Non-Abelian string of a finite length

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(Received 12 June 2015; published 7 July 2015)

We consider world-sheet theories for non-Abelian strings assuming compactification on a cylinder with a finite circumference L and periodic boundary conditions. The dynamics of the orientational modes is described by the two-dimensional $CP(N-1)$ model. We analyze both the nonsupersymmetric (bosonic) model and the $\mathcal{N}=(2,2)$ supersymmetric $CP(N-1)$ emerging in the case of $1/2$-BPS saturated strings (Bogomol'nyi–Prasad–Sommerfteld saturated string that breaks only half of supersymmetry) in $\mathcal{N}=2$ supersymmetric QCD with $N_f=N$. The nonsupersymmetric case was studied previously; technically our results agree with those obtained previously, although our interpretation is totally different. In the large-N limit we detect a phase transition at $L\sim\Lambda_{CP}^{-1}$ (which is expected to become a rapid crossover at finite N). If at large L the $CP(N-1)$ model develops a mass gap and is in the Coulomb/confinement phase, with exponentially suppressed finite-L effects, at small L it is in the deconfinement phase, and the orientational modes contribute to the Lüsher term. The latter becomes dependent on the rank of the bulk gauge group. In the supersymmetric $CP(N-1)$ models at finite L we find a large-N solution which was not known previously. We observe a single phase independently of the value of $L\Lambda_{CP}$. For any value of this parameter a mass gap develops and supersymmetry remains unbroken. So does the $SU(N)$ symmetry of the target space. The mass gap turns out to be independent of the string length. The Lüscher term is absent due to supersymmetry.

DOI: 10.1103/PhysRevD.92.025011 PACS numbers: 11.30.Pb, 12.38.Lg

I. INTRODUCTION

Recently there was considerable progress in studies of long confining strings; see [1]. The energy of the Abrikosov-Nielsen-Olesen (ANO) closed string [2] in the Abelian-Higgs model as a function of the string length L (in the large-L limit) can be written as

$$E(L)=TL-\frac{\gamma}{L}+\frac{c_3}{TL^3}+\cdots, \tag{1.1}$$

where T is the string tension and the ellipses stand for terms of the higher order in $1/L$. This $1/L$ expansion is determined by the low-energy effective two-dimensional theory on the string world sheet. For the Abrikosov-Nielsen-Olesen string the world-sheet theory is given by the Nambu-Goto action plus higher derivative corrections. It is plausible to assume that a similar structure applies to QCD confining strings. Recently significant progress occurred in measuring the spectrum of long confining QCD strings in lattice simulations; see, for example, [3].

The $1/L$ term in (1.1) is referred to as the Lüscher term [4]. The coefficient γ is universal. Its value is determined by the number of massless (light) degrees of freedom on the string world sheet. The Abelian strings possess only two massless excitations due to two translational zero modes; the Lüscher term is, correspondingly, $\gamma=\pi/3$.

In this paper we will study the energy of a finite-L closed *non*-Abelian string assuming that L is much larger than the string transverse size.

The main feature of the non-Abelian strings is the occurrence of extra (quasi)moduli: orientational zero modes associated with their color flux rotation in the internal space. Dynamics of these orientational moduli is described by the two-dimensional $CP(N-1)$ model on the string world sheet. If the bulk theory supporting such string vortices is supersymmetric,[1] the world-sheet $CP(N-1)$ model will have various degrees of supersymmetry. Non-Abelian strings were first found in $\mathcal{N}=2$ supersymmetric gauge theories [5–8]. Later this construction was generalized to a wide class of non-Abelian gauge theories, both supersymmetric and nonsupersymmetric; see [9–12]. The Lüsher term for nonsupersymmetric non-Abelian strings was previously discussed in [13].

Our current task is broader: we want to study the L dependence of $E(L)$ for all values of L, large and small (see below), taking account of the orientational moduli that are described by the two-dimensional $CP(N-1)$ model.

[1]In the simplest version non-Abelian vortex strings are supported in gauge theories with the $U(N)$ gauge group and $N_f=N$ flavors of quarks.

The latter is asymptotically free and develops its own dynamical scale Λ_{CP}. This modifies the expansion in (1.1). Assuming that

$$\Lambda_{CP} \ll \sqrt{T} \tag{1.2}$$

we can write

$$E(L) = TL + \frac{f(\Lambda_{CP}L)}{L} + O\left(\frac{1}{TL^3}\right). \tag{1.3}$$

Below we will present a detailed calculation of the string energy for strings with

$$L \gg 1/\sqrt{T}. \tag{1.4}$$

For these values of L higher derivative corrections to the effective world-sheet theory can be ignored, and we use a $CP(N-1)$-based description to calculate the function $f(\Lambda_{CP}L)$ (which is already known [13] in the limits $L \gg \Lambda_{CP}^{-1}$ and $L \ll \Lambda_{CP}^{-1}$). To solve the $CP(N-1)$ model we use the large-N approximation [14]. Given the constraint (1.4) which is also assumed, we call the string "large" if $L \gg \Lambda_{CP}^{-1}$, and "small" otherwise.

Now, when we have two free parameters in the problem under consideration, N and L, and both can be large, the ordering of taking limits is of paramount importance and a source of a number of paradoxes. We will *always* take first the limit $N \to \infty$. In this limit the number of dynamical degrees of freedom is infinite (even in the quantum-mechanical limit $L \to 0$) and, moreover, all interactions die off. This makes possible phase transitions.

For a nonsupersymmetric case we find a phase transition in the $CP(N-1)$ model on the string world sheet. Its origin is intuitively clear: at large L the theory is strongly coupled while at small L it is weakly coupled, and its behavior should be close to that given by perturbation theory. Correspondingly, at a large string length this theory develops a mass gap and is in the Coulomb/confinement phase. Finite-length effects coming from orientational moduli are exponentially suppressed. We find that at $L \gg \Lambda_{CP}$

$$f(\Lambda_{CP}L) = -\frac{\pi}{3} - N\sqrt{\frac{2}{\pi}}\sqrt{\Lambda_{CP}L}\,e^{-\Lambda_{CP}L} + \cdots, \tag{1.5}$$

where the first term is the conventional Lüscher term coming from the translational moduli.

At a small length the $CP(N-1)$ model is in the deconfinement phase. Massless orientational moduli contribute to the Lüscher term, which becomes dependent on the rank of the bulk gauge group. At $\sqrt{T} \ll L \ll \Lambda_{CP}$ we find that

$$f(\Lambda_{CP}L) = -N\frac{\pi}{3}. \tag{1.6}$$

The asymptotic values of the Lüscher coefficient γ associated with the limits of large and small L in (1.5) and (1.6), respectively, were reported earlier in [13] for the open string. Here we confirm these results and derive $f(\Lambda_{CP}L)$ for the closed string. In other words, we impose periodic boundary conditions (on the boson and fermion fields in the case of the supersymmetric model; see below).

If N is large but finite, we expect that the phase transition becomes a rapid crossover. We do not expect strictly massless states to appear in the small-L domain at finite N.

Next, we study a supersymmetric case considering a BPS-saturated (satisfying the BPS equation, which is a condition for energy minimum) non-Abelian string in four-dimensional $\mathcal{N}=2$ SQCD ($\mathcal{N}=2$ extended supersymmetric quantum chromodynamics). In this case the world-sheet theory for orientational modes is the $\mathcal{N}=(2,2)$ supersymmetric $CP(N-1)$ model. Solving this theory in the large-N limit we find a *single* phase with unbroken supersymmetry and a mass gap. The mass gap turns out to be independent of the string length. The chiral Z_{2N} symmetry is broken down to Z_2, in much the same way as for an infinitely long string. The photon field acquires a mass term, and no Coulomb/confining potential is generated. Instead, the theory has N degenerate vacua representing N elementary strings. The Lüscher term vanishes due to the boson-fermion cancellation.

Thus, the dynamical L-behavior of non-Abelian strings, with or without supersymmetry, is drastically different in the large-N solution.

As was mentioned, in both cases we impose periodic boundary conditions on the spatial interval of length L. In the nonsupersymmetric case this is equivalent to endowing the string under consideration with temperature β^{-1},

$$\beta = L. \tag{1.7}$$

Such strings were considered previously; see, e.g., [15–17]. Our results differ from those of [15–17] partly in interpretation and partly in essence.

The paper is organized as follows. In Secs. II and III we briefly review nonsupersymmetric bulk theory supporting non-Abelian strings and the large-N solution of the $CP(N-1)$ model at $L \to \infty$ [14], respectively. In Sec. IV we use the large-N method to study non-Abelian strings of finite length and, in particular, describe the Coulomb/confinement phase. Section V is devoted to the deconfinement phase. In Secs. VI and VII, central in our analysis, we deal with the supersymmetric $\mathcal{N}=(2,2)$ string. In Sec. VIII we calculate the photon mass on the world sheet of the supersymmetric string under consideration as a function of L. Section IX summarizes our conclusions. Appendixes contain details of our calculations.

II. NONSUPERSYMMETRIC NON-ABELIAN STRINGS

In this section we briefly review the simplest four-dimensional nonsupersymmetric model supporting non-Abelian strings [18], give a topological argument for their stability, and outline the effective low-energy theory on the world sheet.

The model suggested in [18] is a bosonic part of $\mathcal{N}=2$ supersymmetric QCD; see [11] for a review. The gauge group of the theory is $SU(N)\times U(1)$. The matter sector of the model consists of $N_f=N$ flavors of complex scalar fields (squarks) charged with respect to $U(1)$, each in the fundamental representation of $SU(N)$. The action of the model is

$$S=\int d^4x\left[-\frac{1}{4g_2^2}(F^a_{\mu\nu})^2-\frac{1}{4g_1^2}(F_{\mu\nu})^2+|\nabla^\mu\varphi^A|^2+\frac{g_2^2}{2}(\bar{\varphi}_AT^a\varphi^A)^2+\frac{g_1^2}{8}(|\varphi^A|^2-N\xi)^2\right], \tag{2.1}$$

where T^a are the generators of $SU(N)$, the covariant derivative is defined as

$$\nabla_\mu=\partial_\mu-\frac{i}{2}A_\mu-iT^aA^a_\mu,$$

A_μ and A^a_μ denote the $U(1)$ and $SU(N)$ gauge fields respectively, and the corresponding coupling constants are g_1 and g_2. The scalar fields φ^{kA} have the color index $k=1,\ldots,N$ and the flavor index $A=1,\ldots,N$. Thus, φ^{kA} can be viewed as an $N\times N$ matrix. The $U(1)$ charges of φ^{kA} are $1/2$.

