PHYSICAL REVIEW D **92,** 024056 (2015)

Post-Tolman-Oppenheimer-Volkoff formalism for relativistic stars

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(Received 17 April 2015; published 30 July 2015)

Besides their astrophysical interest, compact stars also provide an arena for understanding the properties of theories of gravity that differ from Einstein's general relativity. Numerous studies have shown that different modified theories of gravity can modify the bulk properties (such as mass and radius) of neutron stars for given assumptions on the microphysics. What is not usually stressed though is the strong degeneracy in the predictions of these theories for the stellar mass and radius. Motivated by this observation, in this paper we take an alternative route and construct a stellar structure formalism which, without adhering to any particular theory of gravity, describes in a simple parametrized form the departure from compact stars in general relativity. This "post-Tolman-Oppenheimer-Volkoff (TOV)" formalism for spherical static stars is inspired by the well-known parametrized post-Newtonian theory, extended to second post-Newtonian order by adding suitable correction terms to the fully relativistic TOV equations. We show how neutron star properties are modified within our formalism, paying special attention to the effect of each correction term. We also show that the formalism is equivalent to general relativity with an "effective" (gravity-modified) equation of state.

DOI: 10.1103/PhysRevD.92.024056 PACS numbers: 04.40.Dg, 04.50.Kd, 04.25.Nx, 97.60.Jd

I. INTRODUCTION

Neutron stars play a special role among astrophysical objects, because they are excellent laboratories for matter under extreme conditions (unlike black holes) and *also* excellent laboratories to probe strong gravity (unlike ordinary stars or white dwarfs) [1]. For these reasons neutron stars are among the main targets of future observatories, such as SKA [2], NICER [3], LOFT [4], and AXTAR [5]. These experiments have the potential to measure neutron star masses and radii to unprecedented levels [6–8]. If general relativity (GR) is assumed to be the correct theory of gravity, the observed mass-radius relation will constrain the equation of state (EOS) of matter at supranuclear densities, which are inaccessible to laboratory experiments [9–14]. A procedure to reconstruct the EOS from observations of the mass-radius relation (working within GR) was developed in a series of papers by Lindblom and collaborators [15–17]; see Ref. [18] for a review.

Besides their interest for nuclear physics, neutron stars are also unique probes of strong-field gravitational physics. For any given EOS, theories that modify the strong-field dynamics of GR generally predict bulk observable properties (neutron star mass, radius, moment of inertia and higher multipole moments) that are different from those in Einstein's theory. However, a survey of the literature on neutron stars in modified theories of gravity (see e.g., Table 3 of Ref. [1]) reveals a high degree of degeneracy in the salient properties of relativistic stars. As we show in Fig. 1, if we assume a nuclear-physics motivated EOS (specifically, EOS APR [19] in the figure), modifications in the gravity sector are usually equivalent to systematic shifts of the GR mass-radius curves towards either higher masses and larger radii (as in the case of scalar-tensor theories [20,21]), lower masses and smaller radii (as in the case of Einstein-dilaton-Gauss-Bonnet [22,23] and Lorentz-violating theories [24,25]) or both, as in Eddington-inspired-Born-Infeld gravity with different signs of the coupling parameter [26,27].

Systematic shifts in the mass-radius relation could be attributed either to the poorly known physics controlling the high-density EOS, or to modifications in the theory of gravity itself. This EOS/gravity degeneracy is intrinsic in all attempts to constrain strong gravity through astrophysical observations of neutron stars: chapter 4 of Ref. [1] reviews various proposals to solve this problem, e.g., through the recently discovered universal relations between the bulk properties of neutron stars [28–31].

In any case, different gravitational theories span (at least qualitatively) the same parameter space in terms of their

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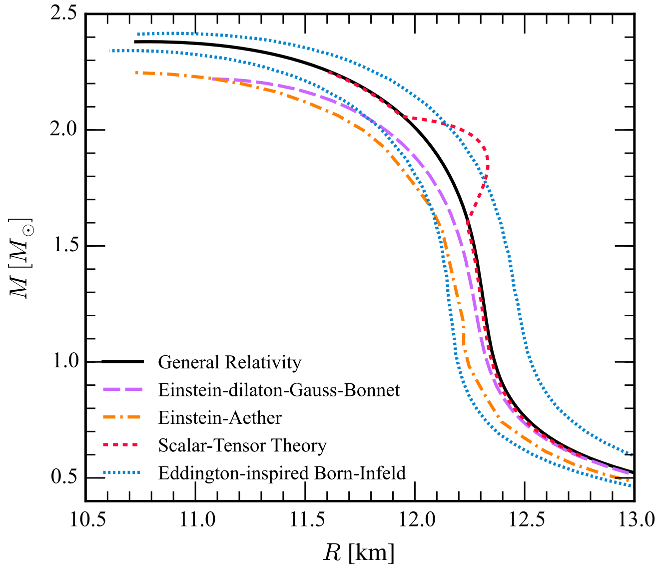

FIG. 1 (color online). *The gravity-theory degeneracy problem.* The mass-radius relations in different modified theories of gravity for EOS APR [19]. Masses are measured in solar masses, and radii in kilometers. The theory parameters used for this plot are $\alpha = 20M_\odot^2$ and $\beta^2 = 1$ (Einstein-dilaton-Gauss-Bonnet [22]), $c_{14} = 0.3$ (Einstein-aether [24]), $\beta = -4.5$ (scalar-tensor theory [20]), and $\kappa = \pm 0.005$ (Eddington-inspired-Born-Infeld gravity [26]). Even if the high-density EOS were known, it would be hard to distinguish the effects of competing theories of gravity on the bulk properties of neutron stars.

predictions for relativistic stellar models. Gravity-induced modifications usually look like smooth deformations of the general relativistic predictions. A notable exception are cases where nonperturbative effects induce phase transitions, as in the "spontaneous scalarization" scenario first proposed in Ref. [32], where modifications only occur in a specific range for the central density.

With the possible exception of nonperturbative phase transitions, these considerations suggest that the broad features of a large class of modified gravity theories can be reproduced, at least for small deviations from GR, by a perturbative expansion around a background solution given by the standard Tolman-Oppenheimer-Volkoff (TOV) equations, which determine the structure of relativistic stellar models in GR [33,34].

Instead of committing to one particular pet theory, in this paper we formulate a parametrized "post-TOV" framework for relativistic stars based on the well-known parametrized post-Newtonian (PPN) theory developed by Nordtvedt and Will [35,36]; see e.g., Refs. [34,37] for introductions to the formalism. The foundations of post-Newtonian (PN) theory for fluid configurations in GR were laid in classic work by Chandrasekhar and collaborators [38,39]. Various authors studied stellar structure using the PN approximation, both in GR [40–43] and in modified theories of gravity, such as scalar-tensor theory [44,45]. To our knowledge, after some early work that will be discussed below [46–48], the investigation of compact stars within the PPN approximation has remained dormant for more than 30 years. In the intervening time the PPN parameters have been extremely well constrained by Solar System and binary pulsar observations at the first PN (1PN) order (see Ref. [49] for a review of current bounds).

In this paper we build a phenomenological post-TOV framework by considering 1PN- and second PN (2PN)-order corrections to the TOV equations. Our strategy is, at heart, quite simple: from a suitable set of PPN hydrostatic equilibrium equations we isolate the purely non-GR pieces. These PPN terms are subsequently added "by hand" to the *full* general relativistic TOV equations, hence producing a set of parametrized post-TOV equations (cf. Ref. [50] for a similar "post-Einsteinian" parametrization in the context of gravitational radiation from binary systems). The formalism introduces a new set of 2PN parameters that are presently unconstrained by weak-field experiments, and that encompass the dominant corrections to the bulk properties of neutron stars in GR in a wide class of modified gravity theories.

A. Executive summary

Since this paper is rather technical, we summarize our main conclusions here. The core of our proposal is to use the following set of "post-TOV" equations of structure for spherically symmetric stars (from now on we use geometrical units $G = c = 1$):

$$\frac{dp}{dr} = \left(\frac{dp}{dr}\right)_{\rm GR} - \frac{\rho m}{r^2}(\mathcal{P}_1 + \mathcal{P}_2), \tag{1a}$$

$$\frac{dm}{dr} = \left(\frac{dm}{dr}\right)_{\rm GR} + 4\pi r^2 \rho(\mathcal{M}_1 + \mathcal{M}_2), \tag{1b}$$

where

$$\mathcal{P}_1 \equiv \delta_1 \frac{m}{r} + 4\pi\delta_2 \frac{r^3 p}{m}, \tag{2a}$$

$$\mathcal{M}_1 \equiv \delta_3 \frac{m}{r} + \delta_4 \Pi, \tag{2b}$$

$$\mathcal{P}_2 \equiv \pi_1 \frac{m^3}{r^5 \rho} + \pi_2 \frac{m^2}{r^2} + \pi_3 r^2 p + \pi_4 \frac{\Pi p}{\rho}, \tag{2c}$$

$$\mathcal{M}_2 \equiv \mu_1 \frac{m^3}{r^5 \rho} + \mu_2 \frac{m^2}{r^2} + \mu_3 r^2 p + \mu_4 \frac{\Pi p}{\rho} + \mu_5 \Pi^3 \frac{r}{m}. \tag{2d}$$

Here r is the circumferential radius, m is the mass function, p is the fluid pressure, ρ is the baryonic rest mass density, ϵ is the total energy density, and $\Pi \equiv (\epsilon - \rho)/\rho$ is the internal energy per unit baryonic mass. A "GR" subscript denotes the standard TOV equations in GR [cf. Eq. (7) below, where we appended a subscript "T" to the mass function for reasons that will become apparent later]; δ_i, π_i $(i = 1, \ldots, 4)$ and μ_i $(i = 1, \ldots, 5)$ are phenomenological post-TOV parameters. The GR limit of the formalism corresponds to setting all of these parameters to zero, i.e., $\delta_i, \pi_i, \mu_i \to 0$.

The dimensionless combinations $\mathcal{P}_1, \mathcal{M}_1$ and $\mathcal{P}_2, \mathcal{M}_2$ represent a parametrized departure from the GR stellar structure and are linear combinations of 1PN- and 2PN-order terms, respectively. In particular, the coefficients δ_i attached to the 1PN terms are simple algebraic combinations of the traditional PPN parameters: see Eqs. (35) and (36) below. As such, they are constrained to be very close to zero by existing Solar System and binary pulsar observations[1]: $|\delta_i| \ll 1$. This result translates to negligibly small 1PN terms in Eq. (1): $\mathcal{P}_1, \mathcal{M}_1 \ll 1$. On the other hand, π_i and μ_i are presently unconstrained, and consequently $\mathcal{P}_2, \mathcal{M}_2$ should be viewed as describing the dominant (significant) departure from GR.

Each of the two combinations $\mathcal{P}_2$ and $\mathcal{M}_2$ involves no more than five dimensionless 2PN terms, but as we show in Sec. III B these terms are representative of five distinct "families" consisting of a large number of 2PN terms. Each family is defined by the property that all of its members lead to approximately self-similar changes in the stellar mass-radius curves when included in $\mathcal{P}_2, \mathcal{M}_2$. In other words, as we verified by numerical calculations, we can account for several terms belonging to the same family by taking just one term from that family (either the dominant one or, when convenient, a much simpler subdominant one) and varying the corresponding post-TOV coefficient π_i or μ_i.

The qualitative effect of each of the 2PN-order post-TOV terms on the mass-radius relation is illustrated in Fig. 2. The values of the π_i and μ_i coefficients in each panel of this figure were chosen for purely illustrative purposes, i.e., we chose these coefficients to be large enough that they can produce visible deviations on the scale of the plot. A first noteworthy feature is that pressure terms *typically* induce corrections that are about an order of magnitude smaller than mass terms.[2] This can be seen by the larger range of π_i's needed to produce visible changes in the mass-radius curve ($|\pi_2| \leq 4$, $|\pi_3| \leq 100$ and $|\pi_4| \leq 10$) when compared to the corresponding corrections in the mass-function equation ($|\mu_2| \leq 1$, $|\mu_3| \leq 1$ and $|\mu_4| \leq 1.5$, respectively). Some terms (such as those proportional to $\pi_2, \pi_3, \pi_4, \mu_3$ and μ_5) induce smooth rigid shifts of the mass-radius curve, similar to those that would be produced by a softening or stiffening of the nuclear EOS. Other terms (like those proportional to μ_1, μ_2 and μ_4) produce more peculiar features that are more or less localized in a finite range of central densities. This is interesting, because (for example) it is plausible to conjecture that some combination of the μ_1 and μ_2 corrections may reproduce the qualitative features of a highly nonperturbative phenomenon such as spontaneous scalarization, despite the intrinsically perturbative nature of our formalism.

The punch line here is that each post-TOV correction is qualitatively different, so we can use the post-TOV formalism as a toolbox to reproduce the mass-radius curves shown in Fig. 1 for various modified theories of gravity. More ambitiously, it would be interesting to address the inverse problem, i.e., to find out how the post-TOV parameters are related to the dominant corrections induced by each different theory. These issues are beyond the scope of this paper, but they are obviously crucial to relate our formalism to experiments, and we plan to address them in future work.

The second main result of this paper has to do with the "completeness" of our post-TOV formalism, in the sense that the stellar structure (1)—if we neglect the small terms $\mathcal{P}_1, \mathcal{M}_1$—can be formally derived by a covariantly conserved perfect fluid stress-energy tensor. That is,

$$\nabla_\nu T^{\mu\nu} = 0, \qquad T^{\mu\nu} = (\epsilon_{\rm eff} + p) u^\mu u^\nu + p g^{\mu\nu}, \quad (3)$$

where the effective, gravity-modified energy density is

$$\epsilon_{\rm eff} = \epsilon + \rho \mathcal{M}_2, \quad (4)$$

and the covariant derivative is compatible with the effective post-TOV metric

$$g_{\mu\nu} = {\rm diag}[-e^{\nu(r)}, (1 - 2m(r)/r)^{-1}, r^2, r^2 \sin^2\theta], \quad (5)$$

with

$$\frac{d\nu}{dr} = \frac{2}{r^2}\left[(1 - \mathcal{M}_2)\frac{m + 4\pi r^3 p}{1 - 2m/r} + m\mathcal{P}_2\right]. \quad (6)$$

Our phenomenological post-TOV formalism is expected to encompass a large number of alternative theories of gravity, but it is not completely general, and future extensions may be possible or even desirable. As we stated earlier, theories which produce nonperturbative phase transitions in their stellar structure equations may not be accurately modeled. The formalism is also limited by the choice of acceptable 2PN terms out of all dimensionally possible combinations, based on criteria that have bearing on the structure of the gravitational field equations (see Sec. III B below).

B. Plan of the paper

The plan of the paper is as follows. In Sec. II we introduce the PPN formalism and review previous applications to relativistic stars (in particular work by Wagoner and Malone [46] as well as Ciufolini and Ruffini [47]). In Sec. III we develop the post-TOV formalism to 1PN order

[1]Using the latest constraints on the PPN parameters [49] we obtain the following upper limits: $|\delta_1| \lesssim 6 \times 10^{-4}$, $|\delta_2| \lesssim 7 \times 10^{-3}$, $|\delta_3| \lesssim 7 \times 10^{-3}$, $|\delta_4| \lesssim 10^{-8}$.

[2]A notable exception to this rule is the π_1 term, for reasons that will be explained in Sec. IV below.

