tag{2.19}$$

which is assumed to be related to the midpoint of the segment $[x^\mu_{\mathcal{W}m}, x^\mu_{\mathcal{W}i}]$. Recall that the coordinates $x^\mu_{\mathcal{W}i}$, just as the differentials (2.16), depend only on vertices but not on the higher-dimensional simplices to which these vertices belong. According to the definition, we have the following chain of equalities:

$$\omega_{\mathcal{W}_1 mi} = \omega_{\mathcal{W}_2 mi} = \dots = \omega_{\mathcal{W}_r mi}. \tag{2.20}$$

It follows from (2.16) and (2.18)–(2.20) that

$$\omega_\mu\left(x_{\mathcal{W}m} + \frac{1}{2}\mathrm{d}x_{\mathcal{W}mi}\right)\mathrm{d}x^\mu_{\mathcal{W}mi} = \omega_{\mathcal{W}mi}. \tag{2.21}$$

The value of the field ω_μ in (2.21) on each one-dimensional simplex is uniquely defined by this simplex.

Next, we assume that the fields ω_μ smoothly depend on the points belonging to the geometric realization of each four-dimensional simplex. In this case, the following formula is valid up to $O((\mathrm{d}x)^2)$ inclusive:

$$\Omega_{\mathcal{W}mi}\Omega_{\mathcal{W}ij}\Omega_{\mathcal{W}jm} = \exp\left[\frac{1}{2}\mathfrak{R}_{\mu\nu}(x_{\mathcal{W}m})\mathrm{d}x^\mu_{\mathcal{W}mi}\mathrm{d}x^\nu_{\mathcal{W}mj}\right], \tag{2.22}$$

where

$$\mathfrak{R}_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu]. \tag{2.23}$$

When deriving formula (2.22), we used the Hausdorff formula.

In exact analogy with (2.18), let us write out the following relations for a tetrad field without explanations:

$$\hat{e}_{\mathcal{W}m\mu}\mathrm{d}x^\mu_{\mathcal{W}mi} = \hat{e}_{\mathcal{W}mi}. \tag{2.24}$$

Using (2.5) and (2.18), we can rewrite the 1-form (2.8) as

$$\hat{\Theta}_{\mathcal{W}ij} = \gamma^a\frac{i}{2}[\psi^\dagger\gamma^a\mathcal{D}_\mu\psi - (\mathcal{D}_\mu\psi)^\dagger\gamma^a\psi]\mathrm{d}x^\mu_{Aij} \equiv \Theta^a\gamma^a \tag{2.25}$$

to within $O(\mathrm{d}x)$; here,

$$\mathcal{D}_\mu\psi = \partial_\mu\psi + \omega_\mu\psi, \tag{2.26}$$

and the smooth field $\psi(x)$ takes the values $\psi(x_{\mathcal{W}i}) = \psi_{\mathcal{W}i}$.

Applying formulas (2.22)–(2.26) to the discrete action (2.7) and changing the summation to integration, we obtain in the continuum limit the well-known gravity action:

$$\mathfrak{A} = \int \varepsilon_{abcd}\left\{-\frac{1}{l_P^2}\mathfrak{R}^{ab}\wedge e^c\wedge e^d - \frac{1}{6}\Theta^a\wedge e^b\wedge e^c\wedge e^d\right\}. \tag{2.27}$$

Here,

$$\begin{gathered} e^a = e^a_\mu\mathrm{d}x^\mu, \qquad \omega \equiv \frac{1}{2}\sigma^{ab}\omega^{ab}_\mu\mathrm{d}x^\mu, \\ \frac{1}{4}\sigma^{ab}\mathfrak{R}^{ab} = \frac{1}{2}\mathfrak{R} \equiv \mathrm{d}\omega + \omega\wedge\omega, \\ \Theta^a = \frac{i}{2}[\psi^\dagger\gamma^a\mathcal{D}_\mu\psi - (\mathcal{D}_\mu\psi)^\dagger\gamma^a\psi]\mathrm{d}x^\mu. \end{gathered} \tag{2.28}$$

Thus, in the naive continuum limit, the action (2.7) proves to be equal to the gravity action in the Palatini form minimally coupled to a Dirac field with Euclidean signature.

We further simplify the problem by assuming that the four-dimensional simplicial complex $\mathfrak{K}$ is embedded into four-dimensional Euclidean space and that the curvature and torsion are equal to zero. This has the following implications:

Let x^a be a set of Cartesian coordinates in the Euclidean space, and $x^a_{\mathcal{W}i}$ be the Cartesian coordinates of the vertex $a_{\mathcal{W}i}$. Then $e^a_{\mathcal{W}ij} = (x^a_{\mathcal{W}j} - x^a_{\mathcal{W}i})$. Now

$$\omega = 0 \to \mathfrak{R} = 0.$$

But instead of a gravity field, the gauge (isotopic) field is introduced into the Dirac part of the action:

$$\Omega_{\mathcal{W}ij} \to U_{\mathcal{W}ij} = U^{-1}_{\mathcal{W}ji} = \exp(ieA_{\mathcal{W}ij}), \qquad A_{\mathcal{W}ij} \in \mathcal{L}, \tag{2.29}$$

where $\mathcal{L}$ is the Lie algebra of the gauge group. The Dirac spinors and the gauge field $A_{\mathcal{W}ij}$ belong to the same representation of algebra $\mathcal{L}$.

Thus, the Dirac part of the action (2.7) acquires the form

$$\begin{aligned}\mathfrak{A}_\psi &= -\frac{1}{3!5!}\sum_{\mathcal{W}}\sum_{i,j,k,l,m}\varepsilon_{\mathcal{W}ijklm}\varepsilon^{abcd}(i\psi^\dagger_{\mathcal{W}m}\gamma^a U_{\mathcal{W}mi}\psi_{\mathcal{W}i})e^b_{\mathcal{W}mj}e^c_{\mathcal{W}mk}e^d_{\mathcal{W}ml}\\ &\equiv \sum_{\mathcal{V}_1\mathcal{V}_2}\psi^\dagger_{\mathcal{V}_1 s_1}[-i\gamma^a_{s_1 s_2}\mathcal{D}^a_{\mathcal{V}_1,\mathcal{V}_2}]\psi_{\mathcal{V}_2 s_2} \equiv \sum_{\mathcal{V}_1\mathcal{V}_2}\psi^\dagger_{\mathcal{V}_1 s_1}[-i\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}]_{s_1 s_2}\psi_{\mathcal{V}_2 s_2}.\end{aligned} \tag{2.30}$$

The indices $s_1, s_2 = 1, 2, 3, 4$ are the Dirac ones. The action (2.30) is invariant under the gauge transformations

$$\begin{aligned}&U_{\mathcal{W}ij} \to S_{\mathcal{W}i}U_{Aij}S^{-1}_{\mathcal{W}j}, \qquad S_{\mathcal{W}i} \in SU(2),\\ &\psi_{\mathcal{W}i} \to S_{\mathcal{W}i}\psi_{\mathcal{W}i}, \qquad \psi^\dagger_{\mathcal{W}i} \to \psi^\dagger_{\mathcal{W}i}S^{-1}_{\mathcal{W}i}.\end{aligned} \tag{2.31}$$

Let

$$v_{\mathcal{W}} = \frac{1}{(4!)(5!)}\varepsilon_{abcd}\varepsilon_{\mathcal{W}ijklm}e^a_{\mathcal{W}mi}e^b_{\mathcal{W}mj}e^c_{\mathcal{W}mk}e^d_{\mathcal{W}ml} \tag{2.32}$$

be the oriented volume of the 4-simplex $s_{\mathcal{W}}$, and $v_{\mathcal{V}}$ be the sum of the volumes $v_{\mathcal{W}}$ for those $\mathcal{W}$-4-simplices which contain the vertex $a_{\mathcal{V}}$. Here, the factor $1/4!$ is required since the volume of a four-dimensional parallelepiped with generatrices $e^a_{\mathcal{W}mi}, e^b_{\mathcal{W}mj}, e^c_{\mathcal{W}mk}$, and $e^d_{\mathcal{W}ml}$ is 4! times larger than the volume of a 4-simplex with the same generatrices, while the factor $1/5!$ is due to the fact that all five vertices of each simplex are taken into account independently. Thus, the spinor space scalar product is given by

$$\langle\psi_1|\psi_2\rangle = \frac{1}{5}\sum_{\mathcal{V}}v_{\mathcal{V}}\psi^\dagger_{(1)\mathcal{V}}\psi_{(2)\mathcal{V}}. \tag{2.33}$$

