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# Computation of the $O(p^6)$ order low-energy constants: An update

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We update our original low-energy constant computations to the $O(p^6)$ order, including two and three flavors, the normal and anomalous ones. Following a comparative analysis, the $O(p^4)$ order results are considered better. In the $O(p^6)$ order, most of our results are consistent with or better than those we have found in the literature, although several are worse.

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## I. INTRODUCTION

Chiral perturbation theory (ChPT) can be effectively used to deal with the interaction of low-energy mesons. It is important to establish the chiral Lagrangian (CL). Until now, the monomials of the CL, including both the normal and anomalous terms, two and three flavors, and the special unitary and unitary groups, have been obtained to the $O(p^6)$ order [1–11]. The CL plays an important role in ChPT and in experiments. To $O(p^6)$ order, the CL seems accurate enough to describe the results of present experiments, in which the coefficients for each of the monomials are called the low-energy constants (LECs). When one wants to give the numerical results for some physical process employing ChPT, the values for LECs are also needed. However, ChPT itself does not provide values for these LECs and, hence, they need to be obtained from other sources. Because low-energy QCD is nonperturbative, which invalidates the standard perturbation computations, exact values for the LECs are difficult to calculate from the underlying QCD theory, especially at $O(p^6)$ order. At $O(p^4)$ order, although exact results have not been obtained, values obtained employing different methods are close. Their signs and magnitudes are the same. At $O(p^6)$ order, there are various methods of obtaining LECs, with common ones being the use of resonance chiral theory [12,13], sum rules [14], lattice QCD [15,16], holographic theory [17], QCD [18], and the global fit [19,20]. Of course, LECs can also be extracted from experimental data without a data fit. Some methods give a value for a single LEC, whereas others provide values for combinations of LECs. Each method has its advantages and disadvantages, and some results do come with large errors.

More than 15 years ago, values of a few LECs obtained by different methods were scattered in various references. Numerous LECs remained unknown in value. Our motivation is to explore a method for obtaining all of the LECs in a single calculation. We previously developed a model-independent method with which to perform our calculation from the underlying QCD theory. Nevertheless, at this earlier time, the theoretical analysis was poor, and we only used simple methods that involved some rough approximations [18,21]. Although the methods were expedient, preliminary results, at least, were obtained.

With rough approximations, the results yield a gauge-invariant, nonlocal, dynamical-quark (GND) model [22]. Reference [19] checks our $O(p^6)$ LECs ($C_i$) via a global fit of the $O(p^4)$ LECs ($L_i$) to $O(p^6)$ order. $\chi^2$ divided by the degrees of freedom ($\chi^2$/DOF) is 4.13/4. If all LECs are multiplied by 0.27, $\chi^2/\mathrm{DOF} = 1.20/3$. A new fit [20] gives $\chi^2/\mathrm{DOF} = 41/8$. Hence, the $C_i$ values from [19,20] appear to be too large. In statistical theory, when there are enough degrees of freedom (a typical choice is more than 30), $\chi^2/\mathrm{DOF} \sim 1$. However, only 4 (or 8), degrees of freedom are not sufficient to assess the reliability of the calculation by $\chi^2$/DOF, and $\chi^2$/DOF is also not as close to 1 as it should be. Furthermore, the $O(p^4)$ and $O(p^6)$ LECs are independent in our calculations. Evaluating the confidence level of $C_i$ by $L_i$ is not suitable, especially when the DOF is small. A more reliable judgment would be comparisons with experiment data. In experiments conducted to date, obviously, the absolute values of our $L_i$ are too large, which indicates the existence of systematic errors in the calculation at $O(p^4)$ and $O(p^6)$ orders. Here, we define the systematic errors as those caused by the rough approximations we have taken (which we briefly review in the next section), such as the large-$N_c$ limit and ladder approximation, the truncation of effective action, a special

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ansatz solution for the Schwinger-Dyson equation, and the modeling of the low-energy behavior of the gluon propagator. The time has now come to improve the calculation precision for the $O(p^4)$ and $O(p^6)$ LECs. While tedious, these values need to be improved step by step. This paper analyzes the origin of the systematic errors and selects the main ones to remedy the problem as precisely as possible.

This paper is organized as follows: In Sec. II, we review our method for obtaining the CL from QCD and introduce a more reliable approximation. In Sec. III, a concrete method for calculating LECs is introduced. In Sec. IV, we list our results for the LECs, both normal and anomalous, up to and including the $O(p^6)$ order. In Sec. V, we compare our results with others in the literature and check for new predictions. Section VI concludes with a summary.

## II. REVIEW AND IMPROVEMENTS OVER PREVIOUS CALCULATIONS OF LECS

In a previous work, we took the large-$N_c$ limit and obtained the action of the effective chiral theory. Derived from first principle QCD, it takes the form [23,24]

$$S_{\rm eff} = -iN_c \mathrm{Tr}\ln[i\partial\!\!\!/ + J_\Omega - \Pi_{\Omega c}] + N_c\int d^4x d^4x' \Phi^{\sigma\rho}_{\Omega c}(x,x')\Pi^{\sigma\rho}_{\Omega c}(x,x') + N_c\sum_{n=2}^{\infty}\int d^4x_1\cdots d^4x'_n \times\frac{(-i)^n(N_c g_s^2)^{n-1}}{n!}\bar{G}^{\sigma_1\cdots\sigma_n}_{\rho_1\cdots\rho_n}(x_1,x'_1,\ldots,x_n,x'_n)\Phi^{\sigma_1\rho_1}_{\Omega c}(x_1,x'_1)\cdots\Phi^{\sigma_n\rho_n}_{\Omega c}(x_n,x'_n) + iN_c\int d^4x\mathrm{tr}_{lf}\left\{\Xi_c(x)\left[-i\sin\frac{\vartheta_c(x)}{N_f}+\gamma_5\cos\frac{\vartheta_c(x)}{N_f}\right]\Phi^T_{\Omega,c}(x,x)\right\}, \quad (1)$$

in which $J_\Omega$ is the external source $J$ including currents [vector ($v^\mu$) and axial-vector ($a^\mu$) currents] and densities [scalar ($s$) and pseudoscalar ($p$) densities] after making a Goldstone-field-dependent chiral rotation $\Omega$:

$$J_\Omega = [\Omega P_R + \Omega^\dagger P_L][J + i\partial\!\!\!/][\Omega P_R + \Omega^\dagger P_L] = v\!\!\!/_\Omega + a\!\!\!/_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5, \qquad J = v\!\!\!/ + a\!\!\!/\gamma_5 - s + ip\gamma_5, \qquad U=\Omega^2, \quad (2)$$

where $P_L$ and $P_R$ are projection operators, U is a special $N_f\times N_f$ unitary matrix parametrizing a pseudoscalar meson field. And $\Phi_{\Omega c}$ and $\Pi_{\Omega c}$ are, respectively, the two-point rotated-quark Green's function and the interaction part of the two-point rotated-quark vertex in the presence of external sources; $\sigma,\rho$ are their spinor and flavor indices; and $\Phi_{\Omega c}$ is defined by

$$\Phi^{\sigma\rho}_{\Omega c}(x,y) \equiv \frac{1}{N_c}\langle\bar{\psi}^\sigma_\Omega(x)\psi^\rho_\Omega(y)\rangle = -i[(i\partial\!\!\!/ + J_\Omega - \Pi_{\Omega c})^{-1}]^{\rho\sigma}(y,x), \qquad \psi_\Omega(x)\equiv[\Omega(x)P_L + \Omega^\dagger(x)P_R]\psi(x), \quad (3)$$

$$\Pi^{\sigma\rho}_{\Omega c}(x,y) = -\tilde{\Xi}^{\sigma\rho}(x)\delta^4(x-y) - \sum_{n=1}^{\infty}\int d^4x_1\cdots d^4x_n d^4x'_1\cdots d^4x'_n\frac{(-i)^{n+1}(N_c g_s^2)^n}{n!} \times\bar{G}^{\sigma\sigma_1\cdots\sigma_n}_{\rho\rho_1\cdots\rho_n}(x,y,x_1,x'_1,\ldots,x_n,x'_n)\Phi^{\sigma_1\rho_1}_{\Omega c}(x_1,x'_1)\cdots\Phi^{\sigma_n\rho_n}_{\Omega c}(x_n,x'_n), \quad (4)$$

with subscript $_c$ denoting the classical field and $\psi(x)$ the light quark fields. $\bar{G}^{\sigma_1\cdots\sigma_n}_{\rho_1\cdots\rho_n}(x_1,x'_1,\ldots,x_n,x'_n)$ is the effective gluon $n$-point Green's function which include pure gluon and heavy quark contributions and $g_s$ is the coupling constant of QCD. $\Phi_{\Omega c}$ and $\Pi_{\Omega c}$ are related by the first equation of (3) and are determined by

$$[\Phi_{\Omega c}+\tilde{\Xi}]^{\sigma\rho} + \sum_{n=1}^{\infty}\int d^4x_1 d^4x'_1\cdots d^4x_n d^4x'_n\frac{(-i)^{n+1}(N_c g_s^2)^n}{n!}\bar{G}^{\sigma\sigma_1\cdots\sigma_n}_{\rho\rho_1\cdots\rho_n}(x,y,x_1,x'_1,\ldots,x_n,x'_n) \times\Phi^{\sigma_1\rho_1}_{\Omega c}(x_1,x'_1)\cdots\Phi^{\sigma_n\rho_n}_{\Omega c}(x_n,x'_n) = O\left(\frac{1}{N_c}\right), \quad (5)$$

where $\vartheta_c$ is the phase angle of the $U(1)$ factor, and $\Xi_c$ and $\tilde{\Xi}$ are two parameters in the calculation, defined by Eqs. (21) and (65) in Ref. [23]. In this work, they have little importance and are neglected. Equation (4) is the Schwinger-Dyson equation (SDE) in the presence of the rotated external source defined by Eq. (2) and with $\Phi_{\Omega c}$ relating to $\Pi_{\Omega c}$ through the first equation of (3). In Ref. [24], we have assumed an approximate ansatz solution of (4) given by

$$\Pi^{\sigma\rho}_{\Omega c}(x,y) = [\Sigma(\bar{\nabla}^2_x)]^{\sigma\rho}\delta^4(x-y), \qquad \bar{\nabla}^\mu_x = \partial^\mu_x - iv^\mu_\Omega(x), \quad (6)$$

where $\Sigma$ is the quark self-energy which satisfies the SDE (4) with a vanishing rotated external source. Under the large $N_c$ limit and the ladder approximation, this SDE in Euclidean space-time is reduced to the standard form

$$\Sigma(p^2) - 3C_2(R)\int\frac{d^4q}{4\pi^3}\frac{\alpha_s[(p-q)^2]}{(p-q)^2}\frac{\Sigma(q^2)}{q^2+\Sigma^2(q^2)} = 0, \qquad (7)$$

where $\alpha_s(p^2) = g_s^2/4\pi$ is the running coupling constant of QCD which depends on $N_C$ and the number of quark flavors, and $C_2(R)$ is the second-order Casimir operator of the quark representation $R$. In this work, the quarks belong to the $SU(N_C)$ fundamental representation and, therefore, $C_2(R) = (N_c^2-1)/2N_c$; in the large $N_C$ limit, the second term is neglected.

In our previous articles, because of the computational complexity, we did not calculate all of the terms in Eq. (1), but we introduced some approximations to truncate the last three terms and left only the first one. The truncation of the last three terms would introduce some systematic errors to all orders of LEC computation. We guess this may be the primary origin of the differences between our previous LECs and those in the literature. The purpose of this paper is to remedy these errors partially and produce a more reliable result.

As was mentioned in Ref. [24], the last term in (1) vanishes in the large-$N_c$ limit. We shall, then, only focus on the second and third terms. The infinite sum in both the third term in (1) and the second term in (4) appear similar but with different coefficients. To fix the primary origin of the systematic errors and to maintain a manageable calculation, we only retain the two-point Green's function contributions, the terms with $n=2$ in (1) and $n=1$ in (4). When we simplify the SDE from (4) to (7), we also only include $n=1$ in (5), which corresponds to the ladder approximation of the SDE. The present approximations and that for the SDE all include one $\Phi_{\Omega c}$ field and one $\Pi_{\Omega c}$ field, omitting multiple fields, or simply more than one field. Then, they are at the same level of accuracy. The other neglected terms, which include at least two $\Phi_{\Omega c}$ fields and describe the meson interactions belonging to higher dimension terms, are less important in the low-energy region in the naive dimension analysis than in the previous ones. In fact, with the second and third ($n=2$) terms of (1) taking the extreme value for $\Pi_{\Omega c}$ in (1), we get (3). Taking the extreme value for $\Phi_{\Omega c}$, we obtain the simplified version of (4), i.e., ladder approximation SDE. In other words, keeping these two extra terms in (1) assures the correct stationary equations–ladder approximation of SDE and makes our approximations more self-consistent compared with our previous calculations.

The final results will substantiate this decision. Hence, we shall add an additional effective action,

$$\Delta S_{\text{eff}} \sim \frac{1}{2}N_c\int d^4x d^4x' \Phi_{\Omega c}^{\sigma\rho}(x,x')\Pi_{\Omega c}^{\sigma\rho}(x,x') \qquad (8)$$

$$\sim -\frac{i}{2}N_c \mathrm{Tr}[(i\partial\!\!\!/ + J_\Omega - \Sigma(\overline{\nabla}^2))^{-1}\Sigma(\overline{\nabla}^2)], \qquad (9)$$

where we have used (5) to combine the second and third (only keeping $n=2$) terms of (1) together and we have further used (6).

With the same considerations for the anomalous parts, although using a different method to introduce the fifth dimensional integral [21], we need not repeat this, as the additional effective Lagrangian is the same as in (9). Therefore, for the present study, we calculate (9) including both the normal and the anomalous LECs.

## III. CALCULATION OF THE ADDITIONAL TERMS

To calculate the additional terms, we first use the Wick rotation to change (9) to Euclidean space-time as in [18,21], and we then expand it as a Taylor series,

$$\Delta S_{\text{eff}} = -\frac{1}{2}N_c \mathrm{Tr}[(D+\Sigma(-\overline{\nabla}^2))^{-1}\Sigma(-\overline{\nabla}^2)] \qquad (10)$$

$$= -\frac{1}{2}N_c \mathrm{tr}[(-i k\!\!\!/ + D + \Sigma((k+i\overline{\nabla})^2))^{-1}\Sigma((k+i\overline{\nabla})^2)] \qquad (11)$$

$$= -\frac{1}{2}N_c\int d^4x\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\Bigg[(\Sigma_k X + i k\!\!\!/ X)\sum_{n=0}^{\infty}(-1)^n \times[(D+\Sigma_1)(\Sigma_k X + i k\!\!\!/ X)]^n\Sigma((k+i\overline{\nabla})^2)\Bigg], \qquad (12)$$

where Tr includes the traces over coordinate space, spinor space, and flavor space, tr includes only the traces over spinor and flavor spaces, $D \equiv \overline{\nabla}\!\!\!/ - i a\!\!\!/_\Omega\gamma_5 - s_\Omega + i p_\Omega\gamma_5$, and $\Sigma_1 \equiv \Sigma((k+i\overline{\nabla})^2) - \Sigma(k^2)$.

To a given order [$O(p^2)$, $O(p^4)$, or $O(p^6)$], after the tracing over the spinor space, the series is

$$\Delta S_{\text{eff}}^{2n} = \int d^4x \sum_{k=1}^{m} a_k \langle O_{\Omega,k}\rangle, \qquad (13)$$

where $a_k$ are coefficients, $O_{\Omega,k}$ are monomials with flavor indices including $\overline{\nabla}^\mu$, $a_\Omega^\mu$, $s_\Omega$ $p_\Omega$, and $\langle\ldots\rangle$ represents trace over flavor space. In (13), we have used the basic relations to simplify the results, including trace relations and the Einstein summation convention and, for the anomalous terms, also including the Schouten identity. Nevertheless, not all of the $O_{\Omega,k}$'s are independent.

Generally, the number of $O_{\Omega,k}$'s is larger than the number of linear independent terms $O_l$. Specifically, the relationship between the two is given by

$$\langle O_l\rangle = \sum_{k=1}^{m} A_{lk}\langle O_{\Omega,k}\rangle, \qquad l = 1,2,3,\ldots,M, \qquad (14)$$

TABLE I. The $p^4$ order LECs, $L_i$ for three-flavor quarks and $\bar{l}_i, l_7$ for two-flavor quarks. $\Lambda_{QCD}$ is in units of MeV, and $L_1, \ldots, L_{10}, l_7$ are in units $10^{-3}$. [18] displays our old results; [2,3] provide the first results from experimental data; [26] gives the LECs from resonance chiral theory; [17] gives the LECs from a class of holographic theories; [15] collects the last lattice results in [27–29]; [20] gives the $L_1, \ldots, L_8$ from the global fit of the $O(p^6)$ LECs; and $L_9, L_{10}$ are in [30–33]. [34] gives some newer fit data at $O(p^6)$ order.

| $N_f = 3$ | $\Lambda_{QCD}$ | $L_1$ | $L_2$ | $L_3$ | $L_4$ | $L_5$ | $L_6$ | $L_7$ | $L_8$ | $L_9$ | $L_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| New | $453^{-6}_{+12}$ | $0.92^{+0.03}_{-0.04}$ | $1.84^{+0.05}_{-0.08}$ | $-4.94^{-0.14}_{+0.21}$ | $0^{+0}_{-0}$ | $1.26^{+0.01}_{-0.06}$ | $0^{+0}_{-0}$ | $-0.42^{+0.04}_{-0.05}$ | $0.84^{-0.05}_{+0.04}$ | $6.53^{+0.24}_{-0.37}$ | $-5.43^{-0.29}_{+0.44}$ |
| Old [18] | $453^{-6}_{+12}$ | $1.23^{+0.03}_{-0.04}$ | $2.46^{+0.05}_{-0.08}$ | $-6.85^{-0.15}_{+0.21}$ | $0^{+0}_{-0}$ | $1.48^{-0.01}_{-0.03}$ | $0^{+0}_{-0}$ | $-0.51^{+0.05}_{-0.06}$ | $1.02^{-0.06}_{+0.06}$ | $8.86^{+0.24}_{-0.37}$ | $-7.40^{-0.29}_{+0.44}$ |
| Ref. [3] | | $0.9 \pm 0.3$ | $1.7 \pm 0.7$ | $-4.4 \pm 2.5$ | $0 \pm 0.5$ | $2.2 \pm 0.5$ | $0 \pm 0.3$ | $-0.4 \pm 0.15$ | $1.1 \pm 0.3$ | $7.4 \pm 0.7$ | $-6.0 \pm 0.7$ |
| Ref. [26] | | $0.4 \pm 0.3$ | $1.4 \pm 0.3$ | $-3.5 \pm 1.1$ | $-0.3 \pm 0.5$ | $1.4 \pm 0.5$ | $-0.2 \pm 0.3$ | $-0.4 \pm 0.2$ | $0.9 \pm 0.3$ | $6.9 \pm 0.7$ | $-5.5 \pm 0.7$ |
| Ref. [17] | | 0.5 | 1.0 | $-3.2$ | | | | | | 6.3 | $-6.3$ |
| Ref. [15] | | | | | $0.09 \pm 0.34$ | $1.19 \pm 0.25$ | $0.16 \pm 0.20$ | | $0.55 \pm 0.15$ | $3.08 \pm 0.23 \pm 0.51$ | $-5.7 \pm 1.1 \pm 0.7$ |
| Ref. [20] | | $0.53 \pm 0.06$ | $0.81 \pm 0.04$ | $-3.07 \pm 0.20$ | $\equiv 0.3$ | $1.01 \pm 0.06$ | $0.14 \pm 0.05$ | $-0.34 \pm 0.09$ | $0.47 \pm 0.10$ | $5.93 \pm 0.43$ | $-3.8 \pm 0.4$ |
| Ref. [34] | | $0.69 \pm 0.18$ | $0.63 \pm 0.13$ | $-2.63 \pm 0.46$ | | | | | | | |
| $N_f = 2$ | $\Lambda_{QCD}$ | $\bar{l}_1$ | $\bar{l}_2$ | $\bar{l}_3$ | $\bar{l}_4$ | $\bar{l}_5$ | $\bar{l}_6$ | $l_7$ | | | |
| New | $465^{-6}_{+12}$ | $-2.33^{-0.17}_{+0.24}$ | $6.85^{+0.09}_{-0.14}$ | $2.33^{+0.28}_{-0.36}$ | $4.24^{+0.00}_{-0.04}$ | $13.55^{+0.53}_{-0.80}$ | $15.60^{+0.44}_{-0.67}$ | $3.61^{-0.55}_{+0.80}$ | | | |
| Old [18] | $465^{-6}_{+12}$ | $-4.77^{-0.17}_{+0.24}$ | $8.01^{+0.09}_{-0.14}$ | $1.97^{+0.29}_{-0.35}$ | $4.34^{-0.01}_{-0.02}$ | $17.35^{+0.53}_{-0.80}$ | $19.98^{+0.44}_{-0.67}$ | $4.18^{-0.65}_{+0.97}$ [a] | | | |
| Ref. [2] | | $-2.3 \pm 3.7$ | $6.0 \pm 1.3$ | $2.9 \pm 2.4$ | $4.3 \pm 0.9$ | $13.9 \pm 1.3$ | $16.5 \pm 1.1$ | $O(5)$ | | | |
| Ref. [20] | | $-0.4 \pm 0.6$ | $4.3 \pm 0.1$ | $2.9 \pm 2.4$ | $4.4 \pm 0.2$ | $12.24 \pm 0.21$ | $16.0 \pm 0.5 \pm 0.7$ | | | | |

[a]There exists a mistake in [18] for $l_7$. This is a correction.

where $M$ is the number of LECs up to a given order.[1] This implies that, generally, $A_{lk}$ is not a square matrix. The reduced row echelon form of $A_{lk}$ is

$$A_{lk} \to B_{lk} = \begin{pmatrix} 1 & C_{12} & O & C_{14} & O & \cdots & \cdots & \cdots \\ & & 1 & C_{24} & O & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & 1 & \cdots & \cdots & \cdots \end{pmatrix}, \tag{15}$$

where the bottom-left corner contains zero elements, with $O$ representing a zero matrix of the appropriate dimension $C$, representing possibly a nonzero matrix, and $\cdots$ also indicating nonzero matrices. The rank of $A_{lk}$ or $B_{lk}$ is equal to the number of independent linear bases with each nonzero row vector in $B_{lk}$ corresponding to a linear basis in $\langle O_\Omega \rangle$. We select those $O_{\Omega,k'}$'s that are independent and set $B_{k'k'} = 1$ and $B_{k'k'}$ to be the first nonzero elements in the $k'$th row in $B_{lk}$. All dependent $O_{\Omega,k}$'s can be replaced by $O_{\Omega,k'}$.

Without using the Cayley-Hamilton relations, the LECs $K_l$ for arbitrary $N_f$ flavors are defined as

$$S^{2n}_{\rm eff} = \int d^4x \sum_{l=1}^{M} K_l \langle O_l \rangle = \int d^4x \sum_{l=1}^{M} \sum_{k=1}^{m} K_l A_{lk} \langle O_{\Omega,k} \rangle. \tag{16}$$

In the second equation, we have used (14). Comparing (13) and (16) because all the relations in $N_f$ flavors have been used, the coefficients in front of $\langle O_{\Omega,k} \rangle$ need to be equal:

$$a_k = \sum_{l=1}^{M} K_l A_{lk}, \qquad k = 1, 2, 3, \ldots m. \tag{17}$$

In (17), there are $m$ equations in $M$ variables with $m > M$. We select the $M$ independent terms $O_{\Omega,k'}$ (with coefficients $a_{k'}$) to solve (17). Hence, the additional LECs are

$$K_l = \sum_{k'} a_{k'} A^{-1}_{k'l}. \tag{18}$$

Replacing $K_l$ in (17), the equations relating to $a_{k'}$ are obviously satisfied because of (18). The other $m - M$ equations relating to $a_k (k \neq k')$ must be held, too. In general, all $a_k$'s in (13) are obtained from (12) by different terms. A small mistake would destroy one of the $m - M$ equations. In the following, we will treat these $m - M$ equations as constraint conditions to check our results.

[1]The building blocks of $O_l$ and $O_\Omega$ are different. Their relations can be found in Table XV in Ref. [18].

TABLE II. $\tilde{Y}_n$ and their relations to $Y_i$. The definitions of the symbols in this table are the same as those in [5].