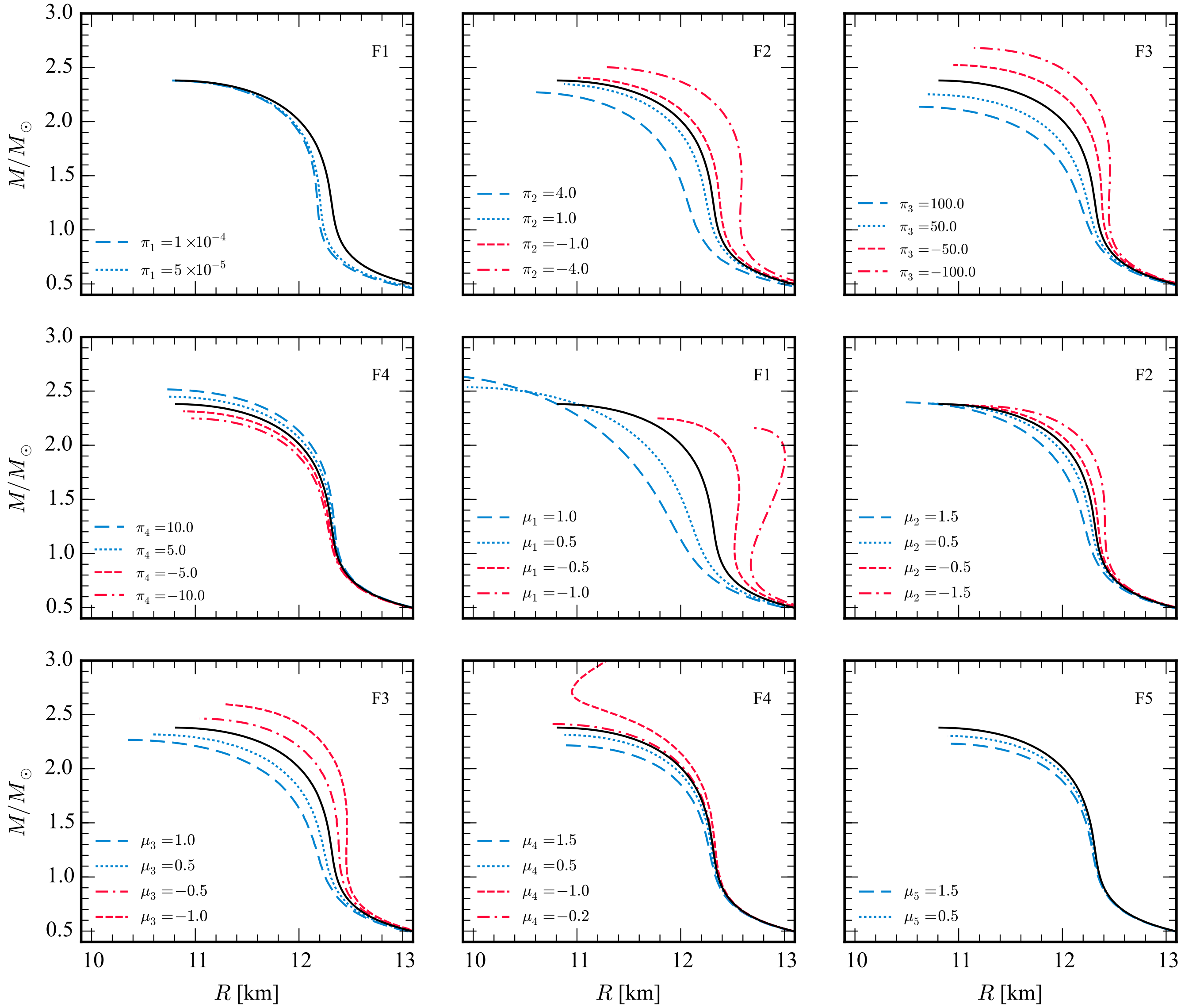

FIG. 2 (color online). *2PN-order post-TOV corrections on the mass-radius curves.* We show the modification induced by different families of post-TOV terms on the general relativistic mass-radius curve, assuming the APR EOS. Left to right and top to bottom, the different panels show the effect of the pressure terms, proportional to π_i $(i = 1, \ldots, 4)$, and of the mass terms, proportional to μ_i $(i = 1, \ldots, 5)$.

(where all parameters are already constrained to be very close to their GR values by Solar System and binary pulsar experiments), and then to 2PN order. We also show the equivalence between the 2PN post-TOV equations and GR with a gravity-modified EOS under a minimal set of reasonable assumptions. In Sec. IV we present some numerical results illustrating the relative importance of the different post-TOV corrections. Some technical material is collected in three appendices. Appendix A gives details of the dimensional analysis arguments used to select the relevant set of 2PN post-TOV coefficients. In Appendix B we present a brief summary of the relativistic Lane-Emden theory, which plays an auxiliary role in the construction of our formalism. Finally, Appendix C shows that certain integral potentials appearing at 1PN order in the stellar structure equations (namely, the gravitational potential U, the internal energy E, and the gravitational potential energy Ω) can be approximated by linear combinations of nonintegral potentials, so these integral potentials are "redundant" and can be discarded when building our post-TOV expansion.

II. SETTING THE STAGE: STELLAR STRUCTURE WITHIN PPN THEORY

A. The TOV equations

A convenient starting point for our analysis is the standard general relativistic TOV equations, describing hydrostatic equilibrium in spherical symmetry [34]. These are given by the familiar formulas

$$\left(\frac{dp}{dr}\right)_{\rm GR} = -\frac{(\epsilon + p)}{r^2}\frac{(m_{\rm T} + 4\pi r^3 p)}{(1 - 2m_{\rm T}/r)}, \tag{7a}$$

$$\left(\frac{dm_{\rm T}}{dr}\right)_{\rm GR} = 4\pi r^2 \epsilon, \tag{7b}$$

where p and ϵ are the fluid's pressure and energy density, respectively, and m_T is the mass function (the subscript is used to distinguish this mass function from similar quantities appearing in PPN theory; see below).

For later convenience we also write down the 1PN-order expansion of these equations (for simplicity the subscript "GR" is omitted):

$$\frac{dp}{dr} = -\frac{m_{\rm T}\rho}{r^2}\left(1 + \Pi + \frac{p}{\rho} + \frac{2m_{\rm T}}{r} + 4\pi\frac{r^3 p}{m_{\rm T}}\right) + \mathcal{O}(2{\rm PN}), \tag{8a}$$

$$\frac{dm_{\rm T}}{dr} = 4\pi r^2 \rho(1 + \Pi), \tag{8b}$$

where we have introduced the baryonic rest-mass density ρ and the dimensionless internal energy per unit mass, $\Pi \equiv (\epsilon - \rho)/\rho$. It can be noticed that the mass function equation only contains 1PN corrections to the Newtonian equations of hydrostatic equilibrium, while higher-order corrections appear in the pressure equation.

B. The PPN stellar structure equations

The PPN formalism [35,36] was first employed for building static, spherically symmetric models of compact stars by Wagoner and Malone [46], and subsequently by Ciufolini and Ruffini [47]. This early work is briefly reviewed here since it will provide the stepping stone towards formulating our post-TOV equations.

A convenient starting point is the set of stellar structure equations derived in Ref. [47] from the original Will-Nordtvedt PPN theory [35,36]. These are [cf. Eqs. (14) of Ref. [47]]

$$\frac{dp}{dr} = -\frac{\epsilon\bar{m}}{r^2}\left[1 + (5 + 3\gamma - 6\beta + \zeta_2)\frac{\bar{m}}{r} + \frac{p}{\epsilon} + \zeta_3\frac{E}{\bar{m}} + (\gamma + \zeta_4)\frac{4\pi r^3 p}{\bar{m}} + \frac{1}{2}(11 + \gamma - 12\beta + \zeta_2 - 2\zeta_4)\frac{\Omega}{\bar{m}}\right], \tag{9a}$$

$$\frac{d\bar{m}}{dr} = 4\pi r^2 \epsilon, \tag{9b}$$

where we have adopted the standard notation for the nine PPN parameters, $\{\beta, \gamma, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \alpha_1, \alpha_2, \alpha_3\}$. In the GR limit $\beta = \gamma = 1$ and $\zeta_i = \alpha_i = 0$ $(i = 1, \ldots, 4)$ [49].

It should be pointed out that the basic parameters $p, \bar{m}$ (as well as the radial coordinate r) entering Eqs. (9a) and (9b) may not be the same as the corresponding ones in the TOV equations. This is a reflection of the "gauge" freedom in defining these parameters in a number of equivalent ways. Indeed, below we are going to exploit this freedom and obtain an "improved" set of PPN equations by a suitable redefinition of the mass function. On the other hand, following Ref. [47], we will stick to the same p and r throughout this analysis, implicitly assuming that they are the *same* variables as the ones in the TOV equations (7).

The potentials Ω and E appearing in Eq. (9a) obey

$$\frac{d\Omega}{dr} = -4\pi r\rho\bar{m}, \qquad \frac{dE}{dr} = 4\pi r^2\rho\Pi. \tag{10}$$

The more familiar Newtonian gravitational potential U (the solution of $\nabla^2 U = -4\pi\rho$) is not featured in Eqs. (9a) and (9b) as a result of a change of radial coordinate and a redefinition of the mass function $\bar{m}$ with respect to the original PPN theory parameters (see Ref. [47] for details).

The stellar structure equations can be manipulated further by switching to a new mass function:

$$m(r) = \bar{m} + AE + B\Omega + C\frac{\bar{m}^2}{r} + D(4\pi r^3 p), \tag{11}$$

where A, B, C, and D are free constants. As evident, $\bar{m}$ and m differ at 1PN level. The constants A and B can be chosen so that the terms proportional to E and Ω in Eq. (9a) are eliminated. This is achieved for

$$A = \zeta_3, \qquad B = \frac{1}{2}(11 + \gamma - 12\beta + \zeta_2 - 2\zeta_4). \tag{12}$$

The resulting "new" set of PPN stellar structure equations is

$$\frac{dp}{dr} = -\frac{\rho m}{r^2}\left[1 + \Pi + \frac{p}{\rho} + (5 + 3\gamma - 6\beta + \zeta_2 - C)\frac{m}{r} + (\gamma + \zeta_4 - D)4\pi\frac{r^3 p}{m}\right], \tag{13a}$$

$$\frac{dm}{dr} = 4\pi r^2\rho\left[1 + (1 + \zeta_3)\Pi + 3D\frac{p}{\rho} - \frac{C}{4\pi}\frac{m^2}{\rho r^4} - \frac{1}{2}(11 + \gamma - 12\beta + \zeta_2 - 2\zeta_4 - 4C + 2D)\frac{m}{r}\right]. \tag{13b}$$

These expressions still contain the gauge freedom associated with the definition of the mass function m in the form of the yet unspecified constants C and D. In particular, the Wagoner-Malone hydrostatic equilibrium equations [46] represent a special case of these expressions, and it is straightforward to see that they can be recovered for

$$D = \gamma + \zeta_4, \qquad C = \frac{1}{2}(7 + 3\gamma - 8\beta + \zeta_2). \tag{14}$$

Making this choice for the constants on the right-hand side of Eq. (11) leads to a new mass function, say $\tilde{m}$, and to the following structure equations, which match Eqs. (6) and (7) of Ref. [46]:

$$\frac{dp}{dr} = -\frac{\rho\tilde{m}}{r^2}\left(1 + \Pi + \frac{p}{\rho} + a\frac{\tilde{m}}{r}\right), \tag{15a}$$

$$\frac{d\tilde{m}}{dr} = 4\pi r^2\rho\left[1 + (1+\zeta_3)\Pi + a\frac{\tilde{m}}{r} + 3(\gamma+\zeta_4)\frac{p}{\rho} - \frac{b}{4\pi}\frac{\tilde{m}^2}{\rho r^4}\right], \tag{15b}$$

where $a \equiv (3 + 3\gamma - 4\beta + \zeta_2)/2$ and the constant b in the notation of Ref. [46] is our C, i.e., $b = (7 + 3\gamma - 8\beta + \zeta_2)/2$.

A comparison between the two sets of PPN equations (9) and (15) discussed in this section reveals that the Wagoner-Malone equations are simpler, in the sense that they do not depend on the auxiliary potentials Ω and E. This advantage, however, is partially offset by the more complicated expression for the mass function equation. If we compare the GR limit of the Wagoner-Malone equations (15) against the 1PN expansion of the TOV equations [Eqs. (8a) and (8b)], we find that the two sets coincide provided we identify $\bar{m} = m_{\rm T}$, i.e.,

$$\tilde{m} = m_{\rm T} + \frac{m_{\rm T}^2}{r} + 4\pi r^3 p, \tag{16}$$

where the last equation follows by taking the GR limit of Eq. (11) in combination with Eqs. (12) and (14). Clearly, the fact that $\tilde{m} \neq m_{\rm T}$ in the GR limit is an unsatisfactory property of the Wagoner-Malone equations.

It would be desirable to have a set of structure equations that [unlike the set (9)] does not involve integral potentials, and such that [unlike the set (15)] the mass function is compatible with the GR limit. Fortunately, it is not too difficult to find a new set of PPN equations for which $m = m_{\rm T}$. In the following section we will propose an improved set of PPN stellar structure equations that satisfies these requirements.

C. An improved set of PPN equations

We can exploit the degree of freedom associated with the constants C, D in Eqs. (13a) and (13b) and produce a new set of PPN equations that exactly match the 1PN TOV equations in the GR limit with $m = m_{\rm T}$. It is easy to see that this can be achieved by making the trivial choice

$$C = D = 0. \tag{17}$$

Note that the constants A and B are still given by Eq. (12). The resulting PPN equations are

$$\frac{dp}{dr} = -\frac{\rho m}{r^2}\left[1 + \Pi + \frac{p}{\rho} + (5 + 3\gamma - 6\beta + \zeta_2)\frac{m}{r} + (\gamma+\zeta_4)4\pi\frac{r^3 p}{m}\right], \tag{18a}$$

$$\frac{dm}{dr} = 4\pi r^2\rho\left[1 + (1+\zeta_3)\Pi - \frac{1}{2}(11 + \gamma - 12\beta + \zeta_2 - 2\zeta_4)\frac{m}{r}\right]. \tag{18b}$$

As advertised, in the GR limit these equations reduce to Eqs. (8a) and (8b) with $m = m_{\rm T}$. The same equations will be used in Sec. III below in the construction of the desired post-TOV equations.

D. The physical interpretation of the mass function

Within the framework of PPN theory, inertial mass and active/passive gravitational mass are, in general, distinct notions. In the context of compact stars, expressions for all three kinds of mass were given in Ref. [47]:

$$M_{\rm in} = \bar{m}(\bar{R}) + \left(\frac{17}{2} + \frac{3}{2}\gamma - 10\beta + \frac{5}{2}\zeta_2\right)\Omega(\bar{R}), \tag{19}$$

$$M_{\rm a} = M_{\rm in} + \left(4\beta - \gamma - 3 - \frac{1}{2}\alpha_3 - \frac{1}{3}\zeta_1 - 2\zeta_2\right)\Omega(\bar{R}) + \zeta_3 E(\bar{R}) - \left(\frac{3}{2}\alpha_3 - 3\zeta_4 + \zeta_1\right)P, \tag{20}$$

$$M_{\rm p} = M_{\rm in} + \left(4\beta - \gamma - 3 - \alpha_1 + \frac{2}{3}\alpha_2 - \frac{2}{3}\zeta_1 - \frac{1}{3}\zeta_2\right) \times \Omega(\bar{R}), \tag{21}$$

where $\bar{R}$ is the stellar radius associated with the mass function $\bar{m}(r)$ [i.e., with the set of equations (9)] and

$$P = 4\pi\int_0^{\bar{R}} dr r^2 p \tag{22}$$

is the volume-integrated pressure.

In GR the three masses are of course identical, $M_{\rm in} = M_{\rm a} = M_{\rm p}$. As argued in Ref. [47], any theory conserving the four-momentum of an isolated system should incorporate the equality of the two gravitational masses, i.e., $M_{\rm a} = M_{\rm p}$. If adopted, this equality leads to following three algebraic relations for the PPN parameters:

$$\zeta_3 = 0, \tag{23}$$

$$\zeta_1 - 3\zeta_4 + \frac{3}{2}\alpha_3 = 0, \tag{24}$$

$$\zeta_1 + 3\alpha_1 - 2\alpha_2 - 5\zeta_2 - \frac{3}{2}\alpha_3 = 0. \tag{25}$$

We can subsequently write for the common gravitational mass

$$M_{\rm g} = M_{\rm a} = M_{\rm p} = \bar{m}(\bar{R}) + F\Omega(\bar{R}), \tag{26}$$

with

$$F = \frac{1}{2}\left(11 + \gamma - 12\beta - \alpha_3 + \zeta_2 - \frac{2}{3}\zeta_1\right). \tag{27}$$

For our new PPN equations with $C = D = 0$ the mass equality $M_{\rm a} = M_{\rm p}$ implies

$$m(r) = \bar{m}(r) + \frac{1}{2}(11 + \gamma - 12\beta + \zeta_2 - 2\zeta_4)\Omega(r). \tag{28}$$

Then with the help of Eq. (24) it is easy to see that

$$M_{\rm g} = m(\bar{R}) + \left(\zeta_4 - \frac{1}{2}\alpha_3 - \frac{1}{3}\zeta_1\right)\Omega(\bar{R}) = m(\bar{R}). \tag{29}$$

If R is the stellar radius associated with our PPN equations (18), the difference $\delta R = R - \bar{R}$ is a 1PN-order quantity. We can then approximately write

$$m(\bar{R}) \approx m(R) - \frac{dm}{dr}(R)\delta R. \tag{30}$$

However, Eq. (18b) implies that $dm/dr(R) = 0$ if $\rho(R) = 0$ at the stellar surface. This is indeed the case for a realistic EOS. Therefore, we have shown that at 1PN precision the mass of the system is given by

$$M_{\rm g} = m(R). \tag{31}$$

This elegant result is one more attractive property of the new PPN equations.