The operator $[i\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}]$ in (2.30), as well as the operator $[i(v_{\mathcal{V}_1})^{-1/2}\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}(v_{\mathcal{V}_2})^{-1/2}]$, is Hermitian. Thus, the eigenfunction problem

$$\sum_{\mathcal{V}_2}\left[i\left(\frac{1}{\sqrt{v_{\mathcal{V}_1}}}\right)\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}\left(\frac{1}{\sqrt{v_{\mathcal{V}_2}}}\right)\right](\sqrt{v_{\mathcal{V}_2}}\psi_{(\mathfrak{P})\mathcal{V}_2}) = \frac{1}{5}\epsilon_{\mathfrak{P}}(\sqrt{v_{\mathcal{V}_1}}\psi_{(\mathfrak{P})\mathcal{V}_1}) \longleftrightarrow \sum_{\mathcal{V}_2}\left[-\frac{i}{v_{\mathcal{V}_1}}\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}\right]\psi_{(\mathfrak{P})\mathcal{V}_2} = \frac{1}{5}\epsilon_{\mathfrak{P}}\psi_{(\mathfrak{P})\mathcal{V}_1} \tag{2.34}$$

is correct, and the set of eigenfunctions $\{\psi_{(\mathfrak{P})}\}$ forms a complete orthonormal basis in the metric (2.33):

$$\begin{aligned}&\frac{1}{5}\sum_{\mathcal{V}}v_{\mathcal{V}}\psi^\dagger_{(\mathfrak{P}_1)\mathcal{V}}\psi_{(\mathfrak{P}_2)\mathcal{V}} = \delta_{\mathfrak{P}_1\mathfrak{P}_2}\\ &\longleftrightarrow \sum_{\mathfrak{P}}\psi_{(\mathfrak{P})\mathcal{V}_1}\psi^\dagger_{(\mathfrak{P})\mathcal{V}_2} = \frac{5}{v_{\mathcal{V}_1}}\delta_{\mathcal{V}_1\mathcal{V}_2}.\end{aligned} \tag{2.35}$$

Let us expand the Dirac fields in this basis:

$$\psi_{\mathcal{V}} = \sum_{\mathfrak{P}}\eta_{\mathfrak{P}}\psi_{(\mathfrak{P})\mathcal{V}}, \qquad \psi^\dagger_{\mathcal{V}} = \sum_{\mathfrak{P}}\eta^\dagger_{\mathfrak{P}}\psi^\dagger_{(\mathfrak{P})\mathcal{V}}. \tag{2.36}$$

The new dynamic variables $\{\eta_{\mathfrak{P}}, \eta^\dagger_{\mathfrak{P}}\}$ are Grassmann. The scalar product (2.33) in these variables is rewritten as

$$\langle\psi_1|\psi_2\rangle = \sum_{\mathfrak{P}}\eta^\dagger_{(1)\mathfrak{P}}\eta_{(2)\mathfrak{P}}. \tag{2.37}$$

It is important here that

$$\gamma^5 i\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2} = -i\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}\gamma^5. \tag{2.38}$$

The long-wavelength limit of the theory is straightforward (see above). To do this, one should believe the quantities $A_{\mathcal{W}ij}$ and $e^a_{\mathcal{W}ij}$ are the smooth 1-forms

$$A_{\mathcal{W}ij} \to A_a(x)\mathrm{d}x^a, \qquad e^a_{\mathcal{W}ij} \to \mathrm{d}x^a,$$

taking the small values $A_{\mathcal{W}ij}$ and $e^a_{\mathcal{W}ij}$ on the vector $e^a_{\mathcal{W}ij}$, and substitute the smooth Dirac field $\psi(x)$, taking the value $\psi_{\mathcal{V}}$ on the vertex $a_{\mathcal{V}}$ for the set of spinors $\psi_{\mathcal{V}}$. As a result, the action (2.30), the scalar product (2.33), and the eigenvalue problem (2.34) transform to the well-known expressions and equations:

$$\mathfrak{A}_\psi = \int(-i\psi^\dagger\gamma^a\nabla_a\psi)\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}x^4,$$
$$\nabla_a = \partial_a + ieA_a, \tag{2.39}$$

$$\langle\psi_1|\psi_2\rangle = \int\psi_1^\dagger(x)\psi_2(x)\mathrm{d}^{(4)}x, \tag{2.40}$$

$$-i\gamma^a\nabla_a\psi_{(\mathfrak{P})}(x) = \epsilon_{\mathfrak{P}}\psi_{(\mathfrak{P})}(x). \tag{2.41}$$

III. THE GAUGE ANOMALY AND ATIYAH-SINGER INDEX THEOREM

The partition function of the fermion system as the functional of the quantities $\{e^a_{\mathcal{W}ij}\}$ and $\{A_{\mathcal{W}ij}\}$ is given by the integral

$$Z\{e^a_{\mathcal{W}ij}, A_{\mathcal{W}ij}\} = \int(D\psi^\dagger D\psi)\exp\mathfrak{A}_\psi. \tag{3.1}$$

Here the fermion functional measure is defined according to

$$(D\psi^\dagger D\psi) \equiv \prod_{\mathcal{V}}\mathrm{d}\psi^\dagger_{\mathcal{V}}\mathrm{d}\psi_{\mathcal{V}}F\{e^a_{\mathcal{W}ij}\}, \tag{3.2}$$

where

$$d\psi_{\mathcal{V}} = \prod_{\varkappa}\prod_{s=1}^{4}\mathrm{d}\psi_{\mathcal{V}\varkappa s}, \qquad d\psi^\dagger_{\mathcal{V}} = \prod_{\varkappa}\prod_{s=1}^{4}\mathrm{d}\psi^\dagger_{\mathcal{V}\varkappa s}, \tag{3.3}$$

and the index $\varkappa$ enumerates the components of the gauge representation. The functional $F\{e^a_{\mathcal{W}ij}\}$ in (3.2) can be calculated easily with the help of the metric (2.33), but it is not interesting here. The scalar product (2.37) in Grassmann variables $\{\eta_{\mathfrak{P}}, \eta^\dagger_{\mathfrak{P}}\}$ permits us to rewrite the measure (3.2) as below:

$$(D\psi^\dagger D\psi) = \prod_{\mathfrak{P}}\mathrm{d}\eta^\dagger_{\mathfrak{P}}\mathrm{d}\eta_{\mathfrak{P}}. \tag{3.4}$$

Let us study the chiral transformation of the Dirac field

$$\psi_{\mathcal{V}} \to \exp(i\alpha_{\mathcal{V}}\gamma^5)\psi_{\mathcal{V}}, \qquad \psi^\dagger_{\mathcal{V}} \to \psi^\dagger_{\mathcal{V}}\exp(i\alpha_{\mathcal{V}}\gamma^5). \tag{3.5}$$

Obviously, the measure (3.2) is invariant under the transformation (3.5). Moreover, even the factors $(\prod_{s=1}^4 \mathrm{d}\psi_{\mathcal{V}\varkappa s})$ and $(\prod_{s=1}^4 \mathrm{d}\psi^\dagger_{\mathcal{V}\varkappa s})$ of the measure (3.2) are each invariant, since the matrix γ^5 is traceless. It follows from here that the measure on the right-hand side of Eq. (3.4) is also invariant under the chiral transformation and the corresponding Jacobian $J = 1$. The last statement permits us to extract some interesting information.

