Noncommutative spaces and covariant formulation of statistical mechanics

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We study the statistical mechanics of a general Hamiltonian system in the context of symplectic structure of the corresponding phase space. This covariant formalism reveals some interesting correspondences between properties of the phase space and the associated statistical physics. While topology, as a global property, turns out to be related to the total number of microstates, the invariant measure which assigns *a priori* probability distribution over the microstates is determined by the local form of the symplectic structure. As an example of a model for which the phase space has a nontrivial topology, we apply our formulation on the Snyder noncommutative space-time with de Sitter four-momentum space and analyze the results. Finally, in the framework of such a setup, we examine our formalism by studying the thermodynamical properties of a harmonic oscillator system.

I. INTRODUCTION

Statistical mechanics is a bridge between mechanics and thermodynamics. Indeed, it may be assumed as the microscopic mechanical basis for the macroscopic thermodynamical properties of a physical system. The key concept which links these two arenas is the notion of the phase space, that is, the space of all distinct possible microstates. The dynamical evolution is nothing but a smooth transition from one microstate to another and statistical mechanics is about how to count the number of these microstates. In a more technical language, the phase space of a dynamical system is a symplectic manifold which naturally is equipped with a symplectic structure [1]. The symplectic structure covariantly determines the Poisson brackets and Liouville volume and the kinematics is then defined in a coordinate-independent manner on the phase space. The dynamics will be also specified upon taking the Hamiltonian function as the generator of time evolution in this setup. Therefore, the Hamiltonian determines how the system evolves through the microstates and the Liouville volume specifies how one can count and measure them. Consider, for instance, a mechanical system consisting of N particles which are subjected to some forces. Each particle moves in Euclidean three-dimensional space $\mathbf{R}^3$ and the corresponding configuration space is then $\mathbf{R}^{3N}$. Therefore, the phase space will be the cotangent bundle of this space, that is, $\mathbf{R}^{3N} \times \mathbf{R}^{3N}$, which naturally admits a symplectic structure. In terms of the positions and momenta of the particles, the associated symplectic structure takes the standard canonical form. Such a canonical symplectic structure, which is also dynamically invariant, may be used to determine the fundamental canonical Poisson brackets and the kinematics for the system under consideration. Subsequently, based on the canonical symplectic structure one leads to the canonical measure which can be viewed as a mathematical expression for the principle of equal probabilities in standard statistical mechanics [2]. Although it might be thought that the symplectic formulation is nothing but a covariant reformulation of the standard statistical mechanics, its advantages may become more clear when one is dealing with the Hamiltonian systems with nontrivial structures (for instance, when the standard $\mathbf{R}^n$ topology of the phase space is replaced with a more complicated topology or the standard canonical Poisson algebra is replaced with a deformed noncanonical one). Such systems, are investigated in the context of doubly special relativity theories, as a flat limit of quantum gravity, where the four-momentum space has a curved (de Sitter or anti–de Sitter) geometry with nontrivial topology [3]. The space-time structure then turns out to be noncommutative and the associated Poisson algebra between the positions and momenta become noncanonical [4,5]. The space-time with noncommutative coordinates was first proposed by Snyder in Ref. [6] in order to regularize quantum field theory in the ultraviolet regime [7]. In more recent times, it was also used by the stability theory of the Lie algebras to be a minimal candidate for the relativistic quantum mechanics [8] and by string theory at a fundamental level [9]. The duality between curved momentum

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space and noncommutative structure was first pointed out by Majid in Ref. [10]. The direct consequence of space-time with noncommutative coordinates is the modification to the dispersion relation and the associated density of states [11–16]. In symplectic formulation, the density of states is properly determined by the Liouville measure which will be constructed from the symplectic structure [17]. In noncommutative space-time, however, the corresponding symplectic structure seems to be noncanonical, which induces a nonuniform probability distribution over the set of microstates through a deformed Liouville volume element. This dynamically invariant Liouville volume enters in the definition of the Gibbs entropy and partition function from which all the thermodynamical properties of a system can be extracted. Therefore, the topology of the phase space will play an important roll since all integrals are evaluated over the entire phase space. For instance, let us consider a system which has an ultraviolet cutoff due to the existence of a maximal momentum. It is easy to see that the topology of the spatial sector of the momentum space may be compact. Moreover, systems with a totally compact phase space, which can be considered as the classical limit of quantum systems with finite dimensional Hilbert space (e.g., angular momentum for systems with fixed total angular momentum) were recently studied in the context of loop quantum gravity [18]. For such a compact phase space, the Liouville volume is finite and consequently a finite number of microstates naturally arises [19]. This shows the correspondence between the topology of the phase space and the number of microstates which can be realized when the symplectic geometry is implemented. The black hole physics is a good example in which the system obeys the thermodynamical laws and its statistical origin is not thoroughly formulated yet [20,21] in the framework of Hamiltonian formalism (see also Ref. [22]).

In this regard, formulating the statistical mechanics in its most fundamental form is important, at least, for two reasons: (i) finding a more precise and fundamental interpretation for the basis of the statistical mechanics, and (ii) studying the statistical mechanics for the Hamiltonian systems with nontrivial structure, such as those we have mentioned above. Motivated by the stated issues, in this paper we are going to formulate statistical mechanics for a general Hamiltonian system in a covariant (coordinate-independent) manner. The paper is organized as follows. In Sec. II, we deal with the kinematics and dynamics of a many-particle system with general Hamiltonian structure in the context of symplectic geometry. The statistical mechanics for such systems is formulated in Sec. III. In Sec. IV, by means of the constructed setup we study the statistical mechanics in the Snyder noncommutative space-time with curved energy-momentum space as an explicit example of a physical system with nontrivial (noncommutative) Hamiltonian

structure. As a case study, the thermodynamical properties of a harmonic oscillator system in Snyder space is also presented at the end of this section. Section V is devoted to the summary and conclusions.

II. HAMILTONIAN SYSTEMS

In this section, in order to study the statistical mechanics in the context of symplectic geometry, we consider the kinematics and dynamics of an N-particle system.