III. THE POST-TOV FORMALISM

The logic underpinning the formalism we are seeking is that of parametrizing the deviation of the stellar structure equations from their GR counterparts, thus producing a set of post-TOV equations. As already pointed out in the introduction, the post-TOV formalism is merely a useful parametrized framework rather than the product of a specific, self-consistent modified gravity theory (in the spirit of PPN theory). In this sense our formalism is akin to the existing "quasi-Kerr" or "bumpy" Kerr metrics, designed to study deviations from the Kerr spacetime in GR (see, e.g., Refs. [51–53]).

By design the post-TOV formalism should be a more powerful tool for building relativistic stars than the PPN framework; after all, the latter is based on a PN approximation of strong gravity. However (as will become clear from the analysis of this section), our formalism has its own limitations, the most important one being the fact that the deviations from GR are introduced in the form of PN corrections. This could mean that the structure of compact stars with a high degree of departure from GR may not be accurately captured by the formalism.

A. Post-TOV equations: 1PN order

The recipe for formulating leading-order post-TOV equations is rather simple: from a suitable set of PPN hydrostatic equilibrium equations we isolate the purely non-GR pieces. These 1PN terms are subsequently added "by hand" to the full general relativistic TOV equations, hence producing a set of parametrized post-Einsteinian equations. It should be pointed out that this procedure can only be applied at the level of 1PN corrections. Higher-order corrections should by sought by other means, such as dimensional analysis (see Sec. III B).

In principle, either set of equations—Eqs. (9a) and (9b) [47] or Eqs. (15a) and (15b) [46]—could have been used. However, our improved PPN equations (18) seem to be best suited for this task.

Considering Eqs. (18a) and (18b), we first isolate the terms that represent a genuine deviation from GR. These are the terms in the second line in the following equations:

$$\frac{dp}{dr} = -\frac{\rho m}{r^2}\left(1 + \Pi + \frac{p}{\rho} + \frac{2m}{r} + 4\pi\frac{r^3 p}{m}\right) - \frac{\rho m}{r^2}\left[(3 + 3\gamma - 6\beta + \zeta_2)\frac{m}{r} + (\gamma - 1 + \zeta_4)4\pi\frac{r^3 p}{m}\right], \tag{32a}$$

$$\frac{dm}{dr} = 4\pi r^2\rho(1 + \Pi) + 4\pi r^2\rho\left[\zeta_3\Pi - \frac{1}{2}(11 + \gamma - 12\beta + \zeta_2 - 2\zeta_4)\frac{m}{r}\right]. \tag{32b}$$

The second step consists of adding the non-GR terms to the TOV equations (7). We obtain (recall that $m = m_{\rm T}$)

$$\frac{dp}{dr} = -\frac{(\epsilon + p)}{r^2}\left(\frac{m + 4\pi r^3 p}{1 - 2m/r}\right) - \frac{\rho m}{r^2}\left(\delta_1\frac{m}{r} + \delta_2 4\pi\frac{r^3 p}{m}\right), \tag{33}$$

$$\frac{dm}{dr} = 4\pi r^2\left[\epsilon + \rho\left(\delta_3\frac{m}{r} + \delta_4\Pi\right)\right], \tag{34}$$

where we have introduced the constant post-TOV parameters:

$$\delta_1 \equiv 3(1 + \gamma) - 6\beta + \zeta_2, \qquad \delta_2 \equiv \gamma - 1 + \zeta_4, \tag{35}$$

$$\delta_3 \equiv -\frac{1}{2}(11 + \gamma - 12\beta + \zeta_2 - 2\zeta_4), \qquad \delta_4 \equiv \zeta_3. \tag{36}$$

As expected, $\delta_i = 0$ in the limit of GR.

The above equations can be written in a more compact form:

$$\frac{dp}{dr} = \left(\frac{dp}{dr}\right)_{\rm GR} - \frac{\rho m}{r^2}\left(\delta_1 \frac{m}{r} + \delta_2 4\pi \frac{r^3 p}{m}\right), \tag{37a}$$

$$\frac{dm}{dr} = \left(\frac{dm}{dr}\right)_{\rm GR} + 4\pi r^2 \rho \left(\delta_3 \frac{m}{r} + \delta_4 \Pi\right). \tag{37b}$$

These expressions represent our main result for the *leading-order* post-TOV stellar structure equations. They describe the 1PN-level corrections produced by an arbitrary deviation from GR that is compatible with PPN theory. In other words, Eqs. (37a) and (37b) encapsulate the stellar structure physics (at this order) for any member of the PPN family of gravity theories.

We could in principle introduce other 1PN order terms, in the spirit of the general parametrized framework of deviating from GR that we have described in the beginning of this section. But the introduction of such terms would correspond to either redefinitions of coordinates and/or the mass function at the 1PN level, as we have already seen, or deviations from special relativity, which we would prefer not to include.

Unfortunately, it turns out that Eqs. (37a) and (37b) are of limited practical value. As discussed in the executive summary, the modern limits on the PPN parameters suggest that these corrections are very close to their GR values, because $\beta, \gamma \approx 1$ and $\alpha_i, \zeta_i \ll 1$, making all the δ_i parameters very small. We should not therefore expect any notable deviation from GR at the level of the leading-order post-TOV equations. We verified this claim by explicit calculations of neutron star stellar models with different EOSs.

Any significant deviations from compact stars in GR have to be sought at 2PN order and beyond, where the existing observational limits leave much room for the practitioner of alternative theories of gravity. This calls for the formulation of a higher-order set of post-TOV equations, a task to which we now turn.

B. Post-TOV equations: 2PN order

In this section we shall formulate post-TOV equations with 2PN-accurate correction terms. Unlike the calculation of the preceding section, we now have to build these equations "from scratch," given that the general PPN theory has not yet been extended to 2PN order. Inevitably, the procedure for building the various 2PN terms will turn out to be somewhat more complicated than that of the preceding section, heavily relying on dimensional analysis for constructing these terms out of the available fluid parameters. Moreover, at 2PN order we also need to consider terms that involve the integral potentials U, E, and Ω (recall that these were eliminated at 1PN order by a suitable redefinition of the mass function). However, as shown numerically and via analytical arguments in Appendix C, the integral potentials can be approximated to a high precision, and for a variety of EOSs, by simple linear combinations of the nonintegral PN terms. As a result, they do not have to be considered separately in the post-TOV expansion.

To begin with, we can get an idea of the form of some of the 2PN terms we are looking for by expanding the TOV equations (7) to that order. Let us first consider the pressure equation (7a):

$$\frac{dp}{dr} = -\frac{\rho m}{r^2}\left[\left(1+\Pi+\frac{p}{\rho}\right)\left(1+\frac{2m}{r}+4\pi\frac{r^3 p}{m}\right) + \frac{4m^2}{r^2} + 8\pi r^2 p\right] + \mathcal{O}(3\mathrm{PN}). \tag{38}$$

As anticipated, all 1PN terms appearing here are also present in our PPN equation (18a). The produced 2PN corrections are proportional to the following combinations:

$$\frac{m^2}{r^2}, \quad \Pi\frac{m}{r}, \quad r^2 p, \quad \frac{mp}{r\rho}, \quad \Pi\frac{r^3 p}{m}, \quad \frac{r^3 p^2}{\rho m}. \tag{39}$$

Additional 2PN terms that do not appear in the TOV equations can be constructed by forming products of the available 1PN terms. The largest set of 1PN terms can be found in the general PPN equations (13a) and (13b):

$$1\mathrm{PN}\colon\ \Pi, \frac{p}{\rho}, \frac{m}{r}, \frac{r^3 p}{m}, \frac{m^2}{\rho r^4}. \tag{40}$$

We can observe that all terms, except the last one, also appear in our final PPN equations (18a) and (18b). From these we can reproduce the set (39) as well as the additional 2PN terms:

$$\frac{r^6 p^2}{m^2}, \Pi\frac{m^2}{\rho r^4}, \frac{m^3}{\rho r^5}, \Pi^2, \Pi\frac{p}{\rho}, \underbrace{\frac{p^2}{\rho^2}, \frac{m^4}{\rho^2 r^8}, \frac{m^2 p}{\rho^2 r^4}}. \tag{41}$$

We have set apart the last three (underbraced) terms of this set because, as a result of their $\sim 1/\rho^2$ scaling, these terms will be discarded. In fact, the same fate will be shared by any term $\sim\rho^\beta$ with $\beta \le -2$.

There are various reasons why we believe that this selection rule should be imposed. In our opinion these reasons are quite convincing, but they fall short of constituting a watertight argument: in all fairness, if we had a single truly compelling reason, we would not need more than one.

The first line of reasoning to exclude the presence of negative powers of ρ (and of the other fluid parameters) in the PN terms is based on the regularity of these terms at the stellar surface, where $p, \rho, \Pi \to 0$ for any realistic EOS. A PN term like the second one in the underbraced group of the set (41) will lead to a term diverging as $\sim 1/\rho$ at the stellar surface in the stellar structure equations, and therefore it is not an acceptable PN correction. Although this surface regularity argument is powerful, it obviously works only for terms that do not scale with positive powers of p or Π.

The second (heuristic) argument applies to gravity theories with the following (symbolic) structure:

$$\{\text{geometry}\} = 8\pi T^{\mu\nu}, \tag{42}$$

$$\nabla_\nu T^{\mu\nu} = 0 \to \frac{dp}{dr} = (\epsilon + p)\{\text{geometry}\}, \tag{43}$$

where "geometry" stands for combinations of the metric and its derivatives, and the last equation assumes a perfect fluid stress-energy tensor. The stress-energy tensor and the right-hand side of Eq. (43) feature $\epsilon + p = \rho(1 + \Pi + p/\rho)$ and p *linearly*. It can then be argued that the solution of the field equations for the metric and its derivatives will display a

$$\{\text{geometry}\} \sim (\epsilon + \tau p)^n \sim \rho^n \left(1 + \Pi + \tau \frac{p}{\rho}\right)^n \tag{44}$$

dependence with respect to the fluid variables [where τ and n are $\mathcal{O}(1)$ numbers]. Such a solution should lead to pressure-dependent PN terms of the form

$$\text{PN term} \sim (r^2\rho)^{n-1} \left(\frac{p}{\rho}\right)^k, \quad k = n, n-1, \ldots, \tag{45}$$

where one ρ factor has been removed and absorbed in the Newtonian prefactor of the structure equations, while at the same time the r^2 factor has been added in order to produce a dimensionless quantity. A key observation is that the form (45) assumes a theory that does *not* depend on *dimensional* coupling constants. Now, according to Eq. (45) the highest negative power of ρ corresponds to $k = n$, which means that the scaling with respect to the density should be

$$\text{PN term} \sim \rho^\beta, \qquad \beta \geq -1. \tag{46}$$

Based on these arguments, we deem acceptable those PN terms which scale with ρ as in Eq. (46). This choice is also consistent with the previous PPN formulas; see Eqs. (13a) and (13b). A similar argument can be used to exclude terms with negative powers of p and Π.[3]

Equation (39) and the top row of Eq. (41) represent a large set of 2PN terms emerging from the expansion of the TOV equation and from products of the various known 1PN terms. This set is large but not necessarily complete. Inevitably, a systematic approach to the problem of "guessing" 2PN terms should involve dimensional analysis. To improve readability we relegate our dimensional analysis considerations to Appendix A, and here we only quote the main result. The *most general* form for 2PN-order terms is given by the dimensionless combination

$$\Lambda_2 \sim \Pi^\theta (r^2 p)^\alpha (r^2\rho)^\beta \left(\frac{m}{r}\right)^{2-2\alpha-\beta-\theta}, \tag{47}$$

where α, β, θ are integers with

$$\beta \geq -1, \tag{48}$$

while different bounds on θ and α apply to the two hydrostatic equilibrium equations:

$$\frac{dp}{dr}: \; 0 \leq \theta \leq 2, \qquad 0 \leq \alpha \leq 2 - \theta, \tag{49}$$

$$\frac{dm}{dr}: \; 0 \leq \theta \leq 3, \qquad 0 \leq \alpha \leq 3 - \theta. \tag{50}$$

The lower bounds on the three parameters α, β, θ are dictated by the same considerations discussed below Eq. (41), namely, regularity at the surface and consistency with the fact that gravitational field equations of the general form (43) are unlikely to generate negative powers higher than $1/\rho$. The upper bounds on α and θ are imposed by the regularity at $r = 0$ of the stellar structure terms arising from Λ_2 (see Appendix A).

From the general expression (47) we can reproduce all previous 1PN and 2PN terms and generate an infinite number of new ones. This possibility could have been a fatal blow to our post-TOV program. Fortunately, the day is saved by the fact that the magnitude of Λ_2 decays rapidly throughout the star as β increases. This trend is clearly visible in the numerical results shown in Fig. 3 (see discussion below).

For all practical purposes these results imply that the first few members of the $\beta = -1, 0, 1, \ldots$ sequence are sufficient to construct accurate post-TOV expansions. A sample set of such dominant 2PN terms is

$$\begin{aligned} \text{2PN}: \; & \frac{m^3}{r^5\rho}, \frac{m^2}{r^2}, r\rho m, \frac{mp}{r\rho}, r^2 p, \frac{r^3p^2}{\rho m}, \frac{r^6p^2}{m^2}, \\ & \frac{r^7p^3}{\rho m^3} \frac{r^{10}p^3}{m^4}, \Pi\frac{m^2}{r^4\rho}, \Pi\frac{m}{r}, \Pi r^2\rho, \Pi\frac{p}{\rho}, \\ & \Pi\frac{r^3p}{m}, \Pi\frac{r^4p^2}{\rho m^2}, \Pi\frac{r^7p^2}{m^3}, \Pi^2\frac{m}{\rho r^3}, \Pi^2, \Pi^2\frac{rp}{m\rho}, \\ & \Pi^2\frac{r^4p}{m^2}, \frac{\Pi^3}{r^2\rho}, \Pi^3\frac{r}{m}. \end{aligned} \tag{51}$$

This set is markedly larger than the previous sets (39) and (41) (whose acceptable terms form a subset of the new set), but a complete post-TOV formalism would have to include all (or almost all) of these terms, with twice the

[3]A related argument for excluding high powers of $1/\rho$ is the following. By virtue of the field equations, the Ricci scalar is usually proportional to the energy density of matter (at least in the Newtonian limit, if the modified theory reproduces GR in the weak-field regime): $R \sim \rho$. If inverse powers of ρ are produced by gravity modifications, they should therefore originate from terms $\sim 1/R^n$ in the action of the theory. These terms are usually associated with ghosts or instabilities [54], and therefore their presence is problematic.