| $n$ | $\tilde{Y}_n$ | Relations | $n$ | $\tilde{Y}_i$ | Relations |
|---|---|---|---|---|---|
| 1 | $\langle u^\mu u_\mu h^{\nu\lambda} h_{\nu\lambda}\rangle$ | $Y_1$ | 35 | $\langle i f_+^{\mu\nu} u^\lambda u_\mu u_\lambda u_\nu + i f_+^{\mu\nu} u_\mu u^\lambda u_\nu u_\lambda\rangle$ | $Y_{68}$ |
| 2 | $\langle h^{\mu\nu} u^\lambda h_{\mu\nu} u_\lambda\rangle$ | $Y_3$ | 36 | $\langle u^\mu u_\mu f_+^{\nu\lambda} f_{+\nu\lambda}\rangle$ | $Y_{71}$ |
| 3 | $\langle h^{\mu\nu}u^\lambda h_{\mu\lambda}u_\nu + h^{\mu\nu}u_\nu h_\mu{}^\lambda u_\lambda\rangle$ | $Y_5$ | 37 | $\langle f_+^{\mu\nu} u^\lambda f_{+\mu\nu} u_\lambda\rangle$ | $Y_{73}$ |
| 4 | $\langle u^\mu u_\mu u^\nu u_\nu \chi_+\rangle$ | $Y_7$ | 38 | $\langle f_+^{\mu\nu} f_{+\mu}{}^\lambda u_\nu u_\lambda\rangle$ | $Y_{75}$ |
| 5 | $\langle u^\mu u_\mu u^\nu \chi_+ u_\nu\rangle$ | $Y_{11}$ | 39 | $\langle f_+^{\mu\nu} f_{+\mu}{}^\lambda u_\lambda u_\nu\rangle$ | $Y_{76}$ |
| 6 | $\langle \chi_+ u^\mu u^\nu u_\mu u_\nu\rangle$ | $Y_{13}$ | 40 | $\langle f_+^{\mu\nu}u^\lambda f_{+\mu\lambda}u_\nu + f_+^{\mu\nu} u_\nu f_{+\mu}{}^\lambda u_\lambda\rangle$ | $Y_{78}$ |
| 7 | $\langle\chi_+ h^{\mu\nu}h_{\mu\nu}\rangle$ | $Y_{17}$ | 41 | $\langle \chi_+ f_+^{\mu\nu} f_{+\mu\nu}\rangle$ | $Y_{81}$ |
| 8 | $\langle u^\mu u_\mu \chi_+\chi_+\rangle$ | $Y_{19}$ | 42 | $\langle i f_+^{\mu\nu}\chi_+ u_\mu u_\nu + i f_+^{\mu\nu} u_\mu u_\nu \chi_+\rangle$ | $Y_{83}$ |
| 9 | $\langle \chi_+ u^\mu \chi_+ u_\mu\rangle$ | $Y_{23}$ | 43 | $\langle i f_+^{\mu\nu} u_\mu \chi_+ u_\nu\rangle$ | $Y_{85}$ |
| 10 | $\langle\chi_+\chi_+\chi_+\rangle$ | $Y_{25}$ | 44 | $\langle f_-^{\mu\nu} h_\nu{}^\lambda u_\lambda u_\mu + f_-^{\mu\nu} u_\mu u^\lambda h_{\nu\lambda}\rangle$ | $Y_{86}$ |
| 11 | $\langle i\chi_- h^{\mu\nu}u_\mu u_\nu + i\chi_- u^\mu u^\nu h_{\mu\nu}\rangle$ | $Y_{28}$ | 45 | $\langle f_-^{\mu\nu} u_\mu h_\nu{}^\lambda u_\lambda + f_-^{\mu\nu} u^\lambda h_{\nu\lambda} u_\mu\rangle$ | $Y_{89}$ |
| 12 | $\langle h^\lambda{}_\lambda h^{\mu\nu} u_\mu u_\nu + h^\lambda{}_\lambda u^\mu u^\nu h_{\mu\nu}\rangle$ | $Y_{28} - \frac{2}{N_f} Y_{30}$ | 46 | $\langle u^\mu u_\mu f_-^{\nu\lambda} f_{-\nu\lambda}\rangle$ | $Y_{90}$ |
| 13 | $\langle i h^{\mu\nu} u_\mu \chi_- u_\nu\rangle$ | $Y_{31}$ | 47 | $\langle f_-^{\mu\nu}u^\lambda f_{-\mu\nu}u_\lambda\rangle$ | $Y_{92}$ |
| 14 | $\langle h^{\mu\nu} u_\mu h^\lambda{}_\lambda u_\nu\rangle$ | $Y_{31} - \frac{1}{N_f}Y_{30}$ | 48 | $\langle f_-^{\mu\nu} f_{-\mu}{}^\lambda u_\nu u_\lambda\rangle$ | $Y_{94}$ |
| 15 | $\langle u^\mu u_\mu \chi_-\chi_-\rangle$ | $Y_{33}$ | 49 | $\langle f_-^{\mu\nu} f_{-\mu}{}^\lambda u_\lambda u_\nu\rangle$ | $Y_{95}$ |
| 16 | $\langle i u^\mu u_\mu h^\nu{}_\nu \chi_- + i u^\mu u_\mu \chi_- h^\nu{}_\nu\rangle$ | $-2Y_{33} + \frac{2}{N_f}Y_{34}$ | 50 | $\langle f_-^{\mu\nu}u^\lambda f_{-\mu\lambda}u_\nu + f_-^{\mu\nu}u_\nu f_{-\mu}{}^\lambda u_\lambda\rangle$ | $Y_{97}$ |
| 17 | $\langle u^\mu u_\mu h^\nu{}_\nu h^\lambda{}_\lambda\rangle$ | $-Y_{33} + \frac{2}{N_f}Y_{34} - \frac{1}{N_f^2}Y_{36}$ | 51 | $\langle i f_+^{\mu\nu} f_{-\nu}{}^\lambda h_{\mu\lambda} - i f_+^{\mu\nu} h_\mu{}^\lambda f_{-\nu\lambda}\rangle$ | $Y_{100}$ |
| 18 | $\langle u^\mu \chi_- u_\mu \chi_-\rangle$ | $Y_{37}$ | 52 | $\langle i f_+^{\mu\nu} f_{-\nu}{}^\lambda f_{-\mu\lambda} - i f_+^{\mu\nu} f_{-\mu}{}^\lambda f_{-\nu\lambda}\rangle$ | $Y_{101}$ |
| 19 | $\langle i u^\mu h^\nu{}_\nu u_\mu \chi_-\rangle$ | $-Y_{37} + \frac{1}{N_f}Y_{34}$ | 53 | $\langle\chi_+ f_-^{\mu\nu} f_{-\mu\nu}\rangle$ | $Y_{102}$ |
| 20 | $\langle u^\mu h^\nu{}_\nu u_\mu h^\lambda{}_\lambda\rangle$ | $-Y_{37} + \frac{2}{N_f}Y_{34} - \frac{1}{N_f^2}Y_{36}$ | 54 | $\langle f_+^{\mu\nu} f_{-\mu\nu}\chi_- - f_+^{\mu\nu}\chi_- f_{-\mu\nu}\rangle$ | $Y_{104}$ |
| 21 | $\langle\chi_-\chi_-\chi_+\rangle$ | $Y_{39}$ | 55 | $\langle i f_+^{\mu\nu} f_{-\mu\nu} h^\lambda{}_\lambda - i f_+^{\mu\nu} h^\lambda{}_\lambda f_{-\mu\nu}\rangle$ | $-Y_{104}$ |
| 22 | $\langle i h^\mu{}_\mu \chi_-\chi_+ + i h^\mu{}_\mu \chi_+\chi_-\rangle$ | $-2Y_{39} + \frac{2}{N_f}Y_{41}$ | 56 | $\langle i f_-^{\mu\nu}\chi_- u_\mu u_\nu - i f_-^{\mu\nu} u_\mu u_\nu \chi_-\rangle$ | $Y_{105}$ |
| 23 | $\langle h^\mu{}_\mu h^\nu{}_\nu \chi_+\rangle$ | $-Y_{39} + \frac{2}{N_f}Y_{41} - \frac{1}{N_f^2}Y_{42}$ | 57 | $\langle f_-^{\mu\nu} h^\lambda{}_\lambda u_\mu u_\nu - f_-^{\mu\nu} u_\mu u_\nu h^\lambda{}_\lambda\rangle$ | $Y_{105}$ |
| 24 | $\langle i\chi_- \chi_+^\mu u_\mu + i\chi_- u^\mu \chi_{+\mu}\rangle$ | $Y_{43}$ | 58 | $\langle f_-^{\mu\nu}\chi_{+\mu}u_\nu + f_-^{\mu\nu} u_\nu \chi_{+\mu}\rangle$ | $Y_{107}$ |
| 25 | $\langle h^\nu{}_\nu \chi_+^\mu u_\mu + h^\nu{}_\nu u^\mu \chi_{+\mu}\rangle$ | $Y_{43} - \frac{2}{N_f}Y_{44}$ | 59 | $\langle\nabla^\mu f_-^{\nu\lambda}\nabla_\mu f_{-\nu\lambda}\rangle$ | $Y_{109}$ |
| 26 | $\langle\chi_+^\mu\chi_{+\mu}\rangle$ | $Y_{47}$ | 60 | $\langle i\nabla^\lambda f_+^{\mu\nu} h_{\mu\lambda} u_\nu - i\nabla^\lambda f_+^{\mu\nu} u_\nu h_{\mu\lambda}\rangle$ | $Y_{110}$ |
| 27 | $\langle u^\mu u_\mu u^\nu u_\nu u^\lambda u_\lambda\rangle$ | $Y_{49}$ | 61 | $\langle i\nabla^\mu f_{+\mu}{}^\nu f_{-\nu}{}^\lambda u_\lambda - i\nabla^\mu f_{+\mu}{}^\nu u^\lambda f_{-\nu\lambda}\rangle$ | $Y_{111}$ |
| 28 | $\langle u^\mu u_\mu u^\nu u^\lambda u_\lambda u_\nu\rangle$ | $Y_{52}$ | 62 | $\langle i\nabla^\mu f_{+\mu}{}^\nu h_\nu{}^\lambda u_\lambda - i\nabla^\mu f_{+\mu}{}^\nu u^\lambda h_{\nu\lambda}\rangle$ | $Y_{112}$ |
| 29 | $\langle u^\mu u_\mu u^\nu u^\lambda u_\nu u_\lambda\rangle$ | $Y_{54}$ | 63 | $\langle i\chi_-^\mu \nabla_\mu h^\nu{}_\nu\rangle$ | $Z_1$ [a] |
| 30 | $\langle u^\mu u^\nu u^\lambda u_\mu u_\nu u_\lambda\rangle$ | $Y_{58}$ | 64 | $\langle\nabla^\mu h^\nu{}_\nu \nabla_\mu h^\lambda{}_\lambda\rangle$ | $Z_2$ [b] |
| 31 | $\langle u^\mu u^\nu u^\lambda u_\mu u_\lambda u_\nu\rangle$ | $Y_{60}$ | 65 | $\langle f_+^{\mu\nu} u_\mu \chi_{-\nu} + f_+^{\mu\nu}\chi_{-\mu} u_\nu\rangle$ | $Z_3$ [c] |
| 32 | $\langle i f_+^{\mu\nu} u^\lambda u_\lambda u_\mu u_\nu + i f_+^{\mu\nu} u_\mu u_\nu u^\lambda u_\lambda\rangle$ | $Y_{64}$ | 66 | $\langle\chi_-^\mu\chi_{-\mu}\rangle$ | (2.15) in [5] |
| 33 | $\langle i f_+^{\mu\nu} u^\lambda u_\mu u_\nu u_\lambda\rangle$ | $Y_{66}$ | 67 | $\langle i f_+^{\mu\nu} f_{+\nu}{}^\lambda f_{+\mu\lambda}\rangle$ | (2.15) in [5] |
| 34 | $\langle i f_+^{\mu\nu} u_\mu u^\lambda u_\lambda u_\nu\rangle$ | $Y_{67}$ | 68 | $\langle\nabla^\mu f_+^{\nu\lambda}\nabla_\mu f_{+\nu\lambda}\rangle$ | (2.15) in [5] |

[a] $Z_1 = -\frac{1}{2}Y_{19} - \frac{1}{2}Y_{23} + \frac{1}{N_f}Y_{24} - \frac{1}{2}Y_{33} + \frac{1}{N_f}Y_{34} - \frac{1}{2}Y_{37} - \frac{1}{2}Y_{39} + \frac{1}{N_f}Y_{41} - \frac{1}{2N_f^2}Y_{42} + \frac{1}{2}Y_{43} - \frac{1}{N_f}Y_{44} + \frac{1}{N_f}Y_{46} - Y_{47} + 4Y_{113}$.

[b] $Z_2 = -\frac{1}{2}Y_{19} - \frac{1}{2}Y_{23} + \frac{1}{N_f}Y_{24} - Y_{33} + \frac{2}{N_f}Y_{34} - Y_{37} - Y_{39} + \frac{2}{N_f}Y_{41} - \frac{1}{N_f^2}Y_{42} + Y_{43} - \frac{2}{N_f}Y_{44} + \frac{1}{N_f}Y_{46} - Y_{47} + 4Y_{113}$.

[c] $Z_3 = -Y_{66} - Y_{67} + Y_{68} + \frac{1}{2}Y_{71} - \frac{1}{2}Y_{73} + Y_{75} - 2Y_{76} + \frac{1}{2}Y_{78} - \frac{1}{2}Y_{83} - Y_{85} + \frac{1}{2}Y_{90} - \frac{1}{2}Y_{92} - Y_{94} + \frac{1}{2}Y_{97} - \frac{1}{2}Y_{100} + \frac{1}{2}Y_{101} - \frac{1}{4}Y_{104} + Y_{110} + Y_{112}$.

TABLE III. The $p^6$ order LECs. $C_i$ for three flavors and $c_j$ for two flavors. Old ones are our old results from [18]. They are in units of $10^{-3}$ GeV$^{-2}$. The value $\equiv 0$ means that the LECs vanish at the large-$N_C$ limit.

| $i$ | New $C_i$ | Old $C_i$ | $j$ | New $c_j$ | Old $c_j$ | $i$ | New $C_i$ | Old $C_i$ | $j$ | New $c_j$ | Old $c_j$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $2.98^{+0.07}_{-0.13}$ | $3.79^{+0.10}_{-0.17}$ | 1 | $3.04^{+0.07}_{-0.12}$ | $3.58^{+0.09}_{-0.15}$ | 46 | $-1.67^{-0.05}_{+0.09}$ | $-0.60^{-0.02}_{+0.04}$ | 26 | $-4.82^{-0.15}_{+0.24}$ | $-1.14^{-0.05}_{+0.07}$ |
| 2 | $\equiv 0$ | $\equiv 0$ | | | | 47 | $3.10^{+0.09}_{-0.15}$ | $0.08^{+0.01}_{-0.00}$ | | | |
| 3 | $-0.05^{+0.01}_{-0.01}$ | $-0.05^{+0.01}_{-0.01}$ | 2 | $-0.09^{+0.01}_{-0.01}$ | $-0.03^{+0.01}_{-0.01}$ | 48 | $4.74^{+0.10}_{-0.17}$ | $3.41^{+0.06}_{-0.10}$ | | | |
| 4 | $2.13^{+0.06}_{-0.10}$ | $3.10^{+0.09}_{-0.15}$ | 3 | $2.15^{+0.06}_{-0.10}$ | $2.89^{+0.08}_{-0.13}$ | 49 | $\equiv 0$ | $\equiv 0$ | | | |
| 5 | $-1.28^{+0.10}_{-0.13}$ | $-1.01^{+0.08}_{-0.11}$ | 4 | $0.82^{-0.04}_{+0.03}$ | $1.21^{-0.07}_{+0.06}$ | 50 | $10.54^{+0.89}_{-1.29}$ | $8.71^{+0.78}_{-1.12}$ | 27 | $16.70^{+1.61}_{-2.30}$ | $13.57^{+1.41}_{-2.00}$ |
| 6 | $\equiv 0$ | $\equiv 0$ | | | | 51 | $-9.83^{+0.28}_{-0.24}$ | $-11.49^{+0.18}_{-0.09}$ | 28 | $5.51^{+1.22}_{-1.62}$ | $0.93^{+0.98}_{-1.25}$ |
| 7 | $\equiv 0$ | $\equiv 0$ | | | | 52 | $-6.39^{-0.77}_{+1.07}$ | $-5.04^{-0.67}_{+0.93}$ | | | |
| 8 | $2.10^{-0.15}_{+0.16}$ | $2.31^{-0.16}_{+0.18}$ | | | | 53 | $-5.36^{-0.71}_{+1.05}$ | $-11.98^{-0.87}_{+1.33}$ | 29 | $-4.77^{-0.66}_{+0.98}$ | $-11.01^{-0.81}_{+1.23}$ |
| 9 | $\equiv 0$ | $\equiv 0$ | | | | 54 | $\equiv 0$ | $\equiv 0$ | | | |
| 10 | $-0.65^{+0.05}_{-0.06}$ | $-1.05^{+0.08}_{-0.09}$ | 5 | $-0.68^{+0.05}_{-0.06}$ | $-0.98^{+0.07}_{-0.09}$ | 55 | $10.45^{+0.80}_{-1.22}$ | $16.79^{+0.96}_{-1.49}$ | 30 | $9.86^{+0.75}_{-1.14}$ | $15.72^{+0.89}_{-1.38}$ |
| 11 | $\equiv 0$ | $\equiv 0$ | | | | 56 | $4.45^{+0.03}_{-0.18}$ | $19.34^{+0.52}_{-0.98}$ | 31 | $3.27^{-0.04}_{-0.07}$ | $17.57^{+0.42}_{-0.82}$ |
| 12 | $-0.34^{+0.01}_{-0.01}$ | $-0.34^{+0.02}_{-0.01}$ | 6 | $-0.35^{+0.02}_{-0.01}$ | $-0.33^{+0.01}_{-0.01}$ | 57 | $4.72^{+1.36}_{-1.85}$ | $7.92^{+1.34}_{-1.85}$ | 32 | $4.24^{+1.32}_{-1.78}$ | $7.18^{+1.28}_{-1.76}$ |
| 13 | $\equiv 0$ | $\equiv 0$ | | | | 58 | $\equiv 0$ | $\equiv 0$ | | | |
| 14 | $-0.87^{+0.14}_{-0.21}$ | $-0.83^{+0.12}_{-0.19}$ | 7 | $-1.83^{+0.25}_{-0.35}$ | $-1.72^{+0.25}_{-0.35}$ | 59 | $-14.59^{-1.01}_{+1.55}$ | $-22.49^{-1.21}_{+1.89}$ | 33 | $-13.69^{-0.94}_{+1.44}$ | $-21.19^{-1.12}_{+1.76}$ |
| 15 | $\equiv 0$ | $\equiv 0$ | 8 | $0.91^{-0.11}_{+0.13}$ | $0.86^{-0.12}_{+0.15}$ | 60 | $\equiv 0$ | $\equiv 0$ | | | |
| 16 | $\equiv 0$ | $\equiv 0$ | | | | 61 | $2.42^{-0.19}_{+0.22}$ | $2.88^{-0.22}_{+0.26}$ | 34 | $2.40^{-0.19}_{+0.22}$ | $2.84^{-0.22}_{+0.26}$ |
| 17 | $0.17^{+0.01}_{-0.04}$ | $0.01^{-0.01}_{-0.01}$ | 9 | $-0.74^{+0.13}_{-0.18}$ | $-0.84^{+0.12}_{-0.17}$ | 62 | $\equiv 0$ | $\equiv 0$ | | | |
| 18 | $-0.60^{+0.07}_{-0.09}$ | $-0.56^{+0.09}_{-0.11}$ | | | | 63 | $2.48^{-0.21}_{+0.25}$ | $2.99^{-0.24}_{+0.30}$ | | | |
| 19 | $-0.27^{+0.09}_{-0.13}$ | $-0.48^{+0.09}_{-0.13}$ | 10 | $-0.22^{+0.07}_{-0.11}$ | $-0.37^{+0.07}_{-0.10}$ | 64 | $\equiv 0$ | $\equiv 0$ | | | |
| 20 | $0.17^{-0.02}_{+0.03}$ | $0.18^{-0.03}_{+0.04}$ | 11 | $\equiv 0$ | $\equiv 0$ | 65 | $-2.82^{+0.18}_{-0.20}$ | $-2.43^{+0.15}_{-0.16}$ | 35 | $2.16^{-0.23}_{+0.31}$ | $3.39^{-0.32}_{+0.41}$ |
| 21 | $-0.06^{+0.01}_{-0.01}$ | $-0.06^{+0.01}_{-0.01}$ | | | | 66 | $0.80^{+0.04}_{-0.07}$ | $1.71^{+0.07}_{-0.12}$ | 36 | $0.80^{+0.04}_{-0.07}$ | $1.57^{+0.06}_{-0.10}$ |
| 22 | $-0.35^{+0.20}_{-0.26}$ | $0.27^{+0.19}_{-0.25}$ | 12 | $-0.41^{+0.20}_{-0.26}$ | $0.15^{+0.18}_{-0.24}$ | 67 | $\equiv 0$ | $\equiv 0$ | | | |
| 23 | $\equiv 0$ | $\equiv 0$ | | | | 68 | $\equiv 0$ | $\equiv 0$ | | | |
| 24 | $0.87^{+0.02}_{-0.04}$ | $1.62^{+0.04}_{-0.07}$ | | | | 69 | $0.52^{+0.00}_{-0.01}$ | $-0.86^{-0.04}_{+0.06}$ | 38 | $0.60^{+0.00}_{-0.01}$ | $-0.68^{-0.03}_{+0.05}$ |
| 25 | $-3.03^{-0.41}_{+0.59}$ | $-5.98^{-0.49}_{+0.72}$ | 13 | $-3.02^{-0.39}_{+0.56}$ | $-5.39^{-0.45}_{+0.66}$ | 70 | $1.66^{-0.11}_{+0.11}$ | $1.73^{-0.08}_{+0.07}$ | 39 | $1.53^{-0.12}_{+0.12}$ | $1.81^{-0.08}_{+0.07}$ |
| 26 | $2.71^{+0.35}_{-0.54}$ | $3.35^{+0.29}_{-0.47}$ | 14 | $3.39^{+0.36}_{-0.56}$ | $4.17^{+0.30}_{-0.49}$ | 71 | $\equiv 0$ | $\equiv 0$ | | | |
| 27 | $-1.35^{+0.13}_{-0.15}$ | $-1.54^{+0.15}_{-0.18}$ | 15 | $-2.39^{+0.19}_{-0.23}$ | $-2.71^{+0.21}_{-0.25}$ | 72 | $-1.80^{+0.12}_{-0.11}$ | $-3.30^{+0.05}_{-0.00}$ | 40 | $-1.64^{+0.13}_{-0.13}$ | $-3.17^{+0.05}_{-0.02}$ |
| 28 | $0.18^{+0.00}_{-0.01}$ | $0.30^{+0.01}_{-0.01}$ | | | | 73 | $0.15^{+0.48}_{-0.62}$ | $0.50^{+0.43}_{-0.56}$ | 41 | $0.07^{+0.47}_{-0.61}$ | $0.30^{+0.42}_{-0.54}$ |
| 29 | $-0.99^{-0.21}_{+0.24}$ | $-3.08^{-0.26}_{+0.32}$ | 16 | $-0.60^{-0.19}_{+0.21}$ | $-2.22^{-0.22}_{+0.27}$ | 74 | $-3.34^{-0.11}_{+0.19}$ | $-5.07^{-0.16}_{+0.27}$ | 42 | $-3.26^{-0.10}_{+0.17}$ | $-4.74^{-0.14}_{+0.24}$ |
| 30 | $0.37^{+0.01}_{-0.02}$ | $0.60^{+0.02}_{-0.03}$ | | | | 75 | $\equiv 0$ | $\equiv 0$ | | | |
| 31 | $-0.46^{+0.07}_{-0.13}$ | $-0.63^{+0.05}_{-0.09}$ | 17 | $-0.92^{+0.15}_{-0.23}$ | $-1.10^{+0.12}_{-0.19}$ | 76 | $-1.15^{-0.26}_{+0.34}$ | $-1.44^{-0.23}_{+0.31}$ | 43 | $-1.11^{-0.25}_{+0.33}$ | $-1.29^{-0.23}_{+0.30}$ |
| 32 | $0.17^{-0.02}_{+0.03}$ | $0.18^{-0.03}_{+0.04}$ | 18 | $0.42^{-0.06}_{+0.08}$ | $0.43^{-0.07}_{+0.08}$ | 77 | $\equiv 0$ | $\equiv 0$ | | | |
| 33 | $-0.05^{-0.02}_{+0.05}$ | $0.09^{-0.00}_{+0.03}$ | 19 | $0.29^{-0.07}_{+0.13}$ | $0.41^{-0.06}_{+0.10}$ | 78 | $8.82^{+0.80}_{-1.21}$ | $17.51^{+1.02}_{-1.59}$ | 44 | $8.19^{+0.74}_{-1.12}$ | $16.16^{+0.94}_{-1.45}$ |
| 34 | $0.66^{-0.18}_{+0.29}$ | $1.59^{-0.10}_{+0.17}$ | 20 | $0.74^{-0.18}_{+0.28}$ | $1.56^{-0.10}_{+0.17}$ | 79 | $5.86^{-0.14}_{+0.12}$ | $-0.56^{-0.30}_{+0.40}$ | 45 | $6.09^{-0.13}_{+0.11}$ | $0.26^{-0.26}_{+0.34}$ |
| 35 | $0.10^{-0.09}_{+0.12}$ | $0.17^{-0.12}_{+0.17}$ | 21 | $0.22^{-0.14}_{+0.19}$ | $0.29^{-0.18}_{+0.24}$ | 80 | $1.01^{-0.05}_{+0.04}$ | $0.87^{-0.04}_{+0.03}$ | 46 | $1.09^{-0.05}_{+0.05}$ | $0.85^{-0.04}_{+0.02}$ |
| 36 | $\equiv 0$ | $\equiv 0$ | | | | 81 | $\equiv 0$ | $\equiv 0$ | | | |
| 37 | $-0.60^{+0.07}_{-0.09}$ | $-0.56^{+0.09}_{-0.11}$ | | | | 82 | $-4.58^{-0.24}_{+0.38}$ | $-7.13^{-0.32}_{+0.51}$ | 47 | $-4.26^{-0.22}_{+0.35}$ | $-6.73^{-0.29}_{+0.47}$ |
| 38 | $0.47^{-0.04}_{+0.02}$ | $0.41^{-0.08}_{+0.07}$ | 22 | $-1.27^{+0.19}_{-0.26}$ | $-1.32^{+0.18}_{-0.25}$ | 83 | $-1.74^{+0.17}_{-0.22}$ | $0.07^{+0.20}_{-0.27}$ | 48 | $-1.71^{+0.17}_{-0.21}$ | $-0.22^{+0.18}_{-0.25}$ |
| 39 | $\equiv 0$ | $\equiv 0$ | 23 | $0.91^{-0.11}_{+0.13}$ | $0.86^{-0.12}_{+0.15}$ | 84 | $\equiv 0$ | $\equiv 0$ | | | |
| 40 | $-4.98^{-0.14}_{+0.25}$ | $-6.35^{-0.18}_{+0.32}$ | 24 | $0.00^{-0.01}_{+0.03}$ | $-4.84^{-0.14}_{+0.25}$ | 85 | $-0.96^{+0.04}_{-0.03}$ | $-0.82^{+0.03}_{-0.02}$ | 49 | $-1.07^{+0.05}_{-0.04}$ | $-0.78^{+0.03}_{-0.01}$ |
| 41 | $\equiv 0$ | $\equiv 0$ | | | | 86 | $\equiv 0$ | $\equiv 0$ | | | |
| 42 | $1.88^{+0.03}_{-0.06}$ | $0.60^{-0.00}_{+0.00}$ | | | | 87 | $4.79^{+0.29}_{-0.46}$ | $7.57^{+0.37}_{-0.60}$ | 50 | $4.35^{+0.26}_{-0.42}$ | $7.18^{+0.34}_{-0.55}$ |
| 43 | $\equiv 0$ | $\equiv 0$ | | | | 88 | $-1.69^{-0.68}_{+0.93}$ | $-5.47^{-0.73}_{+1.03}$ | 51 | $-1.36^{-0.65}_{+0.89}$ | $-4.85^{-0.69}_{+0.97}$ |
| 44 | $1.73^{+0.07}_{-0.14}$ | $6.32^{+0.20}_{-0.36}$ | 25 | $4.88^{+0.16}_{-0.27}$ | $6.03^{+0.19}_{-0.33}$ | 89 | $17.27^{+1.11}_{-1.77}$ | $34.74^{+1.61}_{-2.62}$ | 52 | $15.84^{+1.00}_{-1.60}$ | $32.19^{+1.46}_{-2.37}$ |
| 45 | $\equiv 0$ | $\equiv 0$ | | | | 90 | $2.32^{-0.44}_{+0.55}$ | $2.44^{-0.38}_{+0.46}$ | 53 | $2.28^{-0.44}_{+0.55}$ | $2.51^{-0.37}_{+0.46}$ |

TABLE IV. The nonzero values of the $p^6$ order anomalous LECs $C_i^W$ in three flavors. They are in units of $10^{-3}$ GeV$^{-2}$. The second column lists our old results [21]. The fourth column to the eighth column contains the results given in [39]: (I) ChPT, (II) VMD, (III) ChPT (extrapolation), (IV) CQM, and (V) CQM (extrapolation). The last two columns contain the results in different references [17,40–47].

| $n$ | New | Old [21] | (I) [39] | (II) [39] | (III) [39] | (IV) [39] | (V) [39] | [17] | [40–47] |
|---|---|---|---|---|---|---|---|---|---|
| 1 | $2.90^{+0.49}_{-0.69}$ | $4.97^{+0.55}_{-0.79}$ | | | | | | | |
| 2 | $-1.79^{+0.09}_{-0.11}$ | $-1.43^{+0.10}_{-0.12}$ | $-0.32\pm 10.4$ | | $0.78\pm 12.7$ | $4.96\pm 9.70$ | $-0.074\pm 13.3$ | | |
| 4 | $-1.89^{+0.19}_{-0.24}$ | $-0.96^{+0.22}_{-0.29}$ | $0.28\pm 9.19$ | | $0.67\pm 10.9$ | $6.32\pm 6.09$ | $-0.55\pm 9.05$ | | |
| 5 | $1.56^{+0.29}_{-0.41}$ | $3.26^{+0.34}_{-0.49}$ | $28.5\pm 28.83$ | | $9.38\pm 152.2$ | $33.05\pm 28.66$ | $34.51\pm 41.13$ | | |
| 6 | $0.72^{+0.02}_{-0.04}$ | $0.91^{+0.03}_{-0.04}$ | | | | | | | |
| 7 | $2.02^{-0.23}_{+0.29}$ | $1.68^{-0.24}_{+0.31}$ | $0.013\pm 1.17$ | | | $0.51\pm 0.06$ | | | $0.1\pm 1.2$ [40] |
| | | | $20.3\pm 18.7$ | | | | | | 1.0[a] |
| | | | | | | | | | $0.35\pm 0.07$ [41] |
| 8 | $0.52^{+0.01}_{-0.03}$ | $0.41^{+0.01}_{-0.02}$ | $0.76\pm 0.18$ | | | | | | $0.58\pm 0.20$ [44] |
| | | | | | | | | | 5.0[a] |
| 9 | $1.21^{-0.03}_{+0.02}$ | $1.15^{-0.03}_{+0.03}$ | | | | | | | |
| 10 | $-0.14^{-0.00}_{+0.01}$ | $-0.18^{-0.01}_{+0.01}$ | | | | | | | |
| 11 | $-1.39^{+0.07}_{-0.09}$ | $-1.15^{+0.08}_{-0.10}$ | $-6.37\pm 4.54$ | | | $-0.00143\pm 0.03$ | | | $0.68\pm 0.21$[b] |
| 12 | $-4.05^{-0.12}_{+0.20}$ | $-5.13^{-0.15}_{+0.25}$ | | | | | | −2.1 | |
| 13 | $-6.81^{-0.20}_{+0.33}$ | $-6.37^{-0.18}_{+0.31}$ | $-74.09\pm 55.89$ | −20.00 | $-8.44\pm 69.9$ | $14.15\pm 15.22$ | $-7.46\pm 19.62$ | −8.8 | |
| 14 | $-2.48^{-0.07}_{+0.12}$ | $-2.00^{-0.06}_{+0.10}$ | $29.99\pm 11.14$ | −6.01 | $0.72\pm 15.3$ | $10.23\pm 7.56$ | $-0.58\pm 9.77$ | −1.3 | |
| 15 | $2.35^{+0.07}_{-0.11}$ | $4.17^{+0.12}_{-0.20}$ | $-25.3\pm 23.93$ | 2.00 | $-3.10\pm 28.6$ | $19.70\pm 7.49$ | $8.89\pm 9.72$ | 4.4 | |
| 16 | $1.79^{+0.05}_{-0.09}$ | $3.58^{+0.10}_{-0.17}$ | | | | | | −0.2 | |
| 17 | $0.71^{+0.02}_{-0.03}$ | $1.98^{+0.06}_{-0.10}$ | | | | | | −0.1 | |
| 19 | $0.56^{+0.02}_{-0.03}$ | $0.29^{+0.01}_{-0.01}$ | | | | | | −7.0 | |
| 20 | $-1.02^{-0.03}_{+0.05}$ | $1.82^{+0.05}_{-0.09}$ | | | | | | −0.4 | |
| 21 | $3.11^{+0.09}_{-0.15}$ | $2.48^{+0.07}_{-0.12}$ | | | | | | 2.6 | |
| 22 | $4.72^{+0.14}_{-0.23}$ | $5.01^{+0.14}_{-0.24}$ | $6.52\pm 0.78$ | 8.01 | | $3.94\pm 0.43$ | | 7.9 | $5.4\pm 0.8$ [40] |
| | | | $5.07\pm 0.71$ | | | $3.94\pm 0.43$ | | | 6.71,6.21,4.45[c] |
| | | | | | | | | | 6.3 [46] |
| | | | | | | | | | $8.4\pm 0.9$ [47] |
| 23 | $2.80^{+0.08}_{-0.14}$ | $2.74^{+0.08}_{-0.13}$ | | | | | | 0.9 | |

[a] This result is just the absolute value given in [43].
[b] [40] gets the result from experimental data [42].
[c] These results are in [45] through different inputs.

Finally, using the Cayley-Hamilton relations, all of the LECs can be obtained for three and two flavors.

## IV. RESULTS

### A. $O(p^2)$ and $O(p^4)$ orders

In our computation, two $O(p^2)$ order LECs, $F_0^2$ and $F_0^2B_0$, are very special, in the sense that we can express them exactly in terms of quark self-energy $\Sigma$ (correspondingly, in ChPT they receive no loop corrections) [23]. For all of the other LECs, however, we can only get their approximate expressions. Therefore, to improve the precisions of the calculations, we need some additional corrections. This is due to the fact that in our earlier works [18,21,22,24], we always took the expressions

$$F_0^2 = N_C\int\frac{d^4k}{(2\pi)^4}e^{-\frac{k^2+\Sigma_k^2}{\Lambda^2}}(4\Sigma_k^2-2k^2\Sigma_k\Sigma_k')\left(X^2+\frac{X}{\Lambda^2}\right),$$

$$F_0^2B_0 = N_C\int\frac{d^4k}{(2\pi)^4}e^{-\frac{k^2+\Sigma_k^2}{\Lambda^2}}4\Sigma_kX,$$

$$X\equiv\frac{1}{k^2+\Sigma_k^2},\qquad \Sigma_k\equiv\Sigma(k^2),\tag{19}$$

TABLE V. The nonzero values of the $p^6$ order anomalous LECs $c_i^W$ in two flavors. They are in units of $10^{-3}$ $\mathrm{GeV}^{-2}$.

| | $c_1^W$ | $c_2^W$ | $c_3^W$ | $c_4^W$ | $c_5^W$ | $c_6^W$ | $c_7^W$ | $c_8^W$ | $c_9^W$ | $c_{10}^W$ | $c_{11}^W$ | $c_{13}^W$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| New | $-1.81^{+0.09}_{-0.11}$ | $-1.61^{+0.08}_{-0.09}$ | $3.61^{-0.19}_{+0.22}$ | $0.80^{-0.04}_{+0.04}$ | $-1.40^{+0.07}_{-0.09}$ | $0.52^{+0.26}_{-0.35}$ | $-0.41^{-0.01}_{+0.02}$ | $0.21^{+0.01}_{-0.01}$ | $6.49^{+0.18}_{-0.30}$ | $-6.15^{-0.17}_{+0.29}$ | $4.63^{+0.13}_{-0.22}$ | $-9.25^{-0.26}_{+0.43}$ |
| Old [21] | $-1.46^{+0.10}_{-0.12}$ | $-1.25^{+0.09}_{-0.11}$ | $2.96^{-0.20}_{+0.25}$ | $0.63^{-0.04}_{+0.05}$ | $-1.17^{+0.08}_{-0.10}$ | $0.77^{+0.26}_{-0.36}$ | $-0.04^{-0.00}_{+0.00}$ | $0.02^{+0.00}_{-0.00}$ | $8.19^{+0.23}_{-0.38}$ | $-8.73^{-0.24}_{+0.41}$ | $4.85^{+0.13}_{-0.23}$ | $-9.70^{-0.27}_{+0.45}$ |

which are set in Euclidean space-time. In this paper, we continue to use the above expressions for $F_0^2$ and $F_0^2B_0$, and we stick with them for both two and three flavors.