Suppose the chiral transformation is infinitesimal: $\alpha_{\mathcal{V}} \to 0$. From the linearized transformations of the Dirac field (3.5), we obtain linearized transformations for the variables $\{\eta_{\mathfrak{P}}, \eta^\dagger_{\mathfrak{P}}\}$:

$$\eta_{\mathfrak{P}} \to \eta_{\mathfrak{P}} + \frac{i}{5}\sum_{\mathfrak{Q}}\eta_{\mathfrak{Q}}\sum_{\mathcal{V}}\alpha_{\mathcal{V}}v_{\mathcal{V}}\psi^\dagger_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{Q}\mathcal{V}},$$
$$\eta^\dagger_{\mathfrak{P}} \to \eta^\dagger_{\mathfrak{P}} + \frac{i}{5}\sum_{\mathfrak{Q}}\eta^\dagger_{\mathfrak{Q}}\sum_{\mathcal{V}}\alpha_{\mathcal{V}}v_{\mathcal{V}}\psi^\dagger_{\mathfrak{Q}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}}. \tag{3.6}$$

The Jacobian of this transformation is equal to

$$J = \left(1 + \frac{2i}{5}\sum_{\mathcal{V}}\alpha_{\mathcal{V}}v_{\mathcal{V}}\sum_{\mathfrak{P}}\psi^\dagger_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}}\right).$$

On the other hand, as was stated before, $J = 1$. Therefore, since the quantities $\alpha_{\mathcal{V}}$ are arbitrary at each vertex, we have

$$\sum_{\mathfrak{P}}\psi^\dagger_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}} = 0. \tag{3.7}$$

For the following analysis, it is necessary to decompose the sum (3.7) into an infrared or long-wavelength part and the rest into an ultraviolet part. This is possible only for a very large number of vertices $\mathfrak{N}$, while the sum rule (3.7) remains valid always.

First, let us consider the infrared part. One must introduce the following scales: the gauge field wavelength order $\sim\lambda$; the scale of ultraviolet cutoff of the long-wavelength sector Λ; the lattice scale $l_P \sim |e^a_{\mathcal{W}ij}|$. The scales satisfy the inequalities

$$\lambda^{-1} \ll \Lambda \ll l_P^{-1}. \tag{3.8}$$

Let us divide the total index set $\{\mathfrak{P}\}$ into three subsets. For the long-wavelength $\psi_{\mathfrak{P}}(x)$:

$$\mathfrak{P} \in \mathcal{S}_{\mathrm{infra}} \Leftrightarrow |\epsilon_{\mathfrak{P}}| < \Lambda_1, \qquad \lambda^{-1} \ll \Lambda_1 \ll l_P^{-1},$$
$$\mathfrak{P} \in \mathcal{S}^{\circledcirc}_{\mathrm{infra}} \Leftrightarrow \Lambda_1 < |\epsilon_{\mathfrak{P}}| < \Lambda_2 \ll l_P^{-1}.$$

The rest of the indices are designated as $\mathcal{I}$, so that

$$\mathcal{S}_{\mathrm{infra}} + \mathcal{S}^{\circledcirc}_{\mathrm{infra}} + \mathcal{I} = \{\mathfrak{P}\}.$$

In consequence of Eq. (2.38), it is evident that for all $\mathfrak{P}$ with $\epsilon_{\mathfrak{P}} \neq 0$ [see Eq. (2.34)],

$$\frac{1}{5}\sum_{\mathcal{V}} v_{\mathcal{V}} \psi^{\dagger}_{\mathfrak{P}\mathcal{V}} \gamma^5 \psi_{\mathfrak{P}\mathcal{V}} = 0. \tag{3.9}$$

Due to Eq. (3.9) and the identity $\gamma^5 \equiv [(1+\gamma^5)/2 - (1-\gamma^5)/2]$, we obtain the relation

$$\frac{1}{5}\sum_{\mathcal{V}} v_{\mathcal{V}} \sum_{\mathfrak{P}\in\mathcal{S}} \psi^{\dagger}_{\mathfrak{P}\mathcal{V}} \gamma^5 \psi_{\mathfrak{P}\mathcal{V}} = n^{\mathcal{S}}_{+} - n^{\mathcal{S}}_{-}, \tag{3.10}$$

where $\mathcal{S}$ is a subset of the index set $\{\mathfrak{P}\}$ and $n^{\mathcal{S}}_{+}(n^{\mathcal{S}}_{-})$ is the number of right (left) zero modes on the index subset $\mathcal{S}$. In any case, the value of the left-hand side of Eq. (3.10) is a whole number $0, \pm 1, \ldots$.

The value of the long-wavelength part of the sum (3.7) is well known:

$$\begin{aligned}
&\sum_{\mathfrak{P}\in\mathcal{S}_{\text{infra}}} \psi^{\dagger}_{\mathfrak{P}}(x)\gamma^5\psi_{\mathfrak{P}}(x) \\
&\quad = -\frac{e^2}{32\pi^2}\varepsilon^{abcd}\mathrm{tr}\{F_{ab}(x)F_{cd}(x)\} \\
&\qquad + \mathrm{O}\left(\frac{1}{(\lambda\Lambda_1)^2}\right)\mathcal{F}_1\{A\} + \mathrm{O}\left(\frac{l_P}{\lambda}\right)\mathcal{F}_2\{A\}, \\
&F_{ab} = \partial_a A_b - \partial_b A_a + ie[A_a, A_b].
\end{aligned} \tag{3.11}$$

Here $\mathcal{F}_1\{A\}$ and $\mathcal{F}_2\{A\}$ are some local gauge invariant functionals of the gauge field. Note that the first summand on the right-hand side of the last equation is generalized easily into a simplicial complex in such a way that the lattice value transforms into the corresponding original continual value in the long-wavelength limit.

The rigorous lattice expression for the left-hand side of Eq. (3.11) looks like

$$\begin{aligned}
&\sum_{\mathfrak{P}\in\mathcal{S}_{\text{infra}}} \psi^{\dagger}_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}} = \mathrm{tr}\gamma^5 K_{\mathcal{V},\mathcal{V}}(\Lambda_1), \\
&K_{\mathcal{V}_1,\mathcal{V}_2}(\Lambda_1) \equiv \sum_{\mathfrak{P}} \exp\left[-\frac{(\epsilon_{\mathfrak{P}})^2}{\Lambda_1^2}\right]\psi_{\mathfrak{P}\mathcal{V}_1}\psi^{\dagger}_{\mathfrak{P}\mathcal{V}_2} \\
&\qquad = \exp\left[-\frac{(i\mathcal{D})^2}{\Lambda_1^2}\right]_{\mathcal{V}_1,\mathcal{V}_2}.
\end{aligned} \tag{3.12}$$

The expansion of the lattice operator $K_{\mathcal{V}_1,\mathcal{V}_2}(\Lambda_1)$ into a power series in $(\lambda\Lambda_1)^{-2} \ll 1$ and $(l_P/\lambda) \ll 1$ leads to the expression on the right-hand side of Eq. (3.11). It is important that this expansion is correct, since the operator $K_{\mathcal{V}_1,\mathcal{V}_2}(\Lambda_1)$ is well defined.

The space integral of the right-hand side of Eq. (3.11) is equal to

$$q + \mathrm{O}\left(\frac{1}{(\lambda\Lambda)^2}\right)c_1 + \mathrm{O}\left(\frac{l_P}{\lambda}\right)c_2, \frac{c_1}{(\lambda\Lambda)^2} \to 0, \qquad \frac{l_P c_2}{\lambda} \to 0. \tag{3.13}$$

Here $q = 0, \pm 1, \ldots$ is the topological charge of the gauge field instanton, and the numbers c_1 and c_2 tend to some finite values in the limit $(1/\lambda\Lambda) \to 0$ and $(l_P/\lambda) \to 0$. Since the value of the left-hand side of Eq. (3.11) is a whole number [see Eq. (3.10)] and the latter two summands in (3.13) are negligible in comparison with 1, one must conclude that $c_1 = c_2 = 0$ [15]. Finally, we have

$$\frac{1}{5}\sum_{\mathcal{V}} v_{\mathcal{V}} \sum_{\mathfrak{P}\in\mathcal{S}_{\text{infra}}} \psi^{\dagger}_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}} = q. \tag{3.14}$$

This equation is rigorous for $(1/\lambda\Lambda) \lll 1$, $(l_P/\lambda) \ggg 1$. Moreover, it follows from Eq. (3.11) that

$$\sum_{\mathfrak{P}\in\mathcal{S}_{\text{infra}}} \psi^{\dagger}_{\mathfrak{P}}(x)\gamma^5\psi_{\mathfrak{P}}(x) = -\frac{e^2}{32\pi^2}\varepsilon^{abcd}\mathrm{tr}\{F_{ab}(x)F_{cd}(x)\} \tag{3.15}$$

in the limit $(1/\lambda\Lambda) \to 0$ and $(l_P/\lambda) \to 0$. It is well known that the right-hand side of Eq. (3.15) is half of the axial vector anomaly. Here the expression for the anomaly is extracted from the fermion measure (3.4). This method was suggested by Vergeles [16] and Fujikawa [17].