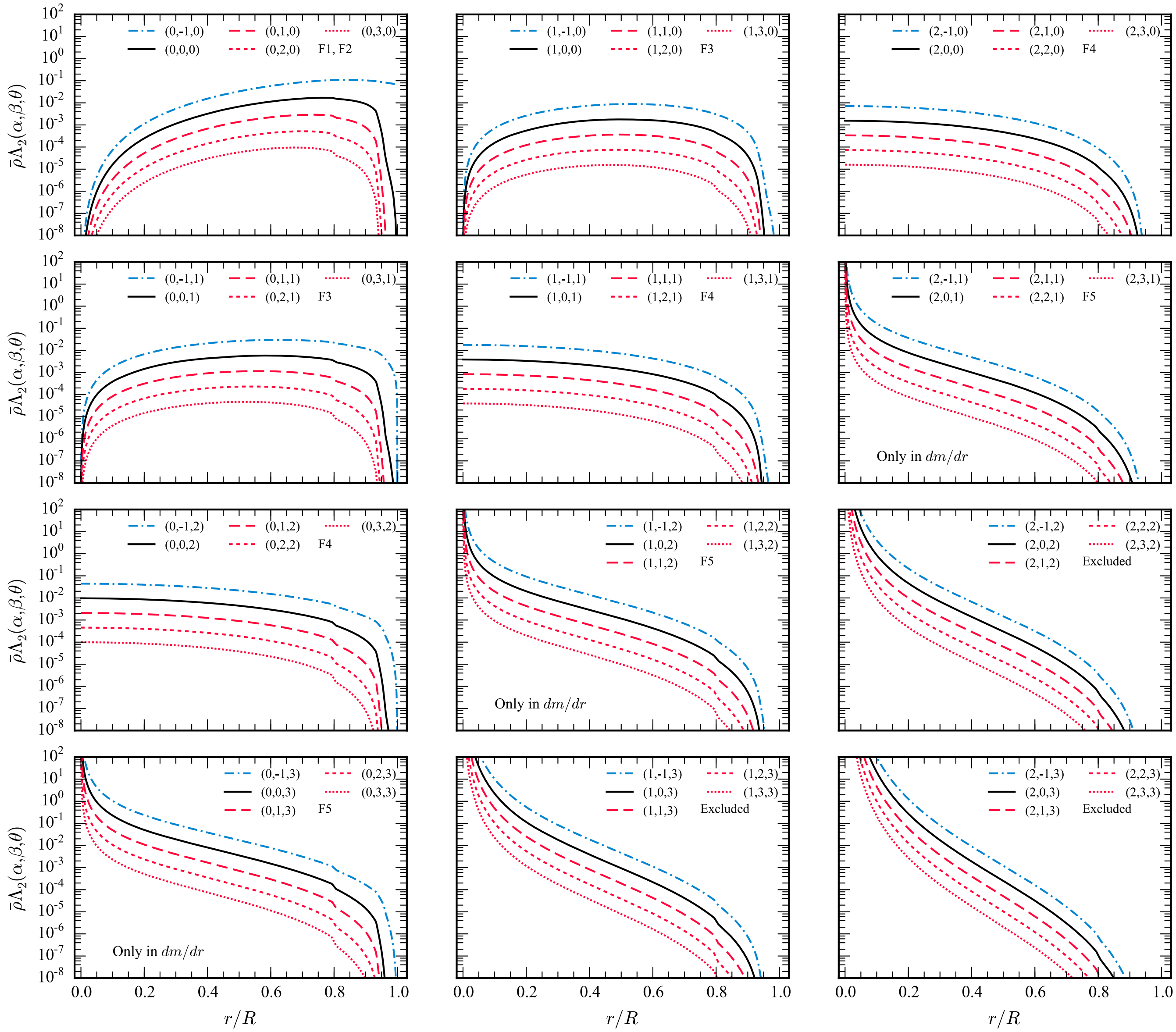

FIG. 3 (color online). *The radial profile of the* Λ_2 *function*. We exhibit the behavior of $\bar{\rho}\Lambda_2$, where $\bar{\rho} = \rho/\rho_c$ and Λ_2 is given in Eq. (47), for a stellar model using the APR EOS, with $\epsilon_c/c^2 = 0.86 \times 10^{15}$ g/cm^3, $M = 1.51 M_\odot$, and $R = 12.3$ km. The curves are labeled according to the respective values of (α, β, θ). From the top row to the bottom row the index θ takes on the values (0,1,2,3), respectively. Despite the multitude of possible dimensionally correct 2PN terms, their self-similarity—which is clear when we compare terms along the bottom-left to top-right diagonals in this "grid" of plots—allows us to group them into a relatively small number of families (see text for details). The contributions plotted in the three panels at the bottom right of the grid (marked as "Excluded") would lead to divergences in the hydrostatic equilibrium equations, and therefore they can be discarded as unphysical.

number of free coefficient in the dp/dr and dm/dr equations. Fortunately, as it turns out, the same job can be done with a much smaller subset of 2PN terms. This is possible because the various 2PN terms can be divided into *five "families*," each family comprising terms with similar profiles. When incorporated in the post-TOV equations, terms belonging to a given family lead to *self-similar* modifications in the mass-radius curves for a given EOS.

Insight into the behavior of the $\Lambda_2(\alpha, \beta, \theta)$ terms can be gained by direct numerical calculations of their radial profiles in relativistic stars. We carried out such calculations for a variety of realistic EOSs as well as relativistic polytropes, and for different choices of central density, verifying that all cases lead to very similar results, as discussed below. More specifically, we considered EOS A [55], FPS [56], SLy4 [57,58], and N [59] in increasing order of stiffness, as well as relativistic polytropes with indices $n = 0.4$, 0.6, and 1.0; see Appendix B, and in particular Eq. (B7). Note that the polytropic models are parametrized by $\lambda = p_c/\epsilon_c$ instead of ϵ_c alone (the subscript "c" indicates a quantity evaluated at the center), but this is equivalent to the central density parametrization. The polytropic models are also invariant with respect to the scale factor $K^{n/2}$; this can be adjusted to generate polytropic models of (say) the same mass (for a given λ) as that of a specific tabulated-EOS model.

Rather than computing Λ_2 itself, from a phenomenological point of view it makes more sense to consider the combination $\rho\Lambda_2$. The reason is that this combination appears in both the pressure and mass equations, and furthermore it has the desirable feature of being regular at the surface for $\beta = -1$. More specifically, in Fig. 3 we plot the dimensionless combination $\bar{\rho}\Lambda_2(\alpha,\beta,\theta)$, where $\bar{\rho} = \rho/\rho_c$. Our sample neutron star model was built using the APR EOS with central energy density $\epsilon_c/c^2 = 0.86\times 10^{15}$ g/cm^3, corresponding to mass $M = 1.51 M_\odot$ and radius $R = 12.3$ km, but we have verified that our qualitative conclusions remain the same for different models and different EOSs.

Figure 3 reveals two key trends: (i) the clear β ordering of the $\rho\Lambda_2$ profiles, with $\beta = -1$ always associated with the dominant term for fixed α and θ, and (ii) the remarkable similarity in the shape of the profiles of terms with dissimilar (α,β,θ) triads along the bottom-left to top-right diagonals in the "grid" of Fig. 3. This property defines distinct families of 2PN terms and implies that the terms of each family cause self-similar changes in the mass-radius curves of the various post-TOV stellar models.

We have identified five 2PN families (labeled "F1", ..., "F5" in the various panels of Fig. 3, and described in more detail in Table I):

(i) F1: This is a single-member family comprising only the $\rho\Lambda_2(0,-1,0)$ term in the top-left panel, which is zero at $r = 0$ but finite at $r = R$.

(ii) F2: The members of this family vanish at $r = 0$ and $r = R$, and have a peak near the surface. These are the $\rho\Lambda_2(0,\beta,0)$ terms with $\beta \geq 0$ in the top-left panel.

(iii) F3: These terms also vanish at both $r = 0$ and $r = R$, but display an approximately flat profile inside the star. They correspond to $\rho\Lambda_2(1,\beta,0)$ (top-middle panel) and $\rho\Lambda_2(0,\beta,1)$ (top-right panel) for $\beta \geq -1$.

(iv) F4: This family comprises terms that are finite at $r = 0$ but zero at $r = R$. These are the $\rho\Lambda_2(2,\beta,0)$ (bottom-left panel) and $\rho\Lambda_2(1,\beta,1)$ (bottom-middle panel) terms with $\beta \geq -1$.

(v) F5: These terms by themselves diverge at $r = 0$ and vanish at $r = R$, but they become well behaved when inserted into the stellar mass-function equation, where they are multiplied by the factor r^2: cf. Eq. (A24). These terms correspond to $\rho\Lambda_2(2,\beta,1)$, and from the constraints (49) and (50) we conclude that *the members of this family can only appear in the mass equation*.

TABLE I. *Taxonomy of the dominant 2PN terms.* The self-similarity between the radial profiles of the various 2PN terms listed in Eq. (51) (and illustrated in Fig. 3) allows us to group them into five distinct families. This table spells out the explicit form of the various terms, and indicates which term in each family is dominant (D) according to our numerical calculations, and which one was chosen (C) as a representative of each family.

Family	2PN term	(α,β,θ)	Dominant/Chosen?
F1	$m^3/(r^5\rho)$	$(0,-1,0)$	D/C
F2	$(m/r)^2$	(0,0,0)	D/C
F2	$rm\rho$	(0,1,0)	–
F3	$mp/(r\rho)$	$(1,-1,0)$	–
F3	$r^2 p$	(1,0,0)	–
F3	$\Pi m^2/(r^4\rho)$	$(0,-1,1)$	D/C
F3	$\Pi m/r$	(0,0,1)	–
F3	$r^2\Pi\rho$	(0,1,1)	–
F4	$r^3p^2/(\rho m)$	$(2,-1,0)$	–
F4	$r^6p^2/(m^2)$	(2,0,0)	–
F4	$\Pi p/\rho$	$(1,-1,1)$	C
F4	$\Pi r^3 p/m$	(1,0,1)	–
F4	$\Pi^2 m/(r^3\rho)$	$(0,-1,2)$	D
F4	Π^2	(0,0,2)	–
F5	$\Pi r^4p^2/(\rho m^2)$	$(2,-1,1)$	–
F5	$\Pi r^7p^2/m^3$	(2,0,1)	–
F5	$\Pi^2 rp/m\rho$	$(1,-1,2)$	–
F5	$\Pi^2 r^4 p/m^2$	(1,0,2)	–
F5	$\Pi^3/(r^2\rho)$	$(0,-1,3)$	D
F5	$\Pi^3 r/m$	(0,0,3)	C

There is an intuitive way to explain the existence of the above families. As an example we consider F3, where the seemingly unrelated terms $\Lambda_2(1,\beta,0)$ and $\Lambda_2(0,\beta,1)$ yield similar profiles. Consider

$$\Lambda_2(0,\beta,1) \sim r^{-1+3\beta}\frac{\Pi\rho^\beta}{m^{\beta-1}}. \tag{52}$$

By means of the approximations $m \sim \rho r^3$, $\Pi \sim p/\rho$ (the latter approximation is motivated by the exact thermodynamical relation $\Pi = np/\rho$ for relativistic polytropes with index n; see Appendix B) we find

$$\Lambda_2(0,\beta,1) \sim r^{2+3\beta}p\left(\frac{\rho}{m}\right)^\beta \sim \Lambda_2(1,\beta,0). \tag{53}$$

Similarly we can show that $\Lambda_2(2,\beta,0) \sim \Lambda_2(1,\beta,1)$ for the F4 family. The argument can be generalized to show that terms along the diagonals of Fig. 3 are equivalent.

Table I summarizes the taxonomy of the most important terms of each family according to the above criteria. The impact of each of these terms as a post-TOV correction has been tested for a variety of EOSs. The results reveal that the members of a given family lead to self-similar modifications to the stellar mass-radius curves. A sample of these numerical results is shown in Fig. 2, which is further discussed in Sec. IV below.

This remarkable self-similarity property means that we can simply select one term from each family and emulate the effect of *all* significant 2PN terms of the same family by simply varying the post-TOV coefficient associated with the selected term.

In doing so, it is reasonable to choose the simplest terms as family representatives. For families F2 and F4 the simplest terms also happen to be the dominant ones (i.e., the ones with the largest $\rho\Lambda_2$), while for F3 and F5 they are the first subdominant ones. The case of the single-member family F1 is trivial. The five terms we select based on this reasoning are

$$\text{F-representatives:}\ \frac{m^3}{r^5\rho}, \frac{m^2}{r^2}, r^2 p, \Pi\frac{p}{\rho}, \Pi^3\frac{r}{m}. \quad (54)$$

The phenomenologically relevant radial profiles of $\bar{\rho}\Lambda_2(\alpha,\beta,\theta)$ produced by these terms are shown in Fig. 4 for three choices of EOS: FPS, APR, and an $n = 0.6$ polytrope. The most striking feature of this figure is the close resemblance of the Λ_2 profiles of identical (α,β,θ) triads for different EOSs, which lends support to the EOS independence of our selection of post-TOV terms.

The family-representative terms (54) are again shown in Fig. 5, where we plot the combinations that appear in the dp/dr and dm/dr equations, i.e., $m\rho\Lambda_2/r^2$ and $r^2\rho\Lambda_2$, respectively (in the latter term we have omitted a trivial prefactor of 4π). We consider two different EOSs: APR and an $n = 0.6$ polytrope. All terms displayed are regular at both $r = 0$ and $r = R$ with the exception of the F5 term in the dp/dr equation, which is divergent at $r = 0$ and must be excluded. Once again, the variations in the radial profiles due to considering different EOSs are extremely mild.

We have thus obtained a *minimum* set of representative 2PN terms, listed in Eq. (54), which in reality encompasses a much larger set, like the one obtained from the combination of Eqs. (41) and (51), as well as terms that involve the integral potentials.

After this admittedly tedious procedure we can finally assemble our 2PN-order post-TOV equations for the pressure and the mass. These are (omitting the negligibly small 1PN corrections)

$$\frac{dp}{dr} = \left(\frac{dp}{dr}\right)_{\rm GR} - \frac{\rho m}{r^2}\left(\pi_1\frac{m^3}{r^5\rho} + \pi_2\frac{m^2}{r^2} + \pi_3 r^2 p + \pi_4\Pi\frac{p}{\rho}\right), \quad (55a)$$

$$\frac{dm}{dr} = \left(\frac{dm}{dr}\right)_{\rm GR} + 4\pi r^2\rho\left(\mu_1\frac{m^3}{r^5\rho} + \mu_2\frac{m^2}{r^2} + \mu_3 r^2 p + \mu_4\Pi\frac{p}{\rho} + \mu_5\Pi^3\frac{r}{m}\right), \quad (55b)$$

where, as anticipated in the executive summary, π_i $(i = 1,\ldots,4)$ and μ_i $(i = 1,\ldots,5)$ are free parameters controlling the size of the corresponding departure from GR.

C. Completing the formalism: the post-TOV metric and stress-energy tensor

So far, our post-TOV formalism comprises no more than a pair of stellar structure equations [Eqs. (55a) and (55b)], which can be used for the description of static and spherically symmetric compact stars. In this section we show that there is more to the formalism than meets the eye: to a high precision it is a "complete" toolkit, in the sense that (i) it can be reformulated in terms of a spherically

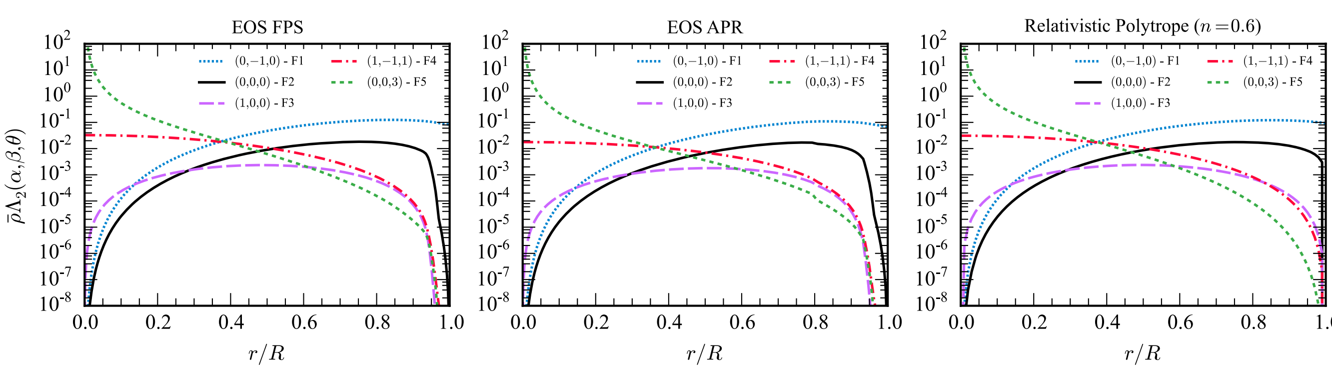

FIG. 4 (color online). *The family-representative 2PN terms.* Here we show the selected representative terms from each of the families depicted in Fig. 3, as listed in Eq. (54), for three different EOSs: FPS (left panel), APR (middle panel) and an $n = 0.6$ polytrope (right panel). Each term illustrates the qualitative behavior of each family of possible 2PN contributions to the structure equations. The high degree of invariance of the Λ_2 profiles with respect to the EOS is evident in this figure. The GR background stellar models utilized in the figure have the following bulk properties: $\epsilon_c = 0.861 \times 10^{15}$ g/cm^3 ($\lambda \equiv p_c/\epsilon_c = 0.165$), $M = 1.51 M_\odot$ and $R = 12.3$ km (left panel); $\epsilon_c = 1.450 \times 10^{15}$ g/cm^3 ($\lambda = 0.198$), $M = 1.50 M_\odot$ and $R = 10.7$ km (center panel); $\lambda = 0.165$, $M = 1.50 M_\odot$ and $R = 11.75$ km (right panel).