At the $O(p^4)$ order, the additional analytical results are

$$\begin{aligned}
\Delta C_2 &= \int \frac{d^4k}{(2\pi)^4}[-\Sigma_k'^2X + 11\Sigma_k^2\Sigma_k'^2X^2 - 34\Sigma_k^4\Sigma_k'^2X^3 - 2\Sigma_k^4X^4 + 40\Sigma_k^6\Sigma_k'^2X^4 + 4\Sigma_k^6X^5 - 16\Sigma_k^8\Sigma_k'^2X^5],\\
\Delta C_3 &= \int \frac{d^4k}{(2\pi)^4}\left[-\frac{1}{2}\Sigma_k'^2X + \frac{49}{6}\Sigma_k^2\Sigma_k'^2X^2 - \frac{2}{3}\Sigma_k^2X^3 - \frac{83}{3}\Sigma_k^4\Sigma_k'^2X^3 - \frac{5}{3}\Sigma_k^4X^4 + \frac{100}{3}\Sigma_k^6\Sigma_k'^2X^4\right.\\
&\quad\left.+\frac{10}{3}\Sigma_k^6X^5 - \frac{40}{3}\Sigma_k^8\Sigma_k'^2X^5\right],\\
\Delta C_4 &= \int \frac{d^4k}{(2\pi)^4}\left[-\frac{26}{3}\Sigma_k^2X^3 + \frac{118}{3}\Sigma_k^4X^4 - \frac{104}{3}\Sigma_k^6X^5\right],\\
\Delta C_5 &= \int \frac{d^4k}{(2\pi)^4}\left[\frac{14}{3}\Sigma_k^2X^3 - \frac{52}{3}\Sigma_k^4X^4 + \frac{44}{3}\Sigma_k^6X^5\right],\\
\Delta C_6 &= \int \frac{d^4k}{(2\pi)^4}[6\Sigma_k^2X^2 - 8\Sigma_k^4X^3],\\
\Delta C_7 &= \int \frac{d^4k}{(2\pi)^4}[2\Sigma_k^2X^2],\\
\Delta C_8 &= \int \frac{d^4k}{(2\pi)^4}[-4\Sigma_kX^2 + 34\Sigma_k^3X^3 - 36\Sigma_k^5X^4],\\
\Delta C_9 &= \int \frac{d^4k}{(2\pi)^4}\left[\Sigma_k'^2X - \frac{2}{3}\Sigma_k\Sigma_k'X^2 - 3\Sigma_k^2\Sigma_k'^2X^2 - \frac{2}{3}\Sigma_k^2X^3 + \frac{2}{3}\Sigma_k^3\Sigma_k'X^3 + 2\Sigma_k^4\Sigma_k'^2X^3\right],\\
\Delta C_{10} &= \int \frac{d^4k}{(2\pi)^4}\left[-2i\Sigma_k'^2X + \frac{82i}{3}\Sigma_k^2\Sigma_k'^2X^2 + \frac{20i}{3}\Sigma_k^2X^3 - \frac{268i}{3}\Sigma_k^4\Sigma_k'^2X^3 - \frac{52i}{3}\Sigma_k^4X^4 + \frac{320i}{3}\Sigma_k^6\Sigma_k'^2X^4\right.\\
&\quad\left.+\frac{32i}{3}\Sigma_k^6X^5 - \frac{128i}{3}\Sigma_k^8\Sigma_k'^2X^5\right],\\
\Delta C_{11} &= \int \frac{d^4k}{(2\pi)^4}[-2\Sigma_kX^2 + 6\Sigma_k^3X^3]. \qquad (20)
\end{aligned}$$

To simplify the results, we have used Gauss formula $\int d^4k\partial_\mu f^\mu(k) = 0$ to remove the high order differentials of $\Sigma_k$, where the $f^\mu(k)$'s are the functions with one index, and their momentum dimensions are lower than −4. The definitions of $C_i$ can be found in Eq. (19) of [22], and the relations between $C_i$ and common $L_i$ can be found in Eq. (24) in [22].

To obtain numerical results, we use the same quark self-energy $\Sigma_k$ as in [18,24], with the running coupling constant $\alpha_s(p^2)$ of [25]. To complete the integral, two other input parameters are needed. One is $F_0$, for which we choose $F_0 = 87$ MeV; the other is a cutoff $\Lambda$ which comes from the calculation of the first term in (1). The details can be found in [18], and therefore we have chosen $\Lambda = 1.0^{+0.1}_{-0.1}$ GeV. The three-flavor numerical results are listed in the second row in Table I. Those LECs that depend on $\Lambda$ are expressible as

$$L_{\Lambda=1\ \mathrm{GeV}}\Big|^{L_{\Lambda=1.1\ \mathrm{GeV}}-L_{\Lambda=1\ \mathrm{GeV}}}_{L_{\Lambda=0.9\ \mathrm{GeV}}-L_{\Lambda=1\ \mathrm{GeV}}}. \qquad (21)$$

TABLE VI. The obtained values for the combinations of the $p^6$ order LECs from $\pi\pi$ scattering and our work. The values in [48] are based on the resonance-saturation (RS) hypothesis and pure dimensional analysis (ND). Those in [50] are in the Proca ($n = 5$) and antisymmetric vector formalism ($n = 3$). [51] gives the results by large-$N_C$ techniques and partial wave dispersion relations. The coefficients in the table are in units of $10^{-4}$.

| | $r_1^r$ | $r_2^r$ | $r_3^r$ | $r_4^r$ | $r_5^r$ | $r_6^r$ |
|---|---|---|---|---|---|---|
| RS in Ref. [48] | −0.6 | 1.3 | −1.7 | −1.0 | 1.1 | 0.3 |
| ND in Ref. [48] | 80 | 40 | 20 | 3 | 6 | 2 |
| Set C ($n = 5$) [50] | $-14 \pm 17 \pm 3$ | $22 \pm 16 \pm 4$ | $-3 \pm 1 \pm 3$ | $-0.22 \pm 0.13 \pm 0.05$ | $0.9 \pm 0.1 \pm 0.5$ | $0.25 \pm 0.01 \pm 0.05$ |
| Set C ($n = 3$) [50] | $-20 \pm 17 \pm 3$ | $7 \pm 10 \pm 4$ | $-4 \pm 1 \pm 3$ | $0.13 \pm 0.13 \pm 0.05$ | $0.9 \pm 0.1 \pm 0.5$ | $0.25 \pm 0.01 \pm 0.05$ |
| Ref. [51] | | 18 | 0.9 | −1.9 | | |
| Old [18] | $-9.32^{-2.62}_{+3.51}$ | $8.93^{+3.12}_{-4.27}$ | $-3.06^{-0.81}_{+1.11}$ | $-0.12^{+0.22}_{-0.29}$ | $0.87^{+0.04}_{-0.06}$ | $0.42^{+0.02}_{-0.03}$ |
| New | $0.11^{-2.50}_{+3.27}$ | $-2.84^{+2.95}_{-3.94}$ | $1.03^{-0.69}_{+0.92}$ | $-0.63^{+0.21}_{-0.28}$ | $0.37^{+0.02}_{-0.04}$ | $0.28^{+0.01}_{-0.02}$ |

For two flavors, we give the usual $\bar{l}_i$, $i = 1, 2, 3, 4, 5, 6$,

$$l_i = \frac{1}{32\pi^2}\gamma_i\left(\bar{l}_i + \ln\frac{M_\pi^2}{\mu^2}\right), \tag{22}$$

where the $\gamma_i$'s are given in Ref. [2]. These results are also listed in Table I. The superscripts and subscripts in the table only indicate how the LECs are sensitive to $\Lambda$. These expressions will also appear in the $p^6$ order results in Table III.

Comparing the "new" and "old" results in Table I, the new absolute values of $L_i(\bar{l}_i)$ are, as expected, smaller than the older ones, except for $\bar{l}_3$. For comparison, we also list the other results obtained by different methods: [2,3] are the first results from experimental data; [26] gives the LECs from resonance chiral theory; [17] gives the LECs from a class of holographic theories; [15] collects the last lattice results in [27–29]; [20] gives the $L_1, \ldots, L_8$ from the global fit of the $O(p^6)$ LECs, $L_9$ is given in [30], and $L_{10}$ is a reasonable average of the values of $-4.06 \pm 0.39$ [31], $-3.1 \pm 0.8$ [32], and $-3.46 \pm 0.32$ [33]; [20] also collects some of the latest results in two flavors; and [34] gives some newer fit data at $O(p^6)$ order. These are the usual methods to obtain LECs at present. On the whole, most of the old results are larger than the others, but the new results are closer. The only one new result that is larger than the old one is $\bar{l}_3$, but it is also much closer than the older one. Our new results are much closer to the experimental results, and most results are within the error uncertainties of the resonance results. Nevertheless, they appear to be a bit far from the results from the global fit [20]. One possible reason is that the global-fitted results do not maintain the large $N_C$ limit, but $L_4$ and $L_6$ have fits that are not very small, the effect of which propagate throughout the calculation and decrease the values of other LECs. So far, our calculation remains valid only in the large $N_C$ limit, and therefore they are not very close to the global fit results.

These observations indicate that our approximation in (8) is reasonable. Although we only selected $n = 2$ in (1) and $n = 1$ in (4), the tendency is clear. The second and third terms in (1) carry part of the systematic error in our original calculations. Table I gives the leading order corrections to date. Hence, we believe that when we extend the calculations to the $O(p^6)$ order, the results will be more credible.

### B. $O(p^6)$ order

Because our method only applies in the large $N_C$ limit, for simplicity, in the $O(p^6)$ order, we only need to calculate the large-$N_C$ limit terms. Without the equations of motion, in the large-$N_C$ limit, the CL is

$$\mathcal{L}_6 = \sum_{n=1}^{68} \tilde{K}_n \tilde{Y}_n. \tag{23}$$

These $\tilde{Y}_n$'s and their relationship to the $Y_i$ defined in [5] are listed in Table II; $\tilde{K}_n$ are some coefficients.

TABLE VII. The obtained values for the combinations of the $p^6$ order LECs from $\pi K$ scattering [53] and our work. [53] gives the results with resonance model estimates. The LECs on the lhs of the table are in units of $10^{-4}$ GeV$^{-2}$.

| | $C_1 + 4C_3$ | $C_2$ | $C_4 + 3C_3$ | $C_1 + 4C_3 + 2C_2$ | | $c_{20}^+ \frac{m_\pi^4}{F_\pi^4}$ | $c_{01}^+ \frac{m_\pi^2}{F_\pi^4}$ | $c_{10}^- \frac{m_\pi^3}{F_\pi^4}$ |
|---|---|---|---|---|---|---|---|---|
| Input $c_{30}^+, c_{11}^+, c_{20}^-$ | $20.7 \pm 4.9$ | $-9.2 \pm 4.9$ | $9.9 \pm 2.5$ | $2.3 \pm 10.8$ | | | | |
| Input $c_{30}^+, c_{11}^+, c_{01}^-$ | $28.1 \pm 4.9$ | $-7.4 \pm 4.9$ | $21.0 \pm 2.5$ | $13.4 \pm 10.8$ | Dispersive | $0.024 \pm 0.006$ | $2.07 \pm 0.10$ | $0.31 \pm 0.01$ |
| $\pi\pi$ amplitude | | | $23.5 \pm 2.3$ | $18.8 \pm 7.2$ | | | | |
| Resonance model | 7.2 | −0.5 | 10.0 | 6.2 | Resonance model | 0.003 | 3.8 | 0.09 |
| Old [18] | $35.9^{+1.3}_{-2.1}$ | $0.0^{+0.0}_{-0.0}$ | $29.5^{+1.1}_{-1.9}$ | $35.9^{+1.3}_{-2.1}$ | Old [18] | $0.006^{-0.002}_{+0.003}$ | $-0.159^{+0.133}_{-0.178}$ | $0.020^{+0.037}_{-0.050}$ |
| New | $27.8^{+1.1}_{-1.8}$ | $0.0^{+0.0}_{-0.0}$ | $19.8^{+0.9}_{-1.4}$ | $27.8^{+1.1}_{-1.8}$ | New | $0.016^{-0.002}_{+0.002}$ | $-0.474^{+0.142}_{-0.186}$ | $-0.082^{+0.036}_{-0.047}$ |

TABLE VIII. The obtained values for the combinations of the $p^6$ order LECs appear in vector and scalar form factors of pion. The results in [48] are based on the resonance-saturation hypothesis. [50] gives the same results in both the Proca and antisymmetric vector formalism. [54] gives a naive estimation of $C^r_{12}$ from scalar meson dominance of the pion scalar form factor and $2C^r_{12}+C^r_{34}$ is estimated through $\lambda_0$ in the $K_{l3}$ measurements. [55] estimates the LECs from the $\pi K$ form factors. The coefficients in the table are in units of $10^{-4}$.

| | New | Old [18] | Ref. [48] | | New | Old [18] | Ref. [48] | Ref. [50] | | New | Old [18] | Ref. [48] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $r^r_{V1}$ | $-1.60^{+0.32}_{-0.41}$ | $-2.13^{+0.30}_{-0.39}$ | $-2.5$ | $r^r_{S2}$ | $-0.86^{+0.00}_{+0.00}$ | $0.07^{+0.05}_{-0.08}$ | $-0.3$ | $1\pm 4\pm 1$ | $r^r_{A1}$ | $1.29^{+0.05}_{-0.06}$ | $1.14^{+0.07}_{-0.09}$ | $-0.5$ |
| $r^r_{V2}$ | $1.10^{+0.07}_{-0.10}$ | $2.23^{+0.10}_{-0.16}$ | $2.6$ | $r^r_{S3}$ | $0.21^{-0.01}_{+0.01}$ | $0.20^{-0.01}_{+0.01}$ | $0.6$ | | $r^r_{A2}$ | $-0.72^{-0.07}_{+0.10}$ | $-0.38^{-0.06}_{+0.08}$ | $1.1$ |

| | New | Old [18] | Ref. [54] | | New | Old [18] | Ref. [55] |
|---|---|---|---|---|---|---|---|
| $C^r_{12}$ | $-0.026^{+0.001}_{-0.001}$ | $-0.026^{+0.001}_{-0.001}$ | $-0.1$ | $C^r_{12}$ | $-0.026^{+0.001}_{-0.001}$ | $-0.026^{+0.001}_{-0.001}$ | $(0.3\pm 5.4)\times 10^{-3}$ |
| $2C^r_{12}+C^r_{34}$ | $-0.001^{-0.012}_{+0.020}$ | $0.068^{-0.006}_{+0.010}$ | $-0.10\pm 0.17$ | $C^r_{12}+C^r_{34}$ | $0.025^{-0.013}_{+0.021}$ | $0.094^{-0.007}_{+0.011}$ | $(3.2\pm 1.5)\times 10^{-2}$ |

As for the $O(p^2)$ and $O(p^4)$ orders, expanding (12) to the $O(p^6)$ order with the method used in Sec. III, treating $\tilde{Y}_n$ as $\langle O_{\Omega,k}\rangle$, and $Y_n$ as $\langle O_l\rangle$ in (16), we can obtain the $\Delta\tilde{K}_i$ coefficients. We have listed the values in (A1) in Appendix A. The final $O(p^6)$ LECs are listed in Table III:

$$C_{\Lambda=1\ \mathrm{GeV}}\Big|^{C_{\Lambda=1.1\ \mathrm{GeV}}-C_{\Lambda=1\ \mathrm{GeV}}}_{C_{\Lambda=0.9\ \mathrm{GeV}}-C_{\Lambda=1\ \mathrm{GeV}}}, \qquad c_{\Lambda=1\ \mathrm{GeV}}\Big|^{c_{\Lambda=1.1\ \mathrm{GeV}}-c_{\Lambda=1\ \mathrm{GeV}}}_{c_{\Lambda=0.9\ \mathrm{GeV}}-c_{\Lambda=1\ \mathrm{GeV}}}.$$

Because of the new relation given in Ref. [6], we remove $c_{37}$, as we did previously. Unlike the $O(p^4)$ order, some absolute values of the new LECs are smaller than the old ones, such as $C_1$ and $C_4$; some are almost unchanged, such as $C_3$ and $C_{12}$; some are larger than the old ones, such as $C_{52}$ and $C_{65}$; and some even change signs, such as $C_{22}$ and $C_{69}$. These arise superficially because of the choice of the independent terms in [5] and the complex relations in (17). It seems to imply that these LECs are largely dependent on $\Lambda$, especially in the $O(p^6)$ order. The basic reason for this phenomenon is that our formulation is still not perfect in dealing with ultraviolet divergence. Since this is a typical nonrenomalizable formulation (although underlying QCD is a perturbatively renormalizable theory, with our nonperturbative calculation formalism, it becomes nonrenomalizable with our present approximations), the $O(p^6)$ order [corresponding to the higher dimension terms in comparison to the $O(p^4)$ terms] certainly causes the higher ultraviolet divergences and then leads to larger cutoff effects. This effect also causes several of the new LECs to have reduced (increased) their central values, but the new errors do not follow the same trends, such as $C_{18}$ and $C_{57}$.

TABLE IX. The LECs come from extrapolating the lattice data on the scalar $K\pi$ form factor [56]. They are in units of $10^{-4}$ $\mathrm{GeV}^{-2}$.

| | $C_{12}$ | $C_{34}$ | $C_{14}$ | $2C_{17}$ |
|---|---|---|---|---|
| Ref. [56] | $5.74\pm 0.95$ | $1.07\pm 0.96$ | $0.71\pm 1.42$ | $1.92\pm 3.36$ |
| Old [18] | $-0.34^{+0.02}_{-0.01}$ | $1.59^{-0.10}_{+0.17}$ | $-0.83^{+0.12}_{-0.19}$ | $0.03^{-0.02}_{-0.02}$ |
| New | $-0.34^{+0.01}_{-0.01}$ | $0.66^{-0.18}_{+0.29}$ | $-0.87^{+0.14}_{-0.21}$ | $0.35^{+0.03}_{-0.08}$ |

The calculations are too complicated. To avoid possible mistakes, the expansion in (12) and most of the other calculations are done by computer. To check the correctness of our results, we examined them in various ways. First, some terms in Table II have two parts, which we calculated separately. $C$, $P$, and Hermitian invariance constrain the two parts of the coefficient as being equal or with a sign difference. Our analytical results for the separate parts must give the same coefficients. Second, if we switch off the quark self-energy, all of the LECs, except the contact terms', must vanish [18]. This places a strong restriction on our results. Third, because of the strict constraint conditions in (17), we have $109-68=41$ constraint conditions, with 109 being the value of $O_\Omega$ in (16). They also impose strong restrictions on our results. With all of the above assessments, we are confident of the reliability of our numerical results for the $O(p^6)$ LECs.

Our choice $F_0 = 87$ MeV is the leading order value of $F_\pi$. In two flavors, the relation between $F_0$ and $F_\pi$ is given in Ref. [35] to the $O(p^6)$ order. With our results listed in Tables I and III, the numerical results to the $O(p^6)$ order yield $F_\pi = 92.76^{+0.01}_{-0.06}$ MeV. Comparing this with the previous result $F_\pi = 92.97^{+0.00}_{-0.04}$, the new result is a bit closer to that in PDG2014 [36], $F_\pi = 92.21$ MeV. From this point of view, the additional terms in (9) improve the results slightly. For three flavors, we also choose $F_0 = 87$ MeV as an input parameter. It is also compatible with a new large-$N_C$ result, $F_0 = 88.1 \pm 4.1$ MeV, in three flavors [37].

### C. Anomaly

Following the same procedures as for the normal terms, the anomalous LECs can also be revised. The $O(p^4)$ CLs are Wess-Zumino terms that had been obtained from the first term in (1) [21,38]. The additional terms in (9) must vanish to the $O(p^4)$ order, thereby imposing another requirement. We checked our calculations and verified this requirement. From another point of view, we checked to see that the terms with $n=2$ in (1) and $n=1$ in (4) were suitable.

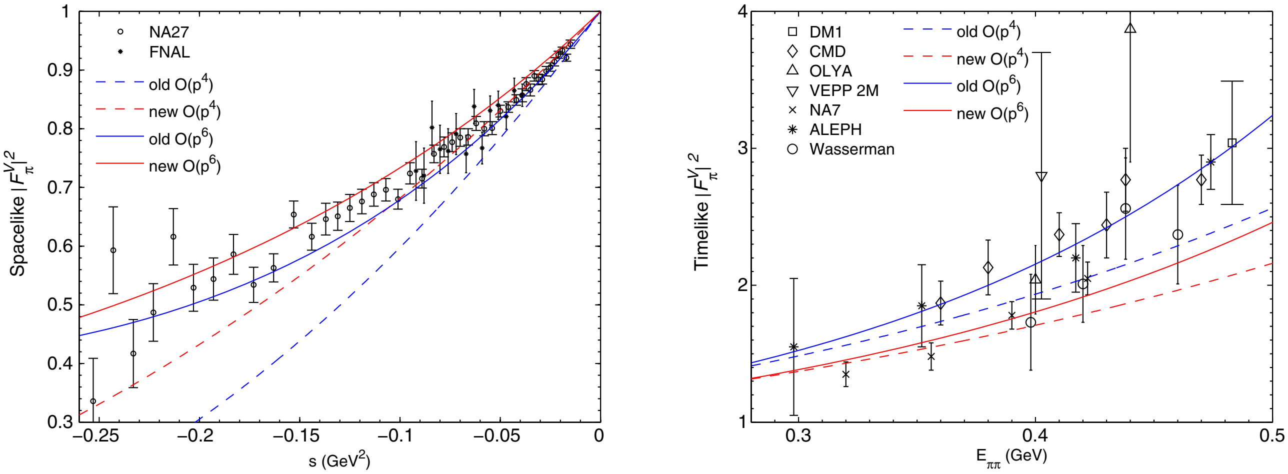


FIG. 1 (color online). The spacelike and timelike data for the vector form factor with the old and new results.

To the $O(p^6)$ order, without the equations of motion, in the large $N_C$ limit, the $n$-flavor CL is

$$\mathcal{L}_6^W = \sum_{n=1}^{23} \tilde{K}_n^W \tilde{O}_n^W. \qquad (24)$$

These $\tilde{O}_n^W$'s and their relations with $O_I^W$ in Ref. [8] are listed in Table VI in Ref. [21]. The analytical results for $\Delta\tilde{K}_n^W$ are listed in (B1) in Appendix B. $C_3^W$, $C_{18}^W$, and $c_{12}^W$ vanish in the large-$N_C$ limit. The nonzero numerical results are listed in the second column of Table IV for three flavors and in the second row of Table V for two flavors:

$$C^W_{\Lambda=1\text{ GeV}}\Big|^{C^W_{\Lambda=1.1\text{ GeV}}-C^W_{\Lambda=1\text{ GeV}}}_{C^W_{\Lambda=0.9\text{ GeV}}-C^W_{\Lambda=1\text{ GeV}}}, \qquad c^W_{\Lambda=1\text{ GeV}}\Big|^{c^W_{\Lambda=1.1\text{ GeV}}-c^W_{\Lambda=1\text{ GeV}}}_{c^W_{\Lambda=0.9\text{ GeV}}-c^W_{\Lambda=1\text{ GeV}}}.$$

## V. COMPARISONS

In this section, we shall gather the LECs to the $O(p^6)$ order given in the literature to provide a means to assess our new results. It is only an update of [18] but also includes some new results. Usually, these LECs are given as dimensionless parameters with the convention of $C_i^r \equiv C_i F_0^2$ or $c_i^r \equiv c_i F_0^2$. The following values of the physical constants come from the central values of PDG2014 [36],

$$m_{\pi^\pm} = 139.57018\text{ MeV}, \qquad m_{\pi^0} = 134.9766\text{ MeV},$$
$$F_\pi = 92.21\text{ MeV}, \qquad m_{K^\pm} = 493.677\text{ MeV}. \qquad (25)$$

Some of these values are needed in certain parts of the calculations.

### A. $\pi\pi$ and $\pi K$ scattering

In $\pi\pi$ scattering, there exist six additional constants, $r_i^r, i = 1, 2, 3, 4, 5, 6$. Their relationship to LECs can be found in Eq. (5.3) in [48]. Reference [48] also gives their values obtained using two theoretical methods, resonance saturation (RS) [49] and pure dimensional analysis (ND) which only accounts for the order of magnitude. References [50,51] also give the $r_i^r$ parameters, all of which are listed in Table VI. Our new $r_{4,5,6}^r$ only change slightly, with the error bars from the references also being small. They are, nonetheless, in good agreement. However, in the references, $r_{1,2,3}^r$ have large error bars, and hence our new results can potentially change signs.

Furthermore, Ref. [52] introduces some coefficients, such as $c_{20}^+$, $c_{01}^+$, $c_{10}^+$, in $\pi K$ scattering. Table VII lists all of these coefficients and some LECs which were obtained in Ref. [53] from different models. Whereas the new results on the left-hand side of Table VII are smaller and approach their results, results on the right-hand side of Table VII seem worse. This may be because of propagation errors from the complex relations between $c^+$ and $C_i$.