A. Kinematics

Let us consider a system of N particles similar to conventional systems in statistical mechanics. The corresponding $6N$-dimensional kinematical phase space may be obtained by coupling N single-particle phase spaces Γ_α with $\alpha = 1, \ldots, N$ as

$$\Gamma = \Gamma_1 \times \cdots \times \Gamma_N. \tag{1}$$

The phase space Γ naturally admits a symplectic structure ω which is a nondegenerate closed 2-form [1]. The classical state of this N-particle system is determined by a point in Γ (phase point). The evolution of phase points is then determined by the Hamiltonian vector field $\mathbf{x}_H$, whose integral curves are trajectories of the phase points (phase trajectories). From the fact that ω is nondegenerate, one can assign a vector field to a function f as $\omega(\mathbf{x}_f) = df$ and the Poisson bracket between two observables (real-valued functions on the phase space) can be defined as

$$\{f, g\} = \omega(\mathbf{x}_f, \mathbf{x}_g). \tag{2}$$

These functions with the above structure constitute a Lie algebra under Poisson brackets on Γ. We also have the so-called Liouville volume on Γ with its standard definition as

$$\omega^{3N} = \frac{1}{(3N)!} \omega \wedge \ldots \wedge \omega \quad (3N \text{ times}). \tag{3}$$

In terms of positions and momenta $x = (\mathbf{q}, \mathbf{p})$ of the particles, the Liouville volume takes the local form

$$\omega^{3N} = \sqrt{\det \boldsymbol{\omega}} \, dq_1^1 \wedge \ldots \wedge dq_N^3 \wedge dp_1^1 \wedge \ldots \wedge dp_3^N$$
$$\equiv \sqrt{\det \boldsymbol{\omega}} \, d^{3N} q \, d^{3N} p, \tag{4}$$

where q_α^i and p_i^α are the ith component of the positions and momenta of the αth particle, respectively, and $\boldsymbol{\omega}$ is the matrix representation of the symplectic structure ω with components $\boldsymbol{\omega}_{ij} = \omega(\partial/\partial x_\alpha^i, \partial/\partial x_\beta^j)$.

Now, one may be able to decompose the symplectic structure as

$$\omega = \sum_{\alpha=1}^{N} \omega_\alpha, \tag{5}$$

where ω_α is the symplectic structure on Γ_α. Although this assumption has no mathematical basis, from a physical point of view it is acceptable since the particles are assumed to be kinematically separated. In other words, the Lie algebras of the functions defined by $\{\omega_\alpha\}$ over $\{\Gamma_\alpha\}$ are separately closed. Therefore, the Liouville volume over Γ takes the form

$$\omega^{3N} = \frac{1}{(3N)!} \left(\sum_{\alpha=1}^{N} \omega_\alpha \right) \wedge \dots \wedge \left(\sum_{\alpha=1}^{N} \omega_\alpha \right), \quad (6)$$

where the wedge products take place $3N$ times. The volume element then works out to be

$$\omega^{3N} = \omega_1^3 \wedge \dots \wedge \omega_N^3 = \left(\prod_{\alpha=1}^{N} \sqrt{\det \boldsymbol{\omega}_\alpha} \right) d^{3N} q \, d^{3N} p, \quad (7)$$

where ω_α^3 is the volume element of the corresponding $\{\Gamma_\alpha\}$, and we have also used the fact that $\omega_\alpha^n = 0$, for $n > 3$. Clearly, $\sqrt{\det \boldsymbol{\omega}}$, that is, a function on Γ, factorizes into products of $\{\sqrt{\det \boldsymbol{\omega}_\alpha}\}$. This cannot be realized in the most general situation, where the symplectic structure cannot be written as (5).

As a special case, consider a system consisting of N particles subject to some forces moving in Euclidean three-dimensional space with standard $\mathbf{R}^3$ topology. Therefore, the phase space will be $\mathbf{R}^{3N} \times \mathbf{R}^{3N}$, which is indeed the cotangent bundle of the configuration space, that is, $\mathbf{R}^{3N}$. Geometrically, the cotangent bundle is a symplectic manifold endowed with a canonical symplectic structure

$$\omega_c = dQ_\alpha^i \wedge dP_i^\alpha, \quad (8)$$

in which Q and P are interpreted as the positions and momenta of particles, respectively. The summation is over both $i = 1, 2, 3$ and $\alpha = 1, \dots, N$. This is the canonical representation of symplectic structure and coordinates (Q_α^i, P_i^α) are known as the canonical coordinates. Substituting the canonical structure (8) into the definition (2), the Poisson bracket will be

$$\{f, g\}_c = \frac{\partial f}{\partial Q_\alpha^i} \frac{\partial g}{\partial P_i^\alpha} - \frac{\partial f}{\partial P_i^\alpha} \frac{\partial g}{\partial Q_\alpha^i}, \quad (9)$$

which leads to the standard canonical Poisson algebra

$$\{Q_\alpha^i, Q_\beta^j\}_c = 0, \qquad \{Q_\alpha^i, P_j^\beta\}_c = \delta_j^i \delta_\alpha^\beta, \qquad \{P_i^\alpha, P_j^\beta\}_c = 0. \quad (10)$$

Also, with the help of (8) we get from (3) the Liouville volume as

$$\omega_c^{3N} = d^{3N}Q \, d^{3N}P, \quad (11)$$

which is nothing but the standard volume element on $\mathbf{R}^{6N}$. Thus, the kinematics of the standard statistical mechanics can be recovered as a special case of this setup. Taking into account the fact that $\det \boldsymbol{\omega}_c = 1$, the measure associated to the standard volume element (11) assigns a uniform probability distribution to the set of microstates. This is justifiable in the light of Laplace's principle of indifference, which states that in the absence of any further information, all outcomes are equally likely [2]. This is the fundamental basis of the statistical mechanics.

In the case where the space-time has a noncommutative structure, apart from the details of the different models, there is always a deformation parameter which can be linked to a minimal length associated with the system under consideration. The direct consequence of this setup is the modification of the dispersion relation and the density of states (see Ref. [8] for the special case of the stable noncommutative algebra for the relativistic statistical mechanics). The density of states is determined by the symplectic structure and the associated Liouville volume. Since the symplectic structure takes a noncanonical form in terms of the physical positions and momenta (q, p) in noncommutative spaces [19], we have $\det \boldsymbol{\omega} = \det \boldsymbol{\omega}(q, p) \neq 1$. This result immediately shows that Laplace's indifference principle is no longer applicable for a noncommutative space-time. This is because of the fact that there is extra information in these setups (minimal length or maximal momentum) and the Liouville measure then assigns a nonuniform probability distribution over the set of microstates by means of the relation (4). This simple consideration of noncommutative space-time as an example of a kinematically deformed system shows the advantage of the coordinate-independent symplectic formulation of the statistical mechanics (see Refs. [17,23] for explicit examples). We will explicitly consider such an example in Sec. IV.

It also should be noted that, according to the Darboux theorem, there is always a local chart on Γ in which any symplectic structure takes the canonical form with $\det \boldsymbol{\omega} = 1$. In the case of kinematically deformed systems such as the noncommutative space-time, however, the new canonical coordinates cannot be interpreted as the positions and momenta of the particles. Moreover, in the case of standard statistical mechanics, one can also find a local chart in which the standard canonical structure ω_c takes a noncanonical form with $\det \boldsymbol{\omega}_c \neq 1$. But again, these new noncanonical coordinates cannot be interpreted as positions and momenta of particles. In all of the above cases, as we will see in the next section, the partition function and consequently resultant thermodynamical properties are independent of the chart in which the system is considered.

