Central charge and entangled gauge fields

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Entanglement entropy of gauge fields is calculated using the partition function in curved spacetime with a boundary. Deriving a Gibbons-Hawking-like term from a Becchi-Rouet-Stora-Tyutin (BRST) action produces a Wald-entropy-like codimension-2 surface term. It is further suggested that boundary degrees of freedom localized on the entanglement surface generated from the gauge redundancy could be used to resolve a subtle mismatch in a universal conformal anomaly-entanglement entropy relation.

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I. INTRODUCTION

Despite being perhaps the weirdest consequence of quantum mechanics [1], the concept of entanglement plays an important role in many areas of physics: it is a key ingredient in quantum information, an order parameter in the phase transition in many-body systems [2], and a measure of renormalization group flow in quantum field theories [3]. The entanglement entropy is also suggested as the origin of the black-hole entropy [4,5]. Decomposing the full Hilbert space into pieces A and its complement B,

$$\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}, \tag{1}$$

the entanglement entropy (or von Neumann entropy) is defined by

$$S_{\rm EE} \equiv -{\rm tr}\rho_A \log \rho_A. \tag{2}$$

The reduced density matrix,

$$\rho_A \equiv {\rm tr}_B \rho, \tag{3}$$

is a partial trace of the full density matrix ρ over the degrees of freedom in the region B. We will be interested in the entanglement entropy of continuous quantum gauge-field theories. For gauge-field theories where the observables are Wilson loops, the partition given in (1) can be an issue since such a partition would cut some loops. The Hilbert space of gauge fields is defined modulo the gauge transformation. The direct factorization as a product of two Hilbert spaces of the subsystems could be troublesome. See [6–9] for attempts to address this issue in lattice gauge theories.

It is often difficult to compute the entanglement entropy directly, in particular for spacetime dimensions higher than two. There are several alternative ways to compute the entanglement entropy. One is the so-called replica (conical) method [10,11] (see also [12]). In this method one calculates the partition function on an n-fold cover of the background spacetime where a conical singularity is introduced. In this paper we will avoid the conical singularity by adopting a new method introduced recently in [13] (see also [14]). We focus on obtaining the universal contribution to the entanglement entropy in Minkowski spacetime with a spherical entangling surface with a radius R. The main observation in [13] is that the full causal development, $\mathcal{D}$, connected to the spherical region with the radius R can be conformally mapped to a new geometry $\mathcal{H} = S^1 \times H^{d-1}$, where H^{d-1} is a $(d-1)$-dimensional hyperbolic plane and S^1 is a circle associated with periodic Euclidean time. (One can also consider a conformal mapping to the static patch of de Sitter space. In this paper we will mostly focus on the hyperbolic geometry $S^1 \times H^{d-1}$.) The vacuum correlators in the causal development of the region inside the spherical surface in d-dimensional flat spacetime $\mathcal{D}$ are mapped to thermal correlators on $\mathcal{H}$. Moreover, the modular flow on $\mathcal{D}$ is shown to correspond to the time translation on $\mathcal{H}$. The correlators on $\mathcal{H}$ are periodic in time under an imaginary shift by a $2\pi R$. The radius R of the circle S^1 then defines the temperature

$$T = \frac{1}{\beta} = \frac{1}{2\pi R}. \tag{4}$$

Therefore, by conformally mapping a vacuum state of a conformal field theory (CFT) onto a thermal state on the hyperbolic spacetime, the computation of the entanglement entropy across the sphere then can be calculated as the thermal entropy of the hyperbolic spacetime via

$$S_{\rm EE} = (1 - \beta\partial_\beta) \log Z(\beta)|_{\mathcal{H},\beta=2\pi R}. \tag{5}$$

On the other hand, AdS/CFT correspondence [15] also provides a way to calculate the entanglement entropy [16,17]. We focus on the field-theory calculation in this paper.

A minor nuisance of the entanglement entropy in continuum quantum field theories is its UV cutoff dependence. However, despite the fact that the coefficients of power-law divergences depend on regularization schemes, the log-divergent term in the entanglement entropy (in even spacetime dimensions) is scheme independent, hence becoming a universal result. Moreover, the log-divergent term in the entanglement entropy with a spherical entangling surface in d-dimensional flat spacetime is shown to be dictated by the central charge [13,17]

$$S_{\mathrm{EE,log}} = (-1)^{\frac{d}{2}-1} 4A \log\left(\frac{R}{\delta}\right), \tag{6}$$

where δ is the divergence cutoff. The type-A central charge "A" is defined as the conformal anomaly coefficient in even spacetime dimensions in

$$\langle T^{\mu}_{\mu} \rangle = \sum_i B_{d(i)} I_{d(i)} - 2(-)^{\frac{d}{2}} A E_d, \tag{7}$$

where E_d is the Euler density and $I_{(d)i}$ are the Weyl invariants that define the type-B anomalies in d dimensions. The type-B central charges do not contribute in our discussion since the spacetime in consideration will be conformally flat. We are interested in the universal contribution to the entanglement entropy from the type-A central charge.