TABLE X. The obtained values for the combinations of the $p^6$ order LECs appear in photon-photon collisions. [64,65] give the results with a resonance model. [66] gives the results with date fitting.

| | New | Old [18] | Ref. [64] | Ref. [66] | | New | Old [18] | Ref. [65] | Ref. [66] |
|---|---|---|---|---|---|---|---|---|---|
| $a_1^r$ | $-8.15^{-0.85}_{+1.17}$ | $-5.65^{-0.91}_{+1.23}$ | $-39 \pm 4$ | $-25.9 \pm 1.6 \pm 3.7$ | $a_1^r$ | $-5.46^{-0.12}_{+0.07}$ | $-5.86^{-0.49}_{+0.58}$ | $-3.2$ | $-25.0 \pm 2.2$ |
| $a_2^r$ | $3.94^{+0.03}_{-0.06}$ | $3.79^{+0.02}_{-0.05}$ | $13 \pm 2$ | $8.6 \pm 0.8 \pm 1.8$ | $a_2^r$ | $-1.73^{-0.09}_{+0.16}$ | $-0.98^{-0.07}_{+0.12}$ | $0.7$ | $1.4 \pm 1.8 \pm 1.4$ |
| $b^r$ | $1.87^{+0.06}_{-0.10}$ | $1.66^{+0.05}_{-0.09}$ | $3 \pm 1$ | $3.4 \pm 0.4 \pm 0.1$ | $b^r$ | $-0.20^{-0.02}_{+0.02}$ | $-0.23^{-0.01}_{+0.02}$ | $0.4$ | $0.2 \pm 0.3 \pm 0.1$ |

TABLE XI. The obtained values for the combinations of the $p^6$ order LECs from pion radiative decay [40] and our work. The coefficients in the table are in units of $10^{-5}$.

| | $C^r_{12}$ | $C^r_{13}$ | $C^r_{61}$ | $C^r_{62}$ | $2C^r_{63}-C^r_{65}$ | $C^r_{64}$ |
|---|---|---|---|---|---|---|
| Ref. [40] | $-0.6\pm0.3$ | $0\pm0.2$ | $1.0\pm0.3$ | $0\pm0.2$ | $1.8\pm0.7$ | $0\pm0.2$ |
| Old [18] | $-0.26^{+0.01}_{-0.01}$ | $0.0^{+0.0}_{-0.0}$ | $2.18^{-0.17}_{+0.20}$ | $0.0^{+0.0}_{-0.0}$ | $6.36^{-0.42}_{+0.56}$ | $0.0^{+0.0}_{-0.0}$ |
| New | $-0.26^{+0.01}_{-0.01}$ | $0.00^{+0.00}_{+0.00}$ | $1.83^{-0.14}_{+0.17}$ | $0.00^{+0.00}_{+0.00}$ | $5.89^{-0.45}_{+0.53}$ | $0.00^{+0.00}_{+0.00}$ |
| | $C^r_{78}$ | $C^r_{80}$ | $C^r_{81}$ | $C^r_{82}$ | $C^r_{87}$ | $C^r_{88}$ |
| Ref. [40] | $10.0\pm3.0$ | $1.8\pm0.4$ | $0\pm0.2$ | $-3.5\pm1.0$ | $3.6\pm1.0$ | $-3.5\pm1.0$ |
| Old [18] | $13.26^{+0.77}_{-1.20}$ | $0.66^{-0.03}_{+0.02}$ | $0.0^{+0.0}_{-0.0}$ | $-5.39^{-0.24}_{+0.39}$ | $5.73^{+0.28}_{-0.45}$ | $-4.14^{-0.55}_{+0.78}$ |
| New | $6.68^{+0.60}_{-0.91}$ | $0.77^{-0.04}_{+0.03}$ | $0.00^{+0.00}_{+0.00}$ | $-3.47^{-0.18}_{+0.29}$ | $3.63^{+0.22}_{-0.35}$ | $-1.28^{-0.52}_{+0.70}$ |

TABLE XII. The $p^6$ order LECs from Cosh holographic models [17] and our work. They are in units of $10^{-3}$ GeV$^{-2}$.

| | $C_1$ | $C_3$ | $C_4$ | $C_{40}$ | $C_{42}$ | $C_{44}$ | $C_{46}$ | $C_{47}$ |
|---|---|---|---|---|---|---|---|---|
| Ref. [17] | $-0.3$ | 0.3 | 0 | 0.2 | 2.2 | $-5.5$ | $-3.2$ | 6.2 |
| Old [18] | $3.79^{+0.10}_{-0.17}$ | $-0.05^{+0.01}_{-0.01}$ | $3.10^{+0.09}_{-0.15}$ | $-6.35^{-0.18}_{+0.32}$ | $0.60^{-0.00}_{+0.00}$ | $6.32^{+0.20}_{-0.36}$ | $-0.60^{-0.02}_{+0.04}$ | $0.08^{+0.01}_{-0.00}$ |
| New | $2.98^{+0.07}_{-0.13}$ | $-0.05^{+0.01}_{-0.01}$ | $2.13^{+0.06}_{-0.10}$ | $-4.98^{-0.14}_{+0.25}$ | $1.88^{+0.03}_{-0.06}$ | $1.73^{+0.07}_{-0.14}$ | $-1.67^{-0.05}_{+0.09}$ | $3.10^{+0.09}_{-0.15}$ |
| | $C_{48}$ | $C_{50}+C_{90}$ | $C_{51}+C_{90}$ | $C_{52}-C_{90}$ | $C_{53}-\frac{1}{2}C_{90}$ | $C_{55}+\frac{1}{2}C_{90}$ | $C_{56}-C_{90}$ | $C_{57}+2C_{90}$ |
| Ref. [17] | 5.8 | 19.1 | 5.2 | $-11.6$ | $-8.8$ | 16.7 | 7.1 | 17.2 |
| Old [18] | $3.41^{+0.06}_{-0.10}$ | $11.16^{+0.40}_{-0.66}$ | $-9.04^{-0.20}_{+0.38}$ | $-7.48^{-0.29}_{+0.47}$ | $-13.21^{-0.68}_{+1.10}$ | $18.01^{+0.77}_{-1.26}$ | $16.89^{+0.89}_{-1.44}$ | $12.80^{+0.58}_{-0.92}$ |
| New | $4.74^{+0.10}_{-0.17}$ | $12.86^{+0.45}_{-0.75}$ | $-7.50^{-0.15}_{+0.30}$ | $-8.71^{-0.33}_{+0.53}$ | $-6.52^{-0.49}_{+0.78}$ | $11.61^{+0.58}_{-0.95}$ | $2.13^{+0.47}_{-0.73}$ | $9.36^{+0.48}_{-0.76}$ |
| | $C_{59}-\frac{1}{2}C_{90}$ | $C_{66}$ | $C_{69}$ | $C_{70}-\frac{1}{2}C_{90}$ | $C_{72}+\frac{1}{2}C_{90}$ | $C_{73}+C_{90}$ | $C_{74}$ | $C_{76}-\frac{1}{2}C_{90}$ |
| Ref. [17] | $-20.1$ | $-0.3$ | 0.3 | 5.3 | $-4.7$ | $-4.4$ | $-19.0$ | 11.1 |
| Old [18] | $-23.71^{-1.02}_{+1.66}$ | $1.71^{+0.07}_{-0.12}$ | $-0.86^{-0.04}_{+0.06}$ | $0.51^{+0.11}_{-0.16}$ | $-2.08^{-0.14}_{+0.23}$ | $2.94^{+0.05}_{-0.10}$ | $-5.07^{-0.16}_{+0.27}$ | $-2.66^{-0.05}_{+0.08}$ |
| New | $-15.75^{-0.79}_{+1.28}$ | $0.80^{+0.04}_{-0.07}$ | $0.52^{+0.00}_{-0.01}$ | $0.49^{+0.11}_{-0.16}$ | $-0.64^{-0.10}_{+0.16}$ | $2.48^{+0.04}_{-0.07}$ | $-3.34^{-0.11}_{+0.19}$ | $-2.31^{-0.04}_{+0.07}$ |
| | $C_{78}+\frac{1}{2}C_{90}$ | $C_{79}-\frac{1}{2}C_{90}$ | $C_{87}$ | $C_{88}-C_{90}$ | $C_{89}$ | | | |
| Ref. [17] | 16.1 | 4.1 | 6.8 | $-5.2$ | 29.2 | | | |
| Old [18] | $18.74^{+0.83}_{-1.36}$ | $-1.78^{-0.11}_{+0.17}$ | $7.57^{+0.37}_{-0.60}$ | $-7.91^{-0.35}_{+0.57}$ | $34.74^{+1.61}_{-2.62}$ | | | |
| New | $9.99^{+0.58}_{-0.93}$ | $4.70^{+0.08}_{-0.15}$ | $4.79^{+0.29}_{-0.46}$ | $-4.01^{-0.24}_{+0.38}$ | $17.27^{+1.11}_{-1.77}$ | | | |

### B. Form factors

Reference [48] also estimates the expressions of the vector form factor, the scalar form factor, and two form factors of $\pi(p)\to e\nu\gamma(q)$ with LECs, and [54,55] give some LECs using measurements of the pion scalar form factor, $K_{l3}$, and the $\pi K$ form factors. Reference [50] gives one form factor from $\pi\pi$ scattering. All of them are listed in Table VIII. Reference [56] extrapolates the lattice data on the scalar $K\pi$ form factor to obtain some LECs; the results are listed in Table IX. Two of these results, $r^r_{S2}$ and $2C^r_{12}+C^r_{34}$, seem better, as their signs have been changed.

Furthermore, in Fig. 1, we compare the experimental data [57–63] for the vector form factors collected in Figs. 4 and 5 in [35] with our old and new results. In obtaining our numerical predictions, we have exploited the formula given by Eq. (3.16) in [35], which depends especially on $O(p^6)$ LECs through the $r^r_{V1}$ and $r^r_{V2}$ defined in [48], and we input the formula with the old and new $O(p^4)$ and $O(p^6)$ LECs.

From Fig. 1, we see that at both the $O(p^4)$ and the $O(p^6)$ order, for the spacelike form factors, the new line is higher than the old ones, whereas for the timelike form factors, the new line is lower than the old ones. The new results are slightly worse than the old ones. Nevertheless, considering

TABLE XIII. The obtained values for the $p^6$ order LECs in Ref. [12] from resonance estimates and our work. They are in units of $10^{-3}$ GeV$^{-2}$.

| | $C_{14}$ | $C_{19}$ | $C_{38}$ | $C_{61}$ | $C_{80}$ | $C_{87}$ |
|---|---|---|---|---|---|---|
| Ref. [12] | $-4.3$ | $-2.8$ | 1.2 | 1.9 | 1.9 | 7.6 |
| Old [18] | $-0.83^{+0.12}_{-0.19}$ | $-0.48^{+0.09}_{-0.13}$ | $0.41^{-0.08}_{+0.07}$ | $2.88^{-0.22}_{+0.26}$ | $0.87^{-0.04}_{+0.03}$ | $7.57^{+0.37}_{-0.60}$ |
| New | $-0.87^{+0.14}_{-0.21}$ | $-0.27^{+0.09}_{-0.13}$ | $0.47^{-0.04}_{+0.02}$ | $2.42^{-0.19}_{+0.22}$ | $1.01^{-0.05}_{+0.04}$ | $4.79^{+0.29}_{-0.46}$ |

TABLE XIV. The obtained values for the $p^6$ order LECs from the resonance Lagrangian given by Ref. [13] and our work. The coefficients in the table are in units of $10^{-4}/F_0^2$.

| | $C_{78}$ | $C_{82}$ | $C_{87}$ | $C_{88}$ | $C_{89}$ | $C_{90}$ |
|---|---|---|---|---|---|---|
| Lowest meson dominance | 1.09 | −0.36 | 0.40 | −0.52 | 1.97 | 0.0 |
| Resonance Lagrangian I | 1.09 | −0.29 | 0.47 | −0.16 | 2.29 | 0.33 |
| Resonance Lagrangian II | 1.49 | −0.39 | 0.65 | −0.14 | 3.22 | 0.51 |
| Old [18] | $1.326^{+0.077}_{-0.120}$ | $-0.539^{-0.024}_{+0.039}$ | $0.573^{+0.028}_{-0.045}$ | $-0.414^{-0.055}_{+0.078}$ | $2.630^{+0.122}_{-0.198}$ | $0.185^{-0.029}_{+0.035}$ |
| New | $0.668^{+0.060}_{-0.091}$ | $-0.347^{-0.018}_{+0.029}$ | $0.363^{+0.022}_{-0.035}$ | $-0.128^{-0.052}_{+0.070}$ | $1.307^{+0.084}_{-0.134}$ | $0.176^{-0.033}_{+0.041}$ |

TABLE XV. The LECs come from the sum rules in [14]. They are in units of $10^{-3}$ GeV$^{-2}$.

| | $C_{12}+C_{61}+C_{80}$ | $C_{12}-C_{61}+C_{80}$ | $C_{61}$ | $C_{12}+C_{80}$ |
|---|---|---|---|---|
| $w_{DK}$ [14] | $2.48\pm0.19$ | $-0.55\pm0.21$ | $1.51\pm0.19$ | $0.97\pm0.11$ |
| $\hat{w}$ [14] | $2.48\pm0.18$ | $-0.46\pm0.19$ | $1.47\pm0.17$ | $1.01\pm0.10$ |
| Old [18] | $3.41^{-0.25}_{+0.28}$ | $-2.36^{+0.20}_{-0.24}$ | $2.88^{-0.22}_{+0.26}$ | $0.53^{-0.02}_{+0.02}$ |
| New | $3.09^{-0.22}_{+0.25}$ | $-1.75^{+0.16}_{-0.19}$ | $2.42^{-0.19}_{+0.22}$ | $0.67^{-0.03}_{+0.03}$ |

TABLE XVI. The obtained values for the $O(p^6)$ order LECs, $C^r_{87}$. [67] gives the result from resonance estimates. [68] gives the result with a sequence of rational approximants and the large-$N_c$ limit. [31] gives the result from semileptonic $\tau$ decays. [32] gives the result from the $\tau$ hadronic spectral functions. They are in units of $10^{-5}$.

| | New | Old [18] | Ref. [67] | Ref. [68] | Ref. [31] | Ref. [32] |
|---|---|---|---|---|---|---|
| $C^r_{87}$ | $3.63^{+0.22}_{-0.35}$ | $5.73^{+0.28}_{-0.45}$ | $3.1\pm1.1$ | $4.3\pm0.4$ | $3.70\pm0.14$ | $3.0\pm0.8$ |

the experimental errors, they are all consistent with the experimental data.

### C. Photon-photon collisions

References [64,65] introduce some parameters by $\gamma\gamma\to\pi^0\pi^0$ and $\gamma\gamma\to\pi^+\pi^-$. These parameters are also all related to LECs. A recent work [66] also gives these results. They are listed in Table X. These results are nearly unchanged, with $a^r_2$ and $b^r$ still having opposite signs.

### D. Radiative pion decay

Reference [40] gives a group of LECs to the $O(p^6)$ order. They are listed in Table XI. Most of the new results are fitted better than the old ones, with the exception of $C^r_{88}$.

### E. Holography

Reference [17] gives almost all of the LECs without scalar and pseudoscalar fields in the large-$N_C$ limit from a class of holographic theories; the results are listed in Table XII. The new results still produce some large differences in these LECs, but some LECs retain the same sign, such as $C_{69}$ and $C_{79}-\frac{1}{2}C_{90}$. Some differences may come from the error associated with $C_{90}$, the source of which needs to be further checked.

### F. Other results

Aside from the above results, there are more LECs given using these different methods. Most of them are from resonance approximations, but our results do not rely on the assumption of the presence of resonances. In this subsection, we list the values we gathered from the literature and compare them with ours.

Table XIII lists some LECs from resonance estimates in [12]; Table XIV lists other resonance-estimated LECs in [13]; Table XV lists the LECs from sum rules in [14]; $C^r_{87}$ values were obtained from many references and are listed in Table XVI; the other LECs obtained from different models in different references are listed in Table XVII. Except for $C^r_{14}+C^r_{15}$ in Table XVII, all of the LECs retain their signs and the same orders of magnitude, and almost all of the new results are in closer correspondence with the others.

TABLE XVII. Some LECs from different references. [69] gives the result from nonleptonic and radiative $K$ decays. [70–73] give the results from resonance chiral theory. [16] gives the result by the extrapolation of lattice data.

| | $10^5(2C^r_{63}-C^r_{65})$ | $10^6C^r_{38}$ | $10^5(C^r_{88}-C^r_{90})$ | $(C_{14}+C_{15})10^3$ GeV$^2$ | $(C_{15}+2C_{17})10^3$ GeV$^2$ |
|---|---|---|---|---|---|
| | $1.8\pm0.7$ [69] | $2\pm6$ [70] | $-4.6\pm0.4$ [72] | $0.37\pm0.08$ [16] | $1.29\pm0.16$ [16] |
| | | $8\pm5$ [71] | $-4.5\pm0.5$ [73] | | |
| Old [18] | $6.36^{-0.48}_{+0.57}$ | $3.08^{-0.62}_{+0.56}$ | $-5.99^{-0.27}_{+0.43}$ | $-0.83^{+0.12}_{-0.19}$ | $0.03^{-0.02}_{-0.02}$ |
| New | $5.89^{-0.45}_{+0.53}$ | $3.57^{-0.34}_{+0.17}$ | $-3.04^{-0.18}_{+0.29}$ | $-0.87^{+0.14}_{-0.21}$ | $0.35^{+0.03}_{-0.08}$ |

TABLE XVIII. The comparison of the relations given in [74] by resonance theory. The coefficients in the table are in units of $10^{-3}$ GeV$^{-2}$.

| | $C_{20}$ | $-3C_{21}$ | $C_{32}$ | $\frac{1}{6}C_{35}$ | $C_{24}$ | $6C_{28}$ | $3C_{30}$ |
|---|---|---|---|---|---|---|---|
| Old [18] | $0.18^{-0.03}_{+0.04}$ | $0.18^{-0.03}_{+0.03}$ | $0.18^{-0.03}_{+0.04}$ | $0.028^{-0.020}_{+0.028}$ | $1.62^{+0.04}_{-0.07}$ | $1.80^{+0.06}_{-0.06}$ | $1.80^{+0.06}_{-0.09}$ |
| New | $0.17^{-0.02}_{+0.03}$ | $0.17^{-0.02}_{+0.03}$ | $0.17^{-0.02}_{+0.03}$ | $0.02^{-0.01}_{+0.02}$ | $0.87^{+0.02}_{-0.04}$ | $1.10^{+0.03}_{-0.05}$ | $1.10^{+0.03}_{-0.05}$ |

Reference [74] gives some LEC relations by some assumptions in the large-$N_c$ limit. We collect our results in Table XVIII to check these relations:

$$C_{20} = -3C_{21} = C_{32} = \frac{1}{6}C_{35}, \qquad C_{24} = 6C_{28} = 3C_{30}. \tag{26}$$

Because we only calculate a part of the large-$N_C$ expression in (1) and some additional assumptions in [74], not all of the LECs satisfy the relations.

For the anomalous LECs, we collect the results in Table IV. The anomalous results are less than the normal ones, and the differences between each are slightly larger than the normal ones.

## VI. SUMMARY

In this research, we updated our original LECs to the $O(p^6)$ order, including two and three flavors, and normal and anomalous ones. The new contributions come from $n = 2$ in (1) and $n = 1$ in (4). This is one small step beyond the GND model. As a check, the $O(p^4)$ order absolute values have decreased and are closer to others, so our updates are plausible. Up to the $O(p^6)$ order, the absolute values of the LECs exhibited varying changes or remained unchanged. We also compared these LECs with the others. Most of them are much closer than the old values, but some combinations of LECs fare badly. The combinations found in the references are directly from phenomenological data, and they can be more precise. However, we have obtained the LECs separately. On the whole, the new LEC values are better than the old ones; one possible reason for the differences is a propagation of errors. In this method, more precise results need a more detailed analysis of (1), which remains as work for the future.

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## APPENDIX A: THE $\Delta\tilde{K}_i$ COEFFICIENTS

$$\begin{aligned}
\Delta\tilde{K}_1 = \int \frac{d^4k}{(2\pi)^4}\Bigg[&\frac{1}{32}\Sigma_k''^2 + \frac{1}{960}\Sigma_k\Sigma_k'''X + \frac{3}{16}\Sigma_k'^4X - \frac{9}{16}\Sigma_k^2\Sigma_k''^2X - \frac{21}{160}\Sigma_k'^2X^2 + \frac{1}{240}\Sigma_k\Sigma_k''X^2 \\
&- \frac{277}{480}\Sigma_k\Sigma_k'^3X^2 - \frac{1}{960}\Sigma_k^3\Sigma_k'''X^2 - \frac{75}{16}\Sigma_k^2\Sigma_k'^4X^2 + \frac{75}{32}\Sigma_k^4\Sigma_k''^2X^2 - \frac{7}{320}X^3 - \frac{7}{80}\Sigma_k\Sigma_k'X^3 \\
&+ \frac{323}{192}\Sigma_k^2\Sigma_k'^2X^3 + \frac{5491}{480}\Sigma_k^3\Sigma_k'^3X^3 + \frac{451}{16}\Sigma_k^4\Sigma_k'^4X^3 - 4\Sigma_k^6\Sigma_k''^2X^3 + \frac{257}{960}\Sigma_k^2X^4 - \frac{771}{80}\Sigma_k^4\Sigma_k'^2X^4 \\
&- \frac{13087}{240}\Sigma_k^5\Sigma_k'^3X^4 - \frac{3467}{48}\Sigma_k^6\Sigma_k'^4X^4 + \frac{49}{16}\Sigma_k^8\Sigma_k''^2X^4 - \frac{11}{12}\Sigma_k^4X^5 + \frac{139}{6}\Sigma_k^6\Sigma_k'^2X^5 + \frac{1259}{12}\Sigma_k^7\Sigma_k'^3X^5 \\
&+ \frac{2201}{24}\Sigma_k^8\Sigma_k'^4X^5 - \frac{7}{8}\Sigma_k^{10}\Sigma_k''^2X^5 + \frac{31}{48}\Sigma_k^6X^6 - \frac{70}{3}\Sigma_k^8\Sigma_k'^2X^6 - \frac{357}{4}\Sigma_k^9\Sigma_k'^3X^6 - \frac{343}{6}\Sigma_k^{10}\Sigma_k'^4X^6 \\
&+ \frac{9}{16}\Sigma_k^8X^7 + \frac{33}{4}\Sigma_k^{10}\Sigma_k'^2X^7 + 28\Sigma_k^{11}\Sigma_k'^3X^7 + 14\Sigma_k^{12}\Sigma_k'^4X^7\Bigg],
\end{aligned}$$

$$
\begin{aligned}
\Delta\tilde{K}_2 = \int \frac{d^4k}{(2\pi)^4} \Big[ & -\frac{1}{96}\Sigma_k''^2 - \frac{31}{432}\Sigma_k'^4 X + \frac{23}{144}\Sigma_k^2\Sigma_k''^2 X + \frac{1}{48}\Sigma_k'^2 X^2 + \frac{11}{48}\Sigma_k\Sigma_k'^3 X^2 + \frac{599}{432}\Sigma_k^2\Sigma_k'^4 X^2 \\
& - \frac{185}{288}\Sigma_k^4\Sigma_k''^2 X^2 - \frac{37}{96}\Sigma_k^2\Sigma_k'^2 X^3 - \frac{245}{72}\Sigma_k^3\Sigma_k'^3 X^3 - \frac{3391}{432}\Sigma_k^4\Sigma_k'^4 X^3 + \frac{13}{12}\Sigma_k^6\Sigma_k''^2 X^3 - \frac{1}{24}\Sigma_k^2 X^4 \\
& + \frac{245}{96}\Sigma_k^4\Sigma_k'^2 X^4 + \frac{2179}{144}\Sigma_k^5\Sigma_k'^3 X^4 + \frac{8509}{432}\Sigma_k^6\Sigma_k'^4 X^4 - \frac{119}{144}\Sigma_k^8\Sigma_k''^2 X^4 + \frac{3}{16}\Sigma_k^4 X^5 - \frac{325}{48}\Sigma_k^6\Sigma_k'^2 X^5 \\
& - \frac{2051}{72}\Sigma_k^7\Sigma_k'^3 X^5 - \frac{5359}{216}\Sigma_k^8\Sigma_k'^4 X^5 + \frac{17}{72}\Sigma_k^{10}\Sigma_k''^2 X^5 - \frac{3}{16}\Sigma_k^6 X^6 + \frac{15}{2}\Sigma_k^8\Sigma_k'^2 X^6 + \frac{289}{12}\Sigma_k^9\Sigma_k'^3 X^6 \\
& + \frac{833}{54}\Sigma_k^{10}\Sigma_k'^4 X^6 + \frac{1}{48}\Sigma_k^8 X^7 - \frac{35}{12}\Sigma_k^{10}\Sigma_k'^2 X^7 - \frac{68}{9}\Sigma_k^{11}\Sigma_k'^3 X^7 - \frac{34}{9}\Sigma_k^{12}\Sigma_k'^4 X^7 \Big],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_3 = \int \frac{d^4k}{(2\pi)^4} \Big[ & \frac{1}{48}\Sigma_k''^2 + \frac{1}{960}\Sigma_k\Sigma_k''' X + \frac{25}{216}\Sigma_k'^4 X - \frac{29}{72}\Sigma_k^2\Sigma_k''^2 X - \frac{7}{120}\Sigma_k'^2 X^2 + \frac{1}{240}\Sigma_k\Sigma_k'' X^2 \\
& - \frac{167}{480}\Sigma_k\Sigma_k'^3 X^2 - \frac{1}{960}\Sigma_k^3\Sigma_k''' X^2 - \frac{713}{216}\Sigma_k^2\Sigma_k'^4 X^2 + \frac{245}{144}\Sigma_k^4\Sigma_k''^2 X^2 - \frac{7}{320} X^3 - \frac{7}{80}\Sigma_k\Sigma_k' X^3 \\
& + \frac{51}{64}\Sigma_k^2\Sigma_k'^2 X^3 + \frac{11573}{1440}\Sigma_k^3\Sigma_k'^3 X^3 + \frac{4393}{216}\Sigma_k^4\Sigma_k'^4 X^3 - \frac{35}{12}\Sigma_k^6\Sigma_k''^2 X^3 + \frac{137}{960}\Sigma_k^2 X^4 - \frac{1513}{240}\Sigma_k^4\Sigma_k'^2 X^4 \\
& - \frac{14183}{360}\Sigma_k^5\Sigma_k'^3 X^4 - \frac{11347}{216}\Sigma_k^6\Sigma_k'^4 X^4 + \frac{161}{72}\Sigma_k^8\Sigma_k''^2 X^4 - \frac{1}{2}\Sigma_k^4 X^5 + \frac{277}{16}\Sigma_k^6\Sigma_k'^2 X^5 + \frac{5503}{72}\Sigma_k^7\Sigma_k'^3 X^5 \\
& + \frac{7225}{108}\Sigma_k^8\Sigma_k'^4 X^5 - \frac{23}{36}\Sigma_k^{10}\Sigma_k''^2 X^5 + \frac{9}{16}\Sigma_k^6 X^6 - \frac{55}{3}\Sigma_k^8\Sigma_k'^2 X^6 - \frac{391}{6}\Sigma_k^9\Sigma_k'^3 X^6 - \frac{1127}{27}\Sigma_k^{10}\Sigma_k'^4 X^6 \\
& + \frac{13}{48}\Sigma_k^8 X^7 + \frac{79}{12}\Sigma_k^{10}\Sigma_k'^2 X^7 + \frac{184}{9}\Sigma_k^{11}\Sigma_k'^3 X^7 + \frac{92}{9}\Sigma_k^{12}\Sigma_k'^4 X^7 \Big],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_4 = \int \frac{d^4k}{(2\pi)^4} \Big[ & \frac{9}{16}\Sigma_k\Sigma_k'^2 X^2 - \frac{1}{16}\Sigma_k X^3 - \frac{271}{48}\Sigma_k^3\Sigma_k'^2 X^3 + \frac{73}{96}\Sigma_k^3 X^4 + \frac{193}{12}\Sigma_k^5\Sigma_k'^2 X^4 - \frac{85}{48}\Sigma_k^5 X^5 \\
& - \frac{53}{3}\Sigma_k^7\Sigma_k'^2 X^5 + \frac{25}{24}\Sigma_k^7 X^6 + \frac{20}{3}\Sigma_k^9\Sigma_k'^2 X^6 \Big],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_5 = \int \frac{d^4k}{(2\pi)^4} \Big[ & \frac{1}{192}\Sigma_k'' X + \frac{1}{96}\Sigma_k' X^2 + \frac{35}{96}\Sigma_k\Sigma_k'^2 X^2 - \frac{7}{96}\Sigma_k X^3 - \frac{1}{48}\Sigma_k^2\Sigma_k' X^3 - \frac{61}{12}\Sigma_k^3\Sigma_k'^2 X^3 \\
& + \frac{19}{24}\Sigma_k^3 X^4 + \frac{377}{24}\Sigma_k^5\Sigma_k'^2 X^4 - \frac{73}{48}\Sigma_k^5 X^5 - \frac{53}{3}\Sigma_k^7\Sigma_k'^2 X^5 + \frac{25}{24}\Sigma_k^7 X^6 + \frac{20}{3}\Sigma_k^9\Sigma_k'^2 X^6 \Big],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_6 = \int \frac{d^4k}{(2\pi)^4} \Big[ & -\frac{1}{192}\Sigma_k'' X - \frac{1}{96}\Sigma_k' X^2 - \frac{89}{96}\Sigma_k\Sigma_k'^2 X^2 + \frac{7}{96}\Sigma_k X^3 + \frac{1}{48}\Sigma_k^2\Sigma_k' X^3 + \frac{515}{48}\Sigma_k^3\Sigma_k'^2 X^3 \\
& - \frac{7}{48}\Sigma_k^3 X^4 - \frac{763}{24}\Sigma_k^5\Sigma_k'^2 X^4 - \frac{49}{48}\Sigma_k^5 X^5 + \frac{106}{3}\Sigma_k^7\Sigma_k'^2 X^5 + \frac{25}{24}\Sigma_k^7 X^6 - \frac{40}{3}\Sigma_k^9\Sigma_k'^2 X^6 \Big],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_7 = \int \frac{d^4k}{(2\pi)^4} \Big[ & -\frac{7}{32}\Sigma_k\Sigma_k'^2 X^2 + \frac{1}{32}\Sigma_k X^3 + \frac{81}{32}\Sigma_k^3\Sigma_k'^2 X^3 - \frac{7}{96}\Sigma_k^3 X^4 - \frac{367}{48}\Sigma_k^5\Sigma_k'^2 X^4 - \frac{2}{3}\Sigma_k^5 X^5 \\
& + \frac{26}{3}\Sigma_k^7\Sigma_k'^2 X^5 + \frac{5}{6}\Sigma_k^7 X^6 - \frac{10}{3}\Sigma_k^9\Sigma_k'^2 X^6 \Big],
\end{aligned}
$$

$$
\Delta\tilde{K}_8 = \int \frac{d^4k}{(2\pi)^4} \Big[ \frac{1}{32}\Sigma_k'^2 X - \frac{3}{32}\Sigma_k^2\Sigma_k'^2 X^2 + \frac{11}{32}\Sigma_k^2 X^3 + \frac{1}{16}\Sigma_k^4\Sigma_k'^2 X^3 - \frac{57}{32}\Sigma_k^4 X^4 + \frac{3}{2}\Sigma_k^6 X^5 \Big],
$$

$$
\Delta\tilde{K}_9 = \int \frac{d^4k}{(2\pi)^4} \Big[ \frac{1}{32}\Sigma_k'^2 X - \frac{3}{32}\Sigma_k^2\Sigma_k'^2 X^2 + \frac{3}{16}\Sigma_k^2 X^3 + \frac{1}{16}\Sigma_k^4\Sigma_k'^2 X^3 - \frac{15}{16}\Sigma_k^4 X^4 + \frac{3}{4}\Sigma_k^6 X^5 \Big],
$$

$$
\Delta\tilde{K}_{10} = \int \frac{d^4k}{(2\pi)^4} \Big[ \frac{1}{32}\Sigma_k X^2 - \frac{1}{4}\Sigma_k^3 X^3 + \frac{1}{4}\Sigma_k^5 X^4 \Big],
$$