B. Dynamics

As we have mentioned above, the time evolution of the system is determined by the Hamiltonian vector field $\mathbf{x}_H$. It can be obtained by solving the dynamical equation

$$\omega(\mathbf{x}_H) = dH, \qquad (12)$$

where H is the Hamiltonian function of the system that is a real-valued function over Γ and $\omega(\mathbf{x}_H)$ is an interior product of ω and $\mathbf{x}_H$. Additionally, $\mathbf{x}_H$ can be expanded as

$$\mathbf{x}_H = \sum_{\alpha=1}^{N} \mathbf{x}_H^{\alpha}, \qquad (13)$$

where $\mathbf{x}_H^{\alpha}$ is the projection of $\mathbf{x}_H$ on Γ_{α}. Note that, in general, the domain of components of $\mathbf{x}_H^{\alpha}$ is the whole phase space Γ and consequently the Lie brackets between $\{\mathbf{x}_H^{\alpha}\}$ will not vanish:

$$[\mathbf{x}_H^{\alpha}, \mathbf{x}_H^{\beta}] \neq 0. \qquad (14)$$

However, as a special case, one can consider a system consisting of noninteracting particles. Therefore, the Hamiltonian function can be written as the sum of individual Hamiltonians, that is,

$$H = \sum_{\alpha=1}^{N} H_{\alpha}. \qquad (15)$$

So, from decomposition (5) for the symplectic structure, the dynamical equation (12) becomes

$$\left(\sum_{\alpha=1}^{N} \omega_{\alpha} \right) \left(\sum_{\beta=1}^{N} \mathbf{x}_H^{\beta} \right) = \sum_{\alpha=1}^{N} dH_{\alpha}. \qquad (16)$$

Using the fact that $\omega_{\alpha}(\mathbf{x}_H^{\beta}) = 0$, for $\alpha \neq \beta$, one is led to a set of N independent differential equations

$$\omega_1(\mathbf{x}_H^1) = dH_1, \ldots, \omega_N(\mathbf{x}_H^N) = dH_N. \qquad (17)$$

Now, it is clear that the domain of the components of $\mathbf{x}_H^{\alpha}$ will be Γ_{α} and the Lie brackets between $\mathbf{x}_H^{\alpha}$ and $\mathbf{x}_H^{\beta}$ vanish:

$$[\mathbf{x}_H^{\alpha}, \mathbf{x}_H^{\beta}] = 0. \qquad (18)$$

This can be seen as a criterion for the *kinematical and dynamical separability* of the particles. In the Appendix, we will see that this benchmark shows itself as a criterion for the statistical independence of the macroscopic subsystems which in turn is a definition of equilibrium.

The other important property of the Hamiltonian systems is the existence of an invariant measure that enables one to construct the equilibrium statistical mechanics through the well-known Liouville theorem as

$$\frac{d}{dt} \omega^{3N} = 0, \qquad (19)$$

where $d/dt = \partial/\partial t + \mathcal{L}_{\mathbf{x}_H}$ is the total time derivative and $\mathcal{L}_{\mathbf{x}_H}$ denotes the Lie derivative with respect to $\mathbf{x}_H$. The

conservation of ω^{3N} can be traced back to the fact that $\mathcal{L}_{\mathbf{x}_H} \omega^{3N} = 3N(\mathcal{L}_{\mathbf{x}_H}\omega) \wedge \omega^{3N-1}$ and $\mathcal{L}_{\mathbf{x}_H}\omega = d\omega(\mathbf{x}_H) + d(\omega(\mathbf{x}_H)) = 0$, where we have used $d(\omega(\mathbf{x}_H)) = d^2H = 0$ and the closure of the symplectic structure $d\omega = 0$. It is obvious that ω^{3N} is not explicitly time dependent. Note also that the Liouville theorem arises from the particular dynamics of a Hamiltonian system.

Consider the standard statistical mechanics as a special case, where the kinematics is given by canonical symplectic structure (8). The dynamical equation (12) then leads to the familiar form of the Hamilton's equations in terms of the physical positions and momenta as

$$\frac{d\mathrm{Q}_{\alpha}^i}{dt} = \frac{\partial H}{\partial \mathrm{P}_i^{\alpha}}, \qquad \frac{d\mathrm{P}_i^{\alpha}}{dt} = -\frac{\partial H}{\partial \mathrm{Q}_{\alpha}^i}. \qquad (20)$$

Note that when the symplectic structure has a noncanonical form, the form of Hamilton's equations would deviate from the relation (20).

The triple (Γ, ω, H) constitutes a general Hamiltonian system and we explore the statistical physics of this general system in the next section by means of an invariant volume element ω^{3N} which is covariantly defined in the relation (3).

III. STATISTICAL MECHANICS

Statistical mechanics links macroscopic properties of a system to microscopic laws that are quantum mechanical at the fundamental level. The basic concept is the von Neumann entropy that is defined as

$$S = -\mathrm{Tr}(\hat{\rho} \ln \hat{\rho}), \qquad (21)$$

where $\hat{\rho}$ is the density operator [24]. Using the principle of maximum entropy and considering specific relevant statistical constraints, we can find the density operator which contains all the statistical information about the system at equilibrium. For example, for a system in contact with a thermal bath at temperature T and therefore with a fixed mean energy, the resultant normalized density operator works out to be

$$\hat{\rho}_c = \frac{1}{\mathcal{Z}} \exp[-\hat{H}/T], \qquad (22)$$

where $\hat{H}$ is the Hamiltonian operator of the system and $\mathcal{Z}$ is called the canonical partition function which is defined as

$$\mathcal{Z} = \mathrm{Tr} \exp[-\hat{H}/T] = \sum_i \exp[-\varepsilon_i/T], \qquad (23)$$

where $\{\varepsilon_i\}$ are the energy eigenvalues of $\hat{H}$ and the summation is taken over all the accessible microstates for the system. At the fundamental level, the microscopic laws are quantum mechanical and the microstates are determined by quantum mechanics. On the other hand,

in the classical limit, the microstate of a system represents itself as a point on the corresponding phase space and due to the infinite resolution of phase space points, the summation over microstates is really not well defined. Consequently, one cannot construct a full classical statistical mechanics in essence. However, quantum mechanics provides the semiclassical approximation in the spirit of the uncertainty principle. Indeed, a phase space with finite resolution and well-defined number of classical microstates can be achieved through the approximation [17]

$$\text{Tr} \to \frac{1}{N!}\int_\Gamma \omega^{3N}, \tag{24}$$

where $1/(N!)$ is due to the fact that the particles may be indistinguishable and ω^{3N} is the dynamically invariant Liouville volume element of the $6N$-dimensional symplectic manifold Γ which is defined in the relation (3). It is important to note that the semiclassical approximation (24) coincides with the full quantum consideration at high temperature regime. In this limit, the von Neumann entropy (21) is replaced with the Gibbs entropy

