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Three-loop correction to the instanton density. I. The quartic double well potential

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This paper deals with quantum fluctuations near the classical instanton configuration. Feynman diagrams in the instanton background are used for the calculation of the tunneling amplitude (the instanton density) at three-loop order for a quartic double well potential. The result for the three-loop contribution coincides, to five significant figures, with the one given long ago by J. Zinn-Justin. Unlike the two-loop contribution where all involved Feynman integrals are rational numbers, in the three-loop case Feynman diagrams can contain irrational contributions.

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I. INTRODUCTION

There is no question that instantons [1], Euclidean classical solutions of the field equations, represent one of the most beautiful phenomena in theoretical physics [2,3]. Instantons in non-Abelian gauge theories of the QCD type are important components of the nonperturbative vacuum structure; in particular, they break chiral symmetries and thus significantly contribute to the nucleon (and our) mass [4]. Instantons in supersymmetric gauge theories lead to derivation of the exact beta function [5] and, in the "Seiberg-Witten" $\mathcal{N} = 2$ case, to the derivation of the superpotential by the exact evaluation of the instanton contributions to all orders [6]. The instanton method now has applications in stochastic settings beyond quantum mechanics or field theories, and even physics—in chemistry and biology—see e.g., the discussion of its usage in the problem of protein folding in [7].

Since the work by Polyakov [1], the problem of a double well potential (DWP) has been considered as the simplest quantum mechanical setting illustrating the role of instantons in more complicated quantum field theories. In the case of the DWP, one can perform certain technical tasks—like we do below—which so far are out of reach in more complicated or realistic settings.

Tunneling in the quantum mechanical context has been studied extensively using WKB and other semiclassical means. The aim of this paper is not to increase the accuracy on these quantum-mechanical results, but rather to develop tools—Feynman diagrams on top of an instanton—which can be used in the context of many dimensions and especially in quantum field theories (QFTs). Therefore, we do *not* use anything stemming from the Schrödinger equation in this work; in particular, we do not use series resulting from recurrence relations or resurgence relations (in general, conjectured) by several authors.

Another reason to study DWP is the existing deep connections between the quantum mechanical instantons—via the Schrödinger equation—with wider mathematical issues, of approximate solutions to differential equations, defined in terms of certain generalized series. A particular form of an exact quantization condition was *conjectured* by J. Zinn-Justin and collaborators (for a review see [8] and references therein), which links a series around the instantons with the usual perturbative series in the perturbative vacuum. Unfortunately, no rigorous proof of such a connection exists, and it remains unknown if it can or cannot be generalized to the field theory cases we are mainly interested in. Recently, for the quartic double well and Sine-Gordon potentials, Dunne and Ünsal (see [9] and also references therein) have presented more arguments for this connection, which they call the resurgent relation.

In [10] the method and key elements (a nontrivial instanton background and new effective vertices) to calculate the two-loop correction to the tunneling amplitude for the DWP were established. In particular, the anharmonic oscillator was considered in order to show how to apply the Feynman diagrams technique. In [11] the Green function in the instanton background was corrected, and an attempt was made to obtain two- and three-loop corrections. Finally, Wöhler and Shuryak [12] corrected some errors made in [11] and reported the exact result for the two-loop correction.

The goal of the present paper is to evaluate the three-loop correction to the tunneling amplitude and compare it with the results obtained in [8] by a completely different method, not available in the field theory settings.

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II. THREE-LOOP CORRECTION TO THE INSTANTON DENSITY

Let us consider the quantum-mechanical problem of a particle of mass $m = 1$ in a double well potential

$$V = \lambda(x^2 - \eta^2)^2. \tag{1}$$

The well-known instanton solution $X_{\rm inst}(t) = \eta \tanh(\frac{1}{2}\omega(t - t_c))$, with $\omega^2 = 8\lambda\eta^2$, describing the barrier tunneling is the path which possesses the minimal action $S_0 = S[X_{\rm inst}(t)] = \frac{\omega^3}{12\lambda}$. Setting $\omega = 1$ and shifting coordinate to the minimum, one gets the anharmonic oscillator potential in a form $V_{\rm anh} = \frac{1}{2}x^2 - \sqrt{2\lambda}x^3 + \lambda x^4$ with one (small) dimensionless parameter λ. J. Zinn-Justin *et al.* [8] use the same potential with $\lambda = g/2$.