$$
\begin{aligned}
\Delta\tilde{K}_{11} = \int \frac{d^4k}{(2\pi)^4}\bigg[ & \frac{1}{48}\Sigma_k'^3 X - \frac{1}{192}\Sigma_k' X^2 + \frac{1}{8}\Sigma_k \Sigma_k'^2 X^2 - \frac{1}{12}\Sigma_k^2 \Sigma_k'^3 X^2 + \frac{7}{192}\Sigma_k X^3 + \frac{1}{48}\Sigma_k^2 \Sigma_k' X^3 \\
& - \frac{7}{12}\Sigma_k^3 \Sigma_k'^2 X^3 + \frac{5}{48}\Sigma_k^4 \Sigma_k'^3 X^3 - \frac{61}{192}\Sigma_k^3 X^4 + \frac{19}{24}\Sigma_k^5 \Sigma_k'^2 X^4 - \frac{1}{24}\Sigma_k^6 \Sigma_k'^3 X^4 + \frac{1}{6}\Sigma_k^5 X^5 - \frac{1}{3}\Sigma_k^7 \Sigma_k'^2 X^5 \bigg],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_{12} = \int \frac{d^4k}{(2\pi)^4}\bigg[ & \frac{1}{576}\Sigma_k''^2 + \frac{1}{1920}\Sigma_k \Sigma_k''' X + \frac{1}{216}\Sigma_k'^4 X - \frac{7}{144}\Sigma_k^2 \Sigma_k''^2 X + \frac{17}{960}\Sigma_k'^2 X^2 + \frac{1}{480}\Sigma_k \Sigma_k'' X^2 \\
& + \frac{83}{960}\Sigma_k \Sigma_k'^3 X^2 - \frac{1}{1920}\Sigma_k^3 \Sigma_k''' X^2 - \frac{10}{27}\Sigma_k^2 \Sigma_k'^4 X^2 + \frac{125}{576}\Sigma_k^4 \Sigma_k''^2 X^2 - \frac{7}{640} X^3 - \frac{7}{160}\Sigma_k \Sigma_k' X^3 \\
& - \frac{169}{384}\Sigma_k^2 \Sigma_k'^2 X^3 + \frac{491}{960}\Sigma_k^3 \Sigma_k'^3 X^3 + \frac{547}{216}\Sigma_k^4 \Sigma_k'^4 X^3 - \frac{109}{288}\Sigma_k^6 \Sigma_k''^2 X^3 - \frac{43}{1920}\Sigma_k^2 X^4 + \frac{203}{320}\Sigma_k^4 \Sigma_k'^2 X^4 \\
& - \frac{6421}{1440}\Sigma_k^5 \Sigma_k'^3 X^4 - \frac{27}{4}\Sigma_k^6 \Sigma_k'^4 X^4 + \frac{7}{24}\Sigma_k^8 \Sigma_k''^2 X^4 + \frac{25}{96}\Sigma_k^4 X^5 + \frac{35}{24}\Sigma_k^6 \Sigma_k'^2 X^5 + \frac{349}{36}\Sigma_k^7 \Sigma_k'^3 X^5 \\
& + \frac{313}{36}\Sigma_k^8 \Sigma_k'^4 X^5 - \frac{1}{12}\Sigma_k^{10} \Sigma_k''^2 X^5 + \frac{1}{24}\Sigma_k^6 X^6 - \frac{35}{12}\Sigma_k^8 \Sigma_k'^2 X^6 - \frac{17}{2}\Sigma_k^9 \Sigma_k'^3 X^6 - \frac{49}{9}\Sigma_k^{10} \Sigma_k'^4 X^6 \\
& - \frac{1}{16}\Sigma_k^8 X^7 + \frac{5}{4}\Sigma_k^{10} \Sigma_k'^2 X^7 + \frac{8}{3}\Sigma_k^{11} \Sigma_k'^3 X^7 + \frac{4}{3}\Sigma_k^{12} \Sigma_k'^4 X^7 \bigg],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_{13} = \int \frac{d^4k}{(2\pi)^4}\bigg[ & -\frac{1}{24}\Sigma_k'^3 X + \frac{1}{96}\Sigma_k' X^2 - \frac{1}{4}\Sigma_k \Sigma_k'^2 X^2 + \frac{1}{6}\Sigma_k^2 \Sigma_k'^3 X^2 + \frac{5}{96}\Sigma_k X^3 - \frac{1}{24}\Sigma_k^2 \Sigma_k' X^3 \\
& + \frac{7}{6}\Sigma_k^3 \Sigma_k'^2 X^3 - \frac{5}{24}\Sigma_k^4 \Sigma_k'^3 X^3 - \frac{59}{96}\Sigma_k^3 X^4 - \frac{19}{12}\Sigma_k^5 \Sigma_k'^2 X^4 + \frac{1}{12}\Sigma_k^6 \Sigma_k'^3 X^4 + \frac{11}{12}\Sigma_k^5 X^5 + \frac{2}{3}\Sigma_k^7 \Sigma_k'^2 X^5 \bigg],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_{14} = \int \frac{d^4k}{(2\pi)^4}\bigg[ & -\frac{13}{288}\Sigma_k''^2 - \frac{1}{320}\Sigma_k \Sigma_k''' X - \frac{13}{54}\Sigma_k'^4 X + \frac{65}{72}\Sigma_k^2 \Sigma_k''^2 X + \frac{49}{480}\Sigma_k'^2 X^2 - \frac{1}{80}\Sigma_k \Sigma_k'' X^2 \\
& + \frac{251}{480}\Sigma_k \Sigma_k'^3 X^2 + \frac{1}{320}\Sigma_k^3 \Sigma_k''' X^2 + \frac{793}{108}\Sigma_k^2 \Sigma_k'^4 X^2 - \frac{1105}{288}\Sigma_k^4 \Sigma_k''^2 X^2 + \frac{21}{320} X^3 + \frac{21}{80}\Sigma_k \Sigma_k' X^3 \\
& - \frac{145}{192}\Sigma_k^2 \Sigma_k'^2 X^3 - \frac{24619}{1440}\Sigma_k^3 \Sigma_k'^3 X^3 - \frac{1235}{27}\Sigma_k^4 \Sigma_k'^4 X^3 + \frac{949}{144}\Sigma_k^6 \Sigma_k''^2 X^3 - \frac{311}{960}\Sigma_k^2 X^4 + \frac{4213}{480}\Sigma_k^4 \Sigma_k'^2 X^4 \\
& + \frac{7017}{80}\Sigma_k^5 \Sigma_k'^3 X^4 + \frac{12805}{108}\Sigma_k^6 \Sigma_k'^4 X^4 - \frac{91}{18}\Sigma_k^8 \Sigma_k''^2 X^4 + \frac{37}{48}\Sigma_k^4 X^5 - \frac{217}{8}\Sigma_k^6 \Sigma_k'^2 X^5 - \frac{689}{4}\Sigma_k^7 \Sigma_k'^3 X^5 \\
& - \frac{4082}{27}\Sigma_k^8 \Sigma_k'^4 X^5 + \frac{13}{9}\Sigma_k^{10} \Sigma_k''^2 X^5 - \frac{1}{6}\Sigma_k^6 X^6 + \frac{175}{6}\Sigma_k^8 \Sigma_k'^2 X^6 + \frac{442}{3}\Sigma_k^9 \Sigma_k'^3 X^6 + \frac{2548}{27}\Sigma_k^{10} \Sigma_k'^4 X^6 \\
& - \frac{43}{24}\Sigma_k^8 X^7 - \frac{61}{6}\Sigma_k^{10} \Sigma_k'^2 X^7 - \frac{416}{9}\Sigma_k^{11} \Sigma_k'^3 X^7 - \frac{208}{9}\Sigma_k^{12} \Sigma_k'^4 X^7 \bigg],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_{15} = \int \frac{d^4k}{(2\pi)^4}\bigg[ & \frac{1}{32}\Sigma_k'^2 X + \frac{13}{32}\Sigma_k^2 \Sigma_k'^2 X^2 + \frac{3}{32}\Sigma_k^2 X^3 - \frac{31}{16}\Sigma_k^4 \Sigma_k'^2 X^3 - \frac{13}{32}\Sigma_k^4 X^4 + \frac{5}{2}\Sigma_k^6 \Sigma_k'^2 X^4 \\
& + \frac{1}{4}\Sigma_k^6 X^5 - \Sigma_k^8 \Sigma_k'^2 X^5 \bigg],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_{16} = \int \frac{d^4k}{(2\pi)^4}\bigg[ & -\frac{5}{96}\Sigma_k'^3 X + \frac{1}{192}\Sigma_k' X^2 + \frac{1}{16}\Sigma_k \Sigma_k'^2 X^2 + \frac{5}{24}\Sigma_k^2 \Sigma_k'^3 X^2 - \frac{1}{192}\Sigma_k X^3 - \frac{1}{48}\Sigma_k^2 \Sigma_k' X^3 \\
& - \frac{59}{48}\Sigma_k^3 \Sigma_k'^2 X^3 - \frac{25}{96}\Sigma_k^4 \Sigma_k'^3 X^3 - \frac{35}{192}\Sigma_k^3 X^4 + \frac{55}{12}\Sigma_k^5 \Sigma_k'^2 X^4 + \frac{5}{48}\Sigma_k^6 \Sigma_k'^3 X^4 + \frac{43}{48}\Sigma_k^5 X^5 \\
& - \frac{71}{12}\Sigma_k^7 \Sigma_k'^2 X^5 - \frac{5}{8}\Sigma_k^7 X^6 + \frac{5}{2}\Sigma_k^9 \Sigma_k'^2 X^6 \bigg],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_{17} = \int \frac{d^4k}{(2\pi)^4}\Big[ & -\frac{5}{144}\Sigma_k''^2 - \frac{1}{480}\Sigma_k\Sigma_k'''X - \frac{85}{432}\Sigma_k'^4X + \frac{95}{144}\Sigma_k^2\Sigma_k''^2X + \frac{31}{480}\Sigma_k'^2X^2 - \frac{1}{120}\Sigma_k\Sigma_k''X^2 \\
& + \frac{97}{240}\Sigma_k\Sigma_k'^3X^2 + \frac{1}{480}\Sigma_k^3\Sigma_k'''X^2 + \frac{2345}{432}\Sigma_k^2\Sigma_k'^4X^2 - \frac{25}{9}\Sigma_k^4\Sigma_k''^2X^2 + \frac{7}{160}X^3 + \frac{7}{40}\Sigma_k\Sigma_k'X^3 \\
& - \frac{5}{96}\Sigma_k^2\Sigma_k'^2X^3 - \frac{997}{80}\Sigma_k^3\Sigma_k'^3X^3 - \frac{14365}{432}\Sigma_k^4\Sigma_k'^4X^3 + \frac{685}{144}\Sigma_k^6\Sigma_k''^2X^3 - \frac{107}{480}\Sigma_k^2X^4 + \frac{121}{40}\Sigma_k^4\Sigma_k'^2X^4 \\
& + \frac{22841}{360}\Sigma_k^5\Sigma_k'^3X^4 + \frac{4115}{48}\Sigma_k^6\Sigma_k'^4X^4 - \frac{175}{48}\Sigma_k^8\Sigma_k''^2X^4 + \frac{23}{96}\Sigma_k^4X^5 - \frac{145}{12}\Sigma_k^6\Sigma_k'^2X^5 - \frac{4475}{36}\Sigma_k^7\Sigma_k'^3X^5 \\
& - \frac{7855}{72}\Sigma_k^8\Sigma_k'^4X^5 + \frac{25}{24}\Sigma_k^{10}\Sigma_k''^2X^5 + \frac{65}{96}\Sigma_k^6X^6 + \frac{85}{6}\Sigma_k^8\Sigma_k'^2X^6 + \frac{425}{4}\Sigma_k^9\Sigma_k'^3X^6 + \frac{1225}{18}\Sigma_k^{10}\Sigma_k'^4X^6 \\
& - \frac{59}{32}\Sigma_k^8X^7 - \frac{41}{8}\Sigma_k^{10}\Sigma_k'^2X^7 - \frac{100}{3}\Sigma_k^{11}\Sigma_k'^3X^7 - \frac{50}{3}\Sigma_k^{12}\Sigma_k'^4X^7\Big],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_{18} = \int \frac{d^4k}{(2\pi)^4}\Big[ & \frac{1}{32}\Sigma_k'^2X + \frac{13}{32}\Sigma_k^2\Sigma_k'^2X^2 + \frac{3}{16}\Sigma_k^2X^3 - \frac{31}{16}\Sigma_k^4\Sigma_k'^2X^3 - \frac{11}{16}\Sigma_k^4X^4 + \frac{5}{2}\Sigma_k^6\Sigma_k'^2X^4 \\
& + \frac{1}{4}\Sigma_k^6X^5 - \Sigma_k^8\Sigma_k'^2X^5\Big],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_{19} = \int \frac{d^4k}{(2\pi)^4}\Big[ & -\frac{1}{48}\Sigma_k'^3X - \frac{1}{96}\Sigma_k'X^2 + \frac{5}{8}\Sigma_k\Sigma_k'^2X^2 + \frac{1}{12}\Sigma_k^2\Sigma_k'^3X^2 - \frac{1}{48}\Sigma_kX^3 + \frac{1}{24}\Sigma_k^2\Sigma_k'X^3 \\
& - \frac{115}{24}\Sigma_k^3\Sigma_k'^2X^3 - \frac{5}{48}\Sigma_k^4\Sigma_k'^3X^3 + \frac{1}{48}\Sigma_k^3X^4 + \frac{37}{3}\Sigma_k^5\Sigma_k'^2X^4 + \frac{1}{24}\Sigma_k^6\Sigma_k'^3X^4 + \frac{7}{12}\Sigma_k^5X^5 \\
& - \frac{79}{6}\Sigma_k^7\Sigma_k'^2X^5 - \frac{5}{4}\Sigma_k^7X^6 + 5\Sigma_k^9\Sigma_k'^2X^6\Big],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_{20} = \int \frac{d^4k}{(2\pi)^4}\Big[ & \frac{1}{72}\Sigma_k''^2 + \frac{1}{960}\Sigma_k\Sigma_k'''X + \frac{35}{432}\Sigma_k'^4X - \frac{37}{144}\Sigma_k^2\Sigma_k''^2X - \frac{3}{80}\Sigma_k'^2X^2 + \frac{1}{240}\Sigma_k\Sigma_k''X^2 \\
& - \frac{9}{160}\Sigma_k\Sigma_k'^3X^2 - \frac{1}{960}\Sigma_k^3\Sigma_k'''X^2 - \frac{919}{432}\Sigma_k^2\Sigma_k'^4X^2 + \frac{155}{144}\Sigma_k^4\Sigma_k''^2X^2 - \frac{7}{320}X^3 - \frac{7}{80}\Sigma_k\Sigma_k'X^3 \\
& + \frac{7}{64}\Sigma_k^2\Sigma_k'^2X^3 + \frac{6373}{1440}\Sigma_k^3\Sigma_k'^3X^3 + \frac{5579}{432}\Sigma_k^4\Sigma_k'^4X^3 - \frac{265}{144}\Sigma_k^6\Sigma_k''^2X^3 + \frac{77}{960}\Sigma_k^2X^4 - \frac{1141}{480}\Sigma_k^4\Sigma_k'^2X^4 \\
& - \frac{481}{20}\Sigma_k^5\Sigma_k'^3X^4 - \frac{14341}{432}\Sigma_k^6\Sigma_k'^4X^4 + \frac{203}{144}\Sigma_k^8\Sigma_k''^2X^4 - \frac{1}{48}\Sigma_k^4X^5 + \frac{415}{48}\Sigma_k^6\Sigma_k'^2X^5 + \frac{383}{8}\Sigma_k^7\Sigma_k'^3X^5 \\
& + \frac{9115}{216}\Sigma_k^8\Sigma_k'^4X^5 - \frac{29}{72}\Sigma_k^{10}\Sigma_k''^2X^5 + \frac{1}{6}\Sigma_k^6X^6 - 10\Sigma_k^8\Sigma_k'^2X^6 - \frac{493}{12}\Sigma_k^9\Sigma_k'^3X^6 - \frac{1421}{54}\Sigma_k^{10}\Sigma_k'^4X^6 \\
& + \frac{7}{24}\Sigma_k^8X^7 + \frac{11}{3}\Sigma_k^{10}\Sigma_k'^2X^7 + \frac{116}{9}\Sigma_k^{11}\Sigma_k'^3X^7 + \frac{58}{9}\Sigma_k^{12}\Sigma_k'^4X^7\Big],
\end{aligned}
$$

$$
\Delta\tilde{K}_{21} = \int \frac{d^4k}{(2\pi)^4}\Big[-\frac{1}{32}\Sigma_kX^2 + \frac{1}{8}\Sigma_k^3X^3\Big],
$$

$$
\Delta\tilde{K}_{22} = \int \frac{d^4k}{(2\pi)^4}\Big[-\frac{1}{32}\Sigma_k'^2X + \frac{3}{32}\Sigma_k^2\Sigma_k'^2X^2 - \frac{3}{32}\Sigma_k^2X^3 - \frac{1}{16}\Sigma_k^4\Sigma_k'^2X^3 + \frac{9}{32}\Sigma_k^4X^4\Big],
$$

$$
\begin{aligned}
\Delta\tilde{K}_{23} = \int \frac{d^4k}{(2\pi)^4}\Big[ & -\frac{1}{16}\Sigma_k'^3X + \frac{1}{2}\Sigma_k\Sigma_k'^2X^2 + \frac{1}{4}\Sigma_k^2\Sigma_k'^3X^2 - \frac{1}{32}\Sigma_kX^3 - \frac{37}{8}\Sigma_k^3\Sigma_k'^2X^3 - \frac{5}{16}\Sigma_k^4\Sigma_k'^3X^3 \\
& - \frac{1}{48}\Sigma_k^3X^4 + \frac{317}{24}\Sigma_k^5\Sigma_k'^2X^4 + \frac{1}{8}\Sigma_k^6\Sigma_k'^3X^4 + \frac{53}{48}\Sigma_k^5X^5 - \frac{179}{12}\Sigma_k^7\Sigma_k'^2X^5 - \frac{35}{24}\Sigma_k^7X^6 \\
& + \frac{35}{6}\Sigma_k^9\Sigma_k'^2X^6\Big],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_{24} &= \int \frac{d^4k}{(2\pi)^4}\bigg[-\Sigma_k^2\Sigma_k'^2X^2 - \frac{3}{16}\Sigma_k^2X^3 + 4\Sigma_k^4\Sigma_k'^2X^3 + \frac{13}{16}\Sigma_k^4X^4 - 5\Sigma_k^6\Sigma_k'^2X^4 - \frac{1}{2}\Sigma_k^6X^5 + 2\Sigma_k^8\Sigma_k'^2X^5\bigg],\\
\Delta\tilde{K}_{25} &= \int \frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{16}\Sigma_k'^3X + \frac{3}{8}\Sigma_k\Sigma_k'^2X^2 + \frac{1}{4}\Sigma_k^2\Sigma_k'^3X^2 - \frac{1}{32}\Sigma_kX^3 - \frac{29}{8}\Sigma_k^3\Sigma_k'^2X^3 - \frac{5}{16}\Sigma_k^4\Sigma_k'^3X^3\\
&\quad + \frac{3}{32}\Sigma_k^3X^4 + \frac{43}{4}\Sigma_k^5\Sigma_k'^2X^4 + \frac{1}{8}\Sigma_k^6\Sigma_k'^3X^4 + \frac{7}{8}\Sigma_k^5X^5 - \frac{25}{2}\Sigma_k^7\Sigma_k'^2X^5 - \frac{5}{4}\Sigma_k^7X^6 + 5\Sigma_k^9\Sigma_k'^2X^6\bigg],\\
\Delta\tilde{K}_{26} &= \int \frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{16}\Sigma_k'^2X - \frac{19}{16}\Sigma_k^2\Sigma_k'^2X^2 + \frac{1}{8}\Sigma_k^2X^3 + \frac{33}{8}\Sigma_k^4\Sigma_k'^2X^3 + \frac{1}{4}\Sigma_k^4X^4 - 5\Sigma_k^6\Sigma_k'^2X^4 - \frac{1}{2}\Sigma_k^6X^5 + 2\Sigma_k^8\Sigma_k'^2X^5\bigg],\\
\Delta\tilde{K}_{27} &= \int \frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{32}\Sigma_k''^2 - \frac{1}{720}\Sigma_k\Sigma_k'''X - \frac{37}{216}\Sigma_k'^4X + \frac{11}{18}\Sigma_k^2\Sigma_k''^2X + \frac{43}{360}\Sigma_k'^2X^2 - \frac{1}{180}\Sigma_k\Sigma_k''X^2\\
&\quad + \frac{49}{120}\Sigma_k\Sigma_k'^3X^2 + \frac{1}{720}\Sigma_k^3\Sigma_k'''X^2 + \frac{1079}{216}\Sigma_k^2\Sigma_k'^4X^2 - \frac{745}{288}\Sigma_k^4\Sigma_k''^2X^2 + \frac{47}{1440}X^3 + \frac{47}{360}\Sigma_k\Sigma_k'X^3\\
&\quad - \frac{73}{144}\Sigma_k^2\Sigma_k'^2X^3 - \frac{2113}{180}\Sigma_k^3\Sigma_k'^3X^3 - \frac{6673}{216}\Sigma_k^4\Sigma_k'^4X^3 + \frac{71}{16}\Sigma_k^6\Sigma_k''^2X^3 - \frac{143}{480}\Sigma_k^2X^4 + \frac{197}{120}\Sigma_k^4\Sigma_k'^2X^4\\
&\quad + \frac{21359}{360}\Sigma_k^5\Sigma_k'^3X^4 + \frac{17257}{216}\Sigma_k^6\Sigma_k'^4X^4 - \frac{245}{72}\Sigma_k^8\Sigma_k''^2X^4 + \frac{2}{3}\Sigma_k^4X^5 - \frac{15}{4}\Sigma_k^6\Sigma_k'^2X^5 - \frac{2089}{18}\Sigma_k^7\Sigma_k'^3X^5\\
&\quad - \frac{10993}{108}\Sigma_k^8\Sigma_k'^4X^5 + \frac{35}{36}\Sigma_k^{10}\Sigma_k''^2X^5 + \frac{1}{24}\Sigma_k^6X^6 + \frac{10}{3}\Sigma_k^8\Sigma_k'^2X^6 + \frac{595}{6}\Sigma_k^9\Sigma_k'^3X^6 + \frac{1715}{27}\Sigma_k^{10}\Sigma_k'^4X^6\\
&\quad - \frac{59}{48}\Sigma_k^8X^7 - \frac{5}{6}\Sigma_k^{10}\Sigma_k'^2X^7 - \frac{280}{9}\Sigma_k^{11}\Sigma_k'^3X^7 - \frac{140}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],\\
\Delta\tilde{K}_{28} &= \int \frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{48}\Sigma_k''^2 - \frac{1}{960}\Sigma_k\Sigma_k'''X - \frac{13}{108}\Sigma_k'^4X + \frac{7}{18}\Sigma_k^2\Sigma_k''^2X + \frac{13}{480}\Sigma_k'^2X^2 + \frac{1}{160}\Sigma_k\Sigma_k''X^2\\
&\quad + \frac{137}{480}\Sigma_k\Sigma_k'^3X^2 + \frac{1}{960}\Sigma_k^3\Sigma_k'''X^2 + \frac{347}{108}\Sigma_k^2\Sigma_k'^4X^2 - \frac{235}{144}\Sigma_k^4\Sigma_k''^2X^2 + \frac{1}{960}X^3 + \frac{1}{240}\Sigma_k\Sigma_k'X^3\\
&\quad - \frac{29}{64}\Sigma_k^2\Sigma_k'^2X^3 - \frac{10813}{1440}\Sigma_k^3\Sigma_k'^3X^3 - \frac{2113}{108}\Sigma_k^4\Sigma_k'^4X^3 + \frac{67}{24}\Sigma_k^6\Sigma_k''^2X^3 - \frac{127}{960}\Sigma_k^2X^4 + \frac{931}{480}\Sigma_k^4\Sigma_k'^2X^4\\
&\quad + \frac{26981}{720}\Sigma_k^5\Sigma_k'^3X^4 + \frac{5437}{108}\Sigma_k^6\Sigma_k'^4X^4 - \frac{77}{36}\Sigma_k^8\Sigma_k''^2X^4 + \frac{27}{32}\Sigma_k^4X^5 - \frac{89}{24}\Sigma_k^6\Sigma_k'^2X^5 - \frac{2629}{36}\Sigma_k^7\Sigma_k'^3X^5\\
&\quad - \frac{3457}{54}\Sigma_k^8\Sigma_k'^4X^5 + \frac{11}{18}\Sigma_k^{10}\Sigma_k''^2X^5 - \frac{35}{32}\Sigma_k^6X^6 + \frac{10}{3}\Sigma_k^8\Sigma_k'^2X^6 + \frac{187}{3}\Sigma_k^9\Sigma_k'^3X^6 + \frac{1078}{27}\Sigma_k^{10}\Sigma_k'^4X^6\\
&\quad + \frac{65}{96}\Sigma_k^8X^7 - \frac{7}{6}\Sigma_k^{10}\Sigma_k'^2X^7 - \frac{176}{9}\Sigma_k^{11}\Sigma_k'^3X^7 - \frac{88}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],\\
\Delta\tilde{K}_{29} &= \int \frac{d^4k}{(2\pi)^4}\bigg[\frac{7}{96}\Sigma_k''^2 + \frac{1}{320}\Sigma_k\Sigma_k'''X + \frac{29}{72}\Sigma_k'^4X - \frac{17}{12}\Sigma_k^2\Sigma_k''^2X - \frac{89}{480}\Sigma_k'^2X^2 + \frac{1}{480}\Sigma_k\Sigma_k''X^2\\
&\quad - \frac{391}{480}\Sigma_k\Sigma_k'^3X^2 - \frac{1}{320}\Sigma_k^3\Sigma_k'''X^2 - \frac{835}{72}\Sigma_k^2\Sigma_k'^4X^2 + \frac{575}{96}\Sigma_k^4\Sigma_k''^2X^2 - \frac{11}{320}X^3 - \frac{11}{80}\Sigma_k\Sigma_k'X^3\\
&\quad + \frac{265}{192}\Sigma_k^2\Sigma_k'^2X^3 + \frac{12793}{480}\Sigma_k^3\Sigma_k'^3X^3 + \frac{5153}{72}\Sigma_k^4\Sigma_k'^4X^3 - \frac{493}{48}\Sigma_k^6\Sigma_k''^2X^3 + \frac{431}{960}\Sigma_k^2X^4 - \frac{801}{160}\Sigma_k^4\Sigma_k'^2X^4\\
&\quad - \frac{32801}{240}\Sigma_k^5\Sigma_k'^3X^4 - \frac{4439}{24}\Sigma_k^6\Sigma_k'^4X^4 + \frac{63}{8}\Sigma_k^8\Sigma_k''^2X^4 - \frac{7}{6}\Sigma_k^4X^5 + 9\Sigma_k^6\Sigma_k'^2X^5 + \frac{805}{3}\Sigma_k^7\Sigma_k'^3X^5\\
&\quad + \frac{2827}{12}\Sigma_k^8\Sigma_k'^4X^5 - \frac{9}{4}\Sigma_k^{10}\Sigma_k''^2X^5 - \frac{17}{12}\Sigma_k^6X^6 - \frac{20}{3}\Sigma_k^8\Sigma_k'^2X^6 - \frac{459}{2}\Sigma_k^9\Sigma_k'^3X^6 - 147\Sigma_k^{10}\Sigma_k'^4X^6\\
&\quad + \frac{45}{16}\Sigma_k^8X^7 + \frac{3}{2}\Sigma_k^{10}\Sigma_k'^2X^7 + 72\Sigma_k^{11}\Sigma_k'^3X^7 + 36\Sigma_k^{12}\Sigma_k'^4X^7\bigg],
\end{aligned}
$$

$$\Delta\tilde{K}_{30}=\int\frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{96}\Sigma_k''^2+\frac{1}{2880}\Sigma_k\Sigma_k'''X+\frac{11}{216}\Sigma_k'^4X-\frac{2}{9}\Sigma_k^2\Sigma_k''^2X-\frac{29}{720}\Sigma_k'^2X^2+\frac{1}{720}\Sigma_k\Sigma_k''X^2$$
$$-\frac{59}{480}\Sigma_k\Sigma_k'^3X^2-\frac{1}{2880}\Sigma_k^3\Sigma_k'''X^2-\frac{385}{216}\Sigma_k^2\Sigma_k'^4X^2+\frac{275}{288}\Sigma_k^4\Sigma_k''^2X^2+\frac{29}{2880}X^3+\frac{29}{720}\Sigma_k\Sigma_k'X^3$$
$$+\frac{343}{576}\Sigma_k^2\Sigma_k'^2X^3+\frac{6091}{1440}\Sigma_k^3\Sigma_k'^3X^3+\frac{2447}{216}\Sigma_k^4\Sigma_k'^4X^3-\frac{79}{48}\Sigma_k^6\Sigma_k''^2X^3+\frac{23}{320}\Sigma_k^2X^4-\frac{847}{480}\Sigma_k^4\Sigma_k'^2X^4$$
$$-\frac{15737}{720}\Sigma_k^5\Sigma_k'^3X^4-\frac{6383}{216}\Sigma_k^6\Sigma_k'^4X^4+\frac{91}{72}\Sigma_k^8\Sigma_k''^2X^4-\frac{37}{96}\Sigma_k^4X^5+\frac{37}{24}\Sigma_k^6\Sigma_k'^2X^5+\frac{1549}{36}\Sigma_k^7\Sigma_k'^3X^5$$
$$+\frac{4079}{108}\Sigma_k^8\Sigma_k'^4X^5-\frac{13}{36}\Sigma_k^{10}\Sigma_k''^2X^5-\frac{61}{96}\Sigma_k^6X^6-\frac{221}{6}\Sigma_k^9\Sigma_k'^3X^6-\frac{637}{27}\Sigma_k^{10}\Sigma_k'^4X^6+\frac{29}{32}\Sigma_k^8X^7$$
$$-\frac{1}{3}\Sigma_k^{10}\Sigma_k'^2X^7+\frac{104}{9}\Sigma_k^{11}\Sigma_k'^3X^7+\frac{52}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],$$

$$\Delta\tilde{K}_{31}=\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{32}\Sigma_k''^2-\frac{1}{960}\Sigma_k\Sigma_k'''X-\frac{35}{216}\Sigma_k'^4X+\frac{23}{36}\Sigma_k^2\Sigma_k''^2X+\frac{19}{240}\Sigma_k'^2X^2-\frac{1}{240}\Sigma_k\Sigma_k''X^2$$
$$+\frac{39}{160}\Sigma_k\Sigma_k'^3X^2+\frac{1}{960}\Sigma_k^3\Sigma_k'''X^2+\frac{1117}{216}\Sigma_k^2\Sigma_k'^4X^2-\frac{785}{288}\Sigma_k^4\Sigma_k''^2X^2-\frac{3}{320}X^3-\frac{3}{80}\Sigma_k\Sigma_k'X^3$$
$$-\frac{65}{64}\Sigma_k^2\Sigma_k'^2X^3-\frac{16753}{1440}\Sigma_k^3\Sigma_k'^3X^3-\frac{7007}{216}\Sigma_k^4\Sigma_k'^4X^3+\frac{75}{16}\Sigma_k^6\Sigma_k''^2X^3-\frac{49}{320}\Sigma_k^2X^4+\frac{1531}{480}\Sigma_k^4\Sigma_k'^2X^4$$
$$+\frac{44441}{720}\Sigma_k^5\Sigma_k'^3X^4+\frac{18203}{216}\Sigma_k^6\Sigma_k'^4X^4-\frac{259}{72}\Sigma_k^8\Sigma_k''^2X^4+\frac{73}{96}\Sigma_k^4X^5-\frac{37}{12}\Sigma_k^6\Sigma_k'^2X^5-\frac{2201}{18}\Sigma_k^7\Sigma_k'^3X^5$$
$$-\frac{11615}{108}\Sigma_k^8\Sigma_k'^4X^5+\frac{37}{36}\Sigma_k^{10}\Sigma_k''^2X^5+\frac{133}{96}\Sigma_k^6X^6+\frac{629}{6}\Sigma_k^9\Sigma_k'^3X^6+\frac{1813}{27}\Sigma_k^{10}\Sigma_k'^4X^6-\frac{199}{96}\Sigma_k^8X^7$$
$$+\frac{5}{6}\Sigma_k^{10}\Sigma_k'^2X^7-\frac{296}{9}\Sigma_k^{11}\Sigma_k'^3X^7-\frac{148}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],$$