Notice that Eqs. (6), (7) use a scheme without introducing the so-called type-D trace anomaly. (In 4D, the type-D anomaly reads $\langle T^{\mu}{}_{\mu} \rangle = \gamma \Box R$ with the type-D central charge γ.) This is also the minimal scheme used recently in [18,19] to obtain the general stress tensors from conformal anomalies based on the method discussed in [20]. (See also the recent paper [21] for related discussion. In their footnote 2, it is suggested that the log-divergent term in the entanglement entropy of the 4D $U(1)$ gauge field might be related to this scheme-dependent type-D trace anomaly. It would be interesting to see if the approach considered in this paper can be further identified as the type-D anomaly contribution.)

For 4D free gauge fields with spin $s = 1$, the result predicted by the formula (6) is given by

$$S^{(s=1)}_{\mathrm{EE,log}} = -\frac{31}{45} \log\left(\frac{R}{\delta}\right). \tag{8}$$

This result can be independently confirmed using the vector heat kernel on manifolds with a conical singularity [22,23]. However, to our knowledge, a field-theory calculation via the approach developed in [13] reproducing this result is absent (besides directly adopting the anomaly coefficients). In [24], a direct modification of the stress tensor is suggested to obtain this result. In this paper, we would like to see more closely what new ingredients are needed to give (8) without introducing the subtle conical singularity. Our main motivation is that finding a way to improve an alternative method of calculating the entanglement entropy might shed light on defining the entanglement entropy directly for general quantum gauge-field theories.

We organize this paper as follows. In the next section, starting from the action principle, we revisit the formulation of gauge fields in general curved spacetime with a boundary. We argue that the corresponding Gibbons-Hawking-like term should be derived from the action instead of adding it by hand. Deriving such a Gibbons-Hawking-like term from a BRST action produces a Wald-entropy-like codimension-2 surface term. We will discuss the suitable boundary conditions for our new action. Although we do not adopt in the paper the replica method that introduces the conical contact entropy [25], we point out the similarity of the codimension-2 surface term in the new action with the contact term in Sec. III. In Sec. IV, we calculate the partition function and thermal entropy on $S^1 \times H^3$ using the heat-kernel method. The main result of this section is that the entropy (38) has a mismatch compared to the universal conformal anomaly prediction (8). (We refer readers to [26] for the corresponding calculation of the conformally coupled scalar field and fermion on $S^1 \times H^3$. The resulting entropy results are consistent with the anomaly prediction (6).) A resolution is suggested in Sec. V, where it is argued that one should include the edge-mode contributions due to the gauge symmetry. We also show that adding these edge modes allows us to reproduce the universal log-divergent term of the Rényi entropy of gauge fields. We conclude this paper with some remarks on the entropy mismatch in the static patch of de Sitter space found by [27], the issue of the Hilbert space decomposition, and black-hole entropy.

II. GAUGE FIELDS IN CURVED SPACETIME WITH A BOUNDARY REVISITED

Our starting point is the standard action of the $U(1)$ gauge field A_μ on a general spacetime background $\mathcal{M}$,

$$S = \frac{1}{4} \int_{\mathcal{M}} F_{\mu\nu} F^{\mu\nu}, \tag{9}$$

where $F_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$ is the field strength. Due to the gauge symmetry, $\delta A_\mu = \nabla_\mu \lambda$ where λ is the gauge parameter, we add the Lorenz gauge-fixing term given by

$$S_{\mathrm{gf}} = \frac{1}{2} \int_{\mathcal{M}} (\nabla_\mu A^\mu)^2. \tag{10}$$

The gauge-fixing procedure introduces the standard Fadeev-Popov ghosts $\bar{b}$ and b that are anticommuting scalars. The full gauge-fixed action is

$$S = \int_{\mathcal{M}} \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \int_{\mathcal{M}} \left(\frac{1}{2} (\nabla_\mu A^\mu)^2 + \nabla^\mu \bar{b} \nabla_\mu b \right). \tag{11}$$