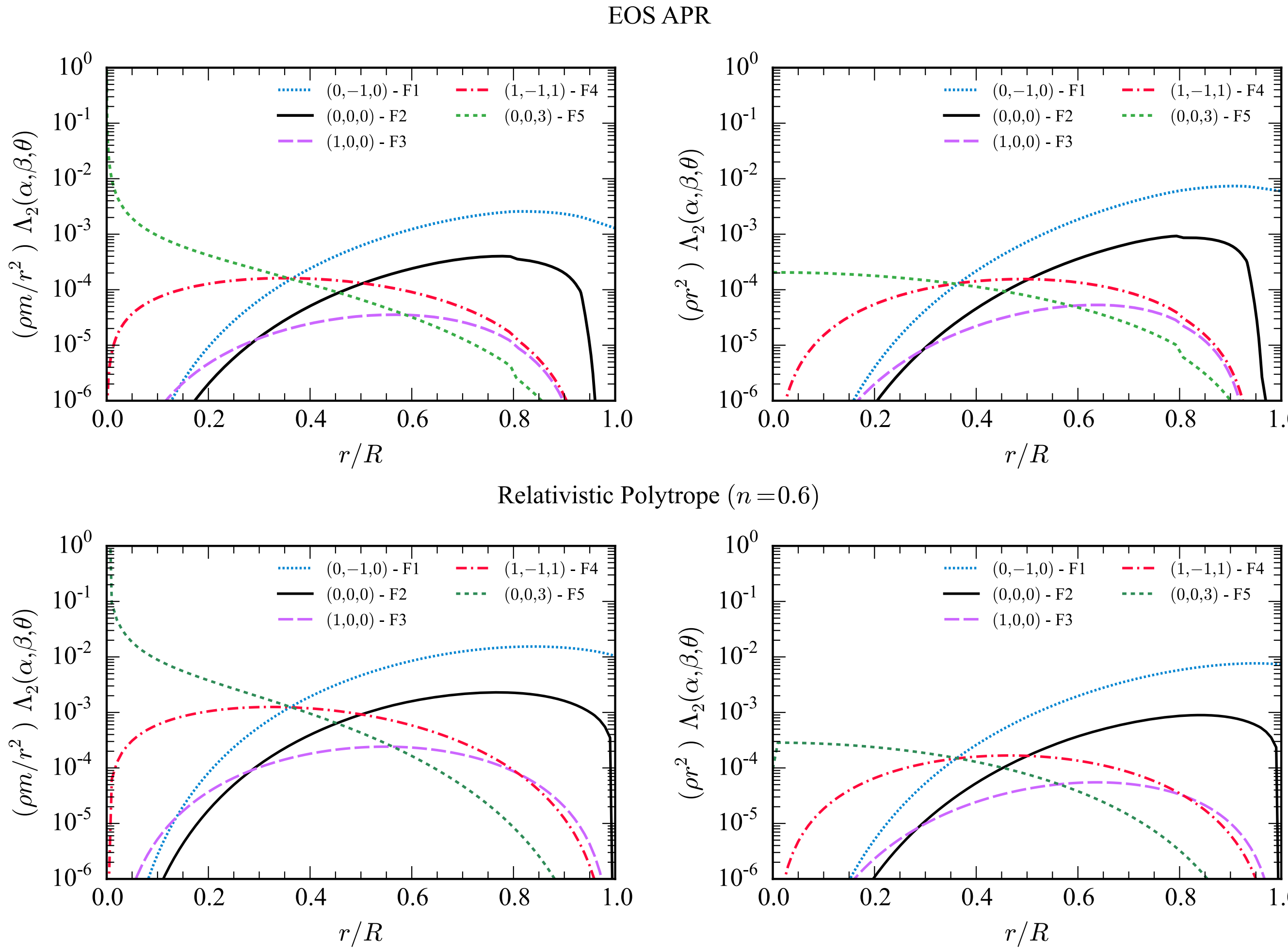

FIG. 5 (color online). *The family-representative terms in the structure equations.* This figure illustrates the behavior of each of the family-representative 2PN terms [see Eq. (54)] multiplied by the Newtonian prefactors in the post-TOV equations. The stellar parameters are identical to the ones used in Fig. 4. Left panel: The combination $(\rho m/r^2)\Lambda_2(\alpha,\beta,\theta)$ appearing in the pressure equation. Right panel: The combination $\rho r^2\Lambda_2(\alpha,\beta,\theta)$ appearing in the mass equation. The top panels correspond to EOS APR; the bottom panels correspond to a relativistic polytrope with polytropic index $n = 0.6$. The divergence at the origin of the F5 term justifies its exclusion from the pressure equation.

symmetric metric $g_{\mu\nu}$ and a perfect fluid stress-energy tensor $T^{\mu\nu}$, and (ii) these two structures are related through the covariant conservation law $\nabla_\nu T^{\mu\nu} = 0$ (where ∇_ν is the metric-compatible covariant derivative), hence respecting the equivalence principle. Remarkably, it also turns out that the metric and matter degrees of freedom can be related as in GR, which implies that the post-TOV formalism is *equivalent* to stellar structure in GR with a *gravity-modified* EOS for matter and an *effective* spacetime geometry.

In order to establish the above statements we begin with the following general result. Assume the static spherically symmetric metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{\nu(r)}dt^2 + \left(1 - \frac{2\mathcal{M}(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad (56)$$

and a perfect fluid stress-energy tensor (with energy density $\mathcal{E}$ and pressure $\mathcal{P}$)

$$T^{\mu\nu} = (\mathcal{E} + \mathcal{P})u^\mu u^\nu + \mathcal{P}g^{\mu\nu}. \quad (57)$$

For a static spherical fluid ball, the energy-momentum conservation equation

$$\nabla_\nu T^{\mu\nu} = 0 \quad (58)$$

leads to

$$\frac{d\mathcal{P}}{dr} = -(\mathcal{E} + \mathcal{P})\Gamma^t_{rt} = -\frac{1}{2}(\mathcal{E} + \mathcal{P})\frac{d\nu}{dr}. \quad (59)$$

As long as we consider theories respecting Eq. (58) with a metric-compatible covariant derivative, this result is independent of the gravitational field equations.

For the mass function $\mathcal{M}(r)$ we can always write a relation of the form

$$\frac{d\mathcal{M}}{dr} = 4\pi r^2 \mathcal{E}[1 + Z(r)], \tag{60}$$

where $Z(r)$ is a theory-dependent function. Einstein's theory is recovered by setting $Z = 0$, as required by the field equations of GR.

To establish the properties described at the beginning of this section we will show that we can successfully map our post-TOV equations onto Eqs. (59) and (60) (with $Z = 0$).

The full post-TOV equations [Eqs. (55a) and (55b)] can be written in the form

$$\frac{dp}{dr} = -\frac{(\epsilon + p)}{r^2}\Gamma(r) - \frac{\rho m}{r^2}\left[\left(1 + \Pi + \frac{p}{\rho}\right)\mathcal{P}_1 + \tilde{\mathcal{P}}_2\right], \tag{61}$$

$$\frac{dm}{dr} = 4\pi r^2 \epsilon + 4\pi r^2 \rho[\mathcal{M}_1 + \mathcal{M}_2], \tag{62}$$

where $\mathcal{P}_1, \mathcal{P}_2$ have been defined in Eqs. (2a)–(2d),

$$\tilde{\mathcal{P}}_2 \equiv \mathcal{P}_2 - \left(\Pi + \frac{p}{\rho}\right)\mathcal{P}_1 \tag{63}$$

is a 2PN-order term, and $\Gamma(r) \equiv (m + 4\pi r^3 p)/(1 - 2m/r)$.

Based on these expressions, we can define the effective energy density

$$\epsilon_{\rm eff} \equiv \epsilon + \rho(\mathcal{M}_1 + \mathcal{M}_2), \tag{64}$$

which implies

$$\frac{dm}{dr} = 4\pi r^2 \epsilon_{\rm eff}. \tag{65}$$

Using Eq. (64) in the pressure equation, we have

$$\frac{dp}{dr} = -[\epsilon_{\rm eff} + p - \rho(\mathcal{M}_1 + \mathcal{M}_2)]\frac{\Gamma}{r^2} - \frac{m}{r^2}[(\epsilon + p)\mathcal{P}_1 + (\epsilon_{\rm eff} + p)\tilde{\mathcal{P}}_2], \tag{66}$$

where we have used the fact that in any 2PN term we can replace the factor ρ by $\epsilon_{\rm eff} + p$, at the cost of introducing 3PN terms. Using Eq. (64) once more in the last term, and after some rearrangement, we obtain

$$\frac{dp}{dr} = -\frac{(\epsilon_{\rm eff} + p)}{r^2}[(1 - \mathcal{M}_2)\Gamma + m(\mathcal{P}_1 + \tilde{\mathcal{P}}_2 - \mathcal{M}_1\mathcal{P}_1)] + \frac{\rho}{r^2}\mathcal{M}_1\Gamma. \tag{67}$$

Given that $\mathcal{M}_1 \ll 1$, the last term can be safely omitted and we are left with

$$\frac{dp}{dr} \approx -\frac{(\epsilon_{\rm eff} + p)}{r^2}[(1 - \mathcal{M}_2)\Gamma + m(\mathcal{P}_1 + \tilde{\mathcal{P}}_2 - \mathcal{M}_1\mathcal{P}_1)] \tag{68}$$

$$\approx -\frac{(\epsilon_{\rm eff} + p)}{r^2}[(1 - \mathcal{M}_2)\Gamma + m\mathcal{P}_2], \tag{69}$$

which is of the form (59). Note that in this and the following expressions the small $\mathcal{M}_1, \mathcal{P}_1$ terms can be omitted.

The resulting mapping is

$$\mathcal{P} = p, \qquad \mathcal{M} = m, \qquad \mathcal{E} = \epsilon_{\rm eff}. \tag{70}$$

It follows that the effective post-TOV metric is

$$g_{\mu\nu} = \mathrm{diag}[-e^{\nu(r)}, (1 - 2m(r)/r)^{-1}, r^2, r^2\sin^2\theta], \tag{71}$$

with

$$\frac{d\nu}{dr} \approx \frac{2}{r^2}[(1 - \mathcal{M}_2)\Gamma + m(\mathcal{P}_1 + \tilde{\mathcal{P}}_2 - \mathcal{M}_1\mathcal{P}_1)] \tag{72}$$

$$\approx \frac{2}{r^2}[(1 - \mathcal{M}_2)\Gamma + m\mathcal{P}_2]. \tag{73}$$

From this result we can see that r represents the circumferential radius of the $r =$ constant spheres and therefore the post-TOV radius R [where $p(R) = 0$] coincides with the circumferential radius of the star.

Finally, the effective post-TOV stress-energy tensor is

$$T^{\mu\nu} = (\epsilon_{\rm eff} + p)u^\mu u^\nu + p g^{\mu\nu}, \tag{74}$$

and it is covariantly conserved with respect to the metric (71).

These expressions clearly demonstrate that our post-TOV formalism is completely equivalent to GR with an effective EOS:

$$p(\epsilon) \to p(\epsilon_{\rm eff}), \tag{75}$$

$$\epsilon_{\rm eff} \approx \epsilon + \rho\mathcal{M}_2. \tag{76}$$

As is evident from this last expression, $\epsilon_{\rm eff}$ represents a gravity-shifted parameter with respect to the physical energy density ϵ. This result highlights a key characteristic of compact relativistic stars when studied in the context of alternative theories of gravity, namely, the intrinsic degeneracy between the physics of the matter and gravity sectors.

Whether the above effective description (and in particular its effective geometry part) can give observables that have a correspondence to observables of an underlying theory or not depends on the nature of that theory. As long as the underlying theory admits a PN expansion, the physical description that arises from the effective formalism should match that of the physical theory. This nontrivial issue will be further discussed elsewhere [60].

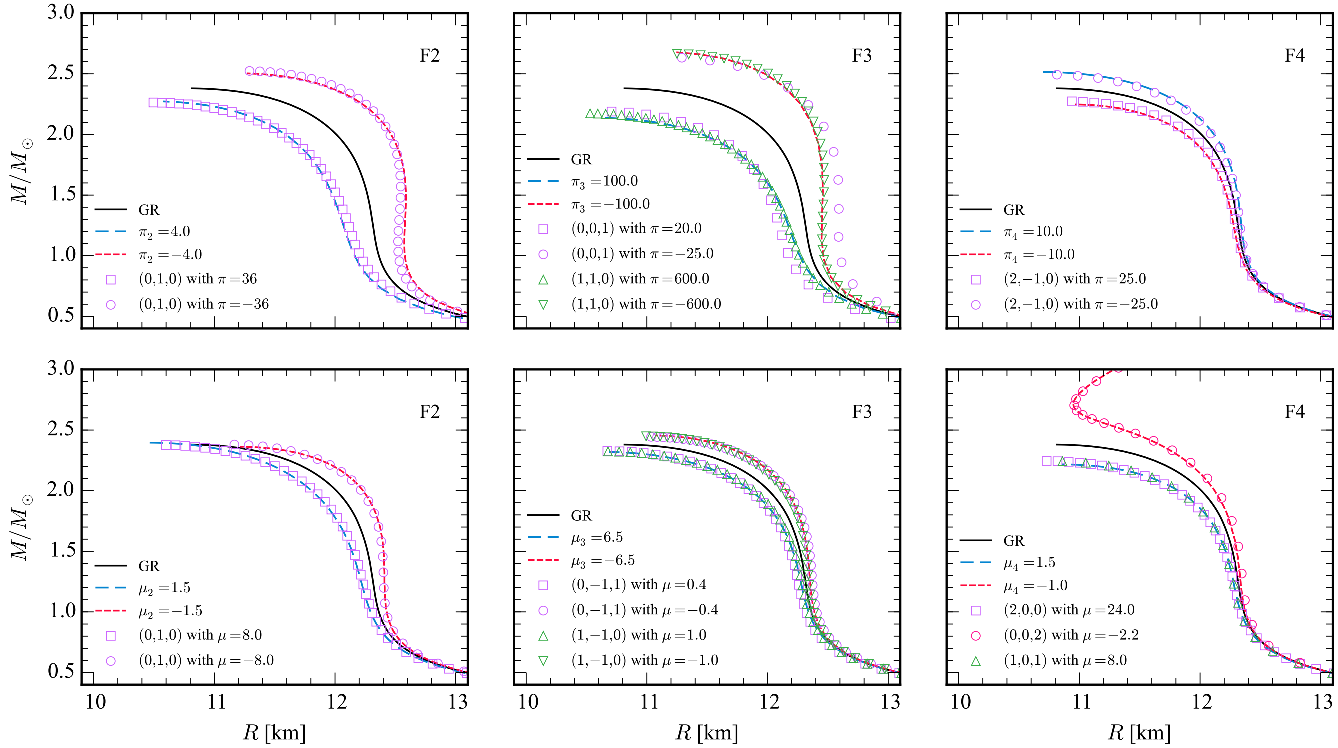

FIG. 6 (color online). *Self-similarity in mass-radius curves—I.* Numerical integrations show that 2PN terms belonging to the same family result in self-similar deviations from GR in the mass-radius relation. This figure illustrates this remarkable property for pressure terms (top row) and mass terms (bottom row) belonging to families F2, F3, and F4 (from left to right). In each panel, the solid line corresponds to GR, the long-dashed line corresponds to a positive-sign correction due to the chosen term in each family, and the short-dashed line corresponds to a negative-sign correction due to the chosen term in each family. The various symbols show that nearly identical corrections can be produced using different terms belonging to the same family, as long as we appropriately rescale their post-TOV coefficients.