$$\Delta\tilde{K}_{32}=\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{32}\Sigma_k''^2-\frac{1}{960}\Sigma_k\Sigma_k'''X-\frac{85}{432}\Sigma_k'^4X+\frac{77}{144}\Sigma_k^2\Sigma_k''^2X+\frac{23}{480}\Sigma_k'^2X^2+\frac{1}{60}\Sigma_k\Sigma_k''X^2$$
$$+\frac{217}{480}\Sigma_k\Sigma_k'^3X^2+\frac{1}{960}\Sigma_k^3\Sigma_k'''X^2+\frac{1949}{432}\Sigma_k^2\Sigma_k'^4X^2-\frac{635}{288}\Sigma_k^4\Sigma_k''^2X^2-\frac{19}{960}X^3-\frac{19}{240}\Sigma_k\Sigma_k'X^3$$
$$-\frac{109}{64}\Sigma_k^2\Sigma_k'^2X^3-\frac{14953}{1440}\Sigma_k^3\Sigma_k'^3X^3-\frac{11509}{432}\Sigma_k^4\Sigma_k'^4X^3+\frac{15}{4}\Sigma_k^6\Sigma_k''^2X^3-\frac{77}{960}\Sigma_k^2X^4+\frac{2033}{240}\Sigma_k^4\Sigma_k'^2X^4$$
$$+\frac{36491}{720}\Sigma_k^5\Sigma_k'^3X^4+\frac{29311}{432}\Sigma_k^6\Sigma_k'^4X^4-\frac{413}{144}\Sigma_k^8\Sigma_k''^2X^4+\frac{1}{48}\Sigma_k^4X^5-\frac{139}{8}\Sigma_k^6\Sigma_k'^2X^5-\frac{883}{9}\Sigma_k^7\Sigma_k'^3X^5$$
$$-\frac{18565}{216}\Sigma_k^8\Sigma_k'^4X^5+\frac{59}{72}\Sigma_k^{10}\Sigma_k''^2X^5+\frac{41}{24}\Sigma_k^6X^6+\frac{50}{3}\Sigma_k^8\Sigma_k'^2X^6+\frac{1003}{12}\Sigma_k^9\Sigma_k'^3X^6+\frac{2891}{54}\Sigma_k^{10}\Sigma_k'^4X^6$$
$$-\frac{11}{12}\Sigma_k^8X^7-\frac{37}{6}\Sigma_k^{10}\Sigma_k'^2X^7-\frac{236}{9}\Sigma_k^{11}\Sigma_k'^3X^7-\frac{118}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],$$

$$\Delta\tilde{K}_{33}=\int\frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{288}\Sigma_k''^2+\frac{1}{480}\Sigma_k\Sigma_k'''X+\frac{7}{108}\Sigma_k'^4X+\frac{5}{72}\Sigma_k^2\Sigma_k''^2X-\frac{1}{30}\Sigma_k'^2X^2+\frac{1}{120}\Sigma_k\Sigma_k''X^2$$
$$+\frac{1}{80}\Sigma_k\Sigma_k'^3X^2-\frac{1}{480}\Sigma_k^3\Sigma_k'''X^2+\frac{17}{54}\Sigma_k^2\Sigma_k'^4X^2-\frac{115}{288}\Sigma_k^4\Sigma_k''^2X^2-\frac{41}{480}X^3-\frac{41}{120}\Sigma_k\Sigma_k'X^3$$
$$-\frac{39}{32}\Sigma_k^2\Sigma_k'^2X^3-\frac{133}{80}\Sigma_k^3\Sigma_k'^3X^3-\frac{455}{108}\Sigma_k^4\Sigma_k'^4X^3+\frac{107}{144}\Sigma_k^6\Sigma_k''^2X^3+\frac{17}{480}\Sigma_k^2X^4-\frac{177}{80}\Sigma_k^4\Sigma_k'^2X^4$$
$$+\frac{1727}{180}\Sigma_k^5\Sigma_k'^3X^4+\frac{115}{9}\Sigma_k^6\Sigma_k'^4X^4-\frac{7}{12}\Sigma_k^8\Sigma_k''^2X^4+\frac{25}{24}\Sigma_k^4X^5+\frac{137}{6}\Sigma_k^6\Sigma_k'^2X^5-\frac{353}{18}\Sigma_k^7\Sigma_k'^3X^5$$
$$-\frac{103}{6}\Sigma_k^8\Sigma_k'^4X^5+\frac{1}{6}\Sigma_k^{10}\Sigma_k''^2X^5+\frac{11}{3}\Sigma_k^6X^6-\frac{100}{3}\Sigma_k^8\Sigma_k'^2X^6+17\Sigma_k^9\Sigma_k'^3X^6+\frac{98}{9}\Sigma_k^{10}\Sigma_k'^4X^6$$
$$-4\Sigma_k^8X^7+14\Sigma_k^{10}\Sigma_k'^2X^7-\frac{16}{3}\Sigma_k^{11}\Sigma_k'^3X^7-\frac{8}{3}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],$$

$$
\begin{aligned}
\Delta\tilde{K}_{34}=\int\frac{d^4k}{(2\pi)^4}\bigg[&-\frac{17}{288}\Sigma_k''^2-\frac{71}{216}\Sigma_k'^4X+\frac{41}{36}\Sigma_k^2\Sigma_k''^2X+\frac{13}{48}\Sigma_k'^2X^2+\frac{11}{12}\Sigma_k\Sigma_k'^3X^2+\frac{2017}{216}\Sigma_k^2\Sigma_k'^4X^2\\
&-\frac{1385}{288}\Sigma_k^4\Sigma_k''^2X^2+\frac{1}{24}X^3+\frac{1}{6}\Sigma_k\Sigma_k'X^3-\frac{59}{24}\Sigma_k^2\Sigma_k'^2X^3-\frac{1615}{72}\Sigma_k^3\Sigma_k'^3X^3-\frac{12419}{216}\Sigma_k^4\Sigma_k'^4X^3\\
&+\frac{1187}{144}\Sigma_k^6\Sigma_k''^2X^3-\frac{1}{2}\Sigma_k^2X^4+\frac{311}{48}\Sigma_k^4\Sigma_k'^2X^4+\frac{2663}{24}\Sigma_k^5\Sigma_k'^3X^4+\frac{32071}{216}\Sigma_k^6\Sigma_k'^4X^4-\frac{455}{72}\Sigma_k^8\Sigma_k''^2X^4\\
&+\frac{1}{3}\Sigma_k^4X^5-\frac{71}{12}\Sigma_k^6\Sigma_k'^2X^5-\frac{1295}{6}\Sigma_k^7\Sigma_k'^3X^5-\frac{20419}{108}\Sigma_k^8\Sigma_k'^4X^5+\frac{65}{36}\Sigma_k^{10}\Sigma_k''^2X^5+\frac{55}{12}\Sigma_k^6X^6\\
&+\frac{1105}{6}\Sigma_k^9\Sigma_k'^3X^6+\frac{3185}{27}\Sigma_k^{10}\Sigma_k'^4X^6-\frac{35}{6}\Sigma_k^8X^7+\frac{5}{3}\Sigma_k^{10}\Sigma_k'^2X^7-\frac{520}{9}\Sigma_k^{11}\Sigma_k'^3X^7-\frac{260}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_{35}=\int\frac{d^4k}{(2\pi)^4}\bigg[&\frac{5}{288}\Sigma_k''^2-\frac{1}{480}\Sigma_k\Sigma_k'''X+\frac{7}{72}\Sigma_k'^4X-\frac{1}{3}\Sigma_k^2\Sigma_k''^2X+\frac{1}{30}\Sigma_k'^2X^2-\frac{1}{120}\Sigma_k\Sigma_k''X^2\\
&-\frac{53}{240}\Sigma_k\Sigma_k'^3X^2+\frac{1}{480}\Sigma_k^3\Sigma_k'''X^2-\frac{197}{72}\Sigma_k^2\Sigma_k'^4X^2+\frac{45}{32}\Sigma_k^4\Sigma_k''^2X^2+\frac{7}{160}X^3+\frac{7}{40}\Sigma_k\Sigma_k'X^3\\
&+\frac{143}{96}\Sigma_k^2\Sigma_k'^2X^3+\frac{4577}{720}\Sigma_k^3\Sigma_k'^3X^3+\frac{1211}{72}\Sigma_k^4\Sigma_k'^4X^3-\frac{347}{144}\Sigma_k^6\Sigma_k''^2X^3-\frac{37}{480}\Sigma_k^2X^4-\frac{133}{30}\Sigma_k^4\Sigma_k'^2X^4\\
&-\frac{11579}{360}\Sigma_k^5\Sigma_k'^3X^4-\frac{9377}{216}\Sigma_k^6\Sigma_k'^4X^4+\frac{133}{72}\Sigma_k^8\Sigma_k''^2X^4-\frac{1}{2}\Sigma_k^4X^5+\frac{1}{24}\Sigma_k^6\Sigma_k'^2X^5+\frac{2267}{36}\Sigma_k^7\Sigma_k'^3X^5\\
&+\frac{5969}{108}\Sigma_k^8\Sigma_k'^4X^5-\frac{19}{36}\Sigma_k^{10}\Sigma_k''^2X^5-\frac{19}{8}\Sigma_k^6X^6+\frac{20}{3}\Sigma_k^8\Sigma_k'^2X^6-\frac{323}{6}\Sigma_k^9\Sigma_k'^3X^6-\frac{931}{27}\Sigma_k^{10}\Sigma_k'^4X^6\\
&+\frac{61}{24}\Sigma_k^8X^7-\frac{23}{6}\Sigma_k^{10}\Sigma_k'^2X^7+\frac{152}{9}\Sigma_k^{11}\Sigma_k'^3X^7+\frac{76}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_{36}=\int\frac{d^4k}{(2\pi)^4}\bigg[&-\frac{5}{288}\Sigma_k''^2-\frac{1}{480}\Sigma_k\Sigma_k'''X-\frac{1}{12}\Sigma_k'^4X+\frac{3}{8}\Sigma_k^2\Sigma_k''^2X+\frac{21}{80}\Sigma_k'^2X^2+\frac{1}{80}\Sigma_k\Sigma_k''X^2\\
&+\frac{37}{240}\Sigma_k\Sigma_k'^3X^2+\frac{1}{480}\Sigma_k^3\Sigma_k'''X^2+3\Sigma_k^2\Sigma_k'^4X^2-\frac{155}{96}\Sigma_k^4\Sigma_k''^2X^2-\frac{3}{160}X^3-\frac{3}{40}\Sigma_k\Sigma_k'X^3\\
&-\frac{367}{96}\Sigma_k^2\Sigma_k'^2X^3-\frac{5003}{720}\Sigma_k^3\Sigma_k'^3X^3-\frac{689}{36}\Sigma_k^4\Sigma_k'^4X^3+\frac{401}{144}\Sigma_k^6\Sigma_k''^2X^3-\frac{127}{480}\Sigma_k^2X^4+\frac{1109}{60}\Sigma_k^4\Sigma_k'^2X^4\\
&+\frac{6613}{180}\Sigma_k^5\Sigma_k'^3X^4+\frac{2699}{54}\Sigma_k^6\Sigma_k'^4X^4-\frac{77}{36}\Sigma_k^8\Sigma_k''^2X^4+\frac{41}{24}\Sigma_k^4X^5-\frac{121}{3}\Sigma_k^6\Sigma_k'^2X^5-\frac{1309}{18}\Sigma_k^7\Sigma_k'^3X^5\\
&-\frac{3451}{54}\Sigma_k^8\Sigma_k'^4X^5+\frac{11}{18}\Sigma_k^{10}\Sigma_k''^2X^5-\frac{8}{3}\Sigma_k^6X^6+40\Sigma_k^8\Sigma_k'^2X^6+\frac{187}{3}\Sigma_k^9\Sigma_k'^3X^6+\frac{1078}{27}\Sigma_k^{10}\Sigma_k'^4X^6\\
&+\frac{11}{6}\Sigma_k^8X^7-\frac{44}{3}\Sigma_k^{10}\Sigma_k'^2X^7-\frac{176}{9}\Sigma_k^{11}\Sigma_k'^3X^7-\frac{88}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],
\end{aligned}
$$

$$
\begin{aligned}
\Delta\tilde{K}_{37}=\int\frac{d^4k}{(2\pi)^4}\bigg[&\frac{5}{288}\Sigma_k''^2+\frac{1}{480}\Sigma_k\Sigma_k'''X+\frac{1}{12}\Sigma_k'^4X-\frac{3}{8}\Sigma_k^2\Sigma_k''^2X-\frac{19}{120}\Sigma_k'^2X^2+\frac{1}{120}\Sigma_k\Sigma_k''X^2\\
&-\frac{37}{240}\Sigma_k\Sigma_k'^3X^2-\frac{1}{480}\Sigma_k^3\Sigma_k'''X^2-3\Sigma_k^2\Sigma_k'^4X^2+\frac{155}{96}\Sigma_k^4\Sigma_k''^2X^2-\frac{11}{480}X^3-\frac{11}{120}\Sigma_k\Sigma_k'X^3\\
&+\frac{191}{96}\Sigma_k^2\Sigma_k'^2X^3+\frac{5003}{720}\Sigma_k^3\Sigma_k'^3X^3+\frac{689}{36}\Sigma_k^4\Sigma_k'^4X^3-\frac{401}{144}\Sigma_k^6\Sigma_k''^2X^3+\frac{37}{480}\Sigma_k^2X^4-\frac{839}{60}\Sigma_k^4\Sigma_k'^2X^4\\
&-\frac{6613}{180}\Sigma_k^5\Sigma_k'^3X^4-\frac{2699}{54}\Sigma_k^6\Sigma_k'^4X^4+\frac{77}{36}\Sigma_k^8\Sigma_k''^2X^4-\frac{7}{12}\Sigma_k^4X^5+\frac{75}{2}\Sigma_k^6\Sigma_k'^2X^5+\frac{1309}{18}\Sigma_k^7\Sigma_k'^3X^5\\
&+\frac{3451}{54}\Sigma_k^8\Sigma_k'^4X^5-\frac{11}{18}\Sigma_k^{10}\Sigma_k''^2X^5+\frac{8}{3}\Sigma_k^6X^6-40\Sigma_k^8\Sigma_k'^2X^6-\frac{187}{3}\Sigma_k^9\Sigma_k'^3X^6-\frac{1078}{27}\Sigma_k^{10}\Sigma_k'^4X^6\\
&-\frac{11}{6}\Sigma_k^8X^7+\frac{44}{3}\Sigma_k^{10}\Sigma_k'^2X^7+\frac{176}{9}\Sigma_k^{11}\Sigma_k'^3X^7+\frac{88}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],
\end{aligned}
$$

$$\Delta\tilde{K}_{38}=\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{144}\Sigma_k''^2-\frac{1}{240}\Sigma_k\Sigma_k'''X-\frac{5}{54}\Sigma_k'^4X-\frac{1}{36}\Sigma_k^2\Sigma_k''^2X+\frac{1}{15}\Sigma_k'^2X^2-\frac{1}{60}\Sigma_k\Sigma_k''X^2$$
$$-\frac{1}{40}\Sigma_k\Sigma_k'^3X^2+\frac{1}{240}\Sigma_k^3\Sigma_k'''X^2+\frac{2}{27}\Sigma_k^2\Sigma_k'^4X^2+\frac{35}{144}\Sigma_k^4\Sigma_k''^2X^2+\frac{7}{80}X^3+\frac{7}{20}\Sigma_k\Sigma_k'X^3$$
$$+\frac{173}{48}\Sigma_k^2\Sigma_k'^2X^3+\frac{397}{360}\Sigma_k^3\Sigma_k'^3X^3+\frac{121}{54}\Sigma_k^4\Sigma_k'^4X^3-\frac{35}{72}\Sigma_k^6\Sigma_k''^2X^3-\frac{77}{240}\Sigma_k^2X^4-\frac{1969}{120}\Sigma_k^4\Sigma_k'^2X^4$$
$$-\frac{63}{10}\Sigma_k^5\Sigma_k'^3X^4-\frac{217}{27}\Sigma_k^6\Sigma_k'^4X^4+\frac{7}{18}\Sigma_k^8\Sigma_k''^2X^4-\frac{9}{4}\Sigma_k^4X^5+\frac{70}{3}\Sigma_k^6\Sigma_k'^2X^5+13\Sigma_k^7\Sigma_k'^3X^5$$
$$+\frac{305}{27}\Sigma_k^8\Sigma_k'^4X^5-\frac{1}{9}\Sigma_k^{10}\Sigma_k''^2X^5+2\Sigma_k^6X^6-\frac{40}{3}\Sigma_k^8\Sigma_k'^2X^6-\frac{34}{3}\Sigma_k^9\Sigma_k'^3X^6-\frac{196}{27}\Sigma_k^{10}\Sigma_k'^4X^6$$
$$-\frac{1}{3}\Sigma_k^8X^7+\frac{8}{3}\Sigma_k^{10}\Sigma_k'^2X^7+\frac{32}{9}\Sigma_k^{11}\Sigma_k'^3X^7+\frac{16}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],$$

$$\Delta\tilde{K}_{39}=\int\frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{120}\Sigma_k\Sigma_k'''X-\frac{5}{108}\Sigma_k'^4X-\frac{5}{36}\Sigma_k^2\Sigma_k''^2X-\frac{11}{120}\Sigma_k'^2X^2+\frac{1}{30}\Sigma_k\Sigma_k''X^2+\frac{23}{60}\Sigma_k\Sigma_k'^3X^2$$
$$-\frac{1}{120}\Sigma_k^3\Sigma_k'''X^2-\frac{95}{108}\Sigma_k^2\Sigma_k'^4X^2+\frac{25}{36}\Sigma_k^4\Sigma_k''^2X^2-\frac{1}{20}X^3-\frac{1}{5}\Sigma_k\Sigma_k'X^3-\frac{59}{24}\Sigma_k^2\Sigma_k'^2X^3$$
$$+\frac{233}{180}\Sigma_k^3\Sigma_k'^3X^3+\frac{835}{108}\Sigma_k^4\Sigma_k'^4X^3-\frac{5}{4}\Sigma_k^6\Sigma_k''^2X^3-\frac{13}{120}\Sigma_k^2X^4+\frac{379}{60}\Sigma_k^4\Sigma_k'^2X^4-\frac{1291}{90}\Sigma_k^5\Sigma_k'^3X^4$$
$$-\frac{2365}{108}\Sigma_k^6\Sigma_k'^4X^4+\frac{35}{36}\Sigma_k^8\Sigma_k''^2X^4+\frac{7}{4}\Sigma_k^4X^5+3\Sigma_k^6\Sigma_k'^2X^5+\frac{289}{9}\Sigma_k^7\Sigma_k'^3X^5+\frac{1555}{54}\Sigma_k^8\Sigma_k'^4X^5$$
$$-\frac{5}{18}\Sigma_k^{10}\Sigma_k''^2X^5-\frac{40}{3}\Sigma_k^8\Sigma_k'^2X^6-\frac{85}{3}\Sigma_k^9\Sigma_k'^3X^6-\frac{490}{27}\Sigma_k^{10}\Sigma_k'^4X^6-\frac{5}{6}\Sigma_k^8X^7+\frac{20}{3}\Sigma_k^{10}\Sigma_k'^2X^7$$
$$+\frac{80}{9}\Sigma_k^{11}\Sigma_k'^3X^7+\frac{40}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],$$

$$\Delta\tilde{K}_{40}=\int\frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{288}\Sigma_k''^2-\frac{1}{480}\Sigma_k\Sigma_k'''X+\frac{5}{72}\Sigma_k'^4X+\frac{1}{12}\Sigma_k^2\Sigma_k''^2X+\frac{53}{240}\Sigma_k'^2X^2-\frac{1}{120}\Sigma_k\Sigma_k''X^2$$
$$-\frac{43}{240}\Sigma_k\Sigma_k'^3X^2+\frac{1}{480}\Sigma_k^3\Sigma_k'''X^2+\frac{29}{72}\Sigma_k^2\Sigma_k'^4X^2-\frac{15}{32}\Sigma_k^4\Sigma_k''^2X^2+\frac{11}{480}X^3+\frac{11}{120}\Sigma_k\Sigma_k'X^3$$
$$-\frac{127}{96}\Sigma_k^2\Sigma_k'^2X^3-\frac{863}{720}\Sigma_k^3\Sigma_k'^3X^3-\frac{359}{72}\Sigma_k^4\Sigma_k'^4X^3+\frac{125}{144}\Sigma_k^6\Sigma_k''^2X^3-\frac{17}{480}\Sigma_k^2X^4+\frac{1301}{240}\Sigma_k^4\Sigma_k'^2X^4$$
$$+\frac{929}{90}\Sigma_k^5\Sigma_k'^3X^4+\frac{3233}{216}\Sigma_k^6\Sigma_k'^4X^4-\frac{49}{72}\Sigma_k^8\Sigma_k''^2X^4-13\Sigma_k^6\Sigma_k'^2X^5-\frac{203}{9}\Sigma_k^7\Sigma_k'^3X^5-\frac{2165}{108}\Sigma_k^8\Sigma_k'^4X^5$$
$$+\frac{7}{36}\Sigma_k^{10}\Sigma_k''^2X^5-\Sigma_k^6X^6+\frac{40}{3}\Sigma_k^8\Sigma_k'^2X^6+\frac{119}{6}\Sigma_k^9\Sigma_k'^3X^6+\frac{343}{27}\Sigma_k^{10}\Sigma_k'^4X^6+\frac{7}{12}\Sigma_k^8X^7$$
$$-\frac{14}{3}\Sigma_k^{10}\Sigma_k'^2X^7-\frac{56}{9}\Sigma_k^{11}\Sigma_k'^3X^7-\frac{28}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],$$

$$\Delta\tilde{K}_{41}=\int\frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{96}\Sigma_k''X+\frac{1}{48}\Sigma_k'X^2-\frac{19}{48}\Sigma_k\Sigma_k'^2X^2-\frac{1}{12}\Sigma_kX^3-\frac{1}{24}\Sigma_k^2\Sigma_k'X^3+\frac{9}{8}\Sigma_k^3\Sigma_k'^2X^3$$
$$+\frac{3}{8}\Sigma_k^3X^4-\frac{3}{4}\Sigma_k^5\Sigma_k'^2X^4\bigg],$$

$$\Delta\tilde{K}_{42}=\int\frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{96}\Sigma_k''X-\frac{1}{24}\Sigma_k'^3X+\frac{1}{32}\Sigma_k'X^2-\frac{1}{12}\Sigma_k\Sigma_k'^2X^2+\frac{1}{6}\Sigma_k^2\Sigma_k'^3X^2-\frac{1}{16}\Sigma_kX^3$$
$$-\frac{1}{12}\Sigma_k^2\Sigma_k'X^3-\frac{161}{48}\Sigma_k^3\Sigma_k'^2X^3-\frac{5}{24}\Sigma_k^4\Sigma_k'^3X^3-\frac{3}{16}\Sigma_k^3X^4+\frac{55}{4}\Sigma_k^5\Sigma_k'^2X^4+\frac{1}{12}\Sigma_k^6\Sigma_k'^3X^4$$
$$+\frac{9}{4}\Sigma_k^5X^5-17\Sigma_k^7\Sigma_k'^2X^5-\frac{5}{3}\Sigma_k^7X^6+\frac{20}{3}\Sigma_k^9\Sigma_k'^2X^6\bigg],$$

$$\Delta\tilde{K}_{43} = \int \frac{d^4k}{(2\pi)^4}\left[-\frac{1}{12}\Sigma_k'^3X + \frac{1}{48}\Sigma_k'X^2 + \frac{5}{8}\Sigma_k\Sigma_k'^2X^2 + \frac{1}{3}\Sigma_k^2\Sigma_k'^3X^2 - \frac{1}{12}\Sigma_kX^3 - \frac{1}{12}\Sigma_k^2\Sigma_k'X^3 - \frac{215}{24}\Sigma_k^3\Sigma_k'^2X^3 - \frac{5}{12}\Sigma_k^4\Sigma_k'^3X^3 - \frac{1}{4}\Sigma_k^3X^4 + 29\Sigma_k^5\Sigma_k'^2X^4 + \frac{1}{6}\Sigma_k^6\Sigma_k'^3X^4 + \frac{7}{2}\Sigma_k^5X^5 - 34\Sigma_k^7\Sigma_k'^2X^5 - \frac{10}{3}\Sigma_k^7X^6 + \frac{40}{3}\Sigma_k^9\Sigma_k'^2X^6\right],$$

$$\Delta\tilde{K}_{44} = \int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{288}\Sigma_k''^2 + \frac{1}{960}\Sigma_k\Sigma_k'''X - \frac{1}{216}\Sigma_k'^4X - \frac{5}{36}\Sigma_k^2\Sigma_k''^2X - \frac{13}{480}\Sigma_k'^2X^2 + \frac{1}{240}\Sigma_k\Sigma_k''X^2 + \frac{1}{160}\Sigma_k\Sigma_k'^3X^2 - \frac{1}{960}\Sigma_k^3\Sigma_k'''X^2 - \frac{217}{216}\Sigma_k^2\Sigma_k'^4X^2 + \frac{185}{288}\Sigma_k^4\Sigma_k''^2X^2 - \frac{1}{960}X^3 - \frac{1}{240}\Sigma_k\Sigma_k'X^3 + \frac{109}{64}\Sigma_k^2\Sigma_k'^2X^3 + \frac{1211}{480}\Sigma_k^3\Sigma_k'^3X^3 + \frac{1595}{216}\Sigma_k^4\Sigma_k'^4X^3 - \frac{163}{144}\Sigma_k^6\Sigma_k''^2X^3 + \frac{37}{960}\Sigma_k^2X^4 - \frac{2147}{160}\Sigma_k^4\Sigma_k'^2X^4 - \frac{10501}{720}\Sigma_k^5\Sigma_k'^3X^4 - \frac{1445}{72}\Sigma_k^6\Sigma_k'^4X^4 + \frac{7}{8}\Sigma_k^8\Sigma_k''^2X^4 - \frac{43}{48}\Sigma_k^4X^5 + \frac{407}{12}\Sigma_k^6\Sigma_k'^2X^5 + \frac{266}{9}\Sigma_k^7\Sigma_k'^3X^5 + \frac{937}{36}\Sigma_k^8\Sigma_k'^4X^5 - \frac{1}{4}\Sigma_k^{10}\Sigma_k''^2X^5 + \frac{73}{24}\Sigma_k^6X^6 - \frac{205}{6}\Sigma_k^8\Sigma_k'^2X^6 - \frac{51}{2}\Sigma_k^9\Sigma_k'^3X^6 - \frac{49}{3}\Sigma_k^{10}\Sigma_k'^4X^6 - \frac{9}{4}\Sigma_k^8X^7 + 12\Sigma_k^{10}\Sigma_k'^2X^7 + 8\Sigma_k^{11}\Sigma_k'^3X^7 + 4\Sigma_k^{12}\Sigma_k'^4X^7\right],$$

$$\Delta\tilde{K}_{45} = \int \frac{d^4k}{(2\pi)^4}\left[-\frac{1}{288}\Sigma_k''^2 - \frac{1}{960}\Sigma_k\Sigma_k'''X + \frac{1}{216}\Sigma_k'^4X + \frac{5}{36}\Sigma_k^2\Sigma_k''^2X - \frac{37}{480}\Sigma_k'^2X^2 - \frac{1}{240}\Sigma_k\Sigma_k''X^2 - \frac{1}{160}\Sigma_k\Sigma_k'^3X^2 + \frac{1}{960}\Sigma_k^3\Sigma_k'''X^2 + \frac{217}{216}\Sigma_k^2\Sigma_k'^4X^2 - \frac{185}{288}\Sigma_k^4\Sigma_k''^2X^2 + \frac{1}{960}X^3 + \frac{1}{240}\Sigma_k\Sigma_k'X^3 + \frac{353}{192}\Sigma_k^2\Sigma_k'^2X^3 - \frac{1211}{480}\Sigma_k^3\Sigma_k'^3X^3 - \frac{1595}{216}\Sigma_k^4\Sigma_k'^4X^3 + \frac{163}{144}\Sigma_k^6\Sigma_k''^2X^3 + \frac{1}{320}\Sigma_k^2X^4 - \frac{1483}{160}\Sigma_k^4\Sigma_k'^2X^4 + \frac{10501}{720}\Sigma_k^5\Sigma_k'^3X^4 + \frac{1445}{72}\Sigma_k^6\Sigma_k'^4X^4 - \frac{7}{8}\Sigma_k^8\Sigma_k''^2X^4 - \frac{7}{16}\Sigma_k^4X^5 + 19\Sigma_k^6\Sigma_k'^2X^5 - \frac{266}{9}\Sigma_k^7\Sigma_k'^3X^5 - \frac{937}{36}\Sigma_k^8\Sigma_k'^4X^5 + \frac{1}{4}\Sigma_k^{10}\Sigma_k''^2X^5 + \frac{19}{8}\Sigma_k^6X^6 - \frac{35}{2}\Sigma_k^8\Sigma_k'^2X^6 + \frac{51}{2}\Sigma_k^9\Sigma_k'^3X^6 + \frac{49}{3}\Sigma_k^{10}\Sigma_k'^4X^6 - \frac{9}{4}\Sigma_k^8X^7 + 6\Sigma_k^{10}\Sigma_k'^2X^7 - 8\Sigma_k^{11}\Sigma_k'^3X^7 - 4\Sigma_k^{12}\Sigma_k'^4X^7\right],$$