The classical action S_0 of the instanton solution is therefore large, and $\frac{1}{S_0}$ is used in the expansion. The ground state energy E_0 within the zero-instanton sector (pure perturbation theory) is written in the form

$$E_0 = \frac{1}{2}\sum_{n=0}^{\infty}\frac{A_n}{S_0^n} \quad (A_0 = 1), \tag{2}$$

Another series to be discussed is the splitting $\delta E = E_{\text{first excited state}} - E_{\text{ground state}}$ related to the so-called instanton density [13] in the one-instanton approximation as

$$\delta E = \Delta E \sum_{n=0}^{\infty}\frac{B_n}{S_0^n} \quad (B_0 = 1), \tag{3}$$

where $\Delta E = 2\sqrt{\frac{6S_0}{\pi}}e^{-S_0}$ is the well-known one-loop semiclassical result [2]. Coefficients A_n in the series (2) can be calculated using the ordinary perturbation theory (see [16]), while many coefficients B_n in the expansion (3) were found by J. Zinn-Justin (see [8] and references therein), obtained via the so-called exact Bohr-Sommerfeld quantization condition.

Alternatively, using the Feynman diagrams technique, Wöhler and Shuryak [12] calculated the two-loop correction $B_1 = -71/72$ in agreement with the result by J. Zinn-Justin [8]. Higher order coefficients B_n in (3) can also be computed in this way. Since we calculate the energy difference, all Feynman diagrams in the instanton background (with the instanton-based vertices and the Green's function) need to be accompanied by subtraction of the same diagrams for the anharmonic oscillator, without the instanton (see [10] for details). For $\frac{1}{\Delta E} \gg \tau \gg 1$ this permits us to evaluate the ratio

$$\frac{\langle -\eta|e^{-H\tau}|\eta\rangle_{\rm inst}}{\langle \eta|e^{-H\tau}|\eta\rangle_{\rm anh}},$$

where the matrix elements $\langle -\eta|e^{-H\tau}|\eta\rangle_{\rm inst}$, $\langle \eta|e^{-H\tau}|\eta\rangle_{\rm anh}$ are calculated using the instanton-based and vacuum diagrams, respectively.

The instanton-based Green's function $G(x, y)$

$$G(x,y) = G^0(x,y)\left[2 - xy + \frac{1}{4}|x-y|(11-3xy) + (x-y)^2\right] + \frac{3}{8}(1-x^2)(1-y^2)\left[\log G^0(x,y) - \frac{11}{3}\right] \tag{4}$$

is expressed in variables $x = \tanh(\frac{t_1}{2}), y = \tanh(\frac{t_2}{2})$, in which the familiar oscillator Green function $e^{-|t_1-t_2|}$ of the harmonic oscillator is

$$G^0(x,y) = \frac{1 - |x-y| - xy}{1 + |x-y| - xy}. \tag{5}$$

In its derivation there were two steps. One was to find a function which satisfies the Green function equation, used via two independent solutions and the standard Wronskian method. The second step is related to a zero mode: one can add a term $\phi_0(t_1)\phi_0(t_2)$ with any coefficient and still satisfy the equation. The coefficient is then fixed from orthogonality to the zero mode (see [11]).