Note that the value of the sum in (3.15) does not depend on the cutoff parameter Λ if it is enclosed in a range of values (3.8). This fact, in turn, means that

$$\sum_{\mathfrak{P}\in\mathcal{S}^{\circledcirc}_{\text{infra}}} \psi^{\dagger}_{\mathfrak{P}}(x)\gamma^5\psi_{\mathfrak{P}}(x) = 0. \tag{3.16}$$

It is clear from here that the decomposition of the sum in (3.7) into long-wavelength and ultraviolet parts is well defined.

The comparison of Eqs. (3.10), (3.7), (3.14), and (3.16) leads to the following equality:

$$\frac{1}{5}\sum_{\mathcal{V}} v_{\mathcal{V}} \sum_{\mathfrak{P}\in\mathcal{I}} \psi^{\dagger}_{\mathfrak{P}\mathcal{V}}\gamma^5\psi_{\mathfrak{P}\mathcal{V}} = n^{\mathcal{I}}_{+} - n^{\mathcal{I}}_{-} = -q. \tag{3.17}$$

Here $n^{\mathcal{I}}_{+}(n^{\mathcal{I}}_{-})$ is the number of the right (left) *irregular* zero modes of Eq. (2.34). The difference between the usual and irregular modes is as follows: For the usual modes and adjacent vertices $a_{\mathcal{W}i}$ and $a_{\mathcal{W}j}$, we have

$$|\psi_{(\mathfrak{P})\mathcal{W}i} - \psi_{(\mathfrak{P})\mathcal{W}j}| \sim l_P\epsilon_{\mathfrak{P}}|\psi_{(\mathfrak{P})\mathcal{W}j}| \to 0. \tag{3.18}$$

By definition, the irregular modes cannot satisfy the estimation (3.18), but they satisfy the estimation

$$|\psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{W}i} - \psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{W}j}| \sim |\psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{W}i}| \tag{3.19}$$

at least at a part of vertices. Thus, the usual and irregular modes are well separated not only by the energy $\epsilon_{\mathfrak{P}}$ but also by the "momentum."

It is important that the relations (3.17) are rigorous.

IV. WILSON FERMION DOUBLING PHENOMENON

Let $q \in \mathbb{Z}$ and $\mathcal{D}^{(q)}_{\mathcal{V}_1,\mathcal{V}_2}$ be the Dirac operator defined on an instanton with the topological charge (q). Denote by $\psi^{\mathcal{I}}_{(0\xi)\mathcal{V}}$ the irregular zero mode of Eq. (2.34):

$$\sum_{\mathcal{V}_2}\left[-\frac{i}{v_{\mathcal{V}_1}}\mathcal{D}^{(q)}_{\mathcal{V}_1,\mathcal{V}_2}\right]\psi^{\mathcal{I}}_{(0\xi)\mathcal{V}_2} = 0. \tag{4.1}$$

The index ξ enumerates the zero modes.

Now, let us denote by $[-(i/v_{\mathcal{V}_1})\mathcal{D}^{(\mathrm{free})}_{\mathcal{V}_1,\mathcal{V}_2}]$ the free lattice Dirac operator. The free Dirac operator is obtained from the general one by the gauge field elimination $U_{\mathcal{W}mi} = \exp(ieA_{\mathcal{W}mi}) \to 1$.

It is easy to obtain the following estimation:

$$\sum_{\mathcal{V}_2}\left[-\frac{i}{v_{\mathcal{V}_1}}\mathcal{D}^{(\mathrm{free})}_{\mathcal{V}_1,\mathcal{V}_2}\right]\psi^{\mathcal{I}}_{(0\xi)\mathcal{V}_2} = \mathrm{O}\left(\frac{e}{\rho}|\psi^{\mathcal{I}}_{(0\xi)\mathcal{V}_1}|\right). \tag{4.2}$$

Here ρ is the scale of the instanton field $A^{(\mathrm{inst})}_{\mathcal{W}mi}$. The proof of (4.2) is based on the estimations

$$A^{(\mathrm{inst})}_{\mathcal{W}mi} \sim (l_P/\rho) \ll 1,$$
$$1 \approx \exp(ieA^{(\mathrm{inst})}_{\mathcal{W}mi}) - ieA^{(\mathrm{inst})}_{\mathcal{W}mi} = U_{\mathcal{W}mi} + \mathrm{O}\left(\frac{el_P}{\rho}\right),$$

and the fact that the lattice Dirac operator is linear in $U_{\mathcal{W}mi}$. Therefore,

$$\mathcal{D}^{(\mathrm{free})}_{\mathcal{V}_1,\mathcal{V}_2} = \mathcal{D}^{(q)}_{\mathcal{V}_1,\mathcal{V}_2} + \mathrm{O}\left(\frac{el_P^4}{\rho}\right).$$

Since $v_{\mathcal{V}_1} \sim l_P^4$, the estimation (4.2) follows from Eq. (4.1).

Let us expand the field configuration $\psi^{\mathcal{I}}_{(0\xi)\mathcal{V}}$ in a series of the free Dirac operator eigenfunctions:

$$\psi^{\mathcal{I}}_{(0\xi)\mathcal{V}} = \sum_{\mathfrak{P}} c_{\mathfrak{P}}\psi^{(\mathrm{free})}_{(\mathfrak{P})\mathcal{V}},$$
$$\sum_{\mathcal{V}_2}\left[-\frac{i}{v_{\mathcal{V}_1}}\mathcal{D}^{(\mathrm{free})}_{\mathcal{V}_1,\mathcal{V}_2}\right]\psi^{(\mathrm{free})}_{(\mathfrak{P})\mathcal{V}_2} = \epsilon_{\mathfrak{P}}\psi^{(\mathrm{free})}_{(\mathfrak{P})\mathcal{V}_1}. \tag{4.3}$$

Here $c_{\mathfrak{P}}$ are some complex numbers.

We are interested in the irregular modes' contribution to the expansion (4.3):

$$\psi^{\mathcal{I}}_{(0\xi)\mathcal{V}} = \sum_{\mathfrak{P}'} c_{\mathfrak{P}'}\psi^{(\mathrm{free})\mathcal{I}}_{(\mathfrak{P}')\mathcal{V}} + \cdots, \tag{4.4}$$

where the indices $\mathfrak{P}'$ enumerate the irregular modes. It is evident that at least some numbers $c_{\mathfrak{P}'}$ in (4.4) are nonzero:

$$c_{\mathfrak{P}'} \neq 0. \tag{4.5}$$

Indeed, the irregular field configuration cannot be expanded in a series of the regular smooth modes only.

The estimation (4.2) and expansion (4.4) allow us to arrive at the final conclusion: the Wilson fermion doubling phenomenon on irregular four-dimensional lattices does exist. Otherwise, the energy gap of the order of $\epsilon^{\mathcal{I}}_{\mathfrak{P}} \sim 1/l_P$ would be expected to take place in the sector of all irregular modes of the free Dirac operator. As was said, in any case the expansion (4.4) contains the irregular modes of the operator. Thus, the additional contributions of the order of $(c_{\mathfrak{P}'}/l_P)$ would be on the right-hand side of the estimation (4.2), the numbers $c_{\mathfrak{P}'} \neq 0$. But the right-hand side of the estimation (4.2) does not depend on the lattice parameter l_P. Thus, there are the soft or low-energy irregular Dirac modes; the index $\mathfrak{P}'$ in the expansion (4.4) enumerates only the soft modes. The soft irregular eigenfunctions of the free Dirac operator are called here doubled fermion modes.

It is necessary to notice that the suggested approach is valid also for the regular lattices or partially regular lattices such as those that are periodic in one dimension and irregular in the other dimensions.