$$\Delta\tilde{K}_{46} = \int \frac{d^4k}{(2\pi)^4}\left[-\frac{1}{24}\Sigma_k''^2 - \frac{1}{960}\Sigma_k\Sigma_k'''X - \frac{13}{48}\Sigma_k'^4X + \frac{11}{16}\Sigma_k^2\Sigma_k''^2X - \frac{7}{480}\Sigma_k'^2X^2 - \frac{1}{240}\Sigma_k\Sigma_k''X^2 + \frac{277}{480}\Sigma_k\Sigma_k'^3X^2 + \frac{1}{960}\Sigma_k^3\Sigma_k'''X^2 + \frac{281}{48}\Sigma_k^2\Sigma_k'^4X^2 - \frac{45}{16}\Sigma_k^4\Sigma_k''^2X^2 + \frac{41}{960}X^3 + \frac{41}{240}\Sigma_k\Sigma_k'X^3 + \frac{589}{192}\Sigma_k^2\Sigma_k'^2X^3 - \frac{2117}{160}\Sigma_k^3\Sigma_k'^3X^3 - \frac{1637}{48}\Sigma_k^4\Sigma_k'^4X^3 + \frac{229}{48}\Sigma_k^6\Sigma_k''^2X^3 - \frac{257}{960}\Sigma_k^2X^4 - \frac{371}{20}\Sigma_k^4\Sigma_k'^2X^4 + \frac{5159}{80}\Sigma_k^5\Sigma_k'^3X^4 + \frac{12449}{144}\Sigma_k^6\Sigma_k'^4X^4 - \frac{175}{48}\Sigma_k^8\Sigma_k''^2X^4 - \frac{43}{48}\Sigma_k^4X^5 + \frac{119}{3}\Sigma_k^6\Sigma_k'^2X^5 - \frac{499}{4}\Sigma_k^7\Sigma_k'^3X^5 - \frac{7871}{72}\Sigma_k^8\Sigma_k'^4X^5 + \frac{25}{24}\Sigma_k^{10}\Sigma_k''^2X^5 + \frac{145}{24}\Sigma_k^6X^6 - \frac{110}{3}\Sigma_k^8\Sigma_k'^2X^6 + \frac{425}{4}\Sigma_k^9\Sigma_k'^3X^6 + \frac{1225}{18}\Sigma_k^{10}\Sigma_k'^4X^6 - \frac{25}{4}\Sigma_k^8X^7 + \frac{25}{2}\Sigma_k^{10}\Sigma_k'^2X^7 - \frac{100}{3}\Sigma_k^{11}\Sigma_k'^3X^7 - \frac{50}{3}\Sigma_k^{12}\Sigma_k'^4X^7\right],$$

$$\Delta\tilde{K}_{47} = \int\frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{48}\Sigma_k''^2 + \frac{67}{432}\Sigma_k'^4X - \frac{41}{144}\Sigma_k^2\Sigma_k''^2X + \frac{1}{24}\Sigma_k'^2X^2 - \frac{11}{48}\Sigma_k\Sigma_k'^3X^2 - \frac{1103}{432}\Sigma_k^2\Sigma_k'^4X^2 + \frac{10}{9}\Sigma_k^4\Sigma_k''^2X^2 - \frac{1}{48}X^3 - \frac{1}{12}\Sigma_k\Sigma_k'X^3 - \frac{223}{96}\Sigma_k^2\Sigma_k'^2X^3 + \frac{187}{36}\Sigma_k^3\Sigma_k'^3X^3 + \frac{5947}{432}\Sigma_k^4\Sigma_k'^4X^3 - \frac{89}{48}\Sigma_k^6\Sigma_k''^2X^3 + \frac{1}{6}\Sigma_k^2X^4 + \frac{1297}{96}\Sigma_k^4\Sigma_k'^2X^4 - \frac{3613}{144}\Sigma_k^5\Sigma_k'^3X^4 - \frac{14653}{432}\Sigma_k^6\Sigma_k'^4X^4 + \frac{203}{144}\Sigma_k^8\Sigma_k''^2X^4 + \frac{13}{16}\Sigma_k^4X^5 - \frac{1367}{48}\Sigma_k^6\Sigma_k'^2X^5 + \frac{3479}{72}\Sigma_k^7\Sigma_k'^3X^5 + \frac{9163}{216}\Sigma_k^8\Sigma_k'^4X^5 - \frac{29}{72}\Sigma_k^{10}\Sigma_k''^2X^5 - \frac{59}{16}\Sigma_k^6X^6 + \frac{155}{6}\Sigma_k^8\Sigma_k'^2X^6 - \frac{493}{12}\Sigma_k^9\Sigma_k'^3X^6 - \frac{1421}{54}\Sigma_k^{10}\Sigma_k'^4X^6 + \frac{161}{48}\Sigma_k^8X^7 - \frac{103}{12}\Sigma_k^{10}\Sigma_k'^2X^7 + \frac{116}{9}\Sigma_k^{11}\Sigma_k'^3X^7 + \frac{58}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],$$

$$\Delta\tilde{K}_{48} = \int\frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{24}\Sigma_k''^2 + \frac{23}{72}\Sigma_k'^4X - \frac{13}{24}\Sigma_k^2\Sigma_k''^2X - \frac{1}{3}\Sigma_k\Sigma_k'^3X^2 - \frac{355}{72}\Sigma_k^2\Sigma_k'^4X^2 + \frac{25}{12}\Sigma_k^4\Sigma_k''^2X^2 - \frac{1}{24}X^3 - \frac{1}{6}\Sigma_k\Sigma_k'X^3 - \frac{9}{2}\Sigma_k^2\Sigma_k'^2X^3 + \frac{28}{3}\Sigma_k^3\Sigma_k'^3X^3 + \frac{1871}{72}\Sigma_k^4\Sigma_k'^4X^3 - \frac{83}{24}\Sigma_k^6\Sigma_k''^2X^3 + \frac{3}{8}\Sigma_k^2X^4 + 33\Sigma_k^4\Sigma_k'^2X^4 - \frac{139}{3}\Sigma_k^5\Sigma_k'^3X^4 - \frac{1523}{24}\Sigma_k^6\Sigma_k'^4X^4 + \frac{21}{8}\Sigma_k^8\Sigma_k''^2X^4 + \frac{13}{8}\Sigma_k^4X^5 - 80\Sigma_k^6\Sigma_k'^2X^5 + \frac{539}{6}\Sigma_k^7\Sigma_k'^3X^5 + \frac{949}{12}\Sigma_k^8\Sigma_k'^4X^5 - \frac{3}{4}\Sigma_k^{10}\Sigma_k''^2X^5 - \frac{77}{8}\Sigma_k^6X^6 + 80\Sigma_k^8\Sigma_k'^2X^6 - \frac{153}{2}\Sigma_k^9\Sigma_k'^3X^6 - 49\Sigma_k^{10}\Sigma_k'^4X^6 + \frac{75}{8}\Sigma_k^8X^7 - \frac{57}{2}\Sigma_k^{10}\Sigma_k'^2X^7 + 24\Sigma_k^{11}\Sigma_k'^3X^7 + 12\Sigma_k^{12}\Sigma_k'^4X^7\bigg],$$

$$\Delta\tilde{K}_{49} = \int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{36}\Sigma_k''^2 - \frac{1}{480}\Sigma_k\Sigma_k'''X - \frac{31}{216}\Sigma_k'^4X + \frac{41}{72}\Sigma_k^2\Sigma_k''^2X + \frac{13}{240}\Sigma_k'^2X^2 - \frac{1}{120}\Sigma_k\Sigma_k''X^2 + \frac{77}{240}\Sigma_k\Sigma_k'^3X^2 + \frac{1}{480}\Sigma_k^3\Sigma_k'''X^2 + \frac{995}{216}\Sigma_k^2\Sigma_k'^4X^2 - \frac{175}{72}\Sigma_k^4\Sigma_k''^2X^2 + \frac{1}{480}X^3 + \frac{1}{120}\Sigma_k\Sigma_k'X^3 + \frac{51}{32}\Sigma_k^2\Sigma_k'^2X^3 - \frac{2591}{240}\Sigma_k^3\Sigma_k'^3X^3 - \frac{6247}{216}\Sigma_k^4\Sigma_k'^4X^3 + \frac{301}{72}\Sigma_k^6\Sigma_k''^2X^3 - \frac{217}{480}\Sigma_k^2X^4 - \frac{1173}{80}\Sigma_k^4\Sigma_k'^2X^4 + \frac{20011}{360}\Sigma_k^5\Sigma_k'^3X^4 + \frac{5411}{72}\Sigma_k^6\Sigma_k'^4X^4 - \frac{77}{24}\Sigma_k^8\Sigma_k''^2X^4 + \frac{1}{6}\Sigma_k^4X^5 + \frac{235}{6}\Sigma_k^6\Sigma_k'^2X^5 - \frac{1967}{18}\Sigma_k^7\Sigma_k'^3X^5 - \frac{1151}{12}\Sigma_k^8\Sigma_k'^4X^5 + \frac{11}{12}\Sigma_k^{10}\Sigma_k''^2X^5 + \frac{139}{24}\Sigma_k^6X^6 - \frac{125}{3}\Sigma_k^8\Sigma_k'^2X^6 + \frac{187}{2}\Sigma_k^9\Sigma_k'^3X^6 + \frac{539}{9}\Sigma_k^{10}\Sigma_k'^4X^6 - \frac{53}{8}\Sigma_k^8X^7 + \frac{31}{2}\Sigma_k^{10}\Sigma_k'^2X^7 - \frac{88}{3}\Sigma_k^{11}\Sigma_k'^3X^7 - \frac{44}{3}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],$$

$$\Delta\tilde{K}_{50} = \int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{36}\Sigma_k''^2 - \frac{11}{54}\Sigma_k'^4X + \frac{7}{18}\Sigma_k^2\Sigma_k''^2X - \frac{1}{32}\Sigma_k'^2X^2 + \frac{17}{48}\Sigma_k\Sigma_k'^3X^2 + \frac{187}{54}\Sigma_k^2\Sigma_k'^4X^2 - \frac{55}{36}\Sigma_k^4\Sigma_k''^2X^2 + \frac{1}{24}X^3 + \frac{1}{6}\Sigma_k\Sigma_k'X^3 + \frac{331}{96}\Sigma_k^2\Sigma_k'^2X^3 - \frac{263}{36}\Sigma_k^3\Sigma_k'^3X^3 - \frac{1019}{54}\Sigma_k^4\Sigma_k'^4X^3 + \frac{23}{9}\Sigma_k^6\Sigma_k''^2X^3 - \frac{3}{16}\Sigma_k^2X^4 - \frac{2129}{96}\Sigma_k^4\Sigma_k'^2X^4 + \frac{1669}{48}\Sigma_k^5\Sigma_k'^3X^4 + \frac{2521}{54}\Sigma_k^6\Sigma_k'^4X^4 - \frac{35}{18}\Sigma_k^8\Sigma_k''^2X^4 - \frac{65}{48}\Sigma_k^4X^5 + \frac{2393}{48}\Sigma_k^6\Sigma_k'^2X^5 - \frac{1601}{24}\Sigma_k^7\Sigma_k'^3X^5 - \frac{1579}{27}\Sigma_k^8\Sigma_k'^4X^5 + \frac{5}{9}\Sigma_k^{10}\Sigma_k''^2X^5 + \frac{295}{48}\Sigma_k^6X^6 - \frac{95}{2}\Sigma_k^8\Sigma_k'^2X^6 + \frac{170}{3}\Sigma_k^9\Sigma_k'^3X^6 + \frac{980}{27}\Sigma_k^{10}\Sigma_k'^4X^6 - \frac{277}{48}\Sigma_k^8X^7 + \frac{197}{12}\Sigma_k^{10}\Sigma_k'^2X^7 - \frac{160}{9}\Sigma_k^{11}\Sigma_k'^3X^7 - \frac{80}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],$$

$$\begin{aligned}
\Delta\tilde{K}_{51} = \int \frac{d^4k}{(2\pi)^4}\bigg[&\frac{11}{288}\Sigma_k''^2 + \frac{1}{480}\Sigma_k\Sigma_k'''X + \frac{2}{9}\Sigma_k'^4X - \frac{17}{24}\Sigma_k^2\Sigma_k''^2X - \frac{43}{240}\Sigma_k'^2X^2 + \frac{1}{120}\Sigma_k\Sigma_k''X^2\\
&- \frac{137}{240}\Sigma_k\Sigma_k'^3X^2 - \frac{1}{480}\Sigma_k^3\Sigma_k'''X^2 - \frac{211}{36}\Sigma_k^2\Sigma_k'^4X^2 + \frac{95}{32}\Sigma_k^4\Sigma_k''^2X^2 - \frac{11}{480}X^3 - \frac{11}{120}\Sigma_k\Sigma_k'X^3\\
&+ \frac{379}{96}\Sigma_k^2\Sigma_k'^2X^3 + \frac{9983}{720}\Sigma_k^3\Sigma_k'^3X^3 + \frac{641}{18}\Sigma_k^4\Sigma_k'^4X^3 - \frac{731}{144}\Sigma_k^6\Sigma_k''^2X^3 + \frac{17}{480}\Sigma_k^2X^4 - \frac{6641}{240}\Sigma_k^4\Sigma_k'^2X^4\\
&- \frac{12313}{180}\Sigma_k^5\Sigma_k'^3X^4 - \frac{9889}{108}\Sigma_k^6\Sigma_k'^4X^4 + \frac{35}{9}\Sigma_k^8\Sigma_k''^2X^4 - \frac{3}{2}\Sigma_k^4X^5 + \frac{847}{12}\Sigma_k^6\Sigma_k'^2X^5 + \frac{1196}{9}\Sigma_k^7\Sigma_k'^3X^5\\
&+ \frac{3143}{27}\Sigma_k^8\Sigma_k'^4X^5 - \frac{10}{9}\Sigma_k^{10}\Sigma_k''^2X^5 + 5\Sigma_k^6X^6 - \frac{220}{3}\Sigma_k^8\Sigma_k'^2X^6 - \frac{340}{3}\Sigma_k^9\Sigma_k'^3X^6 - \frac{1960}{27}\Sigma_k^{10}\Sigma_k'^4X^6\\
&- \frac{10}{3}\Sigma_k^8X^7 + \frac{80}{3}\Sigma_k^{10}\Sigma_k'^2X^7 + \frac{320}{9}\Sigma_k^{11}\Sigma_k'^3X^7 + \frac{160}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],
\end{aligned}$$

$$\begin{aligned}
\Delta\tilde{K}_{52} = \int \frac{d^4k}{(2\pi)^4}\bigg[&-\frac{5}{288}\Sigma_k''^2 - \frac{1}{480}\Sigma_k\Sigma_k'''X - \frac{5}{72}\Sigma_k'^4X + \frac{5}{12}\Sigma_k^2\Sigma_k''^2X - \frac{7}{240}\Sigma_k'^2X^2 - \frac{1}{120}\Sigma_k\Sigma_k''X^2\\
&+ \frac{19}{80}\Sigma_k\Sigma_k'^3X^2 + \frac{1}{480}\Sigma_k^3\Sigma_k'''X^2 + \frac{235}{72}\Sigma_k^2\Sigma_k'^4X^2 - \frac{175}{96}\Sigma_k^4\Sigma_k''^2X^2 + \frac{11}{480}X^3 + \frac{11}{120}\Sigma_k\Sigma_k'X^3\\
&+ \frac{7}{32}\Sigma_k^2\Sigma_k'^2X^3 - \frac{5843}{720}\Sigma_k^3\Sigma_k'^3X^3 - \frac{515}{24}\Sigma_k^4\Sigma_k'^4X^3 + \frac{455}{144}\Sigma_k^6\Sigma_k''^2X^3 + \frac{1}{160}\Sigma_k^2X^4 + \frac{1931}{240}\Sigma_k^4\Sigma_k'^2X^4\\
&+ \frac{3779}{90}\Sigma_k^5\Sigma_k'^3X^4 + \frac{12215}{216}\Sigma_k^6\Sigma_k'^4X^4 - \frac{175}{72}\Sigma_k^8\Sigma_k''^2X^4 - \frac{1}{12}\Sigma_k^4X^5 - \frac{133}{4}\Sigma_k^6\Sigma_k'^2X^5 - \frac{1489}{18}\Sigma_k^7\Sigma_k'^3X^5\\
&- \frac{7835}{108}\Sigma_k^8\Sigma_k'^4X^5 + \frac{25}{36}\Sigma_k^{10}\Sigma_k''^2X^5 - \frac{25}{12}\Sigma_k^6X^6 + \frac{125}{3}\Sigma_k^8\Sigma_k'^2X^6 + \frac{425}{6}\Sigma_k^9\Sigma_k'^3X^6 + \frac{1225}{27}\Sigma_k^{10}\Sigma_k'^4X^6\\
&+ \frac{25}{12}\Sigma_k^8X^7 - \frac{50}{3}\Sigma_k^{10}\Sigma_k'^2X^7 - \frac{200}{9}\Sigma_k^{11}\Sigma_k'^3X^7 - \frac{100}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],
\end{aligned}$$

$$\begin{aligned}
\Delta\tilde{K}_{53} = \int \frac{d^4k}{(2\pi)^4}\bigg[&\frac{33}{32}\Sigma_k\Sigma_k'^2X^2 - \frac{7}{96}\Sigma_kX^3 - \frac{977}{96}\Sigma_k^3\Sigma_k'^2X^3 + \frac{1}{96}\Sigma_k^3X^4 + \frac{1375}{48}\Sigma_k^5\Sigma_k'^2X^4 + \frac{61}{24}\Sigma_k^5X^5\\
&- \frac{187}{6}\Sigma_k^7\Sigma_k'^2X^5 - \frac{35}{12}\Sigma_k^7X^6 + \frac{35}{3}\Sigma_k^9\Sigma_k'^2X^6\bigg],
\end{aligned}$$

$$\begin{aligned}
\Delta\tilde{K}_{54} = \int \frac{d^4k}{(2\pi)^4}\bigg[&\frac{1}{48}\Sigma_k'^3X + \frac{1}{96}\Sigma_k'X^2 + \frac{1}{8}\Sigma_k\Sigma_k'^2X^2 - \frac{1}{12}\Sigma_k^2\Sigma_k'^3X^2 - \frac{1}{24}\Sigma_k^2\Sigma_k'X^3 - \frac{43}{48}\Sigma_k^3\Sigma_k'^2X^3\\
&+ \frac{5}{48}\Sigma_k^4\Sigma_k'^3X^3 - \frac{1}{24}\Sigma_k^3X^4 + \frac{77}{48}\Sigma_k^5\Sigma_k'^2X^4 - \frac{1}{24}\Sigma_k^6\Sigma_k'^3X^4 + \frac{5}{24}\Sigma_k^5X^5 - \frac{5}{6}\Sigma_k^7\Sigma_k'^2X^5\bigg],
\end{aligned}$$

$$\begin{aligned}
\Delta\tilde{K}_{55} = \int \frac{d^4k}{(2\pi)^4}\bigg[&-\frac{7}{576}\Sigma_k''^2 + \frac{1}{960}\Sigma_k\Sigma_k'''X - \frac{17}{144}\Sigma_k'^4X + \frac{1}{12}\Sigma_k^2\Sigma_k''^2X - \frac{11}{160}\Sigma_k'^2X^2 + \frac{1}{240}\Sigma_k\Sigma_k''X^2\\
&+ \frac{143}{480}\Sigma_k\Sigma_k'^3X^2 - \frac{1}{960}\Sigma_k^3\Sigma_k'''X^2 + \frac{139}{144}\Sigma_k^2\Sigma_k'^4X^2 - \frac{15}{64}\Sigma_k^4\Sigma_k''^2X^2 - \frac{11}{960}X^3 - \frac{11}{240}\Sigma_k\Sigma_k'X^3\\
&+ \frac{83}{192}\Sigma_k^2\Sigma_k'^2X^3 - \frac{2917}{1440}\Sigma_k^3\Sigma_k'^3X^3 - \frac{493}{144}\Sigma_k^4\Sigma_k'^4X^3 + \frac{97}{288}\Sigma_k^6\Sigma_k''^2X^3 + \frac{37}{960}\Sigma_k^2X^4 - \frac{431}{480}\Sigma_k^4\Sigma_k'^2X^4\\
&+ \frac{1051}{180}\Sigma_k^5\Sigma_k'^3X^4 + \frac{2911}{432}\Sigma_k^6\Sigma_k'^4X^4 - \frac{35}{144}\Sigma_k^8\Sigma_k''^2X^4 - \frac{1}{6}\Sigma_k^4X^5 - \frac{7}{24}\Sigma_k^6\Sigma_k'^2X^5 - \frac{323}{36}\Sigma_k^7\Sigma_k'^3X^5\\
&- \frac{1639}{216}\Sigma_k^8\Sigma_k'^4X^5 + \frac{5}{72}\Sigma_k^{10}\Sigma_k''^2X^5 + \frac{5}{24}\Sigma_k^6X^6 + \frac{5}{2}\Sigma_k^8\Sigma_k'^2X^6 + \frac{85}{12}\Sigma_k^9\Sigma_k'^3X^6 + \frac{245}{54}\Sigma_k^{10}\Sigma_k'^4X^6\\
&+ \frac{5}{24}\Sigma_k^8X^7 - \frac{5}{3}\Sigma_k^{10}\Sigma_k'^2X^7 - \frac{20}{9}\Sigma_k^{11}\Sigma_k'^3X^7 - \frac{10}{9}\Sigma_k^{12}\Sigma_k'^4X^7\bigg],
\end{aligned}$$

$$\begin{aligned}\Delta\tilde{K}_{56}=\int\frac{d^4k}{(2\pi)^4}\Big[&\frac{1}{12}\Sigma_k'^3X-\frac{1}{192}\Sigma_k'X^2-\frac{1}{4}\Sigma_k\Sigma_k'^2X^2-\frac{1}{3}\Sigma_k^2\Sigma_k'^3X^2+\frac{1}{192}\Sigma_kX^3+\frac{1}{48}\Sigma_k^2\Sigma_k'X^3\\&+\frac{113}{24}\Sigma_k^3\Sigma_k'^2X^3+\frac{5}{12}\Sigma_k^4\Sigma_k'^3X^3+\frac{27}{64}\Sigma_k^3X^4-\frac{133}{8}\Sigma_k^5\Sigma_k'^2X^4-\frac{1}{6}\Sigma_k^6\Sigma_k'^3X^4-\frac{19}{8}\Sigma_k^5X^5\\&+\frac{41}{2}\Sigma_k^7\Sigma_k'^2X^5+\frac{25}{12}\Sigma_k^7X^6-\frac{25}{3}\Sigma_k^9\Sigma_k'^2X^6\Big],\end{aligned}$$

$$\begin{aligned}\Delta\tilde{K}_{57}=\int\frac{d^4k}{(2\pi)^4}\Big[&\frac{7}{192}\Sigma_k''^2+\frac{1}{640}\Sigma_k\Sigma_k'''X+\frac{11}{48}\Sigma_k'^4X-\frac{5}{8}\Sigma_k^2\Sigma_k''^2X-\frac{3}{320}\Sigma_k'^2X^2+\frac{1}{160}\Sigma_k\Sigma_k''X^2\\&-\frac{157}{320}\Sigma_k\Sigma_k'^3X^2-\frac{1}{640}\Sigma_k^3\Sigma_k'''X^2-\frac{253}{48}\Sigma_k^2\Sigma_k'^4X^2+\frac{165}{64}\Sigma_k^4\Sigma_k''^2X^2-\frac{23}{1920}X^3-\frac{23}{480}\Sigma_k\Sigma_k'X^3\\&+\frac{341}{384}\Sigma_k^2\Sigma_k'^2X^3+\frac{11513}{960}\Sigma_k^3\Sigma_k'^3X^3+\frac{1495}{48}\Sigma_k^4\Sigma_k'^4X^3-\frac{421}{96}\Sigma_k^6\Sigma_k''^2X^3+\frac{151}{1920}\Sigma_k^2X^4-\frac{3491}{320}\Sigma_k^4\Sigma_k'^2X^4\\&-\frac{28361}{480}\Sigma_k^5\Sigma_k'^3X^4-\frac{11425}{144}\Sigma_k^6\Sigma_k'^4X^4+\frac{161}{48}\Sigma_k^8\Sigma_k''^2X^4-\frac{67}{96}\Sigma_k^4X^5+\frac{265}{8}\Sigma_k^6\Sigma_k'^2X^5+\frac{344}{3}\Sigma_k^7\Sigma_k'^3X^5\\&+\frac{7237}{72}\Sigma_k^8\Sigma_k'^4X^5-\frac{23}{24}\Sigma_k^{10}\Sigma_k''^2X^5+\frac{73}{48}\Sigma_k^6X^6-\frac{445}{12}\Sigma_k^8\Sigma_k'^2X^6-\frac{391}{4}\Sigma_k^9\Sigma_k'^3X^6-\frac{1127}{18}\Sigma_k^{10}\Sigma_k'^4X^6\\&-\frac{5}{8}\Sigma_k^8X^7+14\Sigma_k^{10}\Sigma_k'^2X^7+\frac{92}{3}\Sigma_k^{11}\Sigma_k'^3X^7+\frac{46}{3}\Sigma_k^{12}\Sigma_k'^4X^7\Big],\end{aligned}$$

$$\begin{aligned}\Delta\tilde{K}_{58}=\int\frac{d^4k}{(2\pi)^4}\Big[&-\frac{3}{2}\Sigma_k\Sigma_k'^2X^2+\frac{5}{48}\Sigma_kX^3+\frac{353}{24}\Sigma_k^3\Sigma_k'^2X^3+\frac{1}{48}\Sigma_k^3X^4-\frac{989}{24}\Sigma_k^5\Sigma_k'^2X^4-\frac{11}{3}\Sigma_k^5X^5\\&+\frac{134}{3}\Sigma_k^7\Sigma_k'^2X^5+\frac{25}{6}\Sigma_k^7X^6-\frac{50}{3}\Sigma_k^9\Sigma_k'^2X^6\Big],\end{aligned}$$

$$\begin{aligned}\Delta\tilde{K}_{59}=\int\frac{d^4k}{(2\pi)^4}\Big[&-\frac{1}{48}\Sigma_k''^2-\frac{1}{8}\Sigma_k'^4X+\frac{3}{8}\Sigma_k^2\Sigma_k''^2X+\frac{1}{2}\Sigma_k\Sigma_k'^3X^2+\frac{25}{8}\Sigma_k^2\Sigma_k'^4X^2-\frac{25}{16}\Sigma_k^4\Sigma_k''^2X^2\\&-\frac{3}{4}\Sigma_k^2\Sigma_k'^2X^3-\frac{97}{12}\Sigma_k^3\Sigma_k'^3X^3-\frac{451}{24}\Sigma_k^4\Sigma_k'^4X^3+\frac{8}{3}\Sigma_k^6\Sigma_k''^2X^3+\frac{37}{4}\Sigma_k^4\Sigma_k'^2X^4+\frac{443}{12}\Sigma_k^5\Sigma_k'^3X^4\\&+\frac{3467}{72}\Sigma_k^6\Sigma_k'^4X^4-\frac{49}{24}\Sigma_k^8\Sigma_k''^2X^4+\frac{1}{4}\Sigma_k^4X^5-\frac{59}{2}\Sigma_k^6\Sigma_k'^2X^5-\frac{421}{6}\Sigma_k^7\Sigma_k'^3X^5-\frac{2201}{36}\Sigma_k^8\Sigma_k'^4X^5\\&+\frac{7}{12}\Sigma_k^{10}\Sigma_k''^2X^5-\frac{7}{4}\Sigma_k^6X^6+35\Sigma_k^8\Sigma_k'^2X^6+\frac{119}{2}\Sigma_k^9\Sigma_k'^3X^6+\frac{343}{9}\Sigma_k^{10}\Sigma_k'^4X^6+\frac{7}{4}\Sigma_k^8X^7\\&-14\Sigma_k^{10}\Sigma_k'^2X^7-\frac{56}{3}\Sigma_k^{11}\Sigma_k'^3X^7-\frac{28}{3}\Sigma_k^{12}\Sigma_k'^4X^7\Big],\end{aligned}$$

$$\begin{aligned}\Delta\tilde{K}_{60}=\int\frac{d^4k}{(2\pi)^4}\Big[&\frac{1}{144}\Sigma_k''^2-\frac{1}{240}\Sigma_k\Sigma_k'''X+\frac{7}{54}\Sigma_k'^4X+\frac{5}{36}\Sigma_k^2\Sigma_k''^2X+\frac{1}{40}\Sigma_k'^2X^2-\frac{1}{60}\Sigma_k\Sigma_k''X^2\\&-\frac{83}{120}\Sigma_k\Sigma_k'^3X^2+\frac{1}{240}\Sigma_k^3\Sigma_k'''X^2+\frac{17}{27}\Sigma_k^2\Sigma_k'^4X^2-\frac{115}{144}\Sigma_k^4\Sigma_k''^2X^2+\frac{11}{240}X^3+\frac{11}{60}\Sigma_k\Sigma_k'X^3\\&+\frac{1}{48}\Sigma_k^2\Sigma_k'^2X^3-\frac{61}{120}\Sigma_k^3\Sigma_k'^3X^3-\frac{455}{54}\Sigma_k^4\Sigma_k'^4X^3+\frac{107}{72}\Sigma_k^6\Sigma_k''^2X^3-\frac{17}{240}\Sigma_k^2X^4+\frac{237}{40}\Sigma_k^4\Sigma_k'^2X^4\\&+\frac{709}{45}\Sigma_k^5\Sigma_k'^3X^4+\frac{230}{9}\Sigma_k^6\Sigma_k'^4X^4-\frac{7}{6}\Sigma_k^8\Sigma_k''^2X^4+\frac{1}{6}\Sigma_k^4X^5-\frac{64}{3}\Sigma_k^6\Sigma_k'^2X^5-\frac{341}{9}\Sigma_k^7\Sigma_k'^3X^5\\&-\frac{103}{3}\Sigma_k^8\Sigma_k'^4X^5+\frac{1}{3}\Sigma_k^{10}\Sigma_k''^2X^5-\frac{11}{6}\Sigma_k^6X^6+\frac{70}{3}\Sigma_k^8\Sigma_k'^2X^6+34\Sigma_k^9\Sigma_k'^3X^6+\frac{196}{9}\Sigma_k^{10}\Sigma_k'^4X^6\\&+\Sigma_k^8X^7-8\Sigma_k^{10}\Sigma_k'^2X^7-\frac{32}{3}\Sigma_k^{11}\Sigma_k'^3X^7-\frac{16}{3}\Sigma_k^{12}\Sigma_k'^4X^7\Big],\end{aligned}$$