Let us examine the potential of the theory (2.1) in more detail. It consists of two non-negative terms, and consequently the minimum of the potential is reached when both terms vanish. The last term proportional to g_1^2 forces φ^A to develop a vacuum expectation value. One can choose φ^{kA} to be proportional to the unit matrix, namely,

$$\varphi_{\rm vac}=\sqrt{\xi}\,{\rm diag}(1,1,\ldots,1), \tag{2.2}$$

where we use $N\times N$ matrix notation for φ^{kA}. Then the last but one term vanishes automatically.

The above vacuum field spontaneously breaks both the gauge and flavor $SU(N)$ groups. However, it is invariant under the action of combined color-flavor global $SU(N)_{C+F}$. Therefore, the symmetry breaking pattern is

$$U(N)_{\rm gauge}\times SU(N)_{\rm flavor}\to SU(N)_{C+F}.$$

This setup was suggested in [19] and became known later as the color-flavor locking.

The topological stability of non-Abelian strings in this model is due to the fact that $\pi_1(SU(N)\times U(1)/Z_N)\neq 0$. One combines the Z_N center of $SU(N)$ with elements $e^{2\pi ik/N}$ of $U(1)$ to get windings in both groups simultaneously.

The string solution [18] breaks the global symmetry of the vacuum as follows:

$$SU(N)_{C+F}\to SU(N-1)\times U(1). \tag{2.3}$$

As a result the orientational zero modes appear, making the vortex non-Abelian. As is clear from the symmetry breaking pattern of Eq. (2.3) the orientational moduli belong to the quotient

$$\frac{SU(N)}{SU(N-1)\times U(1)}=CP(N-1). \tag{2.4}$$

Thus, the low-energy effective theory on the string world sheet is described by the $CP(N-1)$ model. The action of the model was derived in [18]; it can be written as

$$S^{(1+1)}=\int d^2x\left[\frac{T_{\rm cl}}{2}(\partial_kz^i)^2+r|\nabla_kn^l|^2\right], \tag{2.5}$$

where

$$T_{\rm cl}=2\pi\xi \tag{2.6}$$

is the classical tension of the string, z^i are two translational moduli in the perpendicular plane, n^l, $l=1,\ldots,N$ are N complex fields subject to the constraint

$$|n^l|^2=1, \tag{2.7}$$

and r is defined below.

The covariant derivative is

$$\nabla_k=\partial_k-iA_k, \tag{2.8}$$

and $k=(1,2)$ labels the world-sheet coordinates. The relation between a two-dimensional coupling r and a four-dimensional coupling g_2 at the scale $\sqrt{\xi}$ is given by

$$r=\frac{4\pi}{g_2^2}. \tag{2.9}$$

The field A_k enters is without a kinetic term and is auxiliary. It can be eliminated by virtue of equations of motion and is introduced to make the $U(1)$ gauge invariance of the model explicit.

Let us count the number of degrees of freedom. The complex scalar fields give $2N$ real degrees of freedom, of which one is eliminated due to the constraint (2.7) and another one due to $U(1)$ gauge invariance. Thus, the total number of degrees of freedom is $2(N-1)$, which is precisely the number of degrees of freedom in the $CP(N-1)$ model.

To conclude this section we note that the formation of non-Abelian strings leads to the confinement of monopoles in the bulk theory. In fact, in the $U(N)$ gauge theories strings are stable and cannot be broken. Therefore, confined monopoles are presented by junctions of two degenerate non-Abelian strings of different kinds; see review [11] for details. In the effective world-sheet theory on the string these confined monopoles are seen as $CP(N-1)$ kinks interpolating between distinct vacua.

III. $CP(N-1)$ MODEL AT ZERO TEMPERATURE

At large N the model was solved [14] in the $1/N$ approximation. Let us outline how this is done. The Lagrangian $\mathcal{L}$ of the $CP(N-1)$ model in the gauged formulation in the Euclidean space-time can be written as

$$\mathcal{L} = |\nabla_k n^l| + \omega(|n^l|^2 - r), \tag{3.1}$$

where we rescale the n^l fields. In addition, we introduce a parameter ω to enforce the constraint. Moreover, we replace the coupling r with the 't Hooft coupling constant λ,

$$\lambda = \frac{N}{r}; \tag{3.2}$$

λ does not scale with N.

Since the n^l fields appear quadratically in the action (3.1) we can perform the Gaussian integration over them, resulting in the equation for the effective potential V,

$$e^{-\hat{T}V} = \int d\omega dA_k \mathrm{det}^{-N}(-(\partial_k - iA_k)^2 + \omega) \times \exp\left(\frac{N}{\lambda}\int d^2x\omega\right), \tag{3.3}$$

where $\hat{T}$ stands for the (asymptotically infinite) Euclidean time.

Since integration over ω and A_k cannot be done exactly, we use a stationary phase approximation. Because of the Lorentz invariance we search for a point such that $A_k = 0$ and $\omega = \mathrm{const}$. To find this stationary point we vary Eq. (3.3) with respect to ω. The resulting equation is

$$\lambda \int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2+\omega} = 1. \tag{3.4}$$

Rewriting the bare coupling constant λ in terms of the scale Λ_{CP} of the $CP(N-1)$ model

$$\frac{4\pi}{\lambda} = \ln\frac{M_{\mathrm{uv}}^2}{\Lambda_{CP}^2}, \tag{3.5}$$

where M_{uv} is the ultraviolet cutoff, we finally find that

$$\omega = \Lambda_{CP}^2. \tag{3.6}$$

Thus, the vacuum value of ω does not vanish. Looking at Eq. (3.1) one can see that a positive value of ω means that a mass for the fields n^l is dynamically generated.

To determine the spectrum of the theory one has to expand the effective action Eq. (3.1) around the saddle point and consider field fluctuations in the quadratic approximation. Linear terms vanish. Terms that are cubic and higher are suppressed by powers of $1/\sqrt{N}$. Two Feynman diagrams in Fig. 1 give rise to the kinetic term for the U(1) gauge field.

Gauge invariance requires the answer to be

$$\Pi_{\mu\nu} = \Pi(p^2)(p^2 g_{\mu\nu} - p_\mu p_\nu). \tag{3.7}$$

The meaning of Eq. (3.7) is simple. It represents the kinetic energy of the gauge field written in momentum space. Thus, what was introduced as an auxiliary field becomes a propagating field. Calculation in Appendix B reproduces Witten's result [14], $\Pi(0) = N/12\pi\Lambda_{CP}^2$, which is interpreted as the inverse of the $U(1)$ charge squared of the n^l fields.

A massless photon in two dimensions produces the Coulomb potential between two charges at separation R,

$$V(R) = \frac{12\pi\Lambda^2}{N}R, \tag{3.8}$$

leading to a linear confinement of the $\bar{n}n$ pairs. Thus, the spectrum of the theory contains $\bar{n}n$ "mesons" rather than free n's.

It is instructive to present an alternative interpretation of this result. In [14] it was shown that n^l fields can be interpreted as kinks interpolating between different vacua. The vacuum structure of the $CP(N-1)$ model was studied in [20]. According to this work the genuine vacuum is

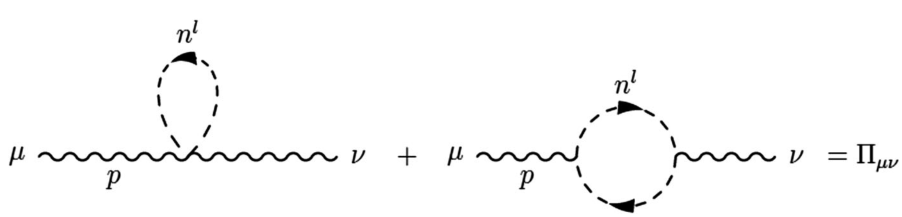

FIG. 1. Feynman diagrams contributing to the kinetic term of the photon field.

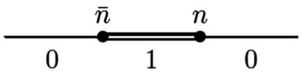

FIG. 2. Configuration of the string with two particles on it. Zero and one represent the true vacuum and the first quasivacuum, respectively.

unique. There are, however, of the order N quasivacua, which become stable in the limit $N \to \infty$, since the energy split between the neighboring quasivacua is $O(1/N)$. Thus, one can imagine the $\bar{n}$ field interpolating between the true vacuum and the first quasivacuum and the n field returning to the true vacuum as in Fig. 2. The linear confining potential between the kink and antikink is associated with the excess in the quasivacuum energy density compared to that in the genuine vacuum.

This two-dimensional confinement of kinks can be interpreted in terms of strings and monopoles of the bulk theory; see [18]. The fine structure of the $CP(N-1)$ vacua on the non-Abelian string means that N elementary strings are split by quantum effects and have slightly different tensions. Therefore, the monopoles, in addition to the four-dimensional confinement (which ensures that they are attached to the string), acquire a two-dimensional confinement along the string. The monopole and antimonopole connected by a string with larger tension form a mesonic bound state.

Consider a monopole-antimonopole pair interpolating between strings 0 and 1; see Fig. 2. The energy of the excited part of the string (labeled as 1) is proportional to the distance as in Eq. (3.8). When it exceeds the mass of two monopoles (which is of the order of Λ_{CP}), then the second monopole-antimonopole pair appear breaking the excited part of the string. This gives an estimate for the typical length of the excited part of the string, $R \sim N/\Lambda_{CP}$.

The above condition guarantees that there is enough energy in the "wrong string" to produce a pair of kinks. However, the probability of this process, string breaking (which can be inferred from the false vacuum decay theory), is proportional to $\exp(-N)$, i.e., dies off exponentially at large N.

IV. THE COULOMB/CONFINEMENT PHASE

To consider closed non-Abelian strings of length L we compactify the space dimension; in other words, we study $CP(N-1)$ model (3.1) on a strip of the finite length L with periodic boundary conditions.

In a Euclidean formulation considering a model at finite length is equivalent to considering the model at finite temperature. The correspondence between the length of the string and the temperature is given by

$$L = \beta, \tag{4.1}$$

where β is the inverse temperature. Thus, the limit of infinite length is the same as the limit of zero temperature.

To solve the $CP(N-1)$ model on a finite strip we use a large-N approximation. The $CP(N-1)$ model at a finite temperature in the large-N approximation was solved previously by Affleck [15]; see also [16] and [17] for reviews. Although we use a different regularization, our results match those obtained in [15]. There are two important differences, however. The first one is related to the interpretation of the photon mass. In [15] the emergence of the photon mass is interpreted as a phase transition into the deconfinement phase already at $L = \infty$. We give a different interpretation of the photon mass (see Sec. IV B); we do not detect any phase transition at $L = \infty$. We interpret the large L phase ($L > 1/\Lambda_{CP}$) as a Coulomb/confinement phase, much in the same way as at infinite L [14].