The two-loop coefficient is given by the two-loop diagrams (see Fig. 1 and [12])

$$\begin{aligned} &B_1 = a + b_1 + b_2 + c, \\ &a = -\frac{97}{1680}, \qquad b_1 = -\frac{53}{1260}, \\ &b_2 = -\frac{39}{560}, \qquad c = -\frac{49}{60}. \end{aligned} \tag{6}$$

The three-loop correction B_2 (3) we are interested in is given by the sum of diagrams, which we group as follows:

$$\begin{aligned} B_2 = {} & B_{2\rm loop} + a_1 + b_{11} + b_{12} + b_{21} + b_{22} + b_{23} + b_{24} \\ & + d + e + f + g + h + c_1 + c_2 + c_3 + c_4 + c_5 + c_6. \end{aligned} \tag{7}$$

All Feynman diagrams in (7) are presented in Figs. 1–3. The rules of constructing the integrals for each should be clear from an example, the explicit expression for the Feynman integral b_{23} in Fig. 2, which is

$$\begin{aligned} b_{23} = {} & \frac{9}{8}\int_{-1}^{1}dx\int_{-1}^{1}dy\int_{-1}^{1}dz\int_{-1}^{1}dw \\ & \times J(x,y,z,w)(xyzwG_{xx}G_{xy}G_{yz}G_{yw}G_{zw}^2 \\ & - G_{xx}^0 G_{xy}^0 G_{yz}^0 G_{yw}^0 (G_{zw}^0)^2), \end{aligned} \tag{8}$$

while for c_4 in Fig. 3 it takes the form

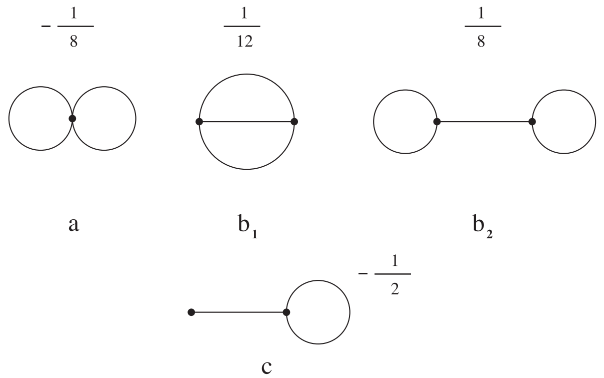

FIG. 1. Diagrams contributing to the two-loop correction $B_1 = a + b_1 + b_2 + c$. They enter into the coefficient B_2 via the term $B_{2\,\mathrm{loop}}$. For the instanton field the effective triple and quartic coupling constants are $V_3 = -\frac{\sqrt{3}}{2}\tanh(t/2) S_0^{-1/2}$ and $V_4 = \frac{1}{2} S_0^{-1}$, respectively, while for the subtracted anharmonic oscillator we have $V_3 = -\frac{\sqrt{3}}{2} S_0^{-1/2}$ and $V_4 = \frac{1}{2} S_0^{-1}$. The tadpole in diagram c, which comes from the zero-mode Jacobian rather than from the action, is effectively represented by the vertex $V_{\mathrm{tad}} = \frac{\sqrt{3}}{4}\frac{\tanh(t/2)}{\cosh^2(t/2)} S_0^{-1/2}$. The signs of the contributions and symmetry factors are indicated.