The above action has the following BRST symmetry parametrized by an infinitesimal anticommuting constant parameter ϵ:

$$\delta A_\mu = (\nabla_\mu b)\epsilon; \qquad \delta b = 0; \qquad \delta\bar{b} = (\nabla^\mu A_\mu)\epsilon, \quad (12)$$

provided that a boundary condition is imposed. Either $\nabla_n b|_{\partial M} = 0$ or $\nabla_\mu A^\mu|_{\partial M} = 0$. (We have denoted $\nabla_n = n^\mu \nabla_\mu$.) In fact, when one writes the bulk ghost action as $-\int_{\mathcal{M}} \bar{b}\Box b$, integration by parts is used and we should also impose a boundary condition: Either $\nabla_n b|_{\partial M} = 0$ or $\bar{b}|_{\partial M} = 0$, the BRST symmetry of which requires $\nabla_\mu A^\mu|_{\partial M} = 0$. We will adopt $\nabla_n b|_{\partial M} = 0$ in the following discussion.

We next emphasize that, different from the action of the nonminimally coupled scalar fields, the original gauge-field action (11) does not have second derivatives of the metric, so one does not really need to add by hand a Gibbons-Hawking-like term in the action. However, as we will discuss later on obtaining the partition function using the heat-kernel method, it is most natural to use an action which involves a second-order differential operator given by

$$D_{\mu\nu} \equiv \Box\delta_{\mu\nu} - R_{\mu\nu}. \quad (13)$$

(We have denoted $\Box = \nabla^\mu\nabla_\mu$ and $R_{\mu\nu}$ is the Ricci curvature tensor generated by $[\nabla_\mu, \nabla_\nu]A^\nu = -R_{\mu\nu}A^\nu$.) This operator is produced from integrating the standard action (11) by parts. The action using the operator $D_{\mu\nu}$ naturally needs a Gibbons-Hawking-like term. However, one should not add a new term during an immediate calculation. Our resolution is that the Gibbons-Hawking-like term should be *derived* in the gauge-field case. More precisely, we consider

$$\begin{aligned}
&\int_{\mathcal{M}} \left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\nabla_\mu A^\mu)^2\right) \\
&= -\int_{\mathcal{M}} \left(\frac{1}{2}A^\mu D_{\mu\nu} A^\nu\right) \\
&\quad -\frac{1}{2}\int_{\partial M} n_\mu [A_\nu \nabla^\nu A^\mu - A^\mu \nabla^\nu A_\nu - A_\nu \nabla^\mu A^\nu] \\
&= -\int_{\mathcal{M}} \left(\frac{1}{2}A^\mu D_{\mu\nu} A^\nu\right) + \frac{1}{2}\int_{\partial M} K_{\mu\nu}A^\mu A^\nu \\
&\quad + \int_{\partial M}\left[A^n(\nabla^\nu A_\nu) + \frac{1}{4}\nabla^n A^2\right] - \frac{1}{2}\int_{\partial M^{d-1}} g^\perp_{\mu\nu} A^\mu A^\nu.
\end{aligned} \quad (14)$$

In the last line we integrate by parts one more time in order to obtain the Gibbons-Hawking-like term that provides the cancellation involving $\nabla_n(\delta g_{\mu\nu})$ when performing the metric variation. $K_{\mu\nu}$ is the extrinsic curvature and n_μ is the (spacelike) outward unit vector normal to $\partial\mathcal{M}$. $g^\perp_{\mu\nu} \equiv n_\mu n_\nu$ denotes a projection onto the directions perpendicular to the codimension-2 boundary. To our knowledge, no literature has mentioned this kind of treatment regarding a gauge-field action in curved spacetime with a boundary. (See, for example, [28,29] for different approaches.)

We consider the following boundary conditions to have a well-defined field equation of A_μ:

$$A_n|_{\partial M} = 0, \qquad \nabla_n \lambda|_{\partial M} = 0, \qquad (\nabla_n A_i + K_{ij}A^j)|_{\partial M} = 0, \quad (15)$$

where n is the normal component while i and j represent the tangential components. This is referred to as the absolute boundary condition in [30] although there the surface action is different from ours. This set of boundary conditions is sufficient for us to have a well-defined field equation of A_μ. The BRST invariance of this absolute boundary condition (15) and its consistency with the gauge choice have been previously mentioned in [30].