The second difference with Ref. [15] is that we find a phase transition at $L \sim 1/\Lambda_{CP}$ into a deconfinement phase in the limit $N \to \infty$; see Sec. V. This is a weak coupling phase. In this phase the global $SU(N)$ is broken and the $CP(N-1)$ model does not develop a mass gap. The gauge field remains auxiliary, and no Coulomb/confining potential is generated.

At large but *finite* N we expect the phase transition to become a rapid crossover. The spontaneous breaking of the global $SU(N)$ symmetry is in contradiction with the Coleman theorem [21], stating that there can be no massless nonsterile particles in $1+1$ dimensions. Therefore we expect that the "would be Goldstone" states of the broken phase acquire small masses suppressed in the large-N limit.

To solve the $CP(N-1)$ model we use the mode expansion with the periodic boundary conditions. The open string setup involves the Dirichlet boundary conditions. For example, for open string the expansion (1.1) is modified. It acquires L-independent terms coming from the energy associated with boundaries. We limit ourselves to a closed string in this paper.

A. Large-N solution

Our starting point is Eq. (3.1). Integrating out n^l fields, one arrives at the same Eq. (3.3) as in the infinite L case. However, now we take into account the gauge holonomy around the compact dimension. Following [15] we choose the gauge

$$A_1 = 0$$

and look for minima of the potential with $A_0 = \text{const}$ and $\omega = \text{const}$. The mode expansion in (3.3) gives for the orientational part of the string energy in (1.3)

$$E_{\text{orient}}(L) = \frac{N}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dq_1 \ln\left\{ q_1^2 + \left(\frac{2\pi k}{L} + A_0\right)^2 + \omega \right\}. \tag{4.2}$$

To calculate (4.2) we follow [22] and use the zeta function regularization. Details of our calculation are presented in Appendix A. Here we give the final result for the string vacuum energy,

$$E_{\rm orient}(L) = \frac{NL\omega}{4\pi}\left[1 - \ln\frac{\omega}{\Lambda_{CP}^2} - 8\sum_{k=1}^{\infty}\frac{K_1(kL\sqrt{\omega})}{kL\sqrt{\omega}}\cos kLA_0\right], \tag{4.3}$$

where K_1 is the modified Bessel function of the second kind (also known as the Macdonald function). An important feature of this expression is the appearance of a nontrivial potential for the photon field. We will dwell on this issue in the next subsection.

To find the saddle point we extremize the expression (4.3) with respect to ω and A_0, which results in the following equations:

$$\frac{\partial E_{\rm orient}}{\partial A_0} = \frac{2NL\sqrt{\omega}}{\pi}\sum_{k=1}^{\infty}K_1(Lk\sqrt{\omega})\sin LkA_0 = 0, \tag{4.4}$$

$$\log\frac{\omega}{\Lambda_{CP}^2} = 4\sum_{k=1}^{\infty}K_0(Lk\sqrt{\omega})\cos LkA_0, \tag{4.5}$$

where the logarithmic term in the left-hand side of Eq. (4.5) is the renormalized inverse coupling $1/\lambda$. The logarithmic integral over momentum is regularized in the infrared by ω.

Equation (4.4) yields the solution of the form $LA_0 = \pi l$, where $l \in \mathbb{Z}$. However, from Eq. (4.3) it is clear that the solution with $LA_0 = 2\pi l$ lies lower in energy than the solution with $LA_0 = (2l-1)\pi$ and is, thus, physical. We take $A_0 = 0$ as a solution of (4.4). Our result for the orientational string energy is shown in Fig. 3, where $\tilde{V} = E_{\rm orient}/L$.

Equation (4.5) yields a nonvanishing value of ω which we interpret—as in the case of zero temperature—as mass generation for the n^l fields. The dependence of the mass on the string length L is shown in Fig. 4 where we put

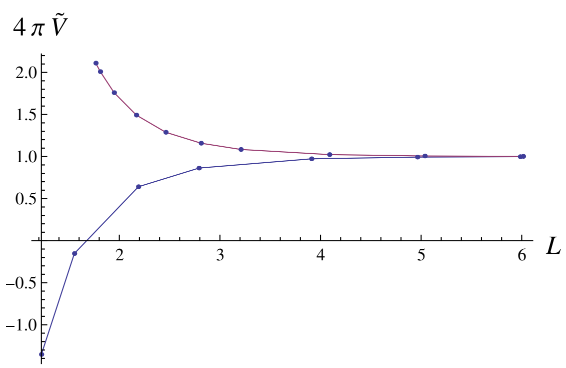

FIG. 3 (color online). Effective potential (in units of Λ_{CP}^2) as a function of length.

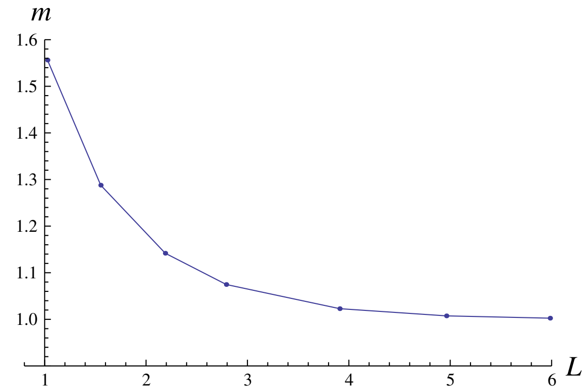

FIG. 4 (color online). Mass (in units of Λ) of fields n^l as a function of L.

$$\sqrt{\omega} \equiv m. \tag{4.6}$$

One can see that the n^l field mass increases with decreasing L.

In the limit $L \gg 1/\Lambda_{CP}$ the modified Bessel functions in (4.3) exhibit exponential falloff at large L. To determine the leading nontrivial correction to the string energy we can use the "zeroth-order" solution $\omega \approx \Lambda_{CP}^2$ of Eq. (4.5) for the vacuum expectation value (VEV) of ω. Clearly this zeroth-order solution coincides with the VEV of ω in the infinite volume; see (3.6). For the total string energy we obtain

$$E(L) = \left(2\pi\xi + \frac{N}{4\pi}\Lambda_{CP}^2\right)L - \frac{\pi}{3}\frac{1}{L} - N\sqrt{\frac{2}{\pi}}\sqrt{\frac{\Lambda_{CP}}{L}}e^{-\Lambda_{CP}L} + \cdots. \tag{4.7}$$

In Eq. (4.7) we included the classical string tension $2\pi\xi L$, its renormalization due to vacuum fluctuations in $CP(N-1)$ [i.e., $(N/4\pi)\Lambda_{CP}^2 L$] and the contribution of the translational modes, which give the standard Lüscher term. This result was quoted in Sec. I; see Eq. (1.5).

We see that the quantum fluctuations of the orientational moduli contribute both to the renormalization of the string tension [the linear in L term in (4.7)] and to the function $f(\Lambda_{CP}L)$ in (1.3). As was expected, in the theory with a mass gap the contribution of orientational moduli to the L-dependent part of the string energy is exponentially suppressed at large L.

Let us note that the case of an open non-Abelian string was previously considered in [23]. The results of [23] show the presence of long range $1/L$ effects coming from the orientational sector even at large L where the theory has a mass gap. We disagree with these results and believe that orientational long range forces in the large-L phase are spurious and are associated with the boundary energy somehow induced [23] by the Dirichlet boundary conditions rather than with the string itself.

B. The photon mass

The A_0 dependence in the potential (4.3) ensures that the gauge field acquires a mass [15]. It is quite natural to expect that the photon becomes massive at nonzero temperature. Physically this means the Debye screening.

Expanding (4.3) at large L we can write down an effective action for the $U(1)$ gauge field,

$$S_{\text{gauge}} = \int d^2x \left\{ \frac{1}{4e^2} F_{kl}^2 - N\sqrt{\frac{2}{\pi}}\sqrt{\frac{\Lambda_{CP}}{L^3}} e^{-\Lambda_{CP}L} \cos A_0 L + \cdots \right\}. \tag{4.8}$$

The kinetic term for the gauge field at nonzero temperature is calculated in Appendix B. To calculate the photon mass to the leading order in $\exp(-\Lambda_{CP}L)$ we need the expression for the gauge coupling e^2 in the limit $L \to \infty$, namely,

$$\frac{1}{e^2} \approx \frac{N}{12\pi\Lambda_{CP}^2}; \tag{4.9}$$

see Sec. III. Expanding (4.8) to the quadratic order in A_0 we arrive at

$$m_A^2 \approx 12\Lambda_{CP}^2 \sqrt{2\pi\Lambda_{CP}L}\, e^{-\Lambda_{CP}L} \tag{4.10}$$

for the photon mass. Note that the nonzero photon mass at finite temperature does not break the gauge invariance since the Lorentz symmetry is explicitly broken; see [15].

The photon becoming massive was the reason for the claim [15] that at nonzero temperature the $CP(N-1)$ model is in the deconfinement phase. We give a different interpretation for this effect.

We treat the quasivacua as the strings of different tensions. Kinks and antikinks interpolate between true vacuum and the first quasivacuum. The Debye screening due to a finite photon mass now can be interpreted as a breaking of the confining string between kink and antikink in the thermal medium (through picking up a kink-antikink pair from the thermal bath). Note that unlike pair production from the vacuum, this process is not suppressed as $\exp(-N)$.

The kink-antikink potential has the form

$$V(R) = e^2 R e^{-m_A R}, \tag{4.11}$$

where R is the kink-antikink separation. It is still linear at small R, while the exponential suppression at large R can be understood as a breaking of the confining string due to the creation of a kink-antikink pair from the thermal bath. Therefore, we still interpret the large L phase as a Coulomb/confinement phase.

A similar question can be addressed in QCD. Do we have confinement of quarks in QCD? We believe that the answer is positive. However, the confining string can be broken by quark-antiquark production. We suggest a similar interpretation for the $CP(N-1)$ model at nonzero temperature.

If L is very large (very low temperatures), the thermal string breaking can be ignored; however, once L reduces below $\log N/\Lambda_{CP}$, the thermal breaking becomes operative.