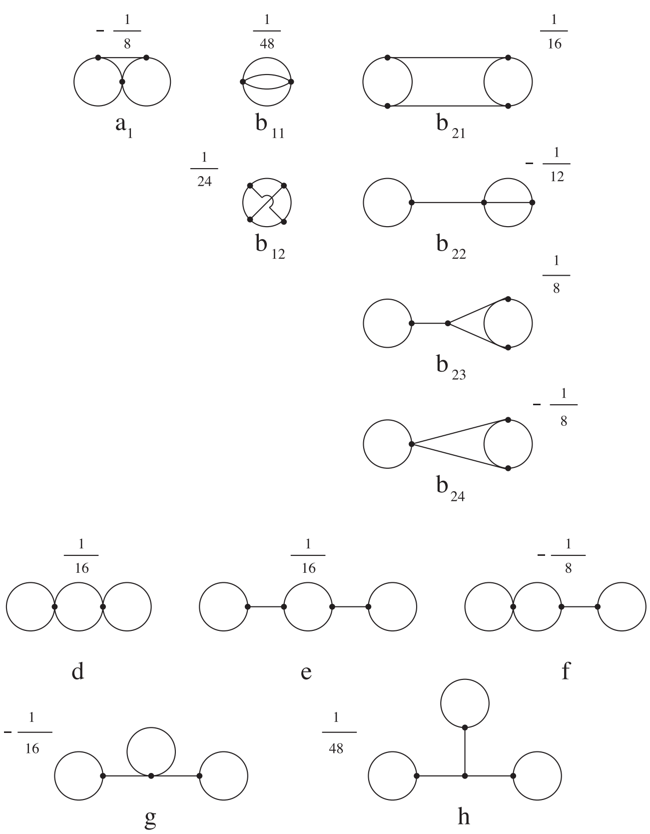

FIG. 2. Diagrams contributing to the coefficient B_2. The signs of the contributions and symmetry factors are indicated.

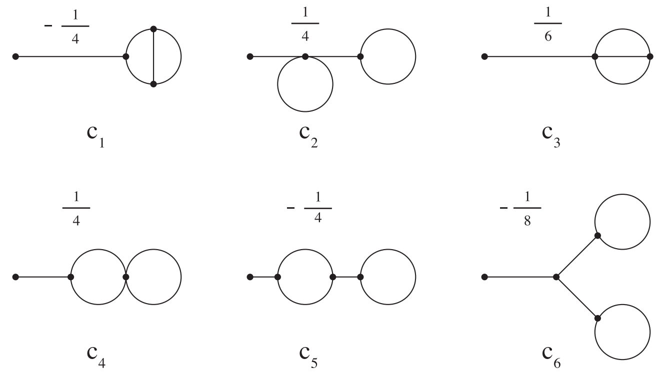

FIG. 3. Diagrams contributing to the coefficient B_2. They come from the Jacobian of the zero mode and have no analogs in the anharmonic oscillator problem. The signs of the contributions and symmetry factors are indicated.

$$c_4 = \frac{3}{8}\int_{-1}^{1} dx \int_{-1}^{1} dy \int_{-1}^{1} dz \frac{xy}{(1-y^2)(1-z^2)} G_{xy} G_{yz}^2 G_{zz}. \tag{9}$$

Here we introduced notations $G_{xy} \equiv G(x,y), G_{xy}^0 \equiv G^0(x,y)$ and $J = \frac{1}{(1-x^2)}\frac{1}{(1-y^2)}\frac{1}{(1-z^2)}\frac{1}{(1-w^2)}$. Note that c-type diagrams come from the Jacobian of the zero mode and have no analogs in the anharmonic oscillator problem.

III. RESULTS

The obtained results are summarized in Table I. All diagrams are of the form of two-dimensional, three-dimensional and four-dimensional integrals. In particular, the diagrams b_{11} and d (see Fig. 2),

$$b_{11} = \frac{1}{48}\int_{-1}^{1} dx \int_{-1}^{1} dy \frac{1}{(1-x^2)(1-y^2)} (G_{xy}^4 - (G_{xy}^0)^4)$$
$$d = \frac{1}{16}\int_{-1}^{1} dx \int_{-1}^{1} dy \frac{1}{(1-x^2)(1-y^2)} \times (G_{xx} G_{xy}^2 G_{yy} - G_{xx}^0 (G_{xy}^0)^2 G_{yy}^0), \tag{10}$$

given by two-dimensional integrals, are the only ones which we are able to calculate analytically,

$$b_{11} = -\frac{1842223}{592704000} - \frac{1}{9800}(367\zeta(2) - 180\zeta(3) - 486\zeta(4)) \equiv b_{11}^{\mathrm{rat}} + b_{11}^{\mathrm{irrat}}$$
$$d = \frac{205441}{2469600} + \frac{525}{411600}\zeta(2) \equiv d^{\mathrm{rat}} + d^{\mathrm{irrat}}. \tag{11}$$