III. CONICAL CONTACT ENTROPY

The main reason we consider in this paper a different approach to study the entanglement entropy of gauge fields instead of adopting the traditional replica method is that one is already able to obtain the expected central charge-entropy relation (8) using the conical method [22]. On the other hand, it is well known that the conical singularity causes subtle issues such as generating a contact term [25]. Notice that a contact term also appears in the case of the nonminimally coupled scalar field [31]. However, as we will discuss later, there is no entropy mismatch regarding the expected central charge-entropy relation in the conformal scalar field's case if using the hyperbolic spacetime approach. In short, in this paper we would like to resolve the entropy mismatch of gauge field's entanglement entropy without touching the contact term issue.

Although we will not focus on the conical approach in this paper, we would like to make a short remark that, as one can see from (14), deriving a Gibbons-Hawking-like term from a BRST action would produce a codimension-2 surface term in general; the purpose of this section is to show the similarity of this codimension-2 surface term with the contact term of gauge fields. Let us briefly review the contact term contribution to the entanglement entropy for gauge fields [25]. (See also [32,33] for recent related discussions.) If all surface terms are dropped, the $U(1)$ action is simply given by

$$S_{\mathcal{M}} = -\frac{1}{2}\int_{\mathcal{M}} (A^\mu D_{\mu\nu} A^\nu) - \int_{\mathcal{M}} \bar{b}\Box b. \quad (16)$$

On a manifold $\mathcal{M}$ with a conical singularity, we are interested in the heat kernel for the first-order change of the conical angle β away from 2π. The conical deficit introduces a singular curvature at the tip of a cone. The curvature can be expanded as [34]

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + (2\pi - \beta)g^{\perp}_{\mu\nu}\delta_\Sigma + \mathcal{O}(2\pi - \beta)^2, \quad (17)$$

where $\bar{R}_{\mu\nu}$ vanishes in flat spacetime. The higher-order terms in (17) do not affect the entanglement entropy. The entropy formula in the conical method reads $S_{\rm cone} = (1 - \beta\partial_\beta)\log Z(\beta)|_{\beta=2\pi}$. The ghosts do not contribute to the contact entropy. The partition function of gauge fields can be written, using (17), as ($\bar{D}_{\mu\nu} \equiv \Box\delta_{\mu\nu} - \bar{R}_{\mu\nu}$)

$$\int \mathcal{D}A \exp\left[\frac{1}{2}\int_{\mathcal{M}} \{A^\mu(\bar{D}_{\mu\nu} - (2\pi - \beta)g^{\perp}_{\mu\nu}\delta_\Sigma)A^\nu\}\right] = \bar{Z}_A - \left(\pi - \frac{\beta}{2}\right)\int_\Sigma \langle g^{\perp}_{\mu\nu}A^\mu A^\nu\rangle, \quad (18)$$

where the first term $\bar{Z}$ denotes the "regular" contribution. The second term leads to the contact entropy. (Σ denotes the codimension-2 surface.) Let us also remark that this term is intimately related to the expectation value of Wald entropy [32,35],

$$\langle S_{\rm Wald}\rangle = -2\pi\left\langle \int_\Sigma \frac{\partial\mathcal{L}}{\partial R_{\mu\nu\lambda\rho}}\epsilon_{\mu\nu}\epsilon_{\lambda\rho}\right\rangle = -\pi\int_\Sigma \langle g^{\perp}_{\mu\nu}A^\mu A^\nu\rangle. \quad (19)$$

We see that the contact term has the same form as the last term in the new action (14). However, in our approach without introducing a conical singularity, such a surface term will be killed by imposing the boundary condition given in the last section. The treatments of the surface terms may be different depending on whether we are considering a physical boundary or a conical singularity. We leave the problem of how our action interacts with the conical singularity as a future problem.

IV. GAUGE FIELDS ON HYPERBOLIC SPACE AND DISCREPANCY

We are interested in the entanglement entropy of gauge fields on $\mathcal{R}^{1,3}$ with the entangling surface S^2 with radius R, at a time slice $t = 0$. Using the approach developed in [13], the computation of the entanglement entropy is mapped to calculating the thermal entropy on the hyperbolic space. The original flat spacetime metric written in polar coordinates is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d^2\Omega_2, \quad (20)$$

where $d^2\Omega_2$ is the metric of the sphere with unit radius. The transformations that map the geometry into the hyperbolic geometry are given by [13]

$$t = R\frac{\sinh(\frac{\tau}{R})}{\cosh u + \cosh(\frac{\tau}{R})}, \quad (21)$$

$$r = R\frac{\sinh(u)}{\cosh u + \cosh(\frac{\tau}{R})}. \quad (22)$$

The metric then becomes

$$ds^2 = \Omega^2(-d\tau^2 + R^2(du^2 + \sinh^2 u d^2\Omega_2)). \quad (23)$$

The prefactor, $\Omega = (\cosh u + \cosh\frac{\tau}{R})^{-1}$, can be eliminated via the conformal transformation and the resulting metric is $S^1 \times H^3$. Notice the limits

$$\tau = \pm\infty \to (t = \pm R, r = 0), \quad (24)$$

$$u = \infty \to (t = 0, r = R) \quad (25)$$

confirm that the full causal development $\mathcal{D}$ is indeed covered in $\mathcal{H}$ after the conformal mapping.