To prove the existence of the Wilson fermion doubling phenomenon on irregular three-dimensional lattices, let us consider the Dirac action on the Cartesian product of a three-dimensional simplicial complex $\mathfrak{K}$ and the set of integers $\mathbb{R}$. As before, I assume that the three-dimensional simplicial complex is embedded into three-dimensional Euclidean space; the vertices of the complex are denoted as $a_{\mathcal{V}}$; the index $\mathcal{V} = 1, 2, \ldots, \mathfrak{N} \to \infty$ enumerates the vertices; and the index $\mathcal{W}$ enumerates 3-simplices. Again it is necessary to use the local enumeration of the vertices $a_{\mathcal{V}}$ attached to a given 3-simplex: all four vertices of a 3-simplex with index $\mathcal{W}$ are enumerated as $a_{\mathcal{W}i}$, $i, j, \ldots = 1, 2, 3, 4$. Later the notations with the extra index $\mathcal{W}$ indicate that the corresponding quantities belong to the 3-simplex with index $\mathcal{W}$. The Levi-Cività symbol within pairs different indices $\varepsilon_{\mathcal{W}lijk} = \pm 1$ depending on whether the order of vertices $a_{\mathcal{W}l}a_{\mathcal{W}i}a_{\mathcal{W}j}a_{\mathcal{W}k}$ defines the positive or negative orientation of this 3-simplex. For each oriented 1-simplex $a_{\mathcal{W}i}a_{\mathcal{W}j}$ of the simplicial complex an elementary vector

$$e^{\alpha}_{\mathcal{W}ij} \equiv -e^{\alpha}_{\mathcal{W}ji}, \qquad \alpha, \beta, \gamma = 1, 2, 3$$

is assigned. The vector $e^{\alpha}_{\mathcal{W}ij}$ connects the vertex $a_{\mathcal{W}i}$ with the vertex $a_{\mathcal{W}j}$ in 3D Euclidean space. The rest of the notations are evident, and they are similar to those in the beginning of Sec. II, but they are supplied here by the additional index $n = 0, \pm 1, \ldots \in \mathbb{R}$, since the dynamic variables are defined now on the discrete set $\mathfrak{K} \times \mathbb{R}$.

The Euclidean Hermitean action of the Dirac field associated with the set $\mathfrak{K} \times \mathbb{R}$ has the form

$$\mathfrak{A}_\psi = -\frac{1}{2!4!)}\sum_n\sum_{\mathcal{W}}\sum_{i,j,k,l}\varepsilon_{\mathcal{W}lijk}\varepsilon^{\alpha\beta\gamma}(i\psi^\dagger_{\mathcal{W}l,n}\gamma^\alpha\psi_{\mathcal{W}i,n})e^\beta_{\mathcal{W}lj,n}e^\gamma_{\mathcal{W}lk,n} - \frac{1}{2}\sum_n\sum_{\mathcal{V}}v_{\mathcal{V}}(i\psi^\dagger_{\mathcal{V},n}\gamma^4(\psi_{\mathcal{V},n+1}-\psi_{\mathcal{V},n-1}))$$
$$= \sum_n\sum_{\mathcal{V}_1\mathcal{V}_2}\psi^\dagger_{\mathcal{V}_1n}[-i\gamma^\alpha\mathcal{D}^\alpha_{\mathcal{V}_1,\mathcal{V}_2}]\psi_{\mathcal{V}_2n} + \sum_{\mathcal{V}}v_{\mathcal{V}}\sum_{n,n'}\psi^\dagger_{\mathcal{V},n}[-i\gamma^4D_{n,n'}]\psi_{\mathcal{V},n'}. \tag{4.6}$$

Here $v_{\mathcal{V}}$ is the total sum of oriented volumes of the adjacent 3-simplices with common vertex $a_{\mathcal{V}}$. The eigenfunction problem (2.34) for irregular modes now looks like

$$\sum_{\mathcal{V}_2}\left[-\frac{i}{v_{\mathcal{V}_1}}\gamma^\alpha\mathcal{D}^\alpha_{\mathcal{V}_1,\mathcal{V}_2}\right]\psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{V}_2n} + \sum_{n'}[-i\gamma^4D_{n,n'}]\psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{V}_1n'} = \epsilon_{\mathfrak{P}}\psi^{\mathcal{I}}_{(\mathfrak{P})\mathcal{V}_1n}, \tag{4.7}$$

or briefly

$$\{\gamma^\alpha(-i/v_{\mathcal{V}})\mathcal{D}^\alpha + \gamma^4(-iD)\}\psi^{\mathcal{I}}_{(\mathfrak{P})} = \epsilon_{\mathfrak{P}}\psi^{\mathcal{I}}_{(\mathfrak{P})}. \tag{4.8}$$

Both operators $(-i/v_{\mathcal{V}_1})\mathcal{D}^\alpha_{\mathcal{V}_1,\mathcal{V}_2}$ and $-iD_{n,n'}$ are Hermitian, and they commute mutually. Therefore, the repeated application of the operator $\{\gamma^\alpha(-i/v_{\mathcal{V}_1})\mathcal{D}^\alpha + \gamma^4(-iD)\}$ to (4.8) leads to the equation

$$\{[(i/v_{\mathcal{V}})\mathcal{D}^\alpha]^2 + [iD]^2\}\psi^{\mathcal{I}}_{(\mathfrak{P})} = \epsilon^2_{\mathfrak{P}}\psi^{\mathcal{I}}_{(\mathfrak{P})} \tag{4.9}$$

due to the fact that $\gamma^\alpha\gamma^4 + \gamma^4\gamma^\alpha = 0$. It has been shown that the soft irregular modes of Eqs. (4.8) and (4.9) do exist, i.e., there exist the eigenvalues of Eq. (4.8) in the subspace of irregular eigenfunctions of the order of $|\epsilon_{\mathfrak{P}}| \ll l_P^{-1}$. Therefore, the spectrum of the operator $[(i/v_{\mathcal{V}})\mathcal{D}^\alpha]$ in the subspace of irregular eigenfunctions contains the eigenvalues of the order of $|\epsilon_{\mathfrak{P}}| \ll l_P^{-1}$. This conclusion follows from Eq. (4.9).

Thus, the doubled fermion modes exist also on three-dimensional irregular lattices.

The classification of the doubled fermion modes should be a subject of future scientific research.

V. THE PROPAGATION OF THE IRREGULAR QUANTA

First, let us fix the necessary properties of the usual Dirac propagators

$$iS_c(x-y) \equiv \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle \tag{5.1}$$

in $(3+1)$ continual space-time with Minkowski signature:

(i) The translational and Lorentz invariance.

(ii) For massless theory,

$$\gamma^5 iS_c(x-y) + iS_c(x-y)\gamma^5 = 0. \tag{5.2}$$

(iii) For $x^0 > z^0 > y^0$,

$$\int d^{(3)}z[iS_c(x-z)]\gamma^0[iS_c(z-y)] = iS_c(x-y). \tag{5.3}$$

(iv) The propagating particles are "good" quasiparticles; i.e., they live indefinitely, have well-defined four-momentum, and their energy is positive.

Property 3 above is the quantum-mechanical superposition principle, and at the same time the property implies that the propagating particle cannot be absorbed or created by vacuum; i.e., the particle is distinguishable against the background of the vacuum.

It is easy to see that all four of the properties uniquely define the particle propagator. Indeed, the most general expression for the propagator in the case $x^0 > y^0$ is

$$iS_c(x-y) = \int\left(\frac{d^{(3)}k}{(2\pi)^3 2|\mathbf{k}|}\right) \times (\gamma^0|\mathbf{k}| - \gamma^\alpha k^\alpha)e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})-i|\mathbf{k}|(x^0-y^0)}f(k^a). \tag{5.4}$$

Here the measure, the expression in the parentheses, and the exponent are Lorentz-invariant. Property 2 above is also fulfilled. Since the propagator (5.4) describes the propagation of the real "good" quasiparticles, all its dependence on the space-time coordinates $(x-y)$ is given by the exponent. The function $f(k^a)$ in (5.4) must also be Lorentz-invariant. This means that it can depend only on $k^ak_a = 0$, and thus it is constant: $f = C$. Property 3 gives $C^2 = C$. Therefore, $f(k^a) = 1$.

If we insist on properties 1–2 only and reject properties 3–4, then the propagator describes the propagation of some irregular quanta and it can acquire other forms. For example,

$$iS^{\mathcal{I}}_c(x-y) \sim l_P^2 i\gamma^a(\partial/\partial x^a)\delta^{(4)}(x-y). \tag{5.5}$$

It is shown below that the propagators of the irregular quanta are similar to the expression (5.5). In order to do this, the structure of the fermion vacuum must be described in general.