$$S = -\frac{1}{N!}\int_\Gamma \rho \ln \rho \, \omega^{3N}, \tag{25}$$

where ρ is now interpreted as a probability density over the space of microstates Γ. The maximum entropy principle for a system in heat bath at temperature T leads to the Gibbs state

$$\rho_c = \frac{1}{\mathcal{Z}}\exp[-H/T], \tag{26}$$

where H is the Hamiltonian function of the system and $\mathcal{Z}$ is the semiclassical canonical partition function

$$\mathcal{Z} = \frac{1}{N!}\int_\Gamma \exp[-H/T]\omega^{3N}. \tag{27}$$

The result of the integral over the phase space for the total partition function (27) is independent of a chart in which the system is considered when the dynamical equation (12) is satisfied in a chart-independent manner. While the local form of the symplectic structure determines the probability distribution over the set of microstates, from the statistical point of view, we expect that the resultant thermodynamical properties extracted from a partition function are independent of a chart in which the physical system is considered (see, for instance, Ref. [17], in which the partition function in two different charts is calculated).

By considering the decompositions (5) and (15) for the symplectic structure ω and Hamiltonian function H, respectively, the probability density will also be decomposed as

$$\rho_c \omega^{3N} = (\rho_c^1 \omega_1^3) \wedge \ldots \wedge (\rho_c^N \omega_N^3), \tag{28}$$

where ρ_c^α is the Gibbs state defined on Γ_α. Hence the integral over Γ can be factorized into the products of the integrals over the phase spaces of the particles as

$$\mathcal{Z} = \frac{1}{N!}\prod_{\alpha=1}^{N}\left(\int_{\Gamma_\alpha} \exp[-H_\alpha/T]\omega_\alpha^3\right). \tag{29}$$

Now, if the particles are influenced by the same kinematics and dynamics, the partition function can be written in the well-known form

$$\mathcal{Z} = \frac{\mathcal{Z}_1^N}{N!}, \tag{30}$$

where we have defined the single-particle state partition function

$$\mathcal{Z}_1 = \int_{\Gamma_1} \exp[-H_1/T]\omega_1^3. \tag{31}$$

It is usually claimed that the relation (30) is applicable when one ignores the quantum correlations between particles. This means that decompositions such as (5) and (15) may be considered as conditions for the absence of quantum correlations. Moreover, the general formalism which is formulated in this section can also support a Hamiltonian system with very nontrivial structure, e.g., a kinematically classical entangled system in which the decomposition (5) is no longer applicable.

IV. APPLICATION IN NONCOMMUTATIVE SPACES

Existence of a minimal length is suggested by any quantum gravity candidate such as string theory and loop quantum gravity [25,26]. It is then widely believed that a nongravitational theory which supports the existence of a minimal length would arise at the flat limit of quantum gravity. Such a theory would be reduced to the standard relativistic quantum mechanics at low energy regime where the effects of a minimal length are negligible. Evidently, taking a minimal length into account naturally leads to the space-time with noncommutative coordinates. The first attempt in this direction was done by Snyder in 1947 who formulated a Lorentz-invariant discrete space-time with noncommutative coordinates [6]. The space-time with noncommutative structure is also suggested by the theory of stability of the Lie algebras [8]. On the other hand, existence of a universal minimal length cannot be supported by the special relativity since any length scale in one inertial frame may be different in another observer's frame through the well-known Lorentz-Fitzgerald contraction. Thus, the doubly special relativity theories are formulated

in order to take into account an observer-independent minimum length as well as the velocity of light [27]. The curved four-momentum space then naturally appears to be a suitable framework to formulate the doubly special relativity theories [3]. Furthermore, the different doubly special relativity theories can be obtained from the different basis of the κ-Poincaré algebra on the associated κ-Minkowski noncommutative space-time [28]. In this respect, it is also shown that the Snyder noncommutative algebra which is proposed in Ref. [6] and the stable algebra of relativistic quantum mechanics that is obtained in Ref. [8] can be obtained from a ten-dimensional phase space with curved geometry for the four-momentum space through the symplectic reduction process [4]. Thus, it seems that the space-time with curved four-momentum space and noncommutative coordinates is a fundamental framework for taking into account a minimal length scale in the flat limit of quantum gravity [29]. It would also be claimed that the Lorentz invariance is an approximate symmetry and will be broken at a high energy regime. Therefore, the Lorentz-violating noncommutative algebra was proposed by Camelia in Ref. [27], which supports the existence of a minimal observer-independent length scale. Magueijo and Smolin showed that the Lorentz invariance can also be preserved by a nonlinear action of the Lorentz group on the momentum space [30]. In order to preserve the Lorentz invariance, the following commutation relations for the generators will be held:

$$\{J_{ab}, J_{cd}\} = J_{ad}\eta_{bc} + J_{bc}\eta_{ad} - J_{bd}\eta_{ac} - J_{ac}\eta_{bd},$$
$$\{J_{ab}, p_c\} = p_a\eta_{bc} - p_b\eta_{ac}, \quad \{J_{ab}, x_c\} = x_a\eta_{bc} - x_b\eta_{ac},$$

$$(32)$$

where $\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$ is the Minkowski metric and $a, b, c, d = 0, \ldots, 3$. The commutation relations between (x_a, p_a) then classify the different Lorentz-invariant algebras such as the Snyder noncommutative algebra [6], the stable algebra [8], and also the standard Lorentz algebra.

While the relativistic algebras are written in the eight-dimensional relativistic phase space, to formulate the statistical mechanics one needs to work in a six-dimensional nonrelativistic subalgebra of the deformed relativistic algebras in which the time parameter is fixed and the corresponding Poisson brackets take a noncanonical form. By means of the resulting deformed Hamiltonian system, one can study the statistical mechanics with the help of the constructed setup in the pervious sections. In recent years, many works have gone in this direction in order to study the effects of an ultraviolet cutoff (minimal length and maximal momentum) on the thermodynamical properties of the physical systems. For instance, thermodynamics of some physical systems in noncommutative spaces is studied in Ref. [31]. For the special case of the

Snyder noncommutative space, see Ref. [32], in which the thermodynamics of the early Universe is explored. It is shown that the Liouville theorem for the deformed phase space with general noncanonical Poisson algebra is satisfied. However, as we have shown in Sec. III, the Liouville theorem can be justified in a very simple manner in the covariant formalism, which also shows the advantage of the setup. Furthermore, thermodynamical ideal gases are studied in the context of doubly special relativity theories [16]. Also, motivated by the string theory, the generalized uncertainty principle is suggested which supports the existence of a minimal length as a nonzero uncertainty in position measurement [25,33]. For the statistical mechanics in this setup, see Refs. [34,35]. Inspired by loop quantum gravity, the polymer quantum mechanics is formulated which supports the existence of a minimal length known as the polymer length scale [36]. Thermodynamical properties of the ideal gases and harmonic oscillator are also studied in Refs. [17,37]. To see the effects of minimal length on the thermodynamics of the black holes in noncommutative space, the generalized uncertainty principle framework, and the polymer quantization scheme, see Refs. [38–40]. Apart from the details of the above-mentioned effective approaches to quantum gravity phenomenology, all of them try to consider the effects of a minimal length in a relevant manner, and implementing the covariant formalism can clarify the consequences and applications in its most fundamental form in the language of symplectic geometry.