We will use the heat-kernel method on $S^1 \times H^3$ to obtain the partition function and the entropy. We denote the kernel as $K(x, y, s)$ on a fixed spacetime background $\mathcal{M}$ satisfying the heat equation

$$(\partial_s + D)K(x, y, s) = 0, \quad (26)$$

where D is a second-order differential kinetic operator. A boundary condition is imposed, $K(x, y, 0) = \delta(x, y)$. The trace of the heat kernel is given by

$$\mathcal{K}(s) = \int_{\mathcal{M}} K(x, x; s) = \sum_i e^{-s\lambda_i}, \quad (27)$$

with summation over all eigenvalues λ_i of the operator D including possible degeneracy. Notice that the parameter s must have dimensions of length squared if the argument of the exponential is to be dimensionless. The partition function can be expressed via the heat kernel,

$$\log Z = -\frac{1}{2}\sum_i \log\lambda_i = \frac{1}{2}\int_0^\infty \frac{ds}{s}\mathcal{K}(s). \quad (28)$$

For the $U(1)$ gauge field on $S^1 \times H^3$, after imposing the boundary condition (15), the action (14) reduces to (16). The partition function can be written as

$$Z = \mathrm{Det}(-\Box_s)\int \mathcal{D}A_\mu \exp\left[\frac{1}{2}\int_{S^1\times H^3} A^\mu D_{\mu\nu}A^\nu\right], \quad (29)$$

where the factor $\mathrm{Det}(-\Box_s)$ stands for the Faddeev-Popov determinant. Factoring out the temporal index and performing a Gaussian integral over A_τ yields

$$Z = \mathrm{Det}(-\Box_s)^{1/2}\int \mathcal{D}A_i \exp\left[\frac{1}{2}\int_{S^1\times H^3}\left(A^i D_{ij}A^j\right)\right]. \quad (30)$$

Next we write

$$\log Z(\beta) = \frac{1}{2}(\mathrm{Det}(-\Box_s) - \mathrm{Det}(D_{ij})) \equiv \frac{1}{2}\int_0^\infty \frac{ds}{s}\mathcal{K}(s), \tag{31}$$

where we have decomposed the total heat kernel by

$$\mathcal{K}(s) = K_{ij}(S^1)K^{ij}(H^3) - K_s(S^1)K_s(H^3), \tag{32}$$

with $K_{ij}(S^1)$ being a shorthand for tr $\int_{S^1} K_{ij}(s,\tau,\tau)$ and $K_{ij}(H^3)$ for tr $\int_{H^3} K_{ij}(s,x,x)$. The same notation applies on K_s (scalar) parts. The volume simply factorizes in the heat kernels because the hyperbolic space is homogeneous.

The heat kernel on S^1 can be evaluated using the method of images preserving the periodic boundary condition. The result is given by an infinite sum on an infinite line shifted by $2\pi Rn(\equiv n\beta)$,

$$K_{ij}(S^1) = \frac{2\beta}{(4\pi s)^{1/2}}\sum_{n=1}^{\infty} e^{-\frac{n^2\beta^2}{4s}} = K_s(S^1). \tag{33}$$

The $n=0$ part is ignored because it will not contribute to the entanglement entropy.

The heat kernels $K^{ij}(H^3)$ and $K_s(H^3)$ can be found in the literature [36,37] and are given by

$$K_{ij}(H^3) = \frac{e^{-\frac{s}{R^2}} + 2 + 4\frac{s}{R^2}}{(4\pi s)^{3/2}}; \qquad K_s(H^3) = \frac{e^{-\frac{s}{R^2}}}{(4\pi s)^{3/2}}. \tag{34}$$

Plugging these results into (31) gives

$$\log Z(\beta) = \frac{2\pi^2 R^2 + 15\beta^2}{90R^2\beta^3}\mathrm{Vol}(H^3). \tag{35}$$

We have to introduce an IR cutoff since the volume of H^3 is divergent. We let [13]

$$\cosh(u_{\max}) = \frac{\mathrm{R}}{\delta}. \tag{36}$$

The scale of the hyperbolic curvature is set to be R. We obtain a log term from

$$\mathrm{Vol}(H^3) = -2\pi R^3 \log\left(\frac{R}{\delta}\right) + \cdots \tag{37}$$

Finally, using (5), we obtain the entropy

$$S_{\mathrm{EE,log}} = -\frac{16}{45}\log\left(\frac{R}{\delta}\right). \tag{38}$$

We find a mismatch when we compare this result with the conformal anomaly prediction (8).