Now I return to the Euclidean metric. For simplicity, the gauge group is assumed to be trivial, so that the index $\varkappa$ will be omitted. Note that from the integration rules

$$\int \mathrm{d}\psi_{\mathcal{V}s} = 0, \int \mathrm{d}\psi_{\mathcal{V}s} \cdot \psi_{\mathcal{V}s'} = \delta_{ss'},$$
$$\int \mathrm{d}\psi^{\dagger}_{\mathcal{V}s} = 0, \int \mathrm{d}\psi^{\dagger}_{\mathcal{V}s} \cdot \psi^{\dagger}_{\mathcal{V}s'} = \delta_{ss'}, \tag{5.6}$$

it follows that the nonzero value of the integral (3.1) is obtained only if the complete products of the fermion variable

$$\left(\prod_{s=1}^{4} \psi_{\mathcal{V}s}\psi^{\dagger}_{\mathcal{V}s}\right) \tag{5.7}$$

are present at each vertex $a_{\mathcal{V}}$. These products can arise only due to the exponent expansion under the integral (3.1). As a consequence of the expansion, the expression $\{\psi^{\dagger}_{\mathcal{V}_1 s_1}[-i\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}]_{s_1 s_2}\psi_{\mathcal{V}_2 s_2}\}$ related to the 1-simplex $a_{\mathcal{V}_1}a_{\mathcal{V}_2}$ can appear [see the Dirac action (2.30)] [18]. Let us assign to the corresponding 1-simplex $a_{\mathcal{V}_1}a_{\mathcal{V}_2}$ an arrow in this case. The arrow is vectored from vertex $a_{\mathcal{V}_2}$ to vertex $a_{\mathcal{V}_1}$ which can be designated as $\overrightarrow{a_{\mathcal{V}_2}a_{\mathcal{V}_1}}$ or $\overleftarrow{a_{\mathcal{V}_1}a_{\mathcal{V}_2}}$. Four arrows come into each vertex, and four arrows come out from each vertex as a result of integration in (3.1). This geometrical picture is realized analytically by assigning to each 1-simplex $\overleftarrow{a_{\mathcal{V}}a_{\mathcal{V}_1}}$ the matrix $[-i\mathcal{D}_{\mathcal{V},\mathcal{V}_1}]_{ss_1}$ and to each 1-simplex $\overrightarrow{a_{\mathcal{V}}a_{\mathcal{V}_1}}$ the matrix $[-i\mathcal{D}_{\mathcal{V}_1,\mathcal{V}}]_{s_1 s}$. Thus, there is the factor

$$\left\{\sum_{s_1,s_2,s_3,s_4=1}^{4} \varepsilon_{s_1s_2s_3s_4}[-i\mathcal{D}_{\mathcal{V},\mathcal{V}_1}]_{s_1s'_1}[-i\mathcal{D}_{\mathcal{V},\mathcal{V}_2}]_{s_2s'_2}[-i\mathcal{D}_{\mathcal{V},\mathcal{V}_3}]_{s_3s'_3}[-i\mathcal{D}_{\mathcal{V},\mathcal{V}_4}]_{s_4s'_4}\right\}$$
$$\times\left\{\sum_{s_5,s_6,s_7,s_8=1}^{4} \varepsilon_{s_5s_6s_7s_8}[-i\mathcal{D}_{\mathcal{V}_5,\mathcal{V}}]_{s'_5s_5}[-i\mathcal{D}_{\mathcal{V}_6,\mathcal{V}}]_{s'_6s_6}[-i\mathcal{D}_{\mathcal{V}_7,\mathcal{V}}]_{s'_7s_7}[-i\mathcal{D}_{\mathcal{V}_8,\mathcal{V}}]_{s'_8s_8}\right\} \tag{5.8}$$

in every vertex $a_{\mathcal{V}}$.

We are interested in the two-point correlator

$$\langle\psi_{s_1}(x^{\mu}(a_{\mathcal{V}_1})\psi^{\dagger}_{s_2}(y^{\mu}(a_{\mathcal{V}_2})\rangle_{\psi,e} \equiv \left\langle\frac{\int(\mathcal{D}\psi^{\dagger}\mathcal{D}\psi)\psi_{\mathcal{V}_1s_1}\psi^{\dagger}_{\mathcal{V}_2s_2}\exp\mathfrak{A}_{\psi}}{\int(\mathcal{D}\psi^{\dagger}\mathcal{D}\psi)\exp\mathfrak{A}_{\psi}}\right\rangle_e = \left\langle\sum\nolimits_{\mathfrak{P}}\frac{1}{\epsilon_{\mathfrak{P}}}\psi_{(\mathfrak{P})}(x_{\mathcal{V}_1})\psi^{\dagger}_{(\mathfrak{P})}(y_{\mathcal{V}_2})\right\rangle_e. \tag{5.9}$$

Here the subscript e means that the quantum e-fluctuations average. These fluctuations are inessential in the long-wavelength case, since the long-wavelength quanta lost the information about lattice. But in the case of the irregular quanta propagation, these fluctuations are crucial. The e-fluctuations averaging implies that the geometrical interval

$$|x-y| = \int_1^2\sqrt{e^a_{\mu}e^a_{\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}}$$

[see (2.13) and (2.28)] between fermion fields in the two-point correlator (5.9) remains constant. Obviously, for a fixed value $|x-y|$, due to the e-fluctuations, the vertices $a_{\mathcal{V}_1}$ and $a_{\mathcal{V}_2}$ will be variable quantities also, so that in the case $|x-y| \gg l_P$ many vertices $a_{\mathcal{V}_1}$ and $a_{\mathcal{V}_2}$ will be taken into account in (5.9). By definition, the irregular modes change much in passing from one vertex to the neighbouring one. Therefore, one can conclude from the above discussion that the irregular quanta propagator decreases very quickly at $(|x-y|/l_P) \to \infty$. From here the estimation (5.5) follows.

Below the problem is considered in more detail.

Further, the sign of e-fluctuations averaging is omitted.

Since there is the external factor $\psi^{\dagger}_{\mathcal{V}_2s_2}$ in the vertex $a_{\mathcal{V}_2}$ [see the numerator in (5.9)], the number of the arrows related with the factors

$$[-i\mathcal{D}_{\mathcal{V}_2,\mathcal{V}'}]_{s_2s'} \tag{5.10}$$

and coming into the vertex $a_{\mathcal{V}_2}$ is reduced up to tree. Mathematically, this fact is realized by the assigning the inverse matrix

$$[-i\mathcal{D}_{\mathcal{V}_2,\mathcal{V}'}]^{-1}_{s's_2}, \sum_{s'}[-i\mathcal{D}_{\mathcal{V}_2,\mathcal{V}'}]^{-1}_{s_1s'}[-i\mathcal{D}_{\mathcal{V}_2,\mathcal{V}'}]_{s's_2} = \delta_{s_1,s_2} \tag{5.11}$$

to the corresponding 1-simplex $a_{\mathcal{V}_2}a_{\mathcal{V}'}$ (see Fig. 2). Therefore, the number of factors $\psi_{\mathcal{V}'s'}$ presented at the vertex $a_{\mathcal{V}'}$ is reduced up to tree also. To compensate for this reduction, one must introduce the additional factor (see Fig. 2)

$$[-i\mathcal{D}_{\mathcal{V}'',\mathcal{V}'}]_{s''s'}. \tag{5.12}$$

Now the condition at the vertex $a_{\mathcal{V}''}$ is the same as at the beginning of the process at the vertex $a_{\mathcal{V}2}$: the additional factor (5.12) gives an additional arrow coming into the vertex $a_{\mathcal{V}''}$. To eliminate one of them, say $\leftarrow a_{\mathcal{V}''}a_{\mathcal{V}'''}$, one should introduce the factor $[-i\mathcal{D}_{\mathcal{V}'',\mathcal{V}'''}]^{-1}_{s'''s''}$, and so on. It is evident that the last link in the chain is $[-i\mathcal{D}_{\mathcal{V}'''',\mathcal{V}_1}]^{-1}_{s_1s''''}$.