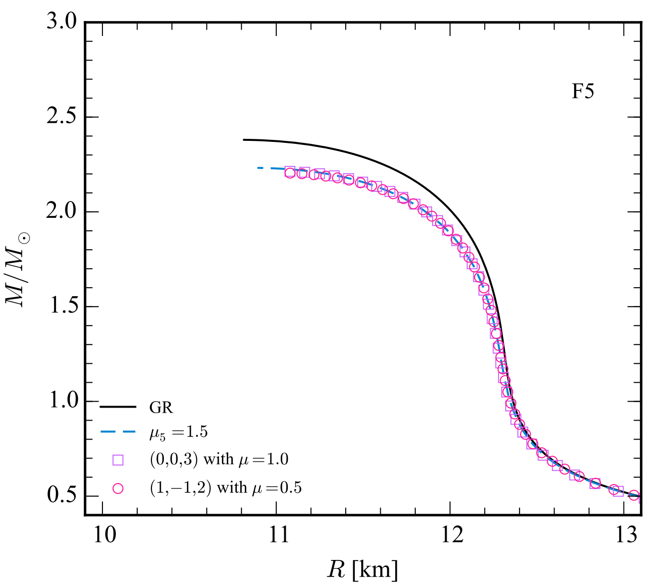

FIG. 7 (color online). *Self-similarity in mass-radius curves—II.* Same as in Fig. 6, but for the F5 family, which only admits post-TOV corrections with $\mu_5 < 0$ (see text).

IV. NUMERICAL RESULTS

In this section we provide a more detailed discussion of our numerical techniques and results, focusing on the mass-radius curves produced by the integration of the post-TOV equations (55a) and (55b) [or equivalently Eqs. (1a) and (1b)].

First, let us briefly summarize the integration procedure we have followed in this paper. We have carried out two kinds of computations: (i) "background" models (these involve the integration of the general relativistic TOV equations) with the purpose of studying the radial profiles of the post-TOV correction terms, and (ii) the integration of the full post-TOV equations, typically including the representative term of a single 2PN family.

The post-TOV structure equations (1a) and (1b) are integrated simultaneously starting at the origin $r = 0$, for fixed values of the coefficients π_i, μ_i, and for a range of central energy density values. The chosen central energy density ϵ_c fixes the central pressure $p_c = p(\epsilon_c)$, the central mass density $\rho_c = m_b n_b(\epsilon_c)$, and the central internal energy $\Pi_c = (\epsilon_c - \rho_c)/\rho_c$, where $m_b = 1.66 \times 10^{-24}$ g is the baryonic mass and n_b is the baryon number density. In general, $p(\epsilon)$ and $n_b(\epsilon)$ are computed using tabulated EOS data. Once the initial conditions have been specified, Eqs. (1a) and (1b)

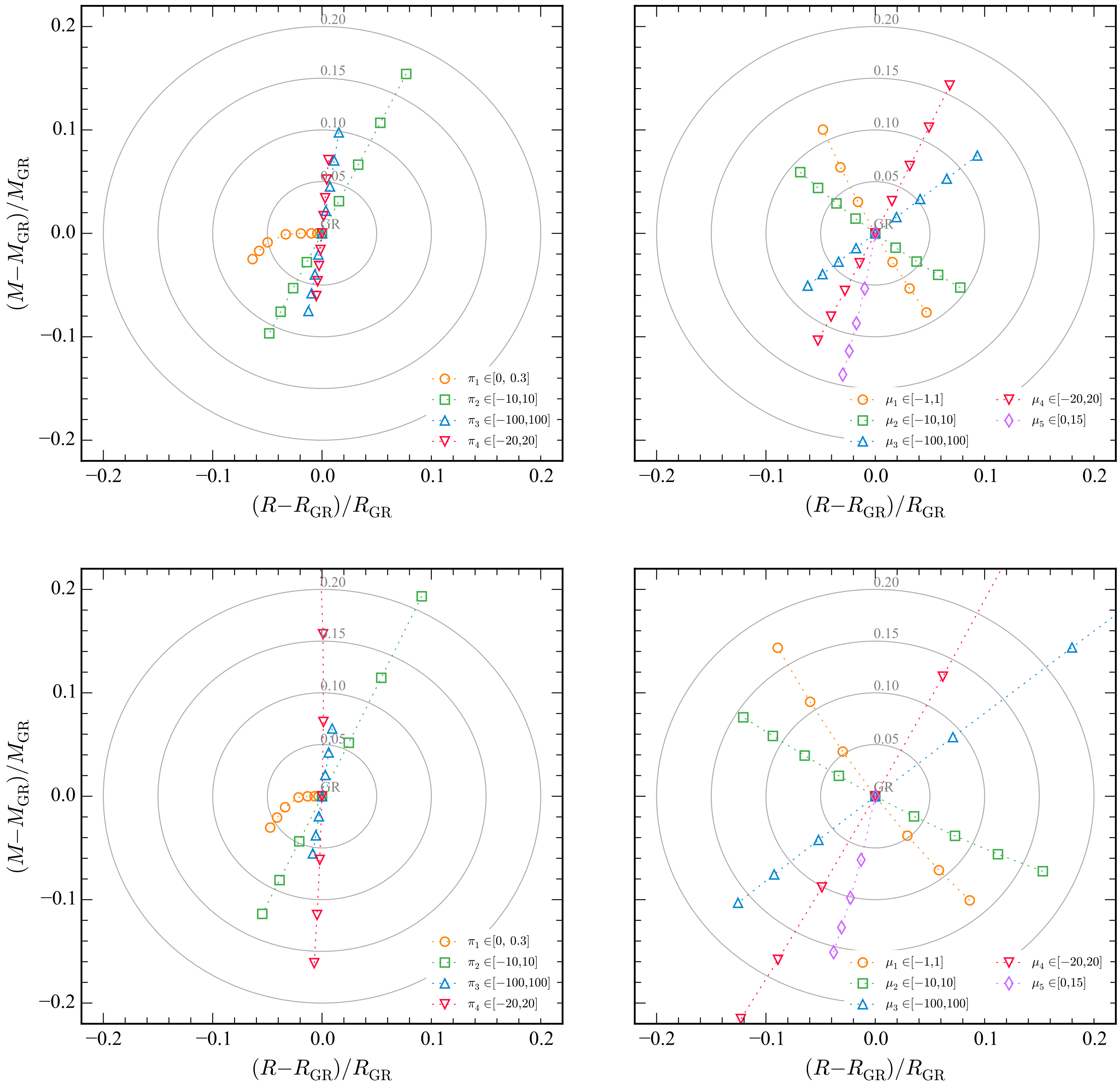

FIG. 8 (color online). *Fractional deviations induced by the post-TOV parameters on the stellar mass and radius.* Here we illustrate the fractional changes caused by the post-TOV parameters in neutron star masses and radii. For a fixed central energy density and EOS APR, we plot the relative deviations from GR in mass and radius that result from varying the post-TOV parameters within the range indicated in the legends. Top row: $\epsilon_c/c^2 = 8.61 \times 10^{14}$ g/cm^3, $M_{\rm GR} = 1.51 M_\odot$, and $R_{\rm GR} = 12.3$ km. Bottom row: $\epsilon_c/c^2 = 1.20 \times 10^{15}$ g/cm^3, $M_{\rm GR} = 2.04 M_\odot$, and $R_{\rm GR} = 11.9$ km. Left panels: Effect of the post-TOV terms that enter in the pressure equation. Right panels: Effect of the post-TOV terms that enter in the mass equation. The circles represent contours of fixed relative deviation from GR.

are integrated outward up to the stellar radius R, where $p(R) = 0$. The gravitational mass is obtained as $M = m(R)$.

The integration procedure for realistic EOS background models is virtually the same as the one just described. We have also employed a number of polytropic background models; for these the integration procedure is slightly different (see Appendix B for details), and it is based on the simpler Lane-Emden formulation, where the pressure is replaced by the density ρ in the structure equations and the stellar model is parametrized by the ratio $\lambda = p_c/\epsilon_c$ rather than ϵ_c alone (this formulation is of course equivalent to the one using tabulated EOSs). The added advantage of this approach is its scale invariance with respect to the polytropic constant K. This means that K can be freely adjusted to generate a model with (say) a specific mass M. This scaling procedure also fixes the radius R.

The main installment of our mass-radius results has already been presented in Fig. 2 of the executive summary (Sec. I A). As discussed there, the various post-TOV correction terms, representing the five 2PN families of Sec. III B, cause qualitatively different modifications to the mass-radius curves.

As a rule of thumb, the corrections to the pressure equation lead to markedly weaker mass-radius modifications than the corrections to the mass equation, for the same magnitude of π_i and μ_i. The effective-metric formulation of the post-TOV formalism suggests a simple qualitative explanation of this observation. The mass corrections $\mathcal{M}_2$ change both the effective EOS and the strength of gravity, as measured by $\nu(r)$, while the pressure corrections $\mathcal{P}_2$ are only associated with a change in the strength of gravity [cf. Eqs. (73) and (76)], and it is well known that changes in the EOS outweigh gravity modifications in terms of their effect on the mass-radius relation.

A notable exception is the single-member family F1, for which the pressure correction term dominates over its mass counterpart. In fact, the F1 pressure term leads to the largest mass-radius changes, as evidenced by the π_1 values used in Fig. 2. It is not too difficult to explain why this happens: near the stellar surface, where all three fluid parameters p, ρ, Π are close to zero, the F1 correction terms remain finite and dominate over all other terms in the post-TOV equations (this can be clearly seen in Fig. 5), thus taking control of the pressure and mass derivatives.

Another noteworthy point is that, when considering individual post-TOV terms, it is not always possible to integrate the equations for both positive and negative values of the corresponding coefficient. This is the case for family F5 in Fig. 2, where the integration fails for $\mu_5 < 0$. We have found that this is caused by an unphysical negative slope dm/dr near the origin.

The remarkable self-similarity in the radial profiles of 2PN terms belonging to the same family has been illustrated in Fig. 3 (see Sec. III B). With hindsight, this property should not come as a total surprise, given the approximate correlations among the fluid variables: $m \sim \rho r^3$, $\Pi \sim p/\rho$.

The emergence of the same self-similarity in the mass-radius curves is something far less anticipated and even more striking. This property, which has allowed us to formulate a practical and versatile set of post-TOV equations, is illustrated in Figs. 6 and 7, where we show mass-radius results for each 2PN family, considering both the pressure and the mass equation and for the same APR EOS stellar model as in Fig. 2. Each panel is devoted to a particular family, and it shows the mass-radius curves resulting from the integration of the post-TOV equations when various terms from Table I are included as corrections (notice that F1 is missing from these plots for the obvious reason that it consists of only one post-TOV correction).

In all cases considered, the terms of the same family are found to cause *nearly identical mass-radius changes* by a suitable rescaling of the relevant coefficient π_i or μ_i. This behavior is most striking for family F4, where different post-TOV corrections in the mass equations lead to the same characteristic back-bending behavior in the mass-radius curve. The only notable exception to this remarkable scaling property is the (0,0,1) member of the F3 family, proportional to $\Pi m/r$, which can be rescaled to agree with other members of the family at high densities but partially fails to capture the behavior of the mass-radius curve at low densities. This partial symmetry breaking can be understood by looking at the leftmost panel in the second row of Fig. 3: the behavior of this term near the surface is not as smooth as for other members of this family. In our opinion this does not warrant extensions of the formalism to include another family, but this is definitely a possibility that could be considered in the future, given the approximate nature of the self-similarity argument.

Another important aspect of the post-TOV results is their "directionality" in the mass-radius plane, in the sense that a given correction term could affect the mass more than the radius, or vice versa. This kind of information cannot be easily extracted from a traditional mass-radius plot such as Fig. 2, but becomes very visible if we display the same results in terms of the fractional changes $\delta M/M_{\rm GR} \equiv (M - M_{\rm GR})/M_{\rm GR}$ and $\delta R/R_{\rm GR} \equiv (R - R_{\rm GR})/R_{\rm GR}$ from the corresponding GR values.

"Dart-board" plots of these fractional changes are shown in Fig. 8. The aforementioned directionality of the various

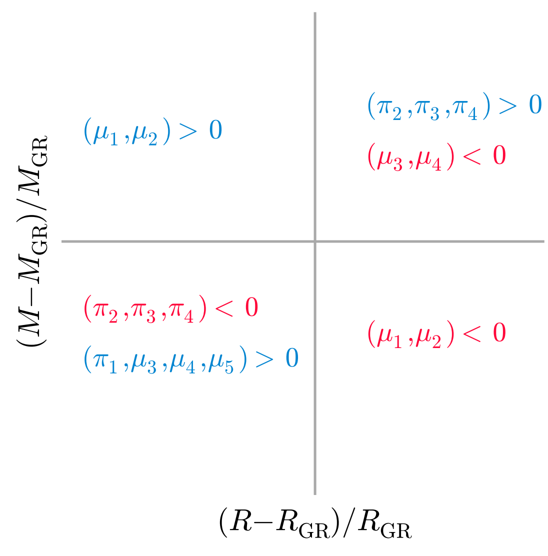

FIG. 9 (color online). *Directions of the post-TOV induced deviations.* This schematic diagram shows which sign of individual post-TOV parameters produces smaller or larger masses/radii with respect to GR, cf. Fig. 8.

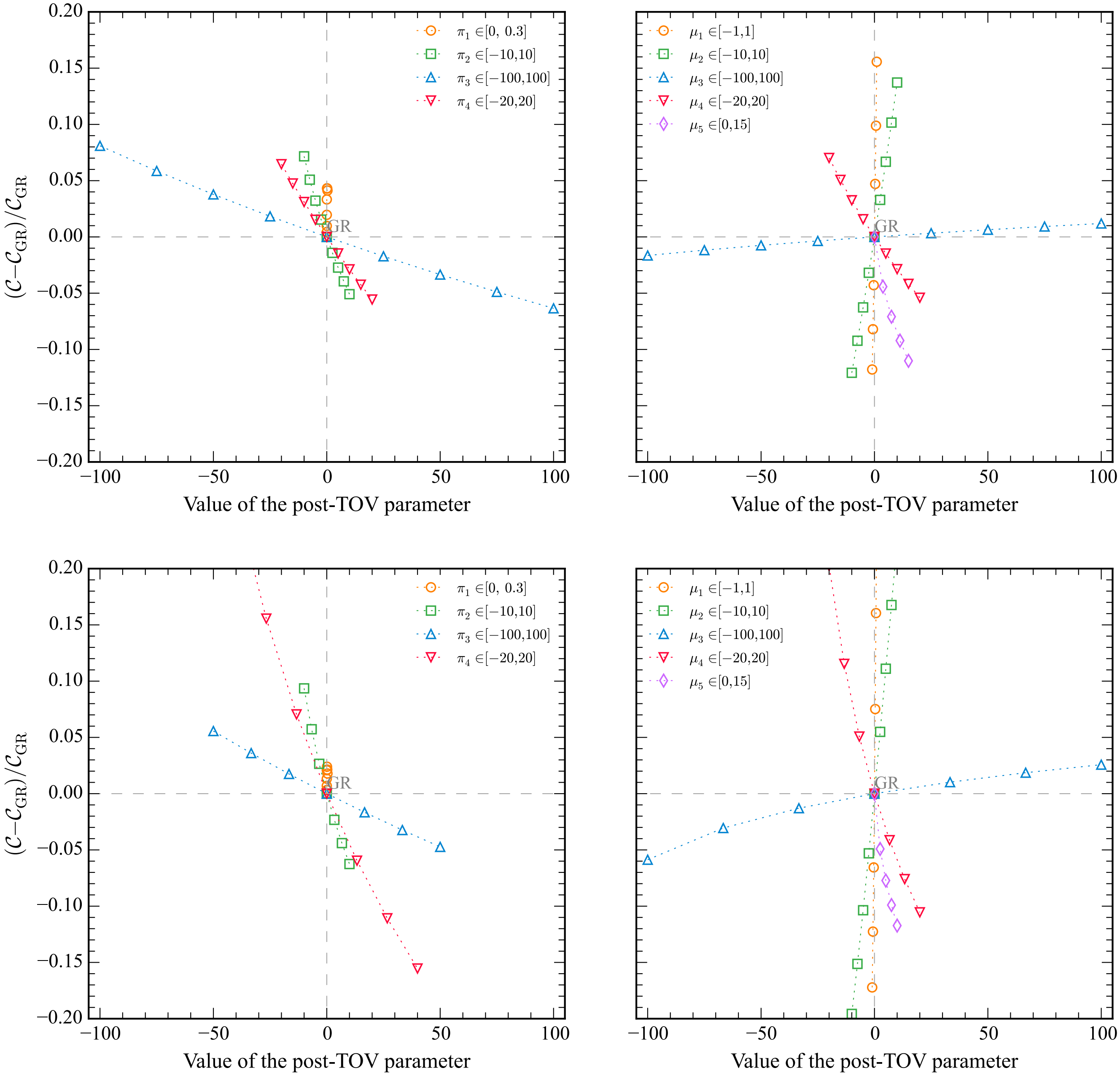

FIG. 10 (color online). *Deviations induced by the post-TOV parameters on the stellar compactness.* Here we consider the influence of the post-TOV parameters on the compactness $\mathcal{C} = M/R$ of neutron stars. Deviations from GR are calculated assuming the same APR EOS models as in the top and bottom rows of Fig. 8. Left panels: Effect of the post-TOV terms appearing in the pressure equation. Right panels: Effect of the post-TOV terms appearing in the mass equation.

post-TOV corrections is clearly visible in this figure. Individual correction terms are seen to drive nearly *linear* departures (at least up to a ~10% level) from the center of the "board." Moreover, certain terms are mutually (nearly) orthogonal, although not aligned with the mass or radius axis. In some cases this happens between the pressure and mass terms of the same family, e.g., family F2. In general, the departures from the GR model are more isotropically scattered when caused by the corrections $\mathcal{M}_2$ in the dm/dr equation, whereas the pressure corrections $\mathcal{P}_2$ are clearly more concentrated near the direction of the mass axis. This behavior fits nicely with the effective-EOS interpretation of how $\mathcal{M}_2$ and $\mathcal{P}_2$ corrections change the mass-radius diagram. As expected, $\mathcal{M}_2$ corrections affect the stiffness of the effective EOS with significant effects on the radius, while $\mathcal{P}_2$ corrections change the strength of gravity, and this mostly affects how much mass a particular model can support.