Here $\zeta(n)$ denotes the Riemann zeta function of argument n (for definition see [17]). They contain a rational and an irrational contribution such that

TABLE I. Contribution of diagrams in Figs. 2 and 3 for the three-loop corrections B_2 (left) and A_2 (right). We write $B_2 = (B_{2\,\mathrm{loop}} + I_{2D} + I_{3D} + I_{4D})$ where I_{2D}, I_{3D}, I_{4D} denote the sum of two-dimensional, three-dimensional and four-dimensional integrals, respectively. Similarly, $A_2 = I_{2D} + I_{3D} + I_{4D}$. The term $B_{2\,\mathrm{loop}} = 39589/259200 \approx 0.152735$ (see text).

Feynman Diagram	Instanton B_2	Vacuum A_2
a_1	−0.0650	$\frac{5}{192}$
b_{12}	0.0257	$-\frac{1}{64}$
b_{21}	0.0496	$-\frac{11}{384}$
b_{22}	−0.1323	$\frac{1}{24}$
b_{23}	0.2807	$-\frac{1}{8}$
b_{24}	−0.1271	$\frac{1}{24}$
e	0.3950	$-\frac{9}{64}$
f	−0.3524	$\frac{3}{32}$
g	−0.3964	$\frac{3}{32}$
h	0.3142	$-\frac{3}{32}$
c_1	−0.3268	···
c_2	0.6333	···
c_3	0.1266	···
c_4	0.2975	···
c_5	−0.7710	···
c_6	−0.8082	···
I_{2D}	0.0963	$-\frac{7}{384}$
I_{3D}	−0.0158	$\frac{19}{64}$
I_{4D}	−0.8408	$-\frac{155}{384}$

$$\frac{b_{11}^{\mathrm{irrat}}}{b_{11}^{\mathrm{rat}}} \approx -4.55, \qquad \frac{d^{\mathrm{irrat}}}{d^{\mathrm{rat}}} \approx 0.025.$$

This shows that the irrational contribution to b_{11} is dominant with respect to the rational part, while for diagram d the situation is the opposite. Other diagrams, see Table I, were evaluated numerically with an absolute accuracy $\sim 10^{-5}$. Surprisingly, almost all of them are of order 10^{-1}, while a few of them (diagrams a_1, b_{12}, b_{21}) are of order 10^{-2}.

J. Zinn-Justin (see [8] and references therein) reports a value of

$$B_2^{\text{Zinn-Justin}} = -\frac{6299}{10368} \approx -0.607542, \tag{12}$$

while the present calculation shows that

$$B_2^{\text{present}} \approx -0.607535, \tag{13}$$

which is in agreement, up to the precision employed in the numerical integration.

Similarly to the two-loop correction B_1, the coefficient B_2 is negative. Note also that for B_1 all diagrams are negative while for B_2 there are diagrams of both signs. For not-so-large barriers ($S_0 \sim 1$), the two-loop and three-loop corrections are of the same order of magnitude.

The dominant contribution comes from the sum of the four-vertex diagrams $b_{12}, b_{21}, b_{23}, e, h, c_1, c_5, c_6$, while the three-vertex diagrams $a_1, b_{22}, b_{24}, f, g, c_2, c_3, c_4$ provide a minor contribution; their sum represents less than 3% of the total correction B_2. It is interesting that for both two- and three-loop cases, the largest contribution comes from diagrams stemming from the Jacobian, c for B_1 and c_5, c_6 for B_2. Those diagrams are absent in the perturbative vacuum series and thus do not have subtractions.