It follows from the above that the correlator (5.9) can be represented in the form

$$\langle\psi_{\mathcal{V}_1s_1}\psi^{\dagger}_{\mathcal{V}_2s_2}\rangle = \sum_{\text{all paths}}\{[-i\mathcal{D}_{\mathcal{V}''''',\mathcal{V}_1}]^{-1}[-i\mathcal{D}_{\mathcal{V}''''',\mathcal{V}''''''}]\ldots[-i\mathcal{D}_{\mathcal{V}'',\mathcal{V}'''}]^{-1}[-i\mathcal{D}_{\mathcal{V}'',\mathcal{V}'}][-i\mathcal{D}_{\mathcal{V}_2,\mathcal{V}'}]^{-1}\}_{s_1s_2}. \tag{5.13}$$

Obviously, the number of the operators $[-i\mathcal{D}]^{-1}$ is greater than the number of the operators $[-i\mathcal{D}]$ by the unity on the right-hand side of Eq. (5.13). Therefore, the total power of the operators $[-i\mathcal{D}]$ and $[-i\mathcal{D}]^{-1}$ on the right-hand side of Eq. (5.13) is odd. Since both these operators are linear in the Dirac matrices γ^a, the expression on the right-hand side of Eq. (5.13) satisfies property 2. But property 3 cannot be fulfilled on the microscopic level—if only because the correlator (5.9) is odd in the total power of the Dirac matrices, while the bilinear form of the correlator is even in this sense. Note that a part of the information is lost in passing from the microscopic description to the long-wavelength limit, and thus property 3 becomes true. Indeed, the information about the lattice is lost completely in the long-wavelength limit, and the lattice action (2.30) transforms to the usual continuum Dirac action (2.39). Therefore, the correlator (5.9) transforms to the expression (5.4) with $f(k^a) = 1$.

Now, let us proceed to the estimation of the irregular quanta correlator. In this case, the information related with the lattice is determinative. Because of this, Eq. (5.13) should be used. Since the direct correlator estimation with the help of Eq. (5.13) is impossible, I apply a simple and adequate computational model which describes the problem in terms of continuum theory. Thus, the model forgets the details of the lattice.

It is supposed here that the microscopic geometry of the lattice is not fixed. This means that the elementary vectors (2.6) connecting the nearest vertices $a_{\mathcal{W}i}$ and $a_{\mathcal{W}j}$ are quantum variables, so that their quantum fluctuations are described by the corresponding wave function. This point of view is necessary in the lattice quantum theory of gravity [8,9]. Though this theory is not satisfactory at present, I hold to the following point of view: if the space-time is discrete on the microscopic level, then the corresponding lattice is irregular, and the geometrical values describing the lattice are quantum variables. Such a lattice is called a “breathing” one.

It seems that the propagation of an irregular fermion on the considered “breathing” lattice is similar in a sense to the dynamics of a Brownian particle: in the process of successive movements of fermions from one vertex to another, the information of a previous jump is forgotten due to the irregularity and “breathing” of the lattice. Thus, the propagation of irregular fermions can be described by a slightly modified Markov process which must model the correlator (5.13) in the four-dimensional Euclidean space.

It is seen from Eqs. (2.30) and (2.32) that

$$\sum_{a=1}^{4} e^a_{\mathcal{V}_1,\mathcal{V}_2}\mathcal{D}^a_{\mathcal{V}_1,\mathcal{V}_2} \sim v_{\mathcal{V}_1,\mathcal{V}_2} \tag{5.14}$$

is the sum of oriented volumes of all 4-simplices with the common 1-simplex $a_{\mathcal{V}_1}a_{\mathcal{V}_2}$. Therefore, the model of the amplitude $[-i\gamma^a\mathcal{D}^a_{\mathcal{V}_1,\mathcal{V}_2}]$ in (5.13) will be the following one:

$$[-i\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}] \to [-i\mathcal{D}(x-y)] \equiv \left[\frac{\rho}{\pi b}(-i\gamma^a\partial_a)\exp\left(-\frac{(x-y)^2}{b^2}\right)\right]. \tag{5.15}$$

The right-hand side of (5.15) is the amplitude of the jump from the point x into the point y. Here the dimensionless Cartesian coordinates $x^a \to x^a/l_P$ are used. The numerical constant $b \sim 1$ is a parameter of the model, and ρ is an unknown normalization constant which is of no importance. It is seen that the direction of the jump vector $(y-x)$ is unconstrained, but the jump step value is constrained by the Gauss distribution. The model of the inverse amplitude $[-i\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}]^{-1}$ is as follows:

$$[-i\mathcal{D}_{\mathcal{V}_1,\mathcal{V}_2}]^{-1} \to [-i\mathcal{D}(x-y)]^{-1} \equiv \left[\frac{1}{\pi\rho b}(-i\gamma^a\partial_a)\exp\left(-\frac{(x-y)^2}{b^2}\right)\right]. \tag{5.16}$$

Now the analog of the relation (5.11) is the equality

$$\int \mathrm{d}^{(4)}y[-i\mathcal{D}(x-y)]^{-1}[-i\mathcal{D}(y-x)] = 1. \tag{5.17}$$

Thereby, the model of the correlator representation (5.13) looks like $(z_0 = y)$

$$\langle\psi(x)\psi^\dagger(y)\rangle^{\mathcal{I}} = \sum_{k=0}^{\infty}\prod_{i=1}^{2k+1}\left\{\int \mathrm{d}^{(4)}z_i\right\}\delta^{(4)}(x-z_{2k+1})[-i\mathcal{D}(z_{2k+1}-z_{2k})]^{-1}[-i\mathcal{D}(z_{2k}-z_{2k-1})]\ldots[-i\mathcal{D}(z_3-z_2)]^{-1} \times[-i\mathcal{D}(z_2-z_1)][-i\mathcal{D}(z_1-y)]^{-1}.$$

Since the operators $[-i\mathcal{D}]$ and $[-i\mathcal{D}]^{-1}$ are coupled, one can set $\rho=1$. This expression is rewritten by passing to the new integration variables $\tilde{z}_i = z_i - z_{i-1}$, $i=1,\ldots,2k+1$:

$$\langle\psi(x)\psi^\dagger(0)\rangle^{\mathcal{I}} = \sum_{k=0}^{\infty}\prod_{i=1}^{2k+1}\int \mathrm{d}^{(4)}z_i\delta^{(4)}\left(x-\sum_{j=1}^{2k+1}z_j\right)[-i\mathcal{D}(z_{2k+1})]^{-1}[-i\mathcal{D}(z_{2k})]\ldots[-i\mathcal{D}(z_2)][-i\mathcal{D}(z_1)]^{-1}.$$

With the help of Eqs. (5.15) and (5.16), the right-hand side of the last relation is rewritten once again:

$$\begin{aligned}\langle\psi(x)\psi^\dagger(0)\rangle^{\mathcal{I}} &= \sum_{k=0}^{\infty}\int\ldots\int \mathrm{d}^{(4)}z_1\ldots\mathrm{d}^{(4)}z_{2k+1}\delta^{(4)}\left(\sum_{i=1}^{2k+1}z_i - x\right)\prod_{i=1}^{2k+1}\left[\frac{1}{\pi b}(-i\gamma^a\partial_a)\exp\left(-\frac{z_i^2}{b^2}\right)\right]\\ &= \sum_{k=0}^{\infty}\int\frac{\mathrm{d}^{(4)}q}{(2\pi)^4}e^{-iqx}\prod_{i=1}^{2k+1}\left[\frac{2}{\pi b^3}\int(i\gamma^a z_i^a)\exp\left(-\frac{z_i^2}{b^2}+iqz_i\right)\mathrm{d}^{(4)}z_i\right]\\ &= \int\frac{\mathrm{d}^{(4)}q}{(2\pi)^4}e^{-iqx}\sum_{k=0}^{\infty}\left[2\pi b\left(\gamma^a\frac{\partial}{\partial q^a}\right)\exp\left(-\frac{q^2b^2}{4}\right)\right]^{2k+1} = \left(-i\gamma^a\frac{\partial}{\partial x^a}\right)\int\frac{\mathrm{d}^{(4)}q}{(2\pi)^4}\frac{\pi b^3\exp\left(-\frac{q^2b^2}{4}-iqx\right)}{1-\pi^2b^6q^2\exp\left(-\frac{q^2b^2}{2}\right)}.\end{aligned} \tag{5.18}$$

The integral on the right-hand side of Eq. (5.18) is determined for

$$0 < b < \left(\frac{e}{2\pi^2}\right)^{1/4} \approx 0,61. \tag{5.19}$$

Integration over the angle variables leads to the expression ($r\equiv|x|$)

$$\langle\psi(x)\psi^\dagger(0)\rangle^{\mathcal{I}} = \left(-i\gamma^a\frac{\partial}{\partial x^a}\right)\left[\left(\frac{b^3}{4\pi r}\right)\int_0^\infty \mathrm{d}q\cdot q^2\frac{J_1(qr)\exp\left(-\frac{q^2b^2}{4}\right)}{1-\pi^2b^6q^2\exp\left(-\frac{q^2b^2}{2}\right)}\right]. \tag{5.20}$$