C. Small length limit

As was already mentioned, we will show in the next section that once L decreases below $1/\Lambda_{CP}$ our $CP(N-1)$ model undergoes a phase transition into the deconfinement phase. To prove this we calculate the vacuum energy in the deconfinement phase in the next section and show that it lies below that in the Coulomb/confinement phase.

To make this comparison we will examine Eqs. (4.3) and (4.5) in the low-L limit. These expressions determine the vacuum energy and the ω expectation value in the Coulomb/confinement phase.

Assuming that $L^2\omega \ll 1$ we can use the following approximation for the sum of the modified Bessel functions [see Eq. (8.526) in [24]]:

$$\sum_{n=1}^{\infty} K_0(ny) \approx \frac{\pi}{2y} + \frac{1}{2}\ln\frac{y}{4\pi} + \frac{\gamma}{2} + O(y^2), \tag{4.12}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. Consequently, we get from (4.5)

$$\ln\frac{\sqrt{\omega}}{\Lambda_{CP}} = 2\left[\frac{\pi}{2L\sqrt{\omega}} + \frac{1}{2}\ln\frac{L\sqrt{\omega}}{4\pi} + \frac{\gamma}{2}\right], \tag{4.13}$$

or approximately

$$\ln\frac{1}{\Lambda_{CP}L} = \frac{\pi}{L\sqrt{\omega}}. \tag{4.14}$$

Now the logarithmic integral that determines the renormalized inverse coupling $1/\lambda$ is regularized in the infrared by $1/L$ rather than by $\sqrt{\omega}$ (which is the case in the large-L limit). This gives us the ω expectation value,

$$\sqrt{\omega} = \frac{\pi}{L}\frac{1}{\ln(1/\Lambda_{CP}L)} + \cdots. \tag{4.15}$$

Equation (4.15) justifies our approximation $L^2\omega \ll 1$ at $L \ll 1/\Lambda_{CP}$. Note also that at $L \ll 1/\Lambda_{CP}$ the coupling constant is small—it is frozen at the scale $1/L$ [the logarithm in the left-hand side of (4.14) is large], so the theory is at weak coupling.

To find the orientational energy in this limit we need to find an approximate expression for the sum of the modified Bessel functions that appears in (4.3),

$$S_E = \frac{2L\sqrt{\omega}}{L\pi}\sum_{k=1}^{\infty}\frac{K_1(kL\sqrt{\omega})}{k}. \tag{4.16}$$

The derivative of the modified Bessel functions satisfies the following relation [see Eq. (9.6.28) in [25]]:

$$K_1'(x) = -K_0(x) - \frac{K_1(x)}{x}. \tag{4.17}$$

Let us introduce a notation,

$$S_1(x) = \sum_{k=1}^{\infty}\frac{K_1(kx)}{k}. \tag{4.18}$$

Then

$$(xS_1(x))' = -x\sum_{k=1}^{\infty}K_0(kx) \overset{\text{(IV.12)}}{\approx} -\frac{\pi}{2} - \frac{x}{2}\ln\frac{x}{4\pi} - \frac{x\gamma}{2} + O(x^3). \tag{4.19}$$

Integrating this expression one finds

$$xS_1(x) \approx -\frac{x\pi}{2} - \frac{x^2}{4}\ln\frac{x}{4\pi} - \frac{x^2}{8}(2\gamma - 1) + \text{const} + O(x^4). \tag{4.20}$$

The behavior of the modified Bessel function at small values of the argument is given by [see Eq. (9.6.9) in [25]]

$$K_1(x) \sim \frac{1}{x}. \tag{4.21}$$

Thus, the sum $S_1(x)$ can be approximated as follows:

$$S_1(x) \approx \sum_{k=1}^{\infty}\frac{1}{xk^2} = \frac{\pi^2}{6x}. \tag{4.22}$$

Hence the constant appears to be $\pi^2/6$. Now we are ready to present the approximate expression we seek for

$$S_E = \frac{2}{L\pi}L\sqrt{\omega}S_1(L\sqrt{\omega}) \approx \frac{\pi}{3L} - \sqrt{\omega} - \frac{L\omega}{2\pi}\ln\frac{L\sqrt{\omega}}{4\pi} - \frac{L\omega}{4\pi}(2\gamma - 1). \tag{4.23}$$

With this approximation we arrive at the orientational energy

$$E_{\text{orient}}(L) = -\frac{\pi}{3}\frac{N}{L} + N\sqrt{\omega} - \frac{N}{2\pi}\omega L\ln\frac{1}{\Lambda_{CP}L} + \cdots \tag{4.24}$$

Substituting here the VEV of ω [see (4.15)], we get

$$E_{\text{orient}}(L) = -\frac{\pi}{3}\frac{N}{L} + \frac{\pi}{2}\frac{N}{L}\frac{1}{\ln(1/\Lambda_{CP}L)} + \cdots. \tag{4.25}$$

The first term here is the Lüscher term proportional to the number of orientational degrees of freedom $2(N-1) \approx 2N$ (in the large N limit). It gets corrected by an infinite series of powers of inverse logarithms $\ln(1/\Lambda_{CP}L)$, if we naively extend the Coulomb/confinement phase into the region of small L. We will show in the next section that in fact the theory undergoes a phase transition into a different phase, with a lower energy.

V. DECONFINEMENT PHASE

Classically the $CP(N-1)$ model has $2(N-1)$ massless states that can be viewed as Goldstone states of the broken $SU(N)$ symmetry. Indeed, classically the vector n^l satisfies a fixed length condition, $|n|^2 = r$; see (3.1). Thus classically n^l acquires a VEV breaking $SU(N)$ symmetry.

However, as was shown above, in the strong coupling large L domain the spontaneous symmetry breaking does not occur, in much the same way as in the infinite-L limit; see [14]. At strong coupling the vector n^l is smeared all over the vacuum manifold due to strong quantum fluctuations. The theory has a mass gap, and moreover the number of the massive n fields becomes $2N$. Effectively the classical constraint $|n|^2 = r$ is lifted; see [14].

At small L the theory enters a weak coupling regime so we expect an occurrence of the classical picture in the limit $N \to \infty$. To study this possibility we assume that one component of the field n^l, say $n_0 \equiv n$, can develop a VEV. Then we integrate over all other components of n^l ($l = 1, 2, \ldots$) keeping the fields n and ω as a background. Note that a similar method was used in [26] for studying phase transitions in the $CP(N-1)$ model with twisted masses.

Now, instead of (4.24), we get

$$E_{\text{orient}}(L) = \omega L|n|^2 - \frac{\pi}{3}\frac{N}{L} - \frac{N}{2\pi}\omega L\ln\frac{1}{\Lambda_{CP}L} + \cdots, \tag{5.1}$$

where the ellipses stand for higher terms in $L^2\omega$. Note that here we drop the contribution associated with the integration over the constant n [the second term in (4.24)] because we introduce n_0 as a constant background field [in other words, we drop the term with $k = 0$ in (4.2)].

Minimizing over ω and n we arrive at the equations

$$|n|^2 = \frac{N}{2\pi}\ln\frac{1}{\Lambda_{CP}L} + \cdots,$$
$$\omega n = 0. \tag{5.2}$$

The solution to these equations with nonzero n_0 read

$$|n|^2 = \frac{N}{2\pi}\ln\frac{1}{\Lambda_{CP}L}, \qquad \omega = 0. \tag{5.3}$$

We see that the mass gap ω is not generated. Substituting this in (5.1) we get that the orientational energy reduces just to the Lüscher term, namely,

$$E_{\rm orient}(L) = -\frac{\pi}{3}\frac{N}{L}. \tag{5.4}$$

This energy is lower than the one in (4.25). Therefore, we conclude that at $L \sim 1/\Lambda_{CP}$ the theory undergoes a phase transition into the phase with the broken $SU(N)$ symmetry. This ensures the presence of $2(N-1)$ Goldstone states n^l, $l = 1, \ldots, (N-1)$. The photon remains an auxiliary field, and no kinetic term is generated for it. As a result, there is no Coulomb/confining linear rising potential between the n states. The phase with the broken $SU(N)$ is a deconfinement phase. Since $|n^l|$ is positively defined, Eq. (5.3) shows that this phase appears at $L < 1/\Lambda_{CP}$.

The results of numerical calculations are in agreement with our conclusions. The relation between orientational energies in both phases is shown in Fig. 5. One can see that the Lüscher term energy is lower and is thus physical.

The phase with the broken symmetry in two dimensions can occur only in the limit $N \to \infty$. As was already explained, if N is large but finite, this would contradict the Coleman theorem [21]. Therefore, we expect that at large but finite N the phase transition becomes a rapid crossover. In particular, we expect that the n^l fields are not strictly massless. They have small masses suppressed by $1/N$.

To conclude this section let us note that the $CP(N-1)$ model compactified on a cylinder with the so-called twisted boundary conditions was studied in [27]. No phase transition was found; moreover, it was shown that the theory has a mass gap that shows no L dependence and is determined entirely by Λ_{CP}. We believe that our results are not in contradiction with those obtained in [27], because at finite L the boundary conditions matter: they can be crucial. In particular, the twisted boundary conditions can be viewed as a gauging of the global $SU(N)$ group with a constant gauge potential. Then the global $SU(N)$ is explicitly broken. This model should be considered as distinct as compared to the $CP(N-1)$ model with the periodic boundary conditions studied in this paper.

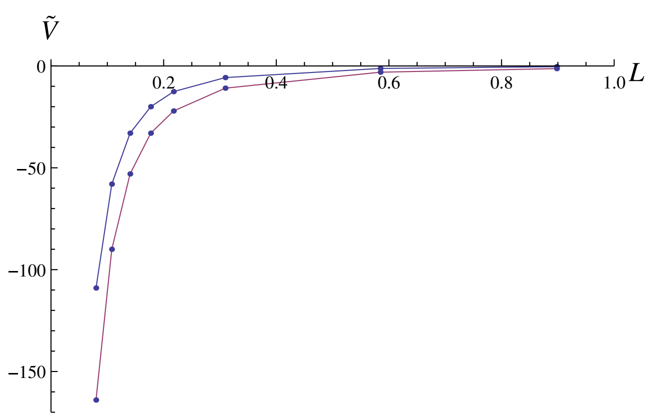

FIG. 5 (color online). Comparison of orientational energies in both phases. The Lüscher term always lies lower. We set $\Lambda_{CP} = 1$.