We already noted that individual three-loop diagrams contain irrational numbers. Since the J. Zinn-Justin's result is a rational number, there must be a cancellation of these irrational contributions in the sum (7). From (11) we note that the term $(b_{11}^{\mathrm{irrat}} + d^{\mathrm{irrat}})$ gives a contribution of order 10^{-2} to the mentioned sum (7), and therefore the coincidence 10^{-5} between the present result (13) and the one of J. Zinn-Justin (12) is an indication that such a cancellation occurs. Now, we evaluate the coefficients A_1, A_2 in (2) using Feynman diagrams (see [16]). In order to do this let us consider the anharmonic oscillator potential $V_{\mathrm{anh}} = \frac{1}{2}x^2 - \sqrt{2\lambda}x^3 + \lambda x^4$ and calculate the transition amplitude $\langle x=0|e^{-H_{\mathrm{anh}}\tau}|x=0\rangle$. All involved Feynman integrals can be evaluated analytically. In the limit $\tau \to \infty$ the coefficients of order S_0^{-1} and S_0^{-2} in front of τ give us the values of A_1 and A_2, respectively. As it was mentioned above the c diagrams do not exist for the anharmonic oscillator problem. The Feynman integrals in Fig. 1 give us the value of A_1; explicitly they are equal to

$$a = \frac{1}{16}, \qquad b_1 = -\frac{1}{24}, \qquad b_2 = -\frac{3}{16}.$$

The diagrams in Fig. 2 determine A_2 and the corresponding values are presented in Table I, $b_{11} = -\frac{1}{384}$ and $d = -\frac{1}{64}$. A straightforward evaluation gives

$$A_1 = -\frac{1}{3}, \qquad A_2 = -\frac{1}{4},$$

which is in agreement with the results obtained in standard multiplicative perturbation theory (see [18]). No irrational numbers appear in the evaluation of A_1 and A_2. It is worth noting that (see Table I) some Feynman integrals give the same contribution,

$$f = g = \frac{3}{32}, \qquad b_{22} = b_{24} = \frac{1}{24}.$$

In the instanton background the corresponding values of these diagrams do not coincide but are very close.

IV. CONCLUSIONS AND DISCUSSIONS

In conclusion, we have calculated the tunneling amplitude (level splitting or the instanton density) up to three loops using Feynman diagrams for quantum perturbations on top of the instanton. Our result for B_2 is found to be in good agreement with the resurgent relation between the perturbative and instanton series suggested by J. Zinn-Justin (for a modern reference, see [8]).

Let us note again that this paper is methodical in nature, and its task was to develop tools to calculate tunneling phenomena in a multidimensional or QFT context, in which any results stemming from the Schrödinger equation are not available. We use a quantum mechanical example as a test of the tools we use, but the tools themselves are expected to work in a much wider context.

One comment on the results is that the final three-loop answer has a rational value. However, unlike the evaluation of the two-loop coefficient B_1 where all Feynman diagrams turned out to be rational numbers, in our case of B_2 at least two diagrams contain irrational parts. What is the origin of these terms and how do they cancel out among themselves? These are questions left unanswered above since several diagrams had resisted our efforts to get the analytic answer, so we used numerical integration methods. Perhaps this can still be improved.

Another intriguing issue is the conjectured relation between the instanton and vacuum series: at the moment we do not understand its origin from the path integral settings. Some diagrams are similar, but the expressions are quite different and unrelated. New diagrams originate from the instanton zero mode Jacobian, and those have no analogues in the vacuum. Surprisingly, they provide the dominant contribution to two- and three-loop corrections B_1 and B_2: ~80% and ~140%, respectively (see Table I).

Finally, we note that, to our knowledge, this is the first three-loop calculation on a nontrivial background of an instanton. Similar calculations for gauge theories would certainly be possible and are of obvious interest. One technical issue to be solved is the gauge Green function orthogonal to all (including gauge change) zero modes.

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Note added in proof.—After this paper was submitted, we obtained a number of new results. We evaluated the contributions of c, c_5-like diagrams, with a maximal number of integrations, to the next order coefficients. Those diagrams still contribute a significant fraction of the total answer, namely, 83%, 127%, 60%, and 20% of two-, three-, four-, and five-loop B_1, B_2, B_3, B_4 contributions, respectively. At the same time, surprisingly, the absolute values of all these diagrams are rather close. An advance in numerical multidimensional integrations leads to an increase in accuracy; the agreement in B_2 is now improved to six significant digits.

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