A. Statistical mechanics in the Snyder noncommutative space

As we have mentioned above, the advantages of the covariant formulation may be revealed when a Hamiltonian system with nontrivial structure is considered. Therefore, in this subsection we study the statistical mechanics in the Snyder noncommutative space as a well-known example of a kinematically deformed Hamiltonian system.

The Snyder relativistic algebra preserves the Lorentz invariance and therefore the commutation relations (32) are satisfied by the corresponding generators. The commutation relations between (x_a, p_a) which define the Snyder algebra are given by [6]

$$\{x_a, x_b\} = \frac{J_{ab}}{\kappa^2}, \qquad \{x_a, p_b\} = \eta_{ab} + \frac{p_a p_b}{\kappa^2},$$
$$\{p_a, p_b\} = 0, \qquad (33)$$

where κ is the deformation parameter with the dimension of the inverse of length, which plays the role of the universal quantum gravity scale in this setup. The four-momentum space of the eight-dimensional relativistic phase space of a test particle which moves on the space-time with non-commutative algebra (33) is a de Sitter space with topology $\mathbf{R} \times \mathbf{S}^3$ [4,6]. Identifying the energy space with $\mathbf{R}$, the

topology for the space of spatial momenta will be $\mathbf{S}^3$. In order to study the statistical mechanics in this setup, one should consider the nonrelativistic subalgebra of (33) and replace the standard canonical Poisson algebra with them. The nonrelativistic Snyder algebra for a particle moves in three-dimensional Euclidean space $\mathbf{R}^3$ and is then given by [41]

$$\{q_i, q_j\} = \frac{J_{ij}}{\kappa^2}, \qquad \{q_i, p_j\} = \delta_{ij} + \frac{p_i p_j}{\kappa^2},$$
$$\{p_i, p_j\} = 0. \tag{34}$$

It is straightforward to show that the above commutation relations can be realized from the symplectic structure [42]

$$\omega = dq^i \wedge dp_i - \frac{1}{2} d(q^i p_i) \wedge d\ln(p^2 + \kappa^2), \quad (35)$$

through the covariant definition (2) for the Poisson brackets. The associated phase space of the particle is then simply the cotangent bundle of the configuration space which now has the nontrivial $\mathbf{R}^3 \times \mathbf{S}^3$ topology. Indeed, the space of the spatial momenta has compact $\mathbf{S}^3$ topology rather than the standard $\mathbf{R}^3$ one (see Refs. [19,41,43] for more details). Since the quantum gravity cutoff (which is defined by the deformation parameter κ) is universal, it will be the same for all of the particles. Thus, for the kinematically deformed system consisting of N particles which obeys the Snyder noncommutative algebra (34), the total symplectic structure can be obtained by the relation (5). The phase space Γ_α for all of the particles is the same and has the nontrivial $\mathbf{R}^3 \times \mathbf{S}^3$ topology. The associated total phase space Γ_S can be easily obtained through the coupling (1), which has the following nontrivial topology:

$$\mathbf{R}^{3N} \times \underbrace{\mathbf{S}^3 \times \cdots \times \mathbf{S}^3}_{N\,\text{times}}. \tag{36}$$

Substituting the symplectic structure (35) into the definition (3), the corresponding Liouville volume takes the form

$$\omega^{3N} = \prod_{\alpha=1}^{N}\left(\frac{d^3 q_\alpha \wedge d^3 p^\alpha}{(1 + (p^\alpha/\kappa)^2)}\right) = \frac{d^{3N} q \wedge d^{3N} p}{\prod_{\alpha=1}^{N}(1 + (p^\alpha/\kappa)^2)}, \tag{37}$$

where q_α^i and p_i^α are the ith component of the positions and momenta of the αth particle, respectively. From this relation it is clear that the probability distribution is nonuniform in Snyder noncommutative space and the standard uniform one can be recovered only at low energy (low temperature) regime $\kappa \to \infty$. Indeed, the existence of a minimal length, as an extra information, changes the probability distribution at the high energy regime. Thus, in all of the kinematically deformed noncommutative phase spaces, the local form of

the symplectic structure (in terms of the positions and momenta of the particles) is noncanonical and therefore the probability distribution will be nonuniform such that the microstates with higher momenta are less probable. This is the advantage of the symplectic covariant formulation of the statistical mechanics which shows that the probability distribution will be nonuniform on any phase space with noncanonical Poisson algebra like the Snyder algebra (34). However, in light of the Darboux theorem, one can always find a local chart through the Darboux transformation $(q_\alpha^i, p_i^\alpha) \to (X_\alpha^i = q_\alpha^i - \frac{q_\alpha^j p_i^\alpha}{(p^\alpha)^2 + \kappa^2}\delta^{ik} p_k^\alpha, Y_i^\alpha = p_i^\alpha)$, in which the symplectic structure (35) takes the canonical form $\omega_\alpha = dX_\alpha^i \wedge dY_i^\alpha$ for all of the particles. Thus, the associated Liouville volume becomes $\omega^{3N} = d^{3N} X \wedge d^{3N} Y$, which induces a uniform probability distribution over the set of microstates. But, it is important to note that (X_α^i, Y_i^α) are different from the standard canonical ones (Q_α^i, P_i^α) which are defined in relations (8) and (11). Indeed, while (Q_α^i, P_i^α) are the positions and momenta of the particles by definition, (X_α^i, Y_i^α) cannot be interpreted as the positions and momenta of the particles. Also, the noncanonical Snyder Liouville volume (37) reduces to the standard canonical one (11) in the limit of $\kappa \to \infty$ and the noncanonical variables (q_α^i, p_i^α) then coincide with standard canonical variables (Q_α^i, P_i^α) in this limit. However, the canonical variables (X_α^i, Y_i^α) are obtained through a Darboux transformation and the functional form of the Hamiltonian function will be modified (compared with the standard functional form) in this chart as $H(X, Y)$ since the dynamical equation (12) will be satisfied in a chart-independent manner.