$$\Delta\tilde{K}_{61}=\int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{48}\Sigma_k''^2+\frac{1}{240}\Sigma_k\Sigma_k'''X+\frac{7}{216}\Sigma_k'^4X-\frac{47}{72}\Sigma_k^2\Sigma_k''^2X-\frac{1}{15}\Sigma_k'^2X^2+\frac{1}{60}\Sigma_k\Sigma_k''X^2\right.$$
$$-\frac{37}{120}\Sigma_k\Sigma_k'^3X^2-\frac{1}{240}\Sigma_k^3\Sigma_k'''X^2-\frac{1055}{216}\Sigma_k^2\Sigma_k'^4X^2+\frac{425}{144}\Sigma_k^4\Sigma_k''^2X^2+\frac{3}{80}X^3+\frac{3}{20}\Sigma_k\Sigma_k'X^3$$
$$+\frac{47}{16}\Sigma_k^2\Sigma_k'^2X^3+\frac{4643}{360}\Sigma_k^3\Sigma_k'^3X^3+\frac{7399}{216}\Sigma_k^4\Sigma_k'^4X^3-\frac{31}{6}\Sigma_k^6\Sigma_k''^2X^3-\frac{21}{80}\Sigma_k^2X^4-\frac{6037}{240}\Sigma_k^4\Sigma_k'^2X^4$$
$$-\frac{6143}{90}\Sigma_k^5\Sigma_k'^3X^4-\frac{19861}{216}\Sigma_k^6\Sigma_k'^4X^4+\frac{287}{72}\Sigma_k^8\Sigma_k''^2X^4-\frac{4}{3}\Sigma_k^4X^5+\frac{205}{3}\Sigma_k^6\Sigma_k'^2X^5+\frac{2437}{18}\Sigma_k^7\Sigma_k'^3X^5$$
$$+\frac{12823}{108}\Sigma_k^8\Sigma_k'^4X^5-\frac{41}{36}\Sigma_k^{10}\Sigma_k''^2X^5+\frac{14}{3}\Sigma_k^6X^6-\frac{220}{3}\Sigma_k^8\Sigma_k'^2X^6-\frac{697}{6}\Sigma_k^9\Sigma_k'^3X^6-\frac{2009}{27}\Sigma_k^{10}\Sigma_k'^4X^6$$
$$\left.-\frac{41}{12}\Sigma_k^8X^7+\frac{82}{3}\Sigma_k^{10}\Sigma_k'^2X^7+\frac{328}{9}\Sigma_k^{11}\Sigma_k'^3X^7+\frac{164}{9}\Sigma_k^{12}\Sigma_k'^4X^7\right],$$

$$\Delta\tilde{K}_{62}=\int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{36}\Sigma_k''^2-\frac{47}{216}\Sigma_k'^4X+\frac{25}{72}\Sigma_k^2\Sigma_k''^2X+\frac{1}{8}\Sigma_k'^2X^2+\frac{1}{3}\Sigma_k\Sigma_k'^3X^2+\frac{691}{216}\Sigma_k^2\Sigma_k'^4X^2\right.$$
$$-\frac{95}{72}\Sigma_k^4\Sigma_k''^2X^2-\frac{47}{24}\Sigma_k^2\Sigma_k'^2X^3-\frac{115}{18}\Sigma_k^3\Sigma_k'^3X^3-\frac{3575}{216}\Sigma_k^4\Sigma_k'^4X^3+\frac{157}{72}\Sigma_k^6\Sigma_k''^2X^3-\frac{1}{6}\Sigma_k^2X^4$$
$$+\frac{551}{48}\Sigma_k^4\Sigma_k'^2X^4+\frac{179}{6}\Sigma_k^5\Sigma_k'^3X^4+\frac{8665}{216}\Sigma_k^6\Sigma_k'^4X^4-\frac{119}{72}\Sigma_k^8\Sigma_k''^2X^4+\frac{2}{3}\Sigma_k^4X^5-\frac{85}{3}\Sigma_k^6\Sigma_k'^2X^5$$
$$-\frac{341}{6}\Sigma_k^7\Sigma_k'^3X^5-\frac{5383}{108}\Sigma_k^8\Sigma_k'^4X^5+\frac{17}{36}\Sigma_k^{10}\Sigma_k''^2X^5-\frac{11}{6}\Sigma_k^6X^6+30\Sigma_k^8\Sigma_k'^2X^6+\frac{289}{6}\Sigma_k^9\Sigma_k'^3X^6$$
$$\left.+\frac{833}{27}\Sigma_k^{10}\Sigma_k'^4X^6+\frac{17}{12}\Sigma_k^8X^7-\frac{34}{3}\Sigma_k^{10}\Sigma_k'^2X^7-\frac{136}{9}\Sigma_k^{11}\Sigma_k'^3X^7-\frac{68}{9}\Sigma_k^{12}\Sigma_k'^4X^7\right],$$

$$\Delta\tilde{K}_{63}=\int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{8}\Sigma_k'^3X+\frac{1}{2}\Sigma_k^2\Sigma_k'^3X^2+\frac{5}{8}\Sigma_k^3\Sigma_k'^2X^3-\frac{5}{8}\Sigma_k^4\Sigma_k'^3X^3-\frac{13}{8}\Sigma_k^5\Sigma_k'^2X^4+\frac{1}{4}\Sigma_k^6\Sigma_k'^3X^4\right.$$
$$\left.-\frac{1}{4}\Sigma_k^5X^5+\Sigma_k^7\Sigma_k'^2X^5\right],$$

$$\Delta\tilde{K}_{64}=\int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{96}\Sigma_k''^2-\frac{13}{144}\Sigma_k'^4X+\frac{5}{48}\Sigma_k^2\Sigma_k''^2X+\frac{1}{4}\Sigma_k\Sigma_k'^3X^2+\frac{149}{144}\Sigma_k^2\Sigma_k'^4X^2-\frac{35}{96}\Sigma_k^4\Sigma_k''^2X^2\right.$$
$$-\frac{3}{8}\Sigma_k^2\Sigma_k'^2X^3-\frac{19}{8}\Sigma_k^3\Sigma_k'^3X^3-\frac{685}{144}\Sigma_k^4\Sigma_k'^4X^3+\frac{7}{12}\Sigma_k^6\Sigma_k''^2X^3+\frac{21}{8}\Sigma_k^4\Sigma_k'^2X^4+\frac{211}{24}\Sigma_k^5\Sigma_k'^3X^4$$
$$+\frac{175}{16}\Sigma_k^6\Sigma_k'^4X^4-\frac{7}{16}\Sigma_k^8\Sigma_k''^2X^4+\frac{1}{8}\Sigma_k^4X^5-\frac{27}{4}\Sigma_k^6\Sigma_k'^2X^5-\frac{185}{12}\Sigma_k^7\Sigma_k'^3X^5-\frac{319}{24}\Sigma_k^8\Sigma_k'^4X^5$$
$$+\frac{1}{8}\Sigma_k^{10}\Sigma_k''^2X^5-\frac{3}{8}\Sigma_k^6X^6+\frac{15}{2}\Sigma_k^8\Sigma_k'^2X^6+\frac{51}{4}\Sigma_k^9\Sigma_k'^3X^6+\frac{49}{6}\Sigma_k^{10}\Sigma_k'^4X^6+\frac{3}{8}\Sigma_k^8X^7$$
$$\left.-3\Sigma_k^{10}\Sigma_k'^2X^7-4\Sigma_k^{11}\Sigma_k'^3X^7-2\Sigma_k^{12}\Sigma_k'^4X^7\right],$$

$$\Delta\tilde{K}_{65}=\int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{12}\Sigma_k'^3X+\frac{1}{48}\Sigma_k'X^2-\frac{1}{2}\Sigma_k\Sigma_k'^2X^2+\frac{1}{3}\Sigma_k^2\Sigma_k'^3X^2-\frac{1}{16}\Sigma_kX^3-\frac{1}{12}\Sigma_k^2\Sigma_k'X^3\right.$$
$$+\frac{7}{3}\Sigma_k^3\Sigma_k'^2X^3-\frac{5}{12}\Sigma_k^4\Sigma_k'^3X^3+\frac{29}{48}\Sigma_k^3X^4-\frac{19}{6}\Sigma_k^5\Sigma_k'^2X^4+\frac{1}{6}\Sigma_k^6\Sigma_k'^3X^4-\frac{1}{3}\Sigma_k^5X^5$$
$$\left.+\frac{4}{3}\Sigma_k^7\Sigma_k'^2X^5\right],$$

$$\Delta\tilde{K}_{66} = \int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{16}\Sigma_k'^2X - \frac{3}{16}\Sigma_k^2\Sigma_k'^2X^2 - \frac{1}{8}\Sigma_k^2X^3 + \frac{1}{8}\Sigma_k^4\Sigma_k'^2X^3\right],$$

$$\Delta\tilde{K}_{67} = \int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{72}\Sigma_k''^2 - \frac{1}{180}\Sigma_k\Sigma_k'''X + \frac{4}{27}\Sigma_k'^4X - \frac{1}{18}\Sigma_k^2\Sigma_k''^2X + \frac{14}{45}\Sigma_k'^2X^2 - \frac{1}{45}\Sigma_k\Sigma_k''X^2\right.$$
$$- \frac{11}{30}\Sigma_k\Sigma_k'^3X^2 + \frac{1}{180}\Sigma_k^3\Sigma_k'''X^2 - \frac{23}{27}\Sigma_k^2\Sigma_k'^4X^2 + \frac{5}{72}\Sigma_k^4\Sigma_k''^2X^2 + \frac{1}{180}X^3 + \frac{1}{45}\Sigma_k\Sigma_k'X^3$$
$$- \frac{31}{36}\Sigma_k^2\Sigma_k'^2X^3 + \frac{8}{5}\Sigma_k^3\Sigma_k'^3X^3 + \frac{46}{27}\Sigma_k^4\Sigma_k'^4X^3 - \frac{1}{36}\Sigma_k^6\Sigma_k''^2X^3 - \frac{7}{30}\Sigma_k^2X^4 + \frac{3}{20}\Sigma_k^4\Sigma_k'^2X^4$$
$$\left.- \frac{191}{90}\Sigma_k^5\Sigma_k'^3X^4 - \frac{13}{9}\Sigma_k^6\Sigma_k'^4X^4 + \frac{1}{3}\Sigma_k^6\Sigma_k'^2X^5 + \frac{8}{9}\Sigma_k^7\Sigma_k'^3X^5 + \frac{4}{9}\Sigma_k^8\Sigma_k'^4X^5\right],$$

$$\Delta\tilde{K}_{68} = \int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{36}\Sigma_k''^2 - \frac{1}{240}\Sigma_k\Sigma_k'''X + \frac{8}{27}\Sigma_k'^4X - \frac{1}{9}\Sigma_k^2\Sigma_k''^2X + \frac{1}{40}\Sigma_k'^2X^2 - \frac{1}{60}\Sigma_k\Sigma_k''X^2\right.$$
$$- \frac{83}{120}\Sigma_k\Sigma_k'^3X^2 + \frac{1}{240}\Sigma_k^3\Sigma_k'''X^2 - \frac{46}{27}\Sigma_k^2\Sigma_k'^4X^2 + \frac{5}{36}\Sigma_k^4\Sigma_k''^2X^2 + \frac{1}{240}X^3 + \frac{1}{60}\Sigma_k\Sigma_k'X^3$$
$$+ \frac{41}{48}\Sigma_k^2\Sigma_k'^2X^3 + \frac{123}{40}\Sigma_k^3\Sigma_k'^3X^3 + \frac{92}{27}\Sigma_k^4\Sigma_k'^4X^3 - \frac{1}{18}\Sigma_k^6\Sigma_k''^2X^3 + \frac{1}{80}\Sigma_k^2X^4 - \frac{103}{40}\Sigma_k^4\Sigma_k'^2X^4$$
$$\left.- \frac{749}{180}\Sigma_k^5\Sigma_k'^3X^4 - \frac{26}{9}\Sigma_k^6\Sigma_k'^4X^4 - \frac{1}{4}\Sigma_k^4X^5 + \frac{5}{3}\Sigma_k^6\Sigma_k'^2X^5 + \frac{16}{9}\Sigma_k^7\Sigma_k'^3X^5 + \frac{8}{9}\Sigma_k^8\Sigma_k'^4X^5\right]. \tag{A1}$$

## APPENDIX B: THE $\Delta\tilde{K}_i^W$ COEFFICIENTS

$$\Delta\tilde{K}_1^W = \int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{2}\Sigma_k^5X^5\right],$$

$$\Delta\tilde{K}_2^W = \int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{8}\Sigma_k^3X^4 + \frac{1}{8}\Sigma_k^5X^5\right],$$

$$\Delta\tilde{K}_3^W = \int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{8}\Sigma_k^3X^4\right],$$

$$\Delta\tilde{K}_4^W = \int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{8}\Sigma_k^3X^4\right],$$

$$\Delta\tilde{K}_5^W = \int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{32}\Sigma_kX^3\right],$$

$$\Delta\tilde{K}_6^W = \int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{32}\Sigma_k^3X^4\right],$$

$$\Delta\tilde{K}_7^W = \int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{64}\Sigma_kX^3 + \frac{1}{64}\Sigma_k^3X^4\right],$$

$$\Delta\tilde{K}_8^W = \int\frac{d^4k}{(2\pi)^4}\left[-\frac{1}{32}\Sigma_k'^2X^2 - \frac{11}{24}\Sigma_k\Sigma_k'^3X^2 + \frac{19}{192}X^3 + \frac{1}{3}\Sigma_k\Sigma_k'X^3 + \frac{5}{24}\Sigma_k^2\Sigma_k'^2X^3 + \frac{13}{6}\Sigma_k^3\Sigma_k'^3X^3\right.$$
$$- \frac{25}{192}\Sigma_k^2X^4 + \frac{59}{96}\Sigma_k^4\Sigma_k'^2X^4 - \frac{95}{24}\Sigma_k^5\Sigma_k'^3X^4 - \frac{1}{24}\Sigma_k^4X^5 - \frac{43}{24}\Sigma_k^6\Sigma_k'^2X^5 + \frac{47}{12}\Sigma_k^7\Sigma_k'^3X^5$$
$$\left.- \frac{3}{4}\Sigma_k^6X^6 - \frac{7}{3}\Sigma_k^9\Sigma_k'^3X^6 - \frac{1}{6}\Sigma_k^8X^7 + \Sigma_k^{10}\Sigma_k'^2X^7 + \frac{2}{3}\Sigma_k^{11}\Sigma_k'^3X^7\right],$$

$$\Delta\tilde{K}_9^W = \int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{32}\Sigma_k'^2X^2 + \frac{5}{8}\Sigma_k\Sigma_k'^3X^2 - \frac{9}{64}X^3 - \frac{1}{2}\Sigma_k\Sigma_k'X^3 - \frac{7}{8}\Sigma_k^2\Sigma_k'^2X^3 - 3\Sigma_k^3\Sigma_k'^3X^3 + \frac{11}{64}\Sigma_k^2X^4 + \frac{181}{96}\Sigma_k^4\Sigma_k'^2X^4 + \frac{45}{8}\Sigma_k^5\Sigma_k'^3X^4 + \frac{3}{8}\Sigma_k^4X^5 - \frac{29}{24}\Sigma_k^6\Sigma_k'^2X^5 - \frac{23}{4}\Sigma_k^7\Sigma_k'^3X^5 + \frac{13}{12}\Sigma_k^6X^6 + \frac{5}{3}\Sigma_k^8\Sigma_k'^2X^6 + \frac{7}{2}\Sigma_k^9\Sigma_k'^3X^6 + \frac{1}{4}\Sigma_k^8X^7 - \frac{3}{2}\Sigma_k^{10}\Sigma_k'^2X^7 - \Sigma_k^{11}\Sigma_k'^3X^7\right],$$

$$\Delta\tilde{K}_{10}^W = \int \frac{d^4k}{(2\pi)^4}\left[\frac{7}{96}\Sigma_k'^2X^2 + \frac{5}{24}\Sigma_k\Sigma_k'^3X^2 - \frac{19}{192}X^3 - \frac{1}{3}\Sigma_k\Sigma_k'X^3 + \frac{1}{3}\Sigma_k^2\Sigma_k'^2X^3 + \frac{5}{6}\Sigma_k^3\Sigma_k'^3X^3 + \frac{17}{192}\Sigma_k^2X^4 - \frac{635}{96}\Sigma_k^4\Sigma_k'^2X^4 - \frac{175}{24}\Sigma_k^5\Sigma_k'^3X^4 - \frac{5}{8}\Sigma_k^4X^5 + \frac{143}{8}\Sigma_k^6\Sigma_k'^2X^5 + \frac{175}{12}\Sigma_k^7\Sigma_k'^3X^5 + \frac{7}{3}\Sigma_k^6X^6 - \frac{50}{3}\Sigma_k^8\Sigma_k'^2X^6 - \frac{35}{3}\Sigma_k^9\Sigma_k'^3X^6 - \frac{5}{6}\Sigma_k^8X^7 + 5\Sigma_k^{10}\Sigma_k'^2X^7 + \frac{10}{3}\Sigma_k^{11}\Sigma_k'^3X^7\right],$$

$$\Delta\tilde{K}_{11}^W = \int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{24}\Sigma_k'^2X^2 + \frac{1}{3}\Sigma_k\Sigma_k'^3X^2 - \frac{13}{192}X^3 - \frac{5}{24}\Sigma_k\Sigma_k'X^3 - \frac{19}{48}\Sigma_k^2\Sigma_k'^2X^3 - \frac{3}{2}\Sigma_k^3\Sigma_k'^3X^3 - \frac{7}{192}\Sigma_k^2X^4 + \frac{17}{48}\Sigma_k^4\Sigma_k'^2X^4 + \frac{5}{2}\Sigma_k^5\Sigma_k'^3X^4 + \frac{25}{48}\Sigma_k^4X^5 + \frac{1}{2}\Sigma_k^6\Sigma_k'^2X^5 - \frac{13}{6}\Sigma_k^7\Sigma_k'^3X^5 + \frac{1}{24}\Sigma_k^6X^6 + \frac{7}{6}\Sigma_k^9\Sigma_k'^3X^6 + \frac{1}{12}\Sigma_k^8X^7 - \frac{1}{2}\Sigma_k^{10}\Sigma_k'^2X^7 - \frac{1}{3}\Sigma_k^{11}\Sigma_k'^3X^7\right],$$

$$\Delta\tilde{K}_{12}^W = \int \frac{d^4k}{(2\pi)^4}\left[-\frac{1}{24}\Sigma_k'^2X^2 - \frac{11}{24}\Sigma_k\Sigma_k'^3X^2 + \frac{1}{12}X^3 + \frac{1}{3}\Sigma_k\Sigma_k'X^3 + \frac{5}{16}\Sigma_k^2\Sigma_k'^2X^3 + \frac{13}{6}\Sigma_k^3\Sigma_k'^3X^3 - \frac{7}{24}\Sigma_k^2X^4 + \frac{7}{16}\Sigma_k^4\Sigma_k'^2X^4 - \frac{95}{24}\Sigma_k^5\Sigma_k'^3X^4 - \frac{41}{24}\Sigma_k^6\Sigma_k'^2X^5 + \frac{47}{12}\Sigma_k^7\Sigma_k'^3X^5 - \frac{1}{2}\Sigma_k^6X^6 - \frac{7}{3}\Sigma_k^9\Sigma_k'^3X^6 - \frac{1}{6}\Sigma_k^8X^7 + \Sigma_k^{10}\Sigma_k'^2X^7 + \frac{2}{3}\Sigma_k^{11}\Sigma_k'^3X^7\right],$$

$$\Delta\tilde{K}_{13}^W = \int \frac{d^4k}{(2\pi)^4}\left[-\frac{1}{8}\Sigma_k^2X^4 + \frac{1}{8}\Sigma_k^4X^5\right],$$

$$\Delta\tilde{K}_{14}^W = \int \frac{d^4k}{(2\pi)^4}\left[\frac{5}{96}\Sigma_k'^2X^2 + \frac{2}{3}\Sigma_k\Sigma_k'^3X^2 - \frac{23}{192}X^3 - \frac{5}{12}\Sigma_k\Sigma_k'X^3 - \frac{31}{48}\Sigma_k^2\Sigma_k'^2X^3 - 3\Sigma_k^3\Sigma_k'^3X^3 + \frac{1}{192}\Sigma_k^2X^4 + \frac{49}{96}\Sigma_k^4\Sigma_k'^2X^4 + 5\Sigma_k^5\Sigma_k'^3X^4 + \frac{13}{24}\Sigma_k^4X^5 + \frac{13}{12}\Sigma_k^6\Sigma_k'^2X^5 - \frac{13}{3}\Sigma_k^7\Sigma_k'^3X^5 + \frac{1}{2}\Sigma_k^6X^6 + \frac{7}{3}\Sigma_k^9\Sigma_k'^3X^6 + \frac{1}{6}\Sigma_k^8X^7 - \Sigma_k^{10}\Sigma_k'^2X^7 - \frac{2}{3}\Sigma_k^{11}\Sigma_k'^3X^7\right],$$

$$\Delta\tilde{K}_{15}^W = \int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{96}\Sigma_k'^2X^2 + \frac{1}{48}\Sigma_k\Sigma_k'^3X^2 + \frac{1}{96}X^3 + \frac{1}{24}\Sigma_k\Sigma_k'X^3 + \frac{19}{96}\Sigma_k^2\Sigma_k'^2X^3 - \frac{7}{12}\Sigma_k^3\Sigma_k'^3X^3 + \frac{5}{48}\Sigma_k^2X^4 - \frac{3}{16}\Sigma_k^4\Sigma_k'^2X^4 + \frac{125}{48}\Sigma_k^5\Sigma_k'^3X^4 - \frac{19}{48}\Sigma_k^4X^5 - \frac{89}{48}\Sigma_k^6\Sigma_k'^2X^5 - \frac{109}{24}\Sigma_k^7\Sigma_k'^3X^5 - \frac{1}{12}\Sigma_k^6X^6 + \frac{10}{3}\Sigma_k^8\Sigma_k'^2X^6 + \frac{7}{2}\Sigma_k^9\Sigma_k'^3X^6 + \frac{1}{4}\Sigma_k^8X^7 - \frac{3}{2}\Sigma_k^{10}\Sigma_k'^2X^7 - \Sigma_k^{11}\Sigma_k'^3X^7\right],$$

$$\Delta\tilde{K}_{16}^W = \int \frac{d^4k}{(2\pi)^4}\left[-\frac{1}{96}\Sigma_k'^2X^2 - \frac{19}{24}\Sigma_k\Sigma_k'^3X^2 + \frac{23}{192}X^3 + \frac{5}{12}\Sigma_k\Sigma_k'X^3 + \frac{19}{24}\Sigma_k^2\Sigma_k'^2X^3 + \frac{9}{2}\Sigma_k^3\Sigma_k'^3X^3 - \frac{11}{64}\Sigma_k^2X^4 - \frac{319}{96}\Sigma_k^4\Sigma_k'^2X^4 - \frac{85}{8}\Sigma_k^5\Sigma_k'^3X^4 - \frac{7}{12}\Sigma_k^4X^5 + \frac{55}{8}\Sigma_k^6\Sigma_k'^2X^5 + \frac{163}{12}\Sigma_k^7\Sigma_k'^3X^5 + \frac{1}{12}\Sigma_k^6X^6 - \frac{25}{3}\Sigma_k^8\Sigma_k'^2X^6 - \frac{28}{3}\Sigma_k^9\Sigma_k'^3X^6 - \frac{2}{3}\Sigma_k^8X^7 + 4\Sigma_k^{10}\Sigma_k'^2X^7 + \frac{8}{3}\Sigma_k^{11}\Sigma_k'^3X^7\right],$$

$$\Delta\tilde{K}_{17}^{W} = \int \frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{24}\Sigma_k'^2X^2 - \frac{17}{24}\Sigma_k\Sigma_k'^3X^2 + \frac{1}{12}X^3 + \frac{1}{3}\Sigma_k\Sigma_k'X^3 - \frac{1}{16}\Sigma_k^2\Sigma_k'^2X^3 + \frac{31}{6}\Sigma_k^3\Sigma_k'^3X^3 - \frac{1}{6}\Sigma_k^2X^4 - \frac{35}{16}\Sigma_k^4\Sigma_k'^2X^4 - \frac{365}{24}\Sigma_k^5\Sigma_k'^3X^4 + \frac{245}{24}\Sigma_k^6\Sigma_k'^2X^5 + \frac{269}{12}\Sigma_k^7\Sigma_k'^3X^5 + \frac{1}{4}\Sigma_k^6X^6 - 15\Sigma_k^8\Sigma_k'^2X^6 - \frac{49}{3}\Sigma_k^9\Sigma_k'^3X^6 - \frac{7}{6}\Sigma_k^8X^7 + 7\Sigma_k^{10}\Sigma_k'^2X^7 + \frac{14}{3}\Sigma_k^{11}\Sigma_k'^3X^7\bigg],$$

$$\Delta\tilde{K}_{18}^{W} = \int \frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{12}\Sigma_k'^2X^2 + \frac{2}{3}\Sigma_k\Sigma_k'^3X^2 - \frac{13}{96}X^3 - \frac{5}{12}\Sigma_k\Sigma_k'X^3 - \frac{19}{24}\Sigma_k^2\Sigma_k'^2X^3 - 3\Sigma_k^3\Sigma_k'^3X^3 - \frac{1}{32}\Sigma_k^2X^4 + \frac{17}{24}\Sigma_k^4\Sigma_k'^2X^4 + 5\Sigma_k^5\Sigma_k'^3X^4 + \frac{7}{12}\Sigma_k^4X^5 + \Sigma_k^6\Sigma_k'^2X^5 - \frac{13}{3}\Sigma_k^7\Sigma_k'^3X^5 + \frac{1}{2}\Sigma_k^6X^6 + \frac{7}{3}\Sigma_k^9\Sigma_k'^3X^6 + \frac{1}{6}\Sigma_k^8X^7 - \Sigma_k^{10}\Sigma_k'^2X^7 - \frac{2}{3}\Sigma_k^{11}\Sigma_k'^3X^7\bigg],$$

$$\Delta\tilde{K}_{19}^{W} = \int \frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{96}\Sigma_k'^2X^2 - \frac{1}{4}\Sigma_k\Sigma_k'^3X^2 + \frac{1}{64}X^3 + \frac{11}{16}\Sigma_k^2\Sigma_k'^2X^3 + 3\Sigma_k^3\Sigma_k'^3X^3 - \frac{3}{64}\Sigma_k^2X^4 - \frac{595}{96}\Sigma_k^4\Sigma_k'^2X^4 - \frac{45}{4}\Sigma_k^5\Sigma_k'^3X^4 - \frac{1}{4}\Sigma_k^4X^5 + \frac{97}{6}\Sigma_k^6\Sigma_k'^2X^5 + \frac{37}{2}\Sigma_k^7\Sigma_k'^3X^5 + \frac{7}{6}\Sigma_k^6X^6 - \frac{50}{3}\Sigma_k^8\Sigma_k'^2X^6 - 14\Sigma_k^9\Sigma_k'^3X^6 - \Sigma_k^8X^7 + 6\Sigma_k^{10}\Sigma_k'^2X^7 + 4\Sigma_k^{11}\Sigma_k'^3X^7\bigg],$$

$$\Delta\tilde{K}_{20}^{W} = \int \frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{24}\Sigma_k'^2X^2 + \frac{5}{12}\Sigma_k\Sigma_k'^3X^2 - \frac{1}{24}X^3 - \frac{1}{6}\Sigma_k\Sigma_k'X^3 - \frac{7}{8}\Sigma_k^2\Sigma_k'^2X^3 - \frac{5}{3}\Sigma_k^3\Sigma_k'^3X^3 + \frac{1}{8}\Sigma_k^2X^4 + \frac{9}{4}\Sigma_k^4\Sigma_k'^2X^4 + \frac{25}{12}\Sigma_k^5\Sigma_k'^3X^4 + \frac{1}{4}\Sigma_k^4X^5 - \frac{17}{12}\Sigma_k^6\Sigma_k'^2X^5 - \frac{5}{6}\Sigma_k^7\Sigma_k'^3X^5\bigg],$$

$$\Delta\tilde{K}_{21}^{W} = \int \frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{4}\Sigma_k\Sigma_k'^3X^2 + \frac{19}{24}\Sigma_k^2\Sigma_k'^2X^3 + 3\Sigma_k^3\Sigma_k'^3X^3 + \frac{1}{24}\Sigma_k^2X^4 - \frac{51}{8}\Sigma_k^4\Sigma_k'^2X^4 - \frac{45}{4}\Sigma_k^5\Sigma_k'^3X^4 - \frac{1}{3}\Sigma_k^4X^5 + \frac{65}{4}\Sigma_k^6\Sigma_k'^2X^5 + \frac{37}{2}\Sigma_k^7\Sigma_k'^3X^5 + \frac{7}{6}\Sigma_k^6X^6 - \frac{50}{3}\Sigma_k^8\Sigma_k'^2X^6 - 14\Sigma_k^9\Sigma_k'^3X^6 - \Sigma_k^8X^7 + 6\Sigma_k^{10}\Sigma_k'^2X^7 + 4\Sigma_k^{11}\Sigma_k'^3X^7\bigg],$$

$$\Delta\tilde{K}_{22}^{W} = \int \frac{d^4k}{(2\pi)^4}\bigg[\frac{1}{16}\Sigma_k'^2X^2 + \frac{1}{16}\Sigma_k\Sigma_k'^3X^2 - \frac{13}{32}\Sigma_k^2\Sigma_k'^2X^3 - \frac{1}{4}\Sigma_k^3\Sigma_k'^3X^3 - \frac{1}{16}\Sigma_k^2X^4 + \frac{21}{32}\Sigma_k^4\Sigma_k'^2X^4 + \frac{5}{16}\Sigma_k^5\Sigma_k'^3X^4 + \frac{1}{16}\Sigma_k^4X^5 - \frac{5}{16}\Sigma_k^6\Sigma_k'^2X^5 - \frac{1}{8}\Sigma_k^7\Sigma_k'^3X^5\bigg],$$

$$\Delta\tilde{K}_{23}^{W} = \int \frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{8}\Sigma_k\Sigma_k'^3X^2 + \frac{9}{16}\Sigma_k^2\Sigma_k'^2X^3 + \frac{3}{2}\Sigma_k^3\Sigma_k'^3X^3 - \frac{63}{16}\Sigma_k^4\Sigma_k'^2X^4 - \frac{45}{8}\Sigma_k^5\Sigma_k'^3X^4 - \frac{1}{4}\Sigma_k^4X^5 + \frac{73}{8}\Sigma_k^6\Sigma_k'^2X^5 + \frac{37}{4}\Sigma_k^7\Sigma_k'^3X^5 + \frac{11}{16}\Sigma_k^6X^6 - \frac{35}{4}\Sigma_k^8\Sigma_k'^2X^6 - 7\Sigma_k^9\Sigma_k'^3X^6 - \frac{1}{2}\Sigma_k^8X^7 + 3\Sigma_k^{10}\Sigma_k'^2X^7 + 2\Sigma_k^{11}\Sigma_k'^3X^7\bigg]. \tag{B1}$$

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