VI. SUPERSYMMETRIC $CP(N-1)$ MODEL WITH NO COMPACTIFICATION

Non-Abelian strings were first found in $\mathcal{N} = 2$ supersymmetric QCD with the $U(N)$ gauge group and $N_f = N$ quark hypermultiplets [5–8]; see [9–12] for reviews. In much the same way as for the nonsupersymmetric case the internal dynamics of orientational zero modes of non-Abelian string is described by two-dimensional $CP(N-1)$ model living on the string world sheet. The string solution is 1/2-BPS saturated; therefore the two-dimensional model under consideration is $\mathcal{N} = (2,2)$ supersymmetric. In this section we briefly review the large-N solution of the $\mathcal{N} = (2,2)$ $CP(N-1)$ model in infinite space [14]. In the next section we will present the large-N solution of the model on a strip of a finite length L (cylindrical compactification).

The bosonic part of the action of the $CP(N-1)$ model is given by

$$S_{\rm bos} = \int d^2x \Big[|\nabla_i n^l|^2 + \frac{1}{4e^2}F_{ij}^2 + \frac{1}{e^2}|\partial_i\sigma|^2 + \frac{1}{2e^2}D^2 + 2|\sigma|^2|n^l|^2 + iD(|n^l|^2 - r_0) \Big], \tag{6.1}$$

where the covariant derivative is defined as $\nabla_i = \partial_i - iA_i$ and σ is a complex scalar field, the scalar superpartner of A_i. Moreover, r_0 is the bare coupling constant. In the limit $e^2 \to \infty$ the gauge field A_i and σ become auxiliary fields. D stands for the D component of the gauge multiplet. The factor i is due to the passage to the Euclidean notation.

The fermionic part of the action takes the form

$$S_{\rm ferm} = \int d^2x \Big[\bar{\xi}_{lR} i(\nabla_0 - i\nabla_3)\xi_R^l + \bar{\xi}_{lL} i(\nabla_0 + i\nabla_3)\xi_L^l + \frac{1}{e^2}\bar{\lambda}_R i(\nabla_0 - i\nabla_3)\lambda_R + \frac{1}{e^2}\bar{\lambda}_L i(\nabla_0 + i\nabla_3)\lambda_L + (i\sqrt{2}\sigma\bar{\xi}_{lR}\xi_L^l + i\sqrt{2}\bar{n}_l(\lambda_R\xi_L^l - \lambda_L\xi_R^l) + \text{H.c.}) \Big], \tag{6.2}$$

where the fields $\xi_{L,R}^l$ are the fermion superpartners of n^l and $\lambda_{L,R}$ belong to the gauge multiplet. In the limit $e^2 \to \infty$ they enforce the following constraints:

$$\bar{n}^l\xi_L^l = 0, \qquad \bar{n}^l\xi_R^l = 0. \tag{6.3}$$

The field σ is auxiliary and can be eliminated, namely,

$$\sigma = -\frac{i}{\sqrt{2}r_0}\bar{\xi}_{lL}\xi^l_R. \tag{6.4}$$

A. Large-N solution

The $\mathcal{N}=(2,2)$ supersymmetric $CP(N-1)$ model was solved in the large-N limit by Witten [14]; see also [28]. In this section we briefly review this solution.

Since both fields n^l and ξ^l appear quadratically, we can integrate them out. This produces two determinants,

$$\det{}^{-N}(-\partial_i^2 + iD + 2|\sigma|^2)\det{}^{N}(-\partial_i^2 + 2|\sigma|^2). \tag{6.5}$$

The first determinant comes from the boson n^l fields, while the second comes from the fermion ξ^l fields. Note that if $D=0$ the two contributions obviously cancel each other, and supersymmetry is unbroken. As before, the nonzero values of $iD+2|\sigma|^2$ and $2|\sigma|^2$ can be interpreted as nonzero values of the mass of n^l and ξ^l fields, and we put $A_k=0$.

The final expression for the effective potential is given by (see, for example, [28])

$$V_{\text{eff}} = \int d^2x \frac{N}{4\pi}\left[-(iD+2|\sigma|^2)\ln\frac{iD+2|\sigma|^2}{\Lambda_{CP}^2} + iD + 2|\sigma|^2\ln\frac{2|\sigma|^2}{\Lambda_{CP}^2}\right], \tag{6.6}$$

where the logarithmic ultraviolet divergence of the coupling constant is traded for the scale Λ_{CP}.

To find a saddle point we minimize the potential with respect to D and σ, which yields the following set of equations:

$$\ln\frac{iD+2|\sigma|^2}{\Lambda_{CP}^2} = 0,$$
$$\ln\frac{iD+2|\sigma|^2}{2|\sigma|^2} = 0. \tag{6.7}$$

The solution to these equations is

$$D = 0, \tag{6.8}$$

which shows that supersymmetry is not broken. The VEV of σ is

$$\sqrt{2}\sigma = \Lambda_{CP}e^{\frac{2\pi k}{N}i}, \qquad k = 0,\ldots,(N-1). \tag{6.9}$$

We see that σ develops a VEV giving masses to the n^l fields and their fermion superpartners ξ^l. The phase factor in the right-hand side of (6.9) does not follow from (6.7). It comes from the broken chiral $U(1)$ symmetry. The axial anomaly breaks it down to Z_{2N}. The field σ has the chiral charge 2. This explains the phase factor in (6.9). Once $|\sigma|$ has a nonzero VEV, the anomalous symmetry breaking ensures that the theory has N vacuum states. Clearly this fine structure cannot be seen in the large N approximation since the phase factor is a $1/N$ effect.

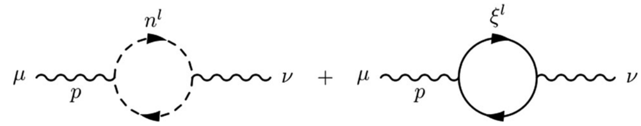

FIG. 6. Feynman diagrams contributing to the kinetic term of the photon.

In full accord with the Witten index, the solution above has N vacua, each with the vanishing energy.

Consider now the vector multiplet. In much the same way as in the nonsupersymmetric case, the photon becomes a propagating field. To find the renormalized gauge coupling one needs to evaluate the two Feynman diagrams shown in Fig. 6. Details of the appropriate calculation are given in Appendix C. The result is

$$\frac{1}{e^2} = \frac{N}{4\pi}\frac{1}{\Lambda_{CP}^2}. \tag{6.10}$$

Through the coupling to the Imσ (due to the chiral anomaly) now the photon acquires a mass. Moreover, the fermion fields $\lambda_{L,R}$ also become propagating, with the same mass as that of the photon, as required by supersymmetry. The masses of the fields of the vector multiplet are as follows [14,28]:

$$m_{ph} = m_{\lambda_{L,R}} = m_{\text{Re}\sigma} = m_{\text{Im}\sigma} = 2\Lambda_{CP}. \tag{6.11}$$

Since the photon became massive there is no linear rising Coulomb potential between the charged states. There is no confinement in the supersymmetric $CP(N-1)$ model even in the infinite volume limit. It has N degenerate vacua that are interpreted as N degenerate elementary non-Abelian strings in the four-dimensional bulk theory. In contrast to the nonsupersymmetric case, the confined monopoles of the bulk theory, which are seen as kinks interpolating between the $CP(N-1)$ vacua, are free to move along the string; see [11] for further details.

VII. SUPERSYMMETRIC $CP(N-1)$ ON A CYLINDER

Now we compactify one space dimension and impose periodic boundary conditions, for both bosons and fermions, in order to preserve $\mathcal{N}=(2,2)$ supersymmetry. We stress that this compactification *cannot* be considered as thermal. Nonzero temperature requires antiperiodic boundary conditions for fermions, which would break supersymmetry explicitly.

The large-N method in the case of the $\mathcal{N} = (2,2)$ $CP(N-1)$ model works similar to that in the nonsupersymmetric case. We compactify now the spatial coordinate x_1 and start from a slightly modified expression for the determinants in Eq. (6.5). Choosing the $A_0 = 0$ gauge and assuming that A_1 is nonzero, we write

$$\det{}^{-N}(-\partial_0^2 - (\partial_1 - iA_1)^2 + m_b^2)\det{}^{N}(-\partial_0^2 - (\partial_1 - iA_1)^2 + m_f^2), \tag{7.1}$$

where we introduced the following notation:

$$m_b^2 = iD + 2|\sigma|^2, \qquad m_f^2 = 2|\sigma|^2. \tag{7.2}$$

The evaluation of each of the determinants is no different from that in the nonsupersymmetric case. Again we use the zeta-function method. Using expressions in Appendix C we can derive the effective potential,

$$E = \frac{LN}{4\pi}\left[-(iD + 2|\sigma|^2)\ln\frac{iD + 2|\sigma|^2}{\Lambda_{CP}^2} + iD + 2|\sigma|^2\ln\frac{2|\sigma|^2}{\Lambda_{CP}^2} - 8m_b^2\sum_{k=1}^{\infty}\frac{K_1(Lm_bk)}{Lm_bk}\cos(LA_1k)\right.$$
$$\left. + 8m_f^2\sum_{k=1}^{\infty}\frac{K_1(Lm_fk)}{Lm_fk}\cos(LA_1k)\right]. \tag{7.3}$$

Here the first three terms are just the effective potential at $L = \infty$, while the second and third lines are the finite-L corrections due to bosons and fermions, respectively.

To find a stationary point we vary the above expression with respect to A_1, D, and σ. The resulting equations are as follows:

$$m_b\sum_{k=1}^{\infty}K_1(Lm_bk)\sin(LA_1k) - m_f\sum_{k=1}^{\infty}K_1(Lm_fk)\sin(LA_1k) = 0,$$
$$2\sigma\left[-\ln\frac{m_b^2}{m_f^2} + 4\sum_{k=1}^{\infty}K_0(Lm_bk)\cos(LA_1k) - 4\sum_{k=1}^{\infty}K_0(Lm_fk)\cos(LA_1k)\right] = 0,$$
$$-\ln\frac{m_b^2}{\Lambda_{CP}^2} + 4\sum_{k=1}^{\infty}K_0(Lm_bk)\cos(LA_1k) = 0. \tag{7.4}$$

Calculation of the gauge coupling constant at finite L is also modified (see Appendix C). As a result, we arrive at

$$\frac{1}{Ne^2} = \frac{1}{4\pi m_b^2} + \frac{L}{2\pi m_b}\sum_{k=1}^{\infty}K_1(Lm_bk)k, \tag{7.5}$$

which reduces to $1/4\pi\Lambda_{CP}^2$ in the limit $L \to \infty$.