Substituting the Liouville volume (37) in the relation (27), the total partition function will be

$$\mathcal{Z} = \frac{1}{N!} \int_{\Gamma_S} \frac{\exp[-H(q, p)/T]}{\prod_{\alpha=1}^{N}(1 + (p^\alpha/\kappa)^2)} d^{3N} q\, d^{3N} p. \tag{38}$$

Since the Snyder Liouville volume (37) is decomposed similar to the relation (7), for the noninteracting systems in which the total Hamiltonian function also decomposes as the relation (15), one can rewrite the total partition function (38) in terms of the single-particle partition function. Taking the decomposition (15) into account and substituting the volume (37) into relation (38), the total partition function can be rewritten in the form of the relation (29) as

$$\mathcal{Z} = \frac{1}{N!} \prod_{\alpha=1}^{N}\left(\int_{\Gamma_\alpha} \frac{\exp[-H_\alpha(q_\alpha, p^\alpha)/T]}{[1 + (p^\alpha/\kappa)^2]} d^3 q_\alpha d^3 p^\alpha\right)$$
$$= \frac{1}{N!} \mathcal{Z}_1^N, \tag{39}$$

where we have defined the single-particle partition function as

$$\mathcal{Z}_1 = \int_{\mathbf{R}^3} d^3q_1 \int_{\mathbf{S}^3} d^3p^1 \frac{\exp[-H_1(q_1, p^1)/T]}{(1 + (p^1/\kappa)^2)}, \quad (40)$$

and we have also assumed that the Hamiltonian functions are the same for all of the particles. Then, all the thermodynamical properties of a system can be deduced from the total partition function (38) or (39) in which, as we have mentioned in the pervious section, the Gibbs factor $1/N!$ is considered for the nonlocalized systems such as the ideal gas, and it should then be removed for the localized systems such as the harmonic oscillator which is the subject of the next subsection.

B. Thermodynamics of 3D harmonic oscillator

Now let us consider a system of N independent and similar three-dimensional harmonic oscillators. Since the oscillators are assumed to be independent, the relations (5) and (15) will be satisfied and one then can implement the relation (39) in order to obtain the partition function of the system. The Hamiltonian of the three-dimensional simple harmonic oscillator is given by $H_1(q, p) = \frac{p^2}{2m} + \frac{1}{2}m\sigma^2 q^2$, where m is the mass of the oscillator, σ is the frequency, $p^2 = p_x^2 + p_y^2 + p_z^2$, and $q^2 = x^2 + y^2 + z^2$. Substituting this Hamiltonian into the relation (40) gives the following single-particle partition function:

$$\mathcal{Z}_1 = \frac{\kappa^2 T^2}{m\hbar^3 \sigma^3} \left(1 - \frac{\sqrt{\pi}\kappa}{\sqrt{2mT}} \operatorname{erfc}\left[\frac{\kappa}{\sqrt{2mT}}\right] \exp\left[\frac{\kappa^2}{2mT}\right]\right). \quad (41)$$

At the high temperature regime, the second term in the right-hand side of the above relation is negligible. Under this condition the dominant contribution is due to the first term and the partition function is approximated by $\mathcal{Z}_1 \approx \frac{\kappa^2 T^2}{m\hbar^3 \sigma^3}$. This shows that in the domain of the high temperature, the number of degrees of freedom will be reduced since the partition function is proportional to $\sim T^2$ rather than the standard one which is proportional to $\sim T^3$. We will show this feature in a more precise manner in this subsection. The total partition function can be obtained from the relation (39) as

$$\mathcal{Z} = \mathcal{Z}_1{}^N, \quad (42)$$

where the Gibbs factor $1/N!$ is dropped since the oscillators are localized. Substituting the single-particle partition function (41) into this relation, the total partition function for N three-dimensional harmonic oscillators can be obtained from which one is able to study the thermodynamical behavior of the system in Snyder space. However, before doing this task, it is useful to consider the limit $\frac{\kappa}{\sqrt{2mT}} \gg 1$ which is related to the first order corrections that would arise from the minimal length

effects. This limit also allows us to compare our result with the full quantum partition function. In this limit, we may use the relation $\operatorname{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}x} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2x^2)^n}$, to arrive at the following asymptotic expression for the partition function,

$$\mathcal{Z}_1 = \left(\frac{T}{\hbar\sigma}\right)^3 \left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)!!}{(2\Theta^2)^n}\right), \quad (43)$$

in which we have defined the dimensionless variable $\Theta \equiv \frac{\kappa}{\sqrt{2mT}}$. In order to obtain the quantum partition function, one should solve the Hamiltonian eigenvalue problem in Snyder space, which is not an easy task at all since the Schrödinger equation takes a complicated form in this setup. However, in the context of the generalized uncertainty principle the problem of a D-dimensional harmonic is exactly solved [44] and the results are applicable also for the Snyder algebra (34) as a particular case. The energy eigenvalues for a three-dimensional harmonic oscillator in Snyder space are given by

$$E_{nl} = \hbar\sigma \left[\left(n + \frac{3}{2}\right)\sqrt{1 + \left(\frac{m\hbar\sigma}{2\kappa^2}\right)^2} + \frac{m\hbar\sigma}{2\kappa^2}\left(\left(n + \frac{3}{2}\right)^2 - \left(l(l+1) + \frac{3}{4}\right)\right)\right], \quad (44)$$

where $n = 0, 1, 2, \ldots$, and $0 \leq l \leq n$ and also we have considered relevant parameter identification to rewrite the result of the Ref. [44] in our notation and units. The associated quantum partition function then will be obtained from the definition (23) as

$$\mathcal{Z}_1 = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \exp\left[-\frac{E_{nl}}{T}\right]. \quad (45)$$

The above summation cannot be evaluated analytically. However, in Ref. [35], it is shown that up to the first order of approximation, in the limit of $\Theta = \frac{\kappa}{\sqrt{2mT}} \gg 1$, it leads to $\mathcal{Z}_1 = (\frac{T}{\hbar\sigma})^3 (1 - \frac{3}{2}\Theta^{-2}) + \mathcal{O}[\Theta^{-4}]$, which is nothing other than the first term in the summation (43). This coincidence shows that the quantum partition function (45) correctly leads to the semiclassical partition function (41) up to the first order of approximation at the high temperature regime. It also reveals the advantages of semiclassical approximation (24) in the sense that although there is not an analytical solution for the case of full quantum partition function (45), the relation (41) provides an analytical expression for the high temperatures. Therefore, in dealing with the statistical considerations of the physical systems in the presence of a minimal length, since one is usually interested in the high temperature regime (such as early Universe
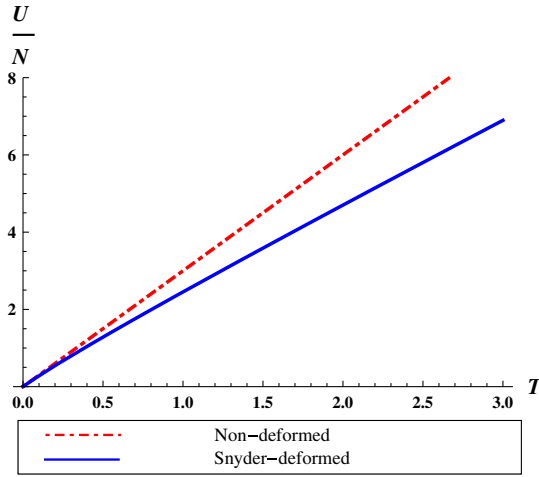
FIG. 1 (color online). Internal energy versus temperature is plotted. The solid and dashed lines represent the internal energy for the Snyder-deformed and nondeformed cases, respectively. The deviation from the standard case arises at the high temperature regime when the minimal length effects dominate. The figure is plotted for $m = 1 = \kappa$.

thermodynamics), the usage of the semiclassical approximation seems to be quite reasonable.