V. EDGE ENTROPY FROM ENTANGLING SURFACE

Here we suggest a way to resolve the mismatch found in (38). Our resolution is based on including edge modes localized on the entangling surface.

Recall that the gauge symmetry results in the gauge-fixing condition $\nabla^\mu_{(\mathcal{M})}A_\mu = 0$. The gauge redundancy is determined by

$$\Box_{(\mathcal{M})}\lambda = 0, \tag{39}$$

where $\Box_{(\mathcal{M})}$ is the d'Alembertian operator on $\mathcal{M}$. In the bulk, the residual gauge freedom is fixed by imposing a boundary condition on the boundary of $\mathcal{M}$, which is $\partial\mathcal{M} = S^1 \times S^2$. (We take $u \to \infty$ as the boundary.) That is, we fix the residual gauge by imposing a constraint on the boundary, $\lambda_{\partial\mathcal{M}} = \bar{\lambda}$, where $\bar{\lambda}$ still satisfies $\Box_{(\partial\mathcal{M})}\bar{\lambda} = 0$. Notice that by taking a large-u limit, the 4D metric (23) (after eliminating the conformal factor) effectively reduces to 2D since the radius of the time circle is much smaller than the radius of S^2, $R \sinh u_{\max}$. Therefore, in the sense of the Kaluza-Klein massive modes decoupling, the effective boundary becomes S^2 and the gauge redundancy condition results in a constraint on S^2 given by

$$\Delta_{(S^2)}\bar{\lambda} = 0, \tag{40}$$

where $\Delta_{(S^2)}$ is the Laplacian operator on S^2. Recall also that in the large-u limit, Eq. (25) shows that it corresponds to the $t=0$ slice that is used to define the original time-independent entanglement entropy with a static entangling surface.

We interpret that the entangling boundary plays a role to encode these boundary redundancy modes. In other words, the freedom of choosing different boundary data, $\bar{\lambda}$, in (40) is interpreted as having edge degrees of freedom on the entangling boundary. We suggest that these boundary modes give additional contributions to the entanglement entropy and can be used to resolve the mismatch (38).

The standard bulk action (14) does not see the boundary modes since all surface actions are set to zero using the boundary condition. We treat the constraint (40) as a field equation on S^2 and define the corresponding partition function again by (28). The question then can be reduced to finding the corresponding eigenvalues using the heat-kernel method.

The heat kernel on S^2 is essentially given by solving the standard eigenvalue problem of the Laplacian on S^2. The eigenvalues are $l(l+1)$ with the orbital quantum number l with the degeneracy given by $(2l+1)$. The eigenfunction is the familiar spherical harmonic. (See [38] for heat kernels in different spacetime manifolds.) The heat kernel (density) that we need is given by

$$K(S^2) = \frac{1}{4\pi r^2}\sum_{l=0}^{\infty}(2l+1)e^{-s\frac{l(l+1)}{r^2}}. \tag{41}$$

We will be interested in the small-s expansion. We use the Euler-MacLaurin formula

$$\sum_{l=0}^{\infty} f(l) = \int_{l=0}^{\infty} dl f(l) + \frac{1}{2}f(0) - \frac{1}{12}f'(0) + \cdots \tag{42}$$

with a function $f(l)$ satisfying $f^{(n)}(\infty) = 0$ for arbitrary n. We focus on the scheme-independent log divergence; the higher-order terms in (42) are irrelevant. From (41) and (42) we obtain

$$K(S^2) = \frac{12r^4 + 4r^2 s + s^2}{48\pi r^4 s} + \cdots \tag{43}$$

We define the partition function on S^2 as

$$\log Z(S^2) = \frac{1}{2}\mathrm{Vol}(S^2)\int_{\epsilon^2}^{\infty}\frac{ds}{s}K(S^2), \tag{44}$$