Consider now the large L limit, $L \gg 1/\Lambda_{CP}$. Assuming that $m_b \sim m_f \sim \Lambda_{CP}$ (we confirm this below), we expand the string energy (7.3), keeping the first exponentially small term

$$E = \frac{LN}{4\pi}\left\{-m_b^2\ln\frac{m_f^2}{\Lambda_{CP}^2} + iD + m_f^2\ln\frac{m_f^2}{\Lambda_{CP}^2}\right\}$$
$$- N\sqrt{\frac{2}{\pi}}\left[\sqrt{\frac{m_b}{L}}e^{-m_bL} - \sqrt{\frac{m_f}{L}}e^{-m_fL}\right]\cos A_1L + \cdots. \tag{7.6}$$

Taking derivatives with respect to D, $\sqrt{2}\bar{\sigma}$, and A_1, we obtain

$$-\frac{N}{4\pi}\log\frac{m_b^2}{\Lambda_{CP}^2} + N\frac{1}{\sqrt{2\pi}}\frac{\exp(-m_bL)}{\sqrt{m_bL}}\cos A_1L + \cdots = 0,$$
$$\sqrt{2}\sigma\left\{\frac{N}{4\pi}\log\frac{m_f^2}{m_b^2} + N\frac{1}{\sqrt{2\pi}}\left[\frac{\exp(-m_bL)}{\sqrt{m_bL}} - \frac{\exp(-m_fL)}{\sqrt{m_fL}}\right]\cos A_1L + \cdots\right\} = 0,$$
$$\left\{\frac{\exp(-m_bL)}{\sqrt{m_bL}} - \frac{\exp(-m_fL)}{\sqrt{m_fL}}\right\}\sin A_1L + \cdots = 0, \tag{7.7}$$

where the ellipses denote next-to-leading corrections in $1/Lm_b$ and $1/Lm_f$.

The solution of these equations is as follows. The second and third equations are satisfied at

$$D = 0, \tag{7.8}$$

which shows that supersymmetry is not broken. A_1 remains undetermined.

With $D = 0$ the first equation determines the σ expectation value, namely,

$$\frac{N}{4\pi}\log\frac{2|\sigma|^2}{\Lambda_{CP}^2} = N\frac{1}{\sqrt{2\pi}}\frac{\exp(-\sqrt{2}|\sigma|L)}{\sqrt{\sqrt{2}|\sigma|L}}\cos A_1 L + \cdots. \tag{7.9}$$

This equation seems to present a puzzle. It shows that the VEV of σ depends on the parameter A_1, which is arbitrary. If this were the case, the theory would have a branch of vacua parametrized by the Polyakov line

$$e^{\int dx_1 A_1} = e^{iA_1 L}, \tag{7.10}$$

which measures the holonomy around the compact dimension. More exactly, the theory would have N branches of vacua, because Z_{2N} symmetry ensures that the overall phase of σ takes N values $2\pi k/N$, $k = 0, \ldots, (N-1)$. This would contradict the Witten index argument, which ensures that the number of vacua is equal to N for the $\mathcal{N} = (2,2)$ supersymmetric $CP(N-1)$ model.

The resolution of this puzzle is that we should quantize the phase variable $A_1 L$ (note that $\int dx_1 A_1$ depends only on time) as a function of the noncompact time. In the emerging quantum mechanics the phase $A_1 L$ is not fixed; instead, it is smeared all over the circle (in the ground state). As a result, the $\cos(A_1 L)$ in (7.9) is averaged to zero and the σ VEVs are given by

$$\sqrt{2}\sigma = \Lambda_{CP} e^{\frac{2\pi k}{N} i}, \qquad k = 0, \ldots, (N-1). \tag{7.11}$$

This is exactly the same result as for $L = \infty$. All cosine functions of $A_1 L$ in the last equation in (7.4) are averaged to zero; therefore the result in (7.11) is exact and does not depend on L.

This result also can be understood by studying the exact twisted superpotential of the $\mathcal{N} = (2,2)$ $CP(N-1)$ model. In the infinite volume it is given by [29–31]

$$W(\sigma) = \frac{N}{4\pi}\left\{\sqrt{2}\sigma\log\frac{\sqrt{2}\sigma}{\Lambda_{CP}} - \sqrt{2}\sigma\right\}. \tag{7.12}$$

This superpotential has correct transformation properties with respect to the chiral $U(1)$ symmetry. Namely, integrated over half of the superspace it is invariant under chiral symmetry up to a term that precisely reproduces the chiral anomaly. Now at finite length this superpotential in principle could have corrections proportional to powers of

$$\exp(-\sqrt{2}\sigma L). \tag{7.13}$$

However, these corrections would spoil the transformation properties of the superpotential with respect to the chiral symmetry. Therefore they are forbidden. As a result, at finite L the exact superpotential of the theory is still given by (7.12). Critical points of this superpotential are given by (7.11) and do not depend on L. This matches our result obtained from the large-N approximation.

In particular, at small L the theory is at weak coupling and can be studied in the quasiclassical approximation. As we already mentioned, the $CP(N-1)$ model compactified on a cylinder with twisted boundary conditions was studied in [27]. It is shown in [27] that the mass gap at weak coupling is produced by fractional instantons and does not depend on L in both supersymmetric and nonsupersymmetric cases. For our case (periodic boundary conditions) the mass gap shows L dependence in the nonsupersymmetric case, while in the supersymmetric case it is L independent. The quasiclassical origin of this behavior needs to be understood in the weak coupling domain of small L. This is left to a future work.

To conclude, in the $\mathcal{N} = (2,2)$ supersymmetric $CP(N-1)$ model we have a *single* phase with the unbroken supersymmetry and N vacua. Each vacuum has vanishing energy and is parametrized by the VEV of σ in Eq. (7.11). Unlike the nonsupersymmetric problem, this VEV is independent of L.

VIII. THE PHOTON MASS

In this section we outline the photon mass calculation.

The effective action for the gauge field can be written as [28]

$$S_{\text{gauge}} = \int d^2x\left\{\frac{1}{4e^2}F_{kl}^2 - \frac{N}{4\pi}\log\frac{\sigma}{\bar{\sigma}}F^*\right\}, \tag{8.1}$$

where the photon mixing with σ is due to the chiral anomaly and

$$F^* = \frac{1}{2}\epsilon_{ij}F^{ij} \tag{8.2}$$

is the dual gauge field strength. In the case of an infinitely long string the gauge coupling and the photon mass were found [28],

$$\frac{1}{e^2} = \frac{N}{4\pi}\frac{1}{\Lambda_{CP}^2}, \tag{8.3}$$

and

$$m_{ph} = 2\Lambda_{CP}, \tag{8.4}$$

respectively. In Sec. VII we derived the expression for the gauge coupling in the case of finite length; see (7.5). The corresponding expression for the photon mass in the limit of $\Lambda_{CP}L \gg 1$ is

$$m_{\rm ph}^2 \approx (2\Lambda_{CP})^2(1 - \sqrt{2\pi\Lambda_{CP}L}e^{-\Lambda_{CP}L}), \tag{8.5}$$

where we used the asymptotic expansion of the modified Bessel functions [see Eq. (9.7.2) in [25]],

$$K_1(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x}. \tag{8.6}$$

Since $K_0'(x) = -K_1(x)$, we can also determine the photon mass in the opposite limit of $\Lambda_{CP}L \ll 1$,

$$\sum_{k=1}^{\infty} K_1(kx)k = -\left(\sum_{k=1}^{\infty} K_0(kx)\right)' \approx \frac{\pi}{2x^2} - \frac{1}{2x},$$
$$m_{\rm ph}^2 \approx \frac{\Lambda_{CP}L}{\pi}(2\Lambda_{CP})^2 \ll (2\Lambda_{CP})^2. \tag{8.7}$$

IX. CONCLUSIONS

We the studied two-dimensional $CP(N-1)$ model [both nonsupersymmetric and $\mathcal{N} = (2,2)$] compactified on a cylinder with circumference L (periodic boundary conditions). We found the large-N solution for any value of L and discussed in detail the large-L and small-L limits.

A drastic difference is detected in passing from the nonsupersymmetric to $\mathcal{N} = (2,2)$ supersymmetric case. In the former case in the large-N limit we observe a phase transition at $L \sim \Lambda_{CP}^{-1}$ (which is expected to become a rapid crossover at finite N). At large L the $CP(N-1)$ model develops a mass gap and is in the Coulomb/confinement phase, with exponentially suppressed finite-L effects. At small L it is in the deconfinement phase; the orientational modes contribute to the Lüsher term. The latter becomes dependent on the rank of the bulk gauge group.

In the supersymmetric $CP(N-1)$ model we have a different picture. Our large-N solution exhibits a single phase independently of the value of $L\Lambda_{CP}$. For any value of this parameter a mass gap develops and supersymmetry remains unbroken. So does the $SU(N)$ symmetry of the target space (i.e., it is restored). The mass gap turns out to be independent of the string length. The Lüscher term is absent due to supersymmetry.

ACKNOWLEDGMENTS

We are grateful to A. Vainshtein for illuminating insights. A. Y. is also grateful to M. Anber, T. Sulejmanpasic, and M. Ünsal for very useful discussions. This work is supported in part by DOE Grant No. DE-SC0011842. The work of A. Y. was supported by William I. Fine Theoretical Physics Institute of the University of Minnesota, by Russian Foundation for Basic Research under Grant No. 13-02-00042a, and by Russian State Grant for Scientific Schools RSGSS-657512010.2. The work of A. Y. was supported by Russian Scientific Foundation under Grant No. 14-22-00281.

APPENDIX A: CALCULATION OF ZETA FUNCTION

We define the zeta function of an operator Ω as follows:

$$\zeta(s) = \mathrm{Tr}\Omega^{-s}. \tag{A1}$$

The operator of interest is given in Eq. (3.3),

$$\Omega = -(\partial_k - iA_k)^2 + m^2, \tag{A2}$$

where instead of ω we write m^2. In the $A_1 = 0$ gauge the expression for the zeta function takes the form

$$\zeta(s) = \frac{\hat{T}}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} dq_1\left(q_1^2 + \left(\frac{2\pi k}{L} + A_0\right)^2 + m^2\right)^{-s}. \tag{A3}$$

Gauge invariance requires invariance under transformation $A_0 \to A_0 + 2\pi k_0/L$, where k_0 is AN integer. This is manifest in (A.3) since the shift can be absorbed in the sum. We always can look for a solution for A_0 in the interval $|A_0| < \pi/L$, say $A_0 = 0$.