Substituting the single-particle partition function (41) into the relation (42) gives the total partition function. The internal energy then can be obtained from the standard definition $U = T^2 (\frac{\partial \ln \mathcal{Z}}{\partial T})_N$ as

$$U = \frac{3NT}{2} \left[1 + \frac{1}{3} (1 - \sqrt{\pi} \Theta \operatorname{erfc}[\Theta] e^{\Theta^2})^{-1} - \Theta^2 \right]. \quad (46)$$

In Fig. 1 we have plotted the internal energy versus temperature in comparison with its nondeformed
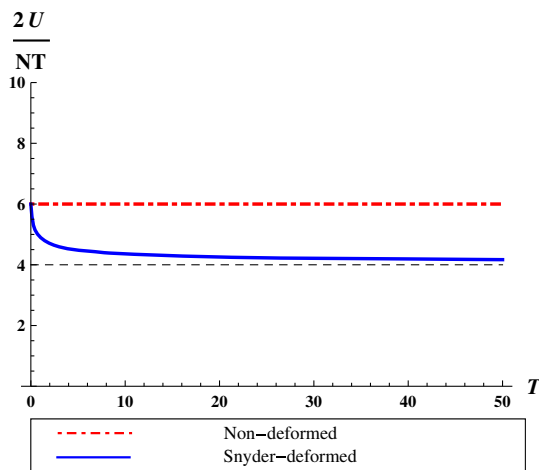

FIG. 2 (color online). The number of degrees of freedom versus temperature. As it is clear from the figure, in Snyder space the number of degrees of freedom for a three-dimensional harmonic oscillator is reduced from 6 to 4 at the high temperature regime.

counterpart. The specific heat can also be deduced from the definition $C_V = (\frac{\partial U}{\partial T})_V$ which leads to a somehow cumbersome expression.

An interesting feature here is the reduction of the number of degrees of freedom in Snyder space. According to the well-known equipartition theorem of energy, for the Hamiltonians of the form $H_1(q, p) = \frac{p^2}{2m} + \frac{1}{2} m\sigma^2 q^2$, each of the 6 degrees of freedom makes a contribution of $\frac{1}{2} T$ towards the internal energy. Thus one may consider $\frac{2U}{NT}$ as the number of degrees of freedom for each three-dimensional harmonic oscillator. In Fig. 2 the number of degrees of freedom for each three-dimensional harmonic oscillator is plotted. As this figure shows, the number of degrees of freedom will be reduced from 6 to 4 in Snyder space. This may be compared with the statistical mechanics of the ideal gases considered in Ref. [45] in which the same result is obtained. However, we would like to emphasize that the reduction of the number of degrees of freedom for the case of ideal gas can be interpreted as an effective dimensional reduction of the space at the high temperature regime which is a common feature of quantum gravity proposals [46] and also phenomenological approaches to the minimal length conjecture [47].

Considering the limit $\Theta = \frac{\kappa}{\sqrt{2mT}} \gg 1$ is useful at least for two reasons. First, we see that the modified thermodynamical relations recover the standard results at the sufficiently low temperature regime when the minimal length effect are negligible (correspondence principle). Second, one can estimate the magnitude of the order of the quantum gravity corrections to the thermodynamical quantities which may potentially lead to observable effects. In this limit, the series in the relation (43) converges and one can use it to obtain all the minimal length (quantum gravity) corrections to the internal energy and specific heat. Substituting the relation (43) into (42) and then using the definition $U = T^2(\frac{\partial \ln \mathcal{Z}}{\partial T})_N$, it is easy to show that the internal energy in the limit of $\Theta \gg 1$ will be

$$\frac{U}{U_0} = 1 + \frac{2}{3} \sum_{n=1}^{\infty} (-1)^n \frac{n(2n+1)!!}{(2\Theta^2)^n}, \quad (47)$$

where $U_0 = 3NT$ is the internal energy for the standard nondeformed N independent three-dimensional harmonic oscillators. In this limit, the specific heat can be also obtained by substituting the above relation into the definition $C_V = (\frac{\partial U}{\partial T})_V$, which gives

$$\frac{C_V}{C_{0_V}} = 1 + \frac{2}{3} \sum_{n=1}^{\infty} (-1)^n \frac{n(n+1)(2n+1)!!}{(2\Theta^2)^n}, \quad (48)$$

where $C_{0_V} = (\frac{\partial U_0}{\partial T})_V = 3N$ is the specific heat of the corresponding nondeformed case.

As is clear from the relations (47) and (48), the first minimal length (quantum gravity) corrections to the internal energy and specific heat of the harmonic oscillator are of the order of Θ^{-2}. To estimate the magnitude of the order of these corrections, consider the vibrational oscillations of a carbon monoxide molecule which may be modeled with the oscillators of mass $m \approx 10^{-26}$ kg and frequency $\sigma \approx 10^{15}$ Hz. With this numerical value for the mass and also considering $\kappa = \mathcal{O}(1)T_{\text{Pl}} \sim 10^{19}$ GeV and $T \sim 1$ TeV, we get $\Theta \sim 10^{17}$. Therefore, the first minimal length corrections to the internal energy and specific heat of the harmonic oscillator in Snyder space are of the order of $\Theta^{-2} \sim 10^{-34}$, which is too small to be experimentally detected for the accessible energy scales.