These trends remain unchanged as the central energy density (and the stellar mass) increases (see bottom panels of Fig. 8). The pressure correction term associated with π_1 (family F1) provides the exception to the rule: a sequence of $\pi_1 > 0$ values leads to a nonlinear trajectory, with initially just the radius decreasing and then followed by a comparable fractional decrease in the mass. Negative values of π_1 are not shown because they lead to unphysical models where in the outer low-density layers of the star dp/dr becomes nearly zero but never negative, thus preventing us from finding the exact location of the surface (as we have pointed out earlier in this section, this behavior is related to the nonzero value of the F1 term at the surface).

Figure 9 provides a schematic chart of the correlation between the sign of the π_i, μ_i coefficients and the sign of the associated variations δM, δR. Interestingly, the π_i terms are limited to just two of the four possible quadrants (note the anticorrelation between the signs of π_1 and the other π_i). This translates to variations that simultaneously make the star bigger (smaller) and heavier (lighter), i.e., $\delta R > 0$, $\delta M > 0$ (or $\delta R < 0$, $\delta M < 0$). In contrast, the μ_i terms occupy all four quadrants, with the F3, F4, and F5 families leading to $\delta R \delta M > 0$ variations, and the F1 and F2 families giving rise to the opposite arrangement, $\delta R \delta M < 0$.

The linear patterns of Fig. 8 suggest that the mass and radius variations, for a given $\sigma_i = \{\pi_i \neq \pi_1, \mu_i\}$, obey the empirical relations

$$\frac{\delta M}{M_{\rm GR}} \approx \sigma_i K_M, \qquad \frac{\delta R}{R_{\rm GR}} \approx \sigma_i K_R, \tag{77}$$

where the structure parameters K_M, K_R are functions of the EOS and of ϵ_c, but they are independent of σ_i. Given the nonlinear character of the post-TOV equations, this conclusion is clearly nontrivial. We can recast this result in terms of the variation of the stellar compactness $\mathcal{C} \equiv M/R$,

$$\frac{\delta \mathcal{C}}{\mathcal{C}_{\rm GR}} \approx \sigma_i (K_M - K_R). \tag{78}$$

This almost linear $\delta\mathcal{C}(\sigma_i)$ dependence[4] can indeed be seen in the numerical results shown in Fig. 10, where we consider the same stellar models as in Fig. 8.

The results presented in this section provide a wealth of information on the character of the post-TOV corrections of stellar structure. It is likely that a more systematic study of the self-similar F families will reveal additional layers of information and provide clues as to why the 2PN terms change the bulk properties of the star the way they do, as a function of the central density. Such a study is beyond the scope of this paper but provides an attractive subject for future work.

V. CONCLUSIONS AND OUTLOOK

This paper is a first step towards establishing a parametrized perturbative framework that should, at least in principle, encompass all modifications to the bulk properties of neutron stars induced by modified theories of gravity. As in the original formulation of the PPN formalism, along the way we were forced to make some reasonable simplifying assumptions in order to reduce the complexity (and increase the practicality) of our parametrization. These reasonable assumptions may well fail to match the well-known creativity of theorists, and it will be interesting to see how the formalism can be extended and improved.

In a follow-up paper we will use our basic post-TOV equations to recover stellar structure calculations in some popular theories of gravity, such as those shown in Fig. 1. It is particularly interesting to compare the formalism against theories that violate some of our basic assumptions, such as scalar-tensor gravity with spontaneous scalarization (which introduces intrinsically nonperturbative effects [32]) or Eddington-inspired-Born-Infeld gravity, with its lack of a Newtonian limit and its unorthodox dependence on the stress-energy tensor [27,62].

We have already obtained some interesting results in this context: for example, our conclusion that the 2PN post-TOV equations are equivalent to an effective modified perfect fluid EOS (see Sec. III C) has an interesting parallel with the results of Delsate *et al.* [62], who reached a similar conclusion for Eddington-inspired-Born-Infeld gravity. We are currently extending the "effective metric" formalism developed in this paper to the exterior spacetime of compact stars [60]. This is necessary to compute physical observables—such as the gravitational redshift of surface atomic lines, the touchdown luminosity of a radius expansion burst, and the apparent surface area of neutron stars [63]—and it is possible that the combination of multiple observables may lift the EOS/gravity degeneracy.

There are several interesting extensions of our work that should be addressed in the future. The most obvious one is to assess whether post-TOV parameters can indeed reproduce the mass-radius curve in various classes of alternative theories, and whether the post-TOV parameters encode specific information on the physical parameters underlying specific theories. This study will hopefully lead to a better understanding of the generality of the EOS/gravity-theory degeneracy.

[4]It is interesting to note that the qualitative effect of the post-TOV terms in the pressure equation can be understood by analogy with the case of anisotropic stars in GR. The post-TOV pressure equation takes the form $dp/dr = (dp/dr)_{\rm GR} - \rho m \pi_i f_i(r)/r^2$, with $f_i(r) > 0$, whereas anisotropic stars obey $dp_r/dr = (dp_r/dr)_{\rm GR} - 2\sigma/r$, with $\sigma = p_r - p_q$ being the difference between the radial and tangential pressure. These two expressions can be matched if $2r\sigma = \rho m \pi_i f_i(r)$. The compactness of anisotropic stars is known to decrease (increase) when σ increases (decreases) [61]. This conclusion is in good qualitative agreement with the results shown in the left panel of Fig. 10.

From a data analysis point of view, it is important to understand whether physical measurements of masses and radii (or perhaps more realistically, measurements of masses and surface redshifts/stellar compactnesses) can lead to constraints on the post-TOV parameters under specific assumptions on the high-density EOS. The answer to this question obviously depends on the relative magnitude of modified gravity effects and EOS uncertainties. It will be interesting to quantify what uncertainties in the EOS are acceptable if we want to experimentally constrain post-TOV parameters at meaningful levels.

Other obvious extensions are (i) the generalization of the post-TOV framework to slowly and possibly fast rotating relativistic stars, and (ii) stability investigations within the post-TOV framework. We hope that our work will stimulate further activity in this field. Stability studies in a post-TOV context may reveal that certain generic features of modified gravity lead to instabilities even for nonrotating stars, possibly excluding whole classes of modified gravity theories.

Last but not least, we would like to point out that our post-TOV toolkit is not (nor was it designed to be) a self-consistent PN expansion, but rather a phenomenological parametrization of the leading-order (unconstrained) deviations from GR. A systematic and self-consistent PPN expansion extending the PN stellar structure works cited in the Introduction [40–43] is an interesting but quite distinct area of investigation that should also be pursued in the future.

ACKNOWLEDGMENTS

K. G. is supported by the Ramón y Cajal Programme of the Spanish Ministerio de Ciencia e Innovación, by the German Science Foundation (DFG) via SFB/TR7 and by NewCompstar (a COST-funded Research Networking Programme). G. P. has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement No. 306425 “Challenging General Relativity”. H. O. S. was supported by NSF CAREER Grant No. PHY-1055103. E. B. was supported by NSF CAREER Grant No. PHY-1055103 and by FCT contract IF/00797/2014/CP1214/CT0012 under the IF2014 Programme. The authors wish to thank Paolo Pani, Hajime Sotani and Kent Yagi for discussions and for sharing or validating some of the numerical results shown in Fig. 1.

APPENDIX A: DIMENSIONAL ANALYSIS OF POST-NEWTONIAN TERMS

In this appendix we develop an algorithm for constructing PN terms using dimensional analysis techniques. Barring PN terms involving the three potentials U, E, Ω, the available parameters for generating PN terms are $\{p, \rho, m, r, \Pi\}$. From these quantities plus the gravitational constant G and the speed of light c we can build the dimensionless combination[5]

$$\Lambda = p^{\alpha}\rho^{\beta}m^{\gamma}r^{\delta}\Pi^{\theta}G^{\kappa}c^{\lambda} \tag{A1}$$

for a suitable choice of integers $\alpha, \beta, \gamma, \delta, \kappa, \lambda$ (these are not to be confused with the PPN parameters of Sec. II). Since Π is already dimensionless, there is no *a priori* dimensional restriction on θ (apart from one coming from the PN order of Λ) and therefore that factor can be omitted in the dimensional analysis. Using the scalings

$$p \sim G\frac{m\rho}{r}, \qquad m \sim \rho r^3, \tag{A2}$$

we obtain the following form for Λ in terms of mass, length, and time dimensions:

$$\Lambda \sim [M]^{\alpha+\beta+\gamma-\kappa}[L]^{-\alpha+\delta-3\beta+\lambda+3\kappa}[T]^{-2\alpha-\lambda-2\kappa}. \tag{A3}$$

Since Λ is required to be dimensionless, we have the three algebraic relations

$$\lambda = -2(\alpha+\kappa), \qquad \kappa = \alpha+\beta+\gamma, \tag{A4}$$

and

$$-\alpha+\delta-3\beta+\lambda+3\kappa = 0. \tag{A5}$$

The first two relations simply express λ and κ in terms of the other parameters. Using them in Eq. (A5), we obtain

$$\gamma+\delta = 2(\alpha+\beta), \tag{A6}$$

which represents the true dimensional degree of freedom. It is straightforward (if tedious) to verify that all PN terms appearing in the PPN equations of Sec. II are consistent with Eq. (A6).

All $\alpha < 0$ terms are divergent at the surface and need not be considered. As we shall shortly see, all terms with $\alpha \geq 4$ are divergent at $r = 0$ in both structure equations, and therefore they should be discarded. The $\alpha = 3$ terms are singular in the dp/dr equation and can be discarded by the same argument; $\alpha = 3$ terms are regular in the dm/dr equation, but they are always dominated in magnitude by the $\alpha < 3$ terms, and therefore they will not be presented in detail here. Therefore our strategy hereafter is to focus on the particular cases $\alpha = 0$ (no pressure dependence) and $\alpha = 1, 2$ (linear and quadratic scaling with the pressure).

[5]Note that this combination is oblivious to the presence of dimensional coupling constants that might appear in modified theories of gravity.

1. Terms with $\alpha = 0$

Starting with the $\alpha = 0$ case we have

$$\gamma + \delta = 2\beta. \tag{A7}$$

The resulting form of Λ in geometric units is

$$\Lambda \sim (r^2\rho)^\beta \left(\frac{m}{r}\right)^\gamma. \tag{A8}$$

Formally, this combination is of order $(m/r)^{\beta+\gamma}$. Therefore, we can generate N-PN terms if $\beta + \gamma = N$. These are of the form

$$\Lambda_N(\beta) \sim (r^2\rho)^\beta \left(\frac{m}{r}\right)^{N-\beta}, \tag{A9}$$

with $\beta = 0, \pm 1, \pm 2, \ldots$ For instance, the first few 1PN and 2PN terms of this series are (we start from $\beta = -1$ for reasons explained below)

$$\Lambda_1(-1) \sim \frac{m^2}{r^4\rho}, \qquad \Lambda_1(0) \sim \frac{m}{r}, \qquad \Lambda_1(1) \sim r^2\rho, \tag{A10}$$

$$\Lambda_2(-1) \sim \frac{m^3}{r^5\rho}, \qquad \Lambda_2(0) \sim \frac{m^2}{r^2}, \qquad \Lambda_2(1) \sim r\rho m. \tag{A11}$$

2. Terms with $\alpha = 1$

The $\alpha = 1$ group of terms can be obtained with the same procedure. We have

$$\gamma + \delta = 2(1 + \beta), \tag{A12}$$

and this leads to terms of the form

$$\Lambda \sim r^2 p (r^2\rho)^\beta \left(\frac{m}{r}\right)^\gamma. \tag{A13}$$

Since $r^2 p$ is a 2PN term, the resulting N-PN combination should take the form

$$\Lambda_N(\beta) \sim r^2 p (r^2\rho)^\beta \left(\frac{m}{r}\right)^{N-2-\beta}. \tag{A14}$$

The first few 1PN and 2PN terms generated from this expression are

$$\Lambda_1(-1) \sim \frac{p}{\rho}, \qquad \Lambda_1(0) \sim \frac{r^3 p}{m}, \qquad \Lambda_1(1) \sim \frac{r^6\rho p}{m^2}, \tag{A15}$$

$$\Lambda_2(-1) \sim \frac{pm}{\rho r}, \qquad \Lambda_2(0) \sim r^2 p, \qquad \Lambda_2(1) \sim \frac{r^5\rho p}{m}. \tag{A16}$$

3. Terms with $\alpha = 2$

Finally, we consider the $\alpha = 2$ terms. The corresponding Λ_N combination is

$$\Lambda_N(\beta) = (r^2 p)^2 (r^2\rho)^\beta \left(\frac{m}{r}\right)^{N-4-\beta}, \tag{A17}$$

and from this we have

$$\Lambda_1(-1) \sim \frac{r^4 p^2}{m^2\rho}, \qquad \Lambda_1(0) \sim \frac{r^7 p^2}{m^3}, \tag{A18}$$

$$\Lambda_2(-1) \sim \frac{r^3 p^2}{\rho m}, \qquad \Lambda_2(0) \sim \frac{r^6 p^2}{m^2}. \tag{A19}$$

4. Generic N-PN-order terms and constraints

It is now not too difficult to see that a N-PN order term with an arbitrary p^α scaling and with Π reintroduced is given by the universal formula

$$\Lambda_N(\alpha, \beta, \theta) \sim \Pi^\theta (r^2 p)^\alpha (r^2\rho)^\beta \left(\frac{m}{r}\right)^{N-2\alpha-\beta-\theta}. \tag{A20}$$

As discussed in Sec. III B, different threads of reasoning lead to the constraint $\beta \geq -1$. The first one has to do with avoiding a divergence at the stellar surface (this already has allowed us to filter out all $\alpha < 0$ terms). An inspection of the two stellar structure equations reveals that terms with $\alpha = \theta = 0$ should scale as

$$\rho \Lambda_N(0, \beta, 0) \sim \rho^{1+\beta} \tag{A21}$$

in the vicinity of the surface, and therefore we ought to take $\beta \geq -1$ in order to avoid a surface singularity. This argument still allows for $\beta < -1$ values in the Λ_N terms with $\alpha, \theta > 0$, since these terms have a smoother profile as a result of the vanishing of p and Π at the surface.

The second thread is no more than a heuristic argument and has to do with the expectation that for a broad family of gravity theories the solution for the metric (and its derivatives) should scale as $\sim(\epsilon + \tau p)^n = \rho^n(1 + \Pi + \tau p/\rho)^n$ with the fluid parameters [where τ and n are $\mathcal{O}(1)$ numbers]. From this it follows that negative powers of ρ will come in the form of dimensionless PN terms $\sim\rho^{n-1}(p/\rho)^k$, where $k = n, n-1, \ldots$ (note that a factor ρ has been absorbed by the Newtonian prefactor in the structure equations). As a consequence, ρ^{-1} is the only possible negative power in a PN expansion. Obviously, this argument automatically takes care of the regularity of any $\Lambda_N(\alpha, \beta, \theta)$ term at the surface.