where $\mathrm{Vol}(S^2) = 4\pi r^2$ is simply the area of the entangle surface. The s-independent term in (43) gives the log divergence. Notice that the gauge parameter λ is understood as the ghost b so it contributes as a *negative* massless scalar field on S^2. We obtain the log-divergent term from the edge modes, $\log Z(S^2) \to -\frac{1}{6}\log(\frac{R^2}{\epsilon^2})$. A dimensional scale R is inserted to have a dimensionless argument. We see that the log-divergent term is independent of the radius of the entangling surface. We identify the UV cutoff ϵ with the cutoff δ in regularizing $\mathrm{Vol}(H^3)$, $\epsilon = \delta$. The edge correction is given by

$$\Delta S^{(s=1)}_{\mathrm{EE,log}} = -\frac{1}{3}\log\left(\frac{R}{\delta}\right), \tag{45}$$

which resolves the mismatch.

Let us generalize our discussion to the Rényi entropy defined by

$$S_q = \frac{\log \mathrm{tr}\rho_A^q}{1-q}. \tag{46}$$

It has the following simple relation to the entanglement entropy (assuming a satisfactory analytic continuation can be performed):

$$S_{\mathrm{EE}} = \lim_{q\to 1} S_q. \tag{47}$$

Having the hyperbolic partition function, we can calculate the Rényi entropy via

$$S_q = \left.\frac{\log Z(q\beta) - q\log Z(\beta)}{(1-q)}\right|_{\beta\to 2\pi R}. \tag{48}$$

Using (35), we obtain

$$S_q = \frac{(q+1)(31q^2+1)}{360\pi q^3 R^3}\mathrm{Vol}(H^3). \tag{49}$$

Let us also include the edge modes. We should view the edge contribution as a universal contribution (the log-divergent term) in the sense that it is independent of β or the radius of the entangling surface. It then should be also independent of the parameter q inserted in temperature $T = \frac{1}{2\pi R q}$. By adding the edge contribution (45), the full log-divergent term of the Rényi entropy becomes

$$S^{(s=1)}_{q,\log} = -\frac{1+q+31q^2+91q^3}{180q^3}\log\left(\frac{R}{\delta}\right). \tag{50}$$

This result (giving the coefficient -31/45 when taking $q=1$) is now consistent with the Rényi entropy result obtained in [22,23] for gauge fields calculated by introducing the conical singularity.

If we use the hyperbolic spacetime heat kernels ([36,37]) to consider the 4D conformally coupled scalar field, the log-divergent term in the Rényi entropy can be obtained directly. We obtain

$$S^{(s=0)}_{q,\log} = -\frac{1+q+q^2+q^3}{360q^3}\log\left(\frac{R}{\delta}\right). \tag{51}$$

Taking $q=1$ gives

$$S^{(s=0)}_{\mathrm{EE,log}} = -\frac{1}{90}\log\left(\frac{R}{\delta}\right), \tag{52}$$

which matches exactly with the expected type-A anomaly prediction. On the other hand, there is no mismatch problem for fermions. The heat-kernel and related algebra can be found in literature (for example, see the appendix in [26]). For useful reference, we list the corresponding result of the 4D Dirac fermion,

$$S^{(s=\frac{1}{2})}_{q,\log} = -\frac{7+7q+37q^2+37q^3}{720q^3}\log\left(\frac{R}{\delta}\right). \tag{53}$$

Taking $q=1$, it gives the expected conformal anomaly prediction

$$S^{(s=\frac{1}{2})}_{\mathrm{EE,log}} = -\frac{11}{90}\log\left(\frac{R}{\delta}\right). \tag{54}$$

In short, the field-theory calculation of the log-divergent terms of the conformally coupled scalar field and massless fermion on $S^1\times H^3$ match directly with the conformal anomaly prediction, without any edge corrections needed. This is consistent with the fact that the boundary modes contribute in the gauge-field case due to the existence of the gauge symmetry.

VI. CONCLUDING REMARKS

Let us make a remark on the contribution of the edge modes to the entropy calculation in the static patch of de Sitter space. Starting again with the flat-space metric (20), one uses the coordinate transformation [13]

$$t = R\frac{\cos\theta\sinh(\tau/R)}{1+\cos\theta\cosh(\tau/R)}, \quad (55)$$

$$r = R\frac{\sin\theta}{1+\cos\theta\cosh(\tau/R)} \quad (56)$$

to obtain

$$ds^2 = \Omega^2[-\cos^2\theta d\tau^2 + R^2(d\theta^2 + \sin^2\theta d\Omega_{d-2}^2)], \quad (57)$$