V. SUMMARY

Appearance of the Hamiltonian systems with nontrivial structure in the context of phenomenological quantum gravity candidates, such as the doubly special relativity theories with deformed noncommutative phase spaces, naively suggests the revision of the statistical mechanics formalism. In the first step, we have formulated the statistical mechanics of a general Hamiltonian system in a covariant (chart-independent) manner by means of the symplectic geometry. The results show that the two properties of the phase space, as a symplectic manifold, play distinguished roles in statistical consideration of a system: The topology of the phase space as a global property and the local form of the symplectic structure that determines the geometry of the phase space. For a topologically trivial phase space with the canonical symplectic structure, the standard statistical mechanics emerges as it is expected. The subtleties arise when the phase space has nontrivial topology or geometry such as the phase spaces with curved momentum space and noncommutative structure. The topology of the phase space turns out to be related to the total number of microstates; for instance, the number of accessible microstates will be finite for a system with compact phase space. The symplectic structure also affects the probability distribution of the microstates. The canonical form of the symplectic structure in terms of the positions and momenta of the particles plays an important role in the standard statistical mechanics. It induces a uniform probability distribution over the microstates by defining a uniform measure on the corresponding phase space. However, the noncommutative phase space always provides a noncanonical symplectic structure in terms of the positions and momenta of the particles and the probability distribution then will be nonuniform. This is a general result for the phase spaces with noncanonical

(noncommutative) structure which can be immediately realized from the covariant formulation of the phase space. We implemented the covariant formalism in order to study the statistical mechanics in Snyder noncommutative space as an explicit example of a phase space with nontrivial topology and geometry. We obtained the associated partition function from which all the thermodynamical properties of the system can be obtained. As a particular example, we obtained the partition function for the three-dimensional harmonic oscillator and we have shown that our result is in good agreement with that which arises from the full quantum consideration at the high temperature regime. While there is no analytical expression for the full quantum partition function of the harmonic oscillator in Snyder space, the semiclassical approximation provides an analytical partition function which is applicable at the high temperature regime. Using the obtained partition function, we have studied the thermodynamical properties of the system of harmonic oscillators in Snyder space and our analysis shows that the number of microstates will be drastically reduced in this setup due to the existence of a minimal length. This result justifies our general claim for the deformed spaces which take into account a minimal length scale. Apart from the details of the models which deal with a universal minimal length such as the noncommutative spaces, doubly special relativity theories, the generalized uncertainty principle, and polymer quantization, the covariant formalism reveals the main role of the minimal length in statistical mechanics: The microstates with higher energy or momenta are less probable when there is a minimal length scale for the system. Also, we calculated the quantum gravity corrections to the internal energy and specific heat of a system of harmonic oscillators. We estimated the order of magnitude of the first minimal length corrections which are of the order of 10^{-34} for both of the internal energy and specific heat. The equilibrium as an essential criterion for the statistical system is also considered in the Appendix by means of the dynamically invariant Liouville volume which enters in the definition of the Gibbs entropy and partition function of the statistical system.

APPENDIX: EQUILIBRIUM

1. Statistical independence

Relation (28) can be seen as a particular case in which the system is divided into statically independent subsystems, and subsystems were taken to be the particles themselves. It is clear that the statistical independence between particles will be spoiled for a system in which the particles interact with each other. However, when the system is in equilibrium, one can still divide it into M macroscopic subsystems $(1 \ll M \ll N)$ which are

statistically independent. In this respect, the total phase space Γ and the associated symplectic structure ω will be

$$\Gamma = \Gamma_1 \times \cdots \times \Gamma_M, \qquad \omega = \sum_{A=1}^{M} \omega_A. \qquad (A1)$$

The corresponding Liouville volume is then given by

$$\omega^{3N} = (\omega_1)^{3n_1} \wedge \ldots \wedge (\omega_M)^{3n_M}, \qquad (A2)$$

where n_A is the number of particles of the subsystems and is assumed to be sufficiently large $(n_A \gg 1)$ to guarantee that the subsystems are macroscopic. The statistical independence of the subsystems is then defined as

$$\rho\omega^{3N} = \rho^1 (\omega_1)^{3n_1} \wedge \ldots \wedge \rho^M (\omega_M)^{3n_M}. \qquad (A3)$$

This criterion is usually employed as an intrinsic characterization of an equilibrium probability distribution [48]. Indeed, the system is in equilibrium if components of the system are in relative equilibrium with respect to each other. Based on this internal definition of the equilibrium, the Gibbs state can be understood as an equilibrium state as follows. Consider the total Hamiltonian function of the system as

$$H = \sum_{A=1}^{M} H_A, \qquad (A4)$$

where $\{H_A\}$ are the Hamiltonian functions of the M macroscopic subsystems and we also disregard the interaction between subsystems based on the fact that the relative number of particles which take part in the interactions is negligible compared to n_A. So, using the relations (26), (A1), and (A4), the Gibbs state turns out to be an equilibrium state based on (A3). Additionally, it is straightforward to show that

$$[\mathbf{x}_H^A, \mathbf{x}_H^B] = 0, \qquad (A5)$$

which is the criterion for the kinematical and dynamical separability of the subsystems. The relation (A5) then can be viewed as the equilibrium criterion when the system is in a Gibbs state.

Note that the decomposition (A4) holds only over not-too-long intervals of time since the effects of interaction of the subsystems will eventually be dominated even if such interactions are too weak such as the universal gravitational effects [49].

2. Liouville equation

Another more common definition for the equilibrium statistical state of the system is

$$\frac{\partial \rho_e}{\partial t} = 0, \qquad (A6)$$

in which ρ_e denotes the density that corresponds to the equilibrium state. The fact that the Gibbs state is an equilibrium state then will be justified through the Liouville equation

$$\frac{d}{dt}(\rho\omega^{3N}) = \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{x}_H}\right)(\rho\omega^{3N}) = 0, \qquad (A7)$$

in which the Liouville theorem (19) is used. The Liouville equation should be satisfied by any statistical state as well as Gibbs one and it is, indeed, the necessary requirement for the consistent probabilistic interpretation of ρ. Using the fact that $\mathcal{L}_{\mathbf{x}_H}\rho = \mathbf{x}_H\rho = \{\rho, H\}$, the Liouville equation (A7) can be rewritten in the more well-known form

$$\frac{\partial \rho}{\partial t} + \{\rho, H\} = 0. \qquad (A8)$$

From the relation (26), it is clear that $\{\rho_c, H\} = 0$, and the Gibbs state then satisfies the relation $\partial \rho_c / \partial t = 0$, which shows that it is indeed an equilibrium state through the definition (A6).

If we use the Liouville equation (A7) and consider the fact that the Liouville volume ω^{3N} does not change explicitly with time, it turns out that

$$\frac{dS}{dt} = -\frac{1}{N!}\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{x}_H}\right)\int_\Gamma \rho \ln \rho \, \omega^{3N}$$
$$= \frac{1}{N!}\int_\Gamma \rho \mathcal{L}_{\mathbf{x}_H}\omega^{3N} = 0, \qquad (A9)$$

which shows that the fine-grained entropy (25) does not change with time as expected for closed systems. The above relation and the Liouville equation (A7) guarantee that the number of microstates does not change through the dynamical evolution of the system, which makes the statistical formulation a consistent setup to give the thermodynamical properties in any ensemble.

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