The exclusion of all $\alpha \geq 4$ terms comes about as a consequence of regularity at the stellar center. Near the origin (where p, ρ, Π take finite nonzero values) a $\Lambda_N(\alpha, \beta, \theta)$ term behaves as

$$\Lambda_N(r \to 0) \sim r^{2(N-\alpha-\theta)}. \tag{A22}$$

The corresponding terms in the stellar structure equations will behave as

$$\frac{dp}{dr} \sim \frac{\rho m}{r^2}\Lambda_N \sim r^{2(N-\alpha-\theta)+1}, \tag{A23}$$

$$\frac{dm}{dr} \sim r^2\rho\Lambda_N \sim r^{2(N-\alpha-\theta+1)}, \tag{A24}$$

and therefore regularity at the center dictates the following limits for each equation:

$$\frac{dp}{dr}:\ 0 \le \alpha \le N-\theta, \tag{A25}$$

$$\frac{dm}{dr}:\ 0 \le \alpha \le N+1-\theta. \tag{A26}$$

We can also see that these conditions entail the following limits for θ:

$$\frac{dp}{dr}:\ 0 \le \theta \le N, \qquad \frac{dm}{dr}:\ 0 \le \theta \le N+1. \tag{A27}$$

For the particular case of 2PN-order terms, we then have

$$\frac{dp}{dr}:\ 0 \le \theta \le 2, \qquad 0 \le \alpha \le 2-\theta, \tag{A28}$$

$$\frac{dm}{dr}:\ 0 \le \theta \le 3, \qquad 0 \le \alpha \le 3-\theta, \tag{A29}$$

which shows that all $\alpha \ge 4$ terms are to be excluded and that $\alpha = 3$ terms can only appear in the mass equation.

APPENDIX B: THE NEWTONIAN AND RELATIVISTIC LANE-EMDEN EQUATIONS

In this appendix we review the nonrelativistic and relativistic Lane-Emden equations. The former equation is classic textbook material (see, e.g., Ref. [64]) and therefore is just sketched here. The somewhat less familiar relativistic extension was developed by Tooper [65,66] and is discussed in a bit more detail below. Our definition for the polytropic EOS, i.e., $p = K\rho^{1+1/n}$, is the same as the one adopted in Ref. [66] but is different than the one used in Tooper's earlier paper [65], i.e., $p = K\epsilon^{1+1/n}$. This subtle difference, combined with the choice between p_c/ρ_c or p_c/ϵ_c (the "c" index refers to the stellar center) for the scale of the system, leads to slightly different Lane-Emden equations.

1. The Newtonian Lane-Emden equation

In Newtonian gravity, one can express the hydrostatic equilibrium equation for spherical nonrotating stars in terms of dimensionless parameters for the pressure, the density, and the radial coordinate. If the EOS is polytropic (i.e., according to our definition, $p = K\rho^{1+1/n}$) the equations governing the dimensionless quantities are scale invariant, depending only on the polytropic index n. By writing the density and the pressure as

$$\theta^n \equiv \frac{\rho}{\rho_c}, \qquad p = K\rho_{\mathrm{c}}^{1+1/n}\theta^{n+1}, \tag{B1}$$

and introducing the dimensionless radial coordinate

$$r = \alpha\xi, \qquad \alpha \equiv \left[\frac{(n+1)K}{4\pi G}\rho_{\mathrm{c}}^{-1+1/n}\right]^{1/2}, \tag{B2}$$

the Newtonian stellar structure equations

$$\frac{dp}{dr} = -\frac{Gm_{\mathrm{N}}}{r^2}\rho, \tag{B3}$$

$$\frac{dm_{\mathrm{N}}}{dr} = 4\pi r^2\rho \tag{B4}$$

lead to

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \tag{B5}$$

This is the famous Lane-Emden equation, and its scale-invariant solutions describe all possible fluid configurations in terms of the single parameter n.

2. The relativistic Lane-Emden equations

Generalizing the Lane-Emden formalism to GR is a straightforward task, but this comes at the price of losing the scale-invariance property of the Newtonian treatment. In relativity we can define the polytropic EOS in the same way as before, where ρ is the baryonic rest mass density. The polytropic exponent is defined as

$$\Gamma = 1 + \frac{1}{n} = \frac{\rho}{p}\frac{dp}{d\rho} = \frac{\epsilon + p}{p}\frac{dp}{d\epsilon}. \tag{B6}$$

Then the energy density ϵ and the internal energy Π are given by

$$\epsilon = \rho + np, \tag{B7}$$

which implies

$$\Pi = n\frac{p}{\rho}. \tag{B8}$$

This observation was used in the argument leading to Eq. (53).

We can now introduce the relativistic version of the Lane-Emden equations. In analogy with the Newtonian case we define $\rho = \rho_{\mathrm{c}}\theta^n$, $r = a\xi$, and $p = K\rho_{\mathrm{c}}^{1+1/n}\theta^{n+1}$.

The ratio between the central pressure and the central energy density,

$$\lambda \equiv \frac{p_c}{\epsilon_c} = \frac{K\rho_c^{1+1/n}}{\rho_c + nK\rho_c^{1+1/n}}, \tag{B9}$$

is a convenient measure of the importance of relativistic effects in the system. Note that our definition deviates from Tooper's [66], who prefers to use the ratio p_c/ρ_c.

The energy density is then

$$\begin{aligned}\epsilon &= \rho_c\theta^n + nK\rho_c^{1+1/n}\theta^{n+1} \\ &= \epsilon_c[1 + n\lambda(\theta - 1)]\theta^n.\end{aligned} \tag{B10}$$

We now want to derive a dimensionless form of the TOV equations (7a) and (7b). The definition of the mass function m_T implies

$$\frac{dm_T}{d\xi} = 4\pi\epsilon_c a^3[1 + n\lambda(\theta - 1)]\theta^n\xi^2. \tag{B11}$$

In terms of the dimensionless mass

$$\bar{m} \equiv \frac{m_T}{a^3\epsilon_c}, \tag{B12}$$

this becomes

$$\frac{d\bar{m}}{d\xi} = 4\pi[1 + n\lambda(\theta - 1)]\theta^n\xi^2. \tag{B13}$$

From the TOV equation for the pressure we similarly obtain, after some manipulations,

$$\begin{aligned}\frac{d\theta}{d\xi} = &-\frac{\bar{m}}{\xi^2}(1 - n\lambda)\left[1 + (n+1)\frac{\lambda}{1 - n\lambda}\theta\right] \\ &\times\left(1 + \lambda\frac{4\pi\xi^3\theta^{n+1}}{\bar{m}}\right)\left[1 - 2(n+1)\lambda\frac{\bar{m}}{\xi}\right]^{-1}.\end{aligned} \tag{B14}$$

In the present case the characteristic length scale is

$$a = [(n+1)K\rho_c^{-1+1/n}(1 - n\lambda)^2]^{1/2}. \tag{B15}$$

At this point we would like to define dimensionless quantities that come from the relativistic Lane-Emden equations. The central baryonic rest-mass density is related to λ as [see Eq. (B9)]

$$\rho_c = K^{-n}\ell^n, \qquad \ell \equiv \frac{\lambda}{1 - n\lambda}. \tag{B16}$$

The factor K^{-n} has units of mass density (or inverse square length in geometrical units), and therefore the dimensionless rest-mass density is

$$\bar{\rho} \equiv \rho K^n = \ell^n\theta^n. \tag{B17}$$

Similarly, the length scale a takes the form

$$a = K^{n/2}\sqrt{(n+1)\ell^{1-n}}(1 - n\lambda), \tag{B18}$$

where $K^{n/2}$ has dimensions of length. The dimensionless radius is defined as

$$\bar{r} \equiv rK^{-n/2} = \sqrt{(n+1)\ell^{1-n}}(1 - n\lambda)\xi. \tag{B19}$$

The remaining dimensionless parameters are

$$\bar{\epsilon} \equiv \epsilon K^n = \left(\frac{\ell^n}{1 - n\lambda}\right)[1 + n\lambda(\theta - 1)]\theta^n, \tag{B20}$$

$$\bar{\mu} \equiv m_T K^{-n/2} \tag{B21}$$

$$= [\sqrt{(n+1)\ell^{1-n}}(1 - n\lambda)]^3\left(\frac{\ell^n}{1 - n\lambda}\right)\bar{m}, \tag{B22}$$

$$\bar{p} \equiv pK^n = \ell^{n+1}\theta^{n+1}, \tag{B23}$$

$$\Pi = n\frac{\bar{p}}{\bar{\rho}} = n\ell\theta. \tag{B24}$$

All of the above dimensionless profiles are functions of ξ, n, and λ. At variance with the Newtonian treatment, the relativistic Lane-Emden formalism does not allow for a simple algebraic mass-radius relation $M(R)$. This is also related to the fact that the system is not scale invariant, due to the presence of λ in the equations.

APPENDIX C: THE PPN POTENTIALS

The goal of this appendix is to study the behavior of the potentials U, E, and Ω appearing in the PPN stellar structure equations (9a) and (9b), first derived by Ciufolini and Ruffini [47]. By means of a mass function redefinition (see Sec. II), these potentials can be eliminated at 1PN order, but they could still appear at 2PN order and higher.

Given the 2PN precision of our calculations we can write these potentials as

$$U(r) = -\int_0^r dr'\frac{m_N}{r'^2} + U(0), \tag{C1a}$$

$$E(r) = 4\pi\int_0^r dr' r'^2\rho\Pi, \tag{C1b}$$

$$\Omega(r) = -4\pi\int_0^r dr' r'\rho m_N, \tag{C1c}$$

where all right-hand side quantities are computed in Newtonian theory. In Eqs. (C1a)–(C1c), $m_N(r)$ denotes the Newtonian mass function

$$m_{\rm N}(r) = 4\pi \int_0^r dr' \rho r'^2 = 4\pi m_{\rm b} \int_0^r dr' n_{\rm b} r'^2, \qquad \text{(C2)}$$

where $n_{\rm b}$ is the baryon number density. The integral quantities U, E, and Ω represent the system's gravitational potential energy, internal energy, and gravitational potential energy, respectively [34]. They appear as dimensionless PN terms in the form of reduced potentials: U, $E/m_{\rm N}$, $\Omega/m_{\rm N}$ [see Eqs. (9a) and (9b)].

The radial profiles of the three potentials inside the star can be determined by first integrating the Newtonian hydrostatic equilibrium equations (B3) and (B4) to find $m_{\rm N}$ and p as functions of r. Using realistic EOS data tables for $p(\rho)$, we can subsequently compute the internal density per unit mass $\Pi(p)$ and the mass density $\rho(p) = m_{\rm b} n_{\rm b}(p)$, and then numerically evaluate the potentials inside the star by integration.

Some insight into the nature of these potentials can be obtained by rewriting Eqs. (C1a)–(C1c) in the form

$$U = \frac{m_{\rm N}}{r} + 4\pi \int_r^R dr' r' \rho, \qquad \text{(C3a)}$$

$$\frac{E}{m_{\rm N}} = \Pi - \frac{1}{m_{\rm N}} \int_0^r dr' m_{\rm N} \frac{d\Pi}{dr'}, \qquad \text{(C3b)}$$

$$\frac{\Omega}{m_{\rm N}} = -\frac{m_{\rm N}}{2r} - \frac{1}{2m_{\rm N}} \int_0^r dr' \left(\frac{m_{\rm N}}{r'}\right)^2 = 4\pi \frac{r^3 p}{m_{\rm N}} - \frac{12\pi}{m_{\rm N}} \int_0^r dr' r'^2 p. \qquad \text{(C3c)}$$

Note that the integration constant for U has been fixed by requiring $U(R) = M/R$ at the stellar surface, while those for E and Ω have been set to zero in order to have regularity of $E/m_{\rm N}$ and $\Omega/m_{\rm N}$ at $r = 0$. The values of the potentials at the stellar center are

$$U(0) = 4\pi \int_0^R dr r \rho, \qquad \frac{\Omega}{m_{\rm N}}(0) = 0, \qquad \frac{E}{m_{\rm N}}(0) = \Pi_{\rm c}. \qquad \text{(C4)}$$

From Eqs. (C3a)–(C3c), we can see that $E/m_{\rm N}$ and $\Omega/m_{\rm N}$ are (partially) expressed in terms of the nonintegral 1PN terms

$$\frac{m_{\rm N}}{r}, \Pi, \frac{r^3 p}{m_{\rm N}}. \qquad \text{(C5)}$$

This suggests the possibility that the behavior of all three potentials could be captured by linear combinations of nonintegral 1PN terms. If true, this would mean that any 2PN term involving $U, E/m_{\rm N}$, or $\Omega/m_{\rm N}$ is effectively accounted for by the presence of the other terms in the post-TOV formulas. For instance, this idea can be demonstrated for U and for the special case of a polytropic system. Starting from Eq. (C1a) and expressing $m_{\rm N}$ in terms of dp/dr, after an integration by parts and the use of Eq. (B6) we arrive at

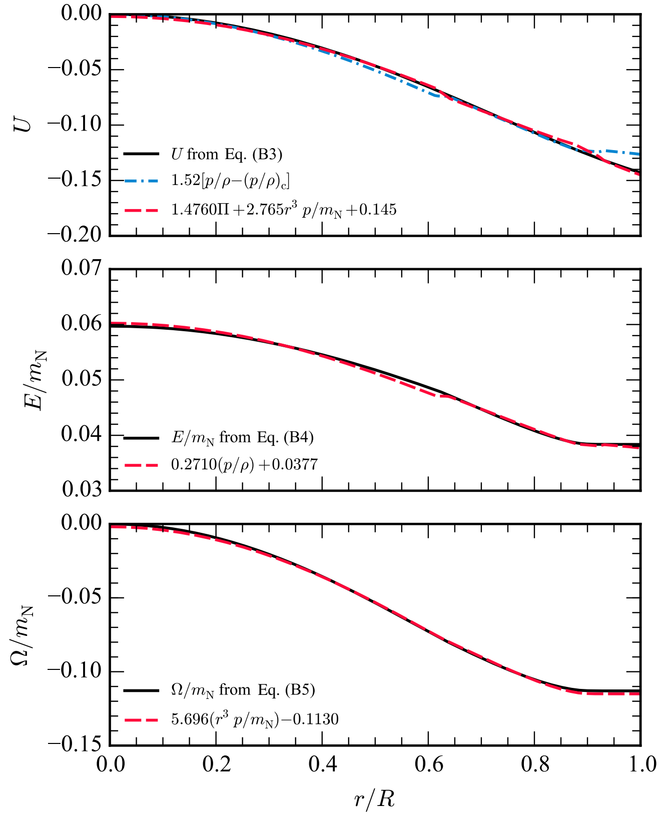

FIG. 11 (color online). *Integral PN potentials and 1PN terms.* The radial profiles of the integral potentials U, $E/m_{\rm N}$, and $\Omega/m_{\rm N}$ are well fitted by linear functions of the nonintegral potentials p/ρ and $r^3 p/m_{\rm N}$. In all plots, the radial coordinate is normalized to the stellar radius R.

$$U = (n+1)\left(\frac{p}{\rho} - \frac{p_{\rm c}}{\rho_{\rm c}}\right) + U(0). \qquad \text{(C6)}$$

We know that for a polytrope $\Pi = np/\rho$, which means that we can also write

$$U = \frac{(n+1)}{n}(\Pi - \Pi_{\rm c}) + U(0). \qquad \text{(C7)}$$

For a polytropic model, therefore, U can be written exactly as a linear function of p/ρ or Π.

We have verified that U, $E/m_{\rm N}$, and $\Omega/m_{\rm N}$ can be approximated by similar linear functions for the case of realistic EOSs. As an illustration, in Fig. 11 we consider a stellar model built using the APR EOS with a central mass density of 0.58×10^{15} g/cm^3, Newtonian mass $m_{\rm N} = 1.50 M_\odot$, and radius $R = 14.8$ km. For this model we plot the radial profiles of U (top panel), $E/m_{\rm N}$ (middle panel), and $\Omega/m_{\rm N}$ (bottom panel). The figure shows that the profiles of the three potentials can be accurately reproduced by linear combinations of the 1PN terms in Eq. (C5), and that U is reasonably well fit by a linear function of p/ρ, as suggested by Eq. (C6). This latter fit breaks down near the surface, but with a different combination of 1PN terms (namely, Π and $r^3 p/m_N$) one can produce a near-perfect fit.

In conclusion, the addition of the integral potentials U, E/m_N, and Ω/m_N in the 2PN terms is unnecessary because their behavior can be captured by linear combinations of the nonintegral PN terms which are already included in the post-TOV equations (1a) and (1b).

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