The characteristic value of the variable q saturating the integral (5.20) is determined by the nearest zero of the denominator in the integral. So $|q|\sim 1$. Since we are interested in the correlator behavior for $r\gg 1$, the argument qr of the Bessel function under the integral (5.20) is effectively large: $qr\gg 1$. Therefore, one can use the asymptotic behavior of the Bessel function:

$$J_1(qr)\to\frac{1}{\sqrt{2\pi qr}}[e^{iqr-3\pi i/4}+e^{-iqr+3\pi i/4}].$$

With the help of the last relation, the integral (5.20) is rewritten as follows:

$$\langle\psi(x)\psi^\dagger(0)\rangle^{\mathcal{I}} = \left(-i\gamma^a\frac{\partial}{\partial x^a}\right)\left[\frac{b^3}{2(2\pi r)^{3/2}}\int_C \mathrm{d}q\cdot q^{3/2}\frac{\exp\left(-\frac{q^2b^2}{4}+iqr-3\pi i/4\right)}{1-\pi^2b^6q^2\exp\left(-\frac{q^2b^2}{2}\right)}\right]. \tag{5.21}$$

The integration contour C is pictured in Fig. 3.

We are interested in the denominator zeros in the upper half-plane of the complex variable $q=q'+iq''$. The zeros are determined by the following set of equations:

$$(q'^2-q''^2) = 2q'q''\mathrm{ctg}(b^2q'q''),\ 2\pi^2b^4\exp[-(b^2q'q'')\mathrm{ctg}(b^2q'q'')] = \frac{\sin(b^2q'q'')}{b^2q'q''}. \tag{5.22}$$

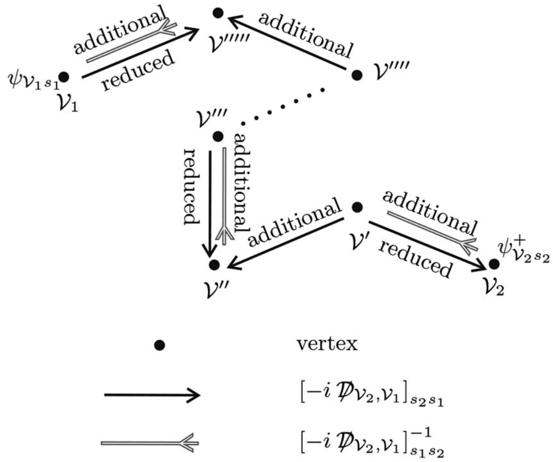

FIG. 2. The graphical representation of the curly brackets on the right-hand side of Eq. (5.13).

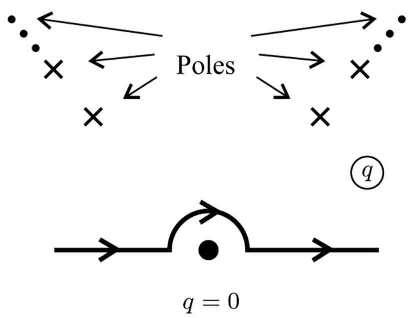

FIG. 3. The integration contour in the integral (5.21) and the location of the integral poles in the complex plane of the q variable.

Since the solutions of the set of equations (5.22) are symmetrized relative to the imaginary axis, it is enough to solve the system for $q' > 0, q'' > 0$. The approximative solution of the last set of equations looks like

$$b^2 q' q'' \approx (2n + 1/2)\pi, n = 0, 1, \ldots,$$
$$q'_n \sim q''_n \approx \frac{\sqrt{(2n + 1/2)\pi}}{b}. \quad (5.23)$$

All zeros of the denominator under the integral (5.21) lead to the simple poles of the expression under the integral sign. Indeed, the derivative of the denominator respect to the integration variable is equal to zero only for $q = 0, \pm\sqrt{2}/b$. Therefore,

$$\text{The denominator} = c_n(q - q_n) \quad \text{at} \quad q \to q_n.$$

Thus, contour C in the integral (5.21) can be deformed upward, so that the integral becomes a sum over poles residue. The sum is saturated by the pair of poles which are nearest to the real axis and placed at $q' = \pm\kappa'/b, q'' = \kappa/b$, where $\kappa', \kappa \sim 1$ [$n = 0$ in (5.23)].

Finally, we have

$$\langle \psi(x)\psi^\dagger(0)\rangle^{\mathcal{I}} \sim \left(i\gamma^a \frac{\partial}{\partial x^a}\right)\left[\frac{1}{r^{3/2}} \exp(-\kappa r/b) \cos\frac{\kappa' r}{b}\right]. \quad (5.24)$$

The right-hand side of the relation (5.5) simulates the obtained result (5.24) in Minkowski space-time with restored dimensionality.

We see that the irregular quanta are "bad" quasiparticles.

The fermion lines, such as in Fig. 1, represent the creation (at $\mathcal{V}_2$), propagation, and annihilation (at $\mathcal{V}_1$) of a fermion quantum, and the quantum creation and annihilation events are induced by the external sources only. If the fermion line is everywhere continuous and endless in the space, then it describes the propagation of a real particle.

VI. SUMMARY

Let us summarize the content of the paper.

At the beginning of the paper, the model of discrete gravity on a simplicial complex is defined. A few years ago, the model was introduced by the author in a series of works (see Refs. [8,9]). Here it is interesting to note that any theory with the action appearing as an integral of the form over the space-time can be generalized easily into a simplicial complex. Thus, the "naive fermions" are introduced which conserve γ^5-invariance.

In the subsequent two sections, the existence of the Wilson fermion doubling phenomenon on an irregular lattice (simplicial complex) is established. It means that the irregular soft (low-energy) fermion quanta are real. The statement is proved on a four-dimensional lattice by means of the Atiyah-Singer index theorem, and then it is extended easily into the cases $D < 4$. By irregular quanta, we mean the quanta with the wave functions essentially depending on the details of the simplicial complex. On the contrary, the long-wavelength quanta are regular in the sense that the corresponding wave functions have lost the information about the irregular lattice.

From there, the fundamental difference between the regular and irregular quanta is established: the irregular quanta cannot propagate in space-time since their propagator decreases exponentially, while the regular quanta propagate "without difficulty" as usual particles. Therefore, the irregular quanta are unphysical. The statement is proved in the last section of the paper.

The term "unphysical" does not indicate that the corresponding quanta are inessential in physics. Some speculations about the possible role of irregular quanta in astrophysics and particle physics are given in the Introduction.

In conclusion, I want to make some remarks.

(1) General regularizations based on the simplicial complex usually break the hypercubic symmetry of the regular lattice. Then, it is reasonable to ask whether or not the breaking of the symmetry is related to the

emergence of the unphysical modes? The answer to this question seems to be negative. Indeed, boson unphysical modes exist besides the fermion unphysical modes on the simplicial complex as well: in our case, the modes are graviton irregular modes, i.e., the graviton quanta with the wave functions essentially depending on the details of the simplicial complex. However, there is no cause for the existence of soft bosonic irregular modes. So the energies of the bosonic irregular quanta are not soft, but they are of the order of l_P^{-1}. This estimation is valid also on the regular hypercubic lattice for the bosonic quanta on the boundary of the Brillouin zone. Thus, the bosonic irregular quanta are doubly unphysical: they cannot propagate, and they cannot be created by physical quanta of any nature. On the contrary, the discussed fermion irregular quanta on the irregular lattice or the fermion quanta on the regular lattice on the boundary of the Brillouin zone are light. It seems that this property is fundamental for the Dirac fermions; this property does not depend on the construction of the lattice.

(2) Another question is concerned with the relation between the topology of the simplicial complex and the existence or absence of the zero or soft irregular fermion modes. The answer to this question seems to be as follows:

Suppose that the topology of the space-time admits the existence of Yang-Mills instantons, and the Dirac field belongs to the corresponding gauge group representation. Thus, our approach guarantees the existence of the irregular zero fermion modes on the background of of the Yang-Mills instanton field for $\mathfrak{N} \to \infty$. It follows from here that the soft irregular fermion modes exist even if the instanton field is absent. Since the wave functions of the irregular modes are local, the soft irregular fermion modes exist independently of the topology of space-time. But if the instanton field is absent, the existence of an irregular local and actually zero mode is not guaranteed.

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