where $\Omega = (1+\cos\theta\cosh(\tau/R))^{-1}$ is the conformal factor to be eliminated. The remaining metric is the static patch of de Sitter space with the scale R. The important limits are

$$\tau = \pm\infty \rightarrow (t, r) = (\pm R, 0), \quad (58)$$

$$\theta = \frac{\pi}{2} \rightarrow (t, r) = (0, R). \quad (59)$$

We see again that the new coordinates cover the causal development $\mathcal{D}$ of the ball $r \leq R$ at $t = 0$. As shown in [13], similar to the hyperbolic space, the modular transformation inside $\mathcal{D}$ again corresponds to the time translation in de Sitter space after the conformal mapping. The state in de Sitter geometry becomes thermal at $T = 1/(2\pi R)$. The thermal entropy in de Sitter space then can also be identified as the entanglement entropy in flat spacetime with a codimension-2 spherical entangling boundary.

In [27], the thermal entropy in 4D de Sitter space is calculated. Interestingly, the same mismatch (38) is found in the log-divergent term. The volume of de Sitter space is finite, so in this case there is no need to introduce an IR cutoff. The log divergence comes from the UV divergence of the partition function. In calculating the partition function of gauge fields, the gauge-fixing process again results in the gauge redundancy that is needed to be fixed by imposing a boundary constraint. Notice that in this case, one identifies the entangling boundary as the cosmological horizon at the boundary of the static patch at $\theta = \pi/2$. From the metric (57) (with $d = 4$), we see that the boundary defined by the limit $\theta \rightarrow \pi/2$ is again effectively (in the spirit of Kaluza-Klein dimensional compactification) a 2D sphere because the prefactor of the time direction shrinks to zero in this limit. The boundary redundancy modes effectively satisfy again (40), which gives the same edge correction (45) that resolves the mismatch found in the de Sitter space calculation as well.

It would be of great interest to better understand the edge modes and explore their potential applications. It has been suggested [6] (see also [7–9]) that one might modify the Hilbert space decomposition as

$$\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{\partial\mathcal{A}}, \quad (60)$$

where $H_{\partial\mathcal{A}}$ denotes a boundary Hilbert space, to have a special treatment of boundary in calculating the entanglement entropy of gauge fields. Let us make an initial attempt to relate this idea to the approach considered here. If one wants to derive the edge contribution (45) starting from a classical surface action, an immediate issue is that a surface action will cause a trouble regarding the variation principle when getting the bulk field equation. If we introduce a boundary Hilbert space $\mathcal{H}_{S^2}$ separately, we might consider a surface action subjected to the path integral quantization in this separated Hilbert space. Then, to incorporate the contribution from the edge, we introduce the following edge partition function:

$$Z(S^2) = (Z(\bar{b}, b))^{\frac{1}{2}}; \qquad Z(\bar{b}, b) = \int \mathcal{D}\bar{b}\mathcal{D}b e^{-\int_{S^2}(\bar{b}\Box b)}. \quad (61)$$

(Since the edge part does not have any gauge field, we simply define $\delta b_{S^2} = \delta\bar{b}_{S^2} = 0$ so that the surface action remains BRST invariant.) Notice that because of the intrinsic asymmetric treatment on ghost fields b and $\bar{b}$ in the BRST transformation, $\delta A = \nabla b$ in (12), the gauge redundancy (and the resulting edge correction on the entropy) is solely determined by b in this framework, so we adopt $(Z(\bar{b}, b))^{\frac{1}{2}}$ as the correct counting.

Finally, we would like to point out that the edge modes are introduced in the context of black-hole entropy in [39], where the authors consider BPS black holes with $AdS_2 \times S^2$ near horizon geometry. They argue that gauge symmetries give rise to physical modes that localize on the boundary of AdS_2 and contribute to the black-hole entropy.[1] The boundary modes in their context come also from supersymmetry and diffeomorphism invariance. It is shown that the boundary modes are needed to obtained the expected log corrections to the black-hole entropy. While it remains to further explore the deeper relation between the black-hole entropy and the entanglement entropy, it would be nice to see that the edge modes are essential in both contexts. It would be interesting to calculate the entanglement entropy of gravitons or other types of gauge fields using the approach considered in this paper, without

[1] I thank Finn Larsen for explaining some concepts in [39].

introducing the conical singularity, and see if the corresponding entanglement edge modes due to gauge redundancies are needed to explain possible discrepancies.

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Note added.—Recently, a paper [40] appeared that considered the entanglement entropy of gauge fields using an approach in which the conical singularity was introduced. The authors of that paper related the 4D mismatch result with the entangling boundary S^2 independently.

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