To evaluate the expression in (A3) we will need the following identities:

$$\Gamma(Z) = \int_0^\infty dt t^{z-1}e^{-t},$$
$$\int_0^\infty dx(x^2)^{(\alpha-1)/2}(x^2 + A^2)^{\beta-1} = \frac{1}{2}(A^2)^{\beta-1+\alpha/2}B(\alpha/2, 1-\beta-\alpha/2), \tag{A4}$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{A5}$$

The definition of the modified Bessel functions of the second kind is

$$\int_0^\infty dx x^{\nu-1}\exp\left(-\frac{a}{x} - bx\right) = 2\left(\frac{a}{b}\right)^{\nu/2}K_\nu(2\sqrt{ab}). \tag{A6}$$

The definition of the theta function (see Chapter 21 of [32]) is

$$\Theta_3(x,\tau) = \sum_{k=-\infty}^{\infty} q^{k^2} e^{2\pi i x} = 1 + 2\sum_{k=1}^{\infty} q^{k^2}\cos 2kx, \qquad q = e^{\pi i\tau}. \tag{A7}$$

Its Jacobi transformation is

$$\Theta_3(x,\tau) = (-i\tau)^{-1/2}\exp\left(\frac{x^2}{i\pi\tau}\right)\Theta_3(x/\tau,-1/\tau). \tag{A8}$$

The evaluation of the zeta function, Eq. (A3), proceeds as follows:

$$\begin{aligned}
\zeta(s) &\overset{(A.5)}{=} \frac{\hat{T}}{2\pi}\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sum_{k=-\infty}^{\infty}\left[\left(\frac{2\pi k}{L}+A_0\right)^2+m^2\right]^{1/2-s} \\
&= \frac{\hat{T}}{2\pi}\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}\left(\frac{2\pi}{L}\right)^{1-2s}\sum_{k=-\infty}^{\infty}\left[\left(k+\frac{LA_0}{2\pi}\right)^2+\epsilon^2\right]^{1/2-s} \\
&\overset{(A.4)}{=} \frac{\hat{T}}{2\pi}\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}\left(\frac{2\pi}{L}\right)^{1-2s}\frac{1}{\Gamma(z)}\int_0^\infty dt t^{z-1}e^{-t\alpha^2}\sum_{k=-\infty}^{\infty}e^{-k^2t-k\beta^2 t} \\
&\overset{(A.7)}{=} \frac{\hat{T}}{2\pi}\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}\left(\frac{2\pi}{L}\right)^{1-2s}\frac{1}{\Gamma(z)}\int_0^\infty dt t^{z-1}e^{-t\alpha^2}\Theta_3\left(\frac{i\beta^2 t}{2},\frac{it}{\pi}\right) \\
&\overset{(A.7),(A.8)}{=} F\frac{\sqrt{\pi}}{\Gamma(z)}\int_0^\infty dt t^{z-3/2}e^{-t\alpha^2+\beta^4 t/4}\left(1+2\sum_{k=1}^{\infty}e^{-\frac{k^2\pi^2}{t}}\cos\pi k\beta^2\right) \\
&\overset{(A.6)}{=} F\frac{\sqrt{\pi}}{\Gamma(z)}\left(\frac{1}{G^2}\right)^{z-\frac{1}{2}}\left(\Gamma\left(z-\frac{1}{2}\right)+4\sum_{k=1}^{\infty}(\pi k G)^{z-\frac{1}{2}}K_{z-\frac{1}{2}}(2\pi k G)\cos\pi k\beta^2\right) \\
&\overset{(A.6)}{=} \frac{\hat{T}L}{4\pi}\frac{1}{m^{2s-2}}\left[\frac{1}{s-1}+\frac{4}{\Gamma(s)}\sum_{k=1}^{\infty}\left(\frac{Lmk}{2}\right)^{s-1}K_{s-1}(Lmk)\cos LA_0 k\right],
\end{aligned} \tag{A9}$$

where we introduced intermediate notations

$$\epsilon = \frac{Lm}{2\pi}, \qquad z = s-\frac{1}{2}, \qquad F = \frac{\hat{T}}{2\pi}\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}\left(\frac{2\pi}{L}\right)^{1-2s}, \tag{A10}$$

and

$$\alpha^2 = \left(\frac{LA_0}{2\pi}\right)^2+\left(\frac{Lm}{2\pi}\right)^2, \qquad \beta^2 = \frac{LA_0}{\pi}, \qquad G^2 = \alpha^2-\beta^4/4. \tag{A11}$$

To find the derivative of the zeta function we will make use of the following properties of Euler's Γ function:

$$\Gamma(z+1) = z\Gamma(z), \qquad \Gamma(0) = \infty. \tag{A12}$$

The derivative is evaluated as follows:

$$\begin{aligned}
\zeta'(0) &= \frac{\hat{T}L}{4\pi}\left[-\frac{1}{m^{2s-2}}\frac{1}{(s-1)^2}-\frac{2\ln m}{m^{2s-2}(s-1)}-\frac{4\Gamma'(s)}{\Gamma^2(s)m^{2s-2}}\sum_{n=1}^{\infty}\left(\frac{Lmk}{2}\right)^{s-1}K_{s-1}(Lmk)\cos LA_0 k\right]\Bigg|_{s=0} \\
&= \frac{\hat{T}Lm^2}{4\pi}\left[-1+\ln m^2+8\sum_{k=1}^{\infty}\frac{K_1(kLm)}{kLm}\cos LA_0 k\right].
\end{aligned} \tag{A13}$$

Following [22] we can write the generating functional,

$$\ln Z = \frac{1}{2}\zeta'(0) + \frac{1}{2}\ln\mu^2\zeta(0), \qquad \text{(A14)}$$

where a normalization constant μ has dimension of mass. Renormalizability requires

$$\mu = M_{\rm uv}.$$

Thus, in terms of the zeta function and its derivative the expression for the effective potential becomes

$$V = -\frac{N}{\hat{T}}(\zeta'(0) + \zeta(0)\ln M_{uv}^2) - \frac{N}{4\pi}Lm^2\ln\frac{M_{\rm uv}^2}{\Lambda^2}. \qquad \text{(A15)}$$

Substituting the expressions for the zeta function and its derivative we obtain

$$V = \frac{NL\omega}{4\pi}\left[1 - \ln\frac{\omega}{\Lambda_{CP}^2} - 8\sum_{k=1}^{\infty}\frac{K_1(kL\sqrt{\omega})}{kL\sqrt{\omega}}\cos kLA_0\right], \qquad \text{(A16)}$$

where we replaced m^2 by ω.

APPENDIX B: KINETIC TERM IN CASE OF BOSONIC THEORY

To find the $U(1)$ charge of the n^l fields one has to consider only the second diagram in Fig. 1. The first diagram is needed only for renormalization. The relevant part of the action written in the Minkowski spacetime takes the form

$$\begin{aligned} iS_B^M &= i\int d^2x[\nabla_\mu\bar{n}_l\nabla^\mu n^l - m^2|n|^2] \\ &= i\int d^2x[\partial_\mu\bar{n}_l\partial^\mu n_l - m^2|n|^2 + iA^\mu(\bar{n}_l\overleftrightarrow{\partial}_\mu n^l) + A^2|n|^2], \end{aligned} \qquad \text{(B1)}$$

where $\overleftrightarrow{\partial}_\mu = \vec{\partial}_\mu - \overleftarrow{\partial}_\mu$. We then pass to Euclidean space,

$$t = -i\tau, \qquad A_0 = i\hat{A}_0, \qquad A_i = \hat{A}_i.$$

The action in Euclidean space is

$$S_B^E = \int d^2\hat{x}[\partial_k\bar{n}_l\partial_k n_l + m^2|n|^2 + i\hat{A}_k(\bar{n}_l\overleftrightarrow{\partial}_k n^l) + \hat{A}^2|n|^2]. \qquad \text{(B2)}$$

Now we can determine the Feynman rules. The results are shown in Fig. 7. Thus for the kinetic term (in the case of an infinitely long string) one can write

$$\Pi_{ij} = N\int\frac{d^2q}{(2\pi)^2}\frac{(p_i + 2q_i)(p_j + 2q_j)}{(m^2 + q^2)(m^2 + (p+q)^2)}. \qquad \text{(B3)}$$

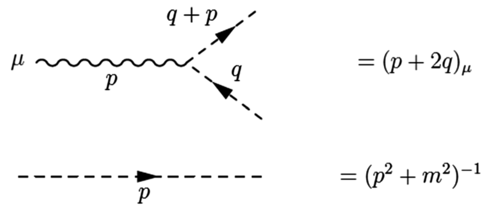

FIG. 7. Feynman rules: vertex and the propagator of n^l field.

Introducing the Feynman parameter to combine the denominators

$$\frac{1}{\alpha(\alpha+\beta)} = \int_0^1 dx\frac{1}{(x\beta+\alpha)^2}, \qquad \text{(B4)}$$

and substituting $l = q + px$ in Eq. (B3) we arrive at

$$\Pi_{ij} = N\int\frac{d^2l dx}{(2\pi)^2}\frac{[p_i p_j(1-2x)^2 - 2x(p_i l_j + p_j l_i) + 4l_i l_j]}{(l^2 + m^2 + p^2x(1-x))^2}. \qquad \text{(B5)}$$

Terms linear in l vanish. To find the $U(1)$ charge we only need to consider the $p_i p_j$ structure. Thus, the expression for the charge is

$$\begin{aligned} \frac{1}{Ne^2} &= \int\frac{d^2l dx}{(2\pi)^2}\frac{(1-2x)^2}{(l^2 + m^2 + p^2x(1-x))^2} \\ &= \int_0^1\frac{dx}{4\pi}\frac{(1-2x)^2}{m^2 + p^2x(1-x)}. \end{aligned} \qquad \text{(B6)}$$

Expanding the last expression to the zeroth power in p one finally finds

$$\frac{1}{Ne^2} = \int_0^1\frac{dx}{4\pi m^2}(1-2x)^2 = \frac{1}{12\pi m^2}. \qquad \text{(B7)}$$

The case of the finite length string is considered along similar lines. We recall (see [15]) that the limit $p_\mu \to 0$ is understood as first putting $p_0 = 0$ and then letting p_1 go continuously to zero. As a result, only $\Pi_{00} \neq 0$. Using the Feynman rules one can derive the following expression:

$$\Pi_{00} = \frac{N}{L}\sum_{k=-\infty}^{\infty}\int\frac{dq}{2\pi}\frac{4\omega_k^2}{(m^2 + q^2 + \omega_k^2)(m^2 + (p+q)^2 + \omega_k^2)}, \qquad \text{(B8)}$$

where we defined $\omega_k = 2\pi k/L$. Introducing again the Feynman parameter and making the same substitution one arrives at

$$\Pi_{00} = \sum_{k=-\infty}^{\infty} \frac{N\omega_k^2}{L} \int_0^1 \frac{dx}{(m^2 + \omega_k^2 + p^2 x(1-x))^{3/2}}. \quad \text{(B9)}$$

We expand this expression and keep only the leading power in p. Then the expression for the charge becomes

$$\frac{1}{Ne^2} = \frac{1}{4L}\left[\sum_{k=-\infty}^{\infty}(m^2+\omega_k^2)^{-3/2} - m^2\sum_{k=-\infty}^{\infty}(m^2+\omega_k^2)^{-5/2}\right] = \frac{L^2}{32\pi^3}\left[\sum_{k=-\infty}^{\infty}(k^2+\alpha^2)^{-3/2} - \alpha^2\sum_{k=-\infty}^{\infty}(k^2+\alpha^2)^{-5/2}\right], \quad \text{(B10)}$$

where $\alpha = Lm/2\pi$. We deal with these sums as follows:

$$\begin{aligned} S_1(z,\alpha) &\equiv \sum_{k=-\infty}^{\infty}(k^2+\alpha^2)^{-z} \\ &\overset{(A.4)}{=} \frac{1}{\Gamma(z)}\int_0^\infty dt t^{z-1}e^{-t\alpha^2}\sum_{k=-\infty}^{\infty}e^{-k^2 t} \\ &\overset{(A.7)}{=} \frac{1}{\Gamma(z)}\int_0^\infty dt t^{z-1}e^{-t\alpha^2}\Theta_3(0, it/\pi) \\ &\overset{(A.8)}{=} \frac{\sqrt{\pi}}{\Gamma(z)}\int_0^\infty dt t^{z-1}e^{-t\alpha^2}\Theta_3(0, -\pi/it) \\ &\overset{(A.6)}{=} \frac{\sqrt{\pi}}{\Gamma(z)}\left[\frac{\Gamma(z-\frac{1}{2})}{\alpha^{2z-1}} + 4\sum_{k=1}^{\infty}\left(\frac{k\pi}{\alpha}\right)^{z-\frac{1}{2}}K_{z-\frac{1}{2}}(2k\pi\alpha)\right]. \end{aligned} \quad \text{(B11)}$$

Thus the expression for the charge can be written as

$$\frac{1}{Ne^2} = \frac{1}{4L}\left(\frac{L}{2\pi}\right)^3[S_1(3/2,\alpha) - \alpha^2 S_1(5/2,\alpha)] = \frac{1}{12\pi m^2} + \frac{L}{2\pi m}\sum_{k=1}^{\infty}K_1(kLm)k - \frac{L^2}{6\pi}\sum_{k=1}^{\infty}K_2(kLm)k^2. \quad \text{(B12)}$$

In the limit $Lm \gg 1$ the contributions from the modified Bessel functions are exponentially small and thus the expression for the charge reduces to that for the infinitely long string.

APPENDIX C: KINETIC TERM IN THE SUPERSYMMETRIC CASE

In Appendix B we calculated the first diagram (the boson part) in Fig. 6. Now we will calculate the second diagram (the fermion part). The relevant part of the fermion action in the Minkowski spacetime is

$$iS_F^M = i\int d^2x\left\{\bar{\xi}i\gamma^\mu\nabla_\mu\xi - i\sqrt{2}\sigma\bar{\xi}\left(\frac{1-\gamma^5}{2}\right)\xi + i\sqrt{2}\sigma^*\bar{\xi}\left(\frac{1+\gamma^5}{2}\right)\xi\right\}, \quad \text{(C1)}$$

where $\nabla_\mu = \partial_\mu - iA_\mu$ is the covariant derivative, and the γ matrices are defined as

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We pass to Euclidean space,

$$t = -i\tau, \quad A_0 = i\hat{A}_0, \quad A_i = \hat{A}_i,$$
$$\hat{\gamma}^0 = \gamma^0, \quad \hat{\gamma}^1 = -i\gamma^1, \quad \hat{\gamma}^5 = \gamma^5,$$

and, since in Euclidean formulation ξ and $\bar{\xi}$ are independent, we define

$$\hat{\xi} = \xi, \quad \hat{\bar{\xi}} = i\bar{\xi}.$$

Thus, the action in Euclidean space can be presented as follows:

$$S_F^E = -\int d^2\hat{x}\left[\hat{\bar{\xi}}i\hat{\gamma}^k\hat{\partial}_k\hat{\xi} + \hat{\bar{\xi}}\hat{\gamma}^k\hat{A}_k\hat{\xi} - \sqrt{2}\sigma\hat{\bar{\xi}}\left(\frac{1-\hat{\gamma}^5}{2}\right)\hat{\xi} + \sqrt{2}\sigma^*\hat{\bar{\xi}}\left(\frac{1+\hat{\gamma}^5}{2}\right)\hat{\xi}\right]. \quad \text{(C2)}$$

Examining this expression in components one can find that it matches that of (6.2). Since from now on all calculations will be carried out in Euclidean space we will drop the caret notation. Using (C2) we find the Feynman rules that are shown in Fig. 8, where we introduced a notation $\sigma = a + ib$ and the mass is $m^2 = 2a^2 + 2b^2$.

We begin from the case of the infinitely long string. The fermion contribution to the kinetic term is

$$\Pi^{ij} = -\int\frac{d^2q}{(2\pi)^2}\frac{1}{(q^2+m^2)[(p+q)^2+m^2]} \times \mathrm{Tr}[\gamma^i(\not{q} + i\sqrt{2}b + \sqrt{2}a\gamma^5)\gamma^j \times (\not{p} + \not{q} + i\sqrt{2}b + \sqrt{2}a\gamma^5)]. \quad \text{(C3)}$$

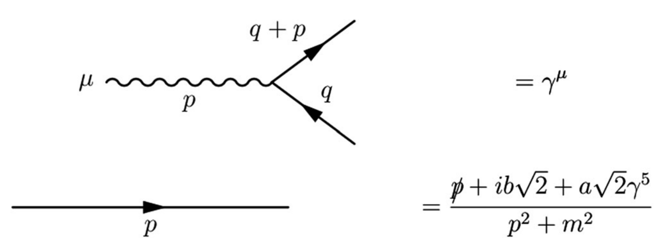

FIG. 8. Feynman rules: vertex and the propagator of ξ^l field.

The Clifford algebra is, as usual,

$$\{\gamma^i\gamma^j\} = 2\delta^{ij}. \tag{C4}$$

As a result, the trace identities for the γ matrices become

$$\begin{aligned} \mathrm{Tr}(\gamma^i\gamma^j) &= 2\delta^{ij}, \\ \mathrm{Tr}(\gamma^i\gamma^j\gamma^k\gamma^l) &= 2\delta^{ij}\delta^{kl} - 2\delta^{ik}\delta^{jl} + 2\delta^{il}\delta^{jk}, \\ \mathrm{Tr}(\text{odd number of } \gamma\text{'s}) &= 0. \end{aligned} \tag{C5}$$

Thus, the expression for the kinetic term takes the form

$$\begin{aligned} \Pi^{ij} &= -\int \frac{d^2q}{(2\pi)^2} \frac{\mathrm{Tr}[\gamma^i \not{q}\gamma^j(\not{p}+\not{q}) - m^2\gamma^i\gamma^j]}{(q^2+m^2)[(p+q)^2+m^2]} \\ &= -\int \frac{d^2q}{(2\pi)^2} \frac{1}{(q^2+m^2)[(p+q)^2+m^2]} \\ &\quad \times [2q^i(p+q)^j + 2q^j(p+q)^i \\ &\quad - 2q\dot{(}p+q)\delta^{ij} - 2m^2\delta^{ij}]. \end{aligned} \tag{C6}$$

Notice that generally speaking $\mathrm{Tr}(\gamma^i\gamma^j\gamma^5) \neq 0$ in two dimensions. However, we find that both such contributions cancel each other.

We proceed as in the bosonic theory, introducing the Feynman parameter and making the same substitution. Linear terms drop out, as usual. Furthermore, considering only p^ip^j structure we obtain

$$\begin{aligned} \Pi_F^{ij} &= p^ip^j \int \frac{d^2l dx}{(2\pi)^2} \frac{1-(1-2x)^2}{(l^2+m^2+p^2x(1-x))^2} \\ &= p^ip^j \int_0^1 \frac{dx}{4\pi} \frac{1-(1-2x)^2}{m^2+p^2x(1-x)}. \end{aligned} \tag{C7}$$

Expanding to zeroth order in p we find the fermion contribution to e^2,

$$\frac{1}{Ne_F^2} = \frac{1}{6\pi m^2}. \tag{C8}$$

Combining this with the result we obtained in the boson theory, we finally arrive at

$$\frac{1}{Ne^2} = \frac{1}{4\pi m^2}. \tag{C9}$$

In the case of the finite length string the starting expression (C6) is modified,

$$\begin{aligned} \Pi^{ij} = -\frac{1}{L}\sum_{k=-\infty}^{\infty} \int \frac{dq}{2\pi} &\frac{1}{(q^2+m^2)[(p+q)^2+m^2]} \\ &\times [2q^i(p+q)^j + 2q^j(p+q)^i \\ &- 2q\dot{(}p+q)\delta^{ij} - 2m^2\delta^{ij}]. \end{aligned} \tag{C10}$$

Again, just as in the boson theory we consider Π_{00}. After we make the same substitution and introduce the Feynman parameter, we obtain

$$\Pi_{00} = \frac{m^2}{L}\sum_{k=-\infty}^{\infty}\int_0^1 \frac{dx}{(p^2x(1-x)+m^2+\omega_k^2)^{3/2}}. \tag{C11}$$

Then we expand this expression and keep only the first nonvanishing power in p. Thus, the fermionic contribution to the charge is

$$\frac{1}{Ne_F^2} = \frac{m^2}{4L}\sum_{k=-\infty}^{\infty}(m^2+\omega_k^2)^{-5/2}. \tag{C12}$$

Summarizing, we obtained a sum identical to that in (B10). Therefore, their evaluation is identical too. Combining the result found in this appendix with that of the boson theory, we obtain for the charge

$$\frac{1}{Ne^2} = \frac{1}{4\pi m^2} + \frac{L}{2\pi m}\sum_{k=1}^{\infty} K_1(Lmk)k. \tag{C13}$$

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