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Test particle motion in the Born-Infeld black hole

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In this paper, we review the classification of the orbits followed by charged massive test particles in the gravitational background of the black hole solutions of Einstein-Born-Infeld spacetime. Even though some features are quite similar to those of Reissner-Nordström spacetime, there are also important differences, particularly those related to the effective potential governing the orbital motion. Explicit solutions involving Weierstrass functions are given for a pair of specific scenarios.

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I. INTRODUCTION

In 1872 James Clerk Maxwell unified electricity and magnetism in a single theory. In Maxwell's theory of electromagnetism, the field of a pointlike charge is singular at the position of the charge. As a consequence, it has infinite self-energy. To avoid this unattractive feature, in 1934 Born and Infeld [1,2] proposed a nonlinear electrodynamics with the goal of obtaining a finite value for the self-energy of a pointlike charge [2]. In this theory, the electric field of a point charge is regular at the origin. Also, its total energy is finite.

In recent years the Born-Infeld action has received considerable attention due to several reasons. In the context of superstring theory, for example, the low energy dynamics of D-branes is governed by the Born-Infeld (BI) action [3]. Also, when analyzing the low-energy effective action for an open superstring, loop calculations lead to BI type actions [4]. For detailed discussions on several aspects of the BI theory in string theory see [5] and [6] for instance.

Other motivations arise from a purely constructive generalization of Einstein-Maxwell systems. In particular the extension of the Reissner-Nordström (RN) black hole solutions in Einstein-Maxwell theory to the charged black hole solutions in Einstein-Born-Infeld (EBI) theory, with or without a cosmological constant, has attracted some attention in recent years. Different aspects of these black holes have been studied including their thermodynamical properties, phase transitions, geodetical motion and higher-dimensional generalizations [7–15].

In this paper, we will focus on the study of the motion of electrically and magnetically charged test particles in BI electrodynamics. The black hole solution for EBI gravity was obtained by García *et al.* [16] in 1984 and two years later Demianski [17] found a static spherically symmetric solution of the EBI equations that is regular at the origin, the so-called EBIon.

The black hole solution we will consider is well known as the nonlinear generalization of the RN black hole solutions characterized by the mass M and the charge Q of the black hole and the BI parameter b, that is related to the strength of the electromagnetic field at the position of the charge, usually to be located at the origin.

In recent years there has been a growing interest in the study of geodesics of certain black holes [18,19]; in particular, the RN solution turns out be the ultimate fate of the gravitational collapse of a very massive star with electric charge. In this context, the properties of a black hole including its geodesics and its generalization to nonlinear electrodynamics is of fundamental interest. Since we already know the black hole solution that generalizes the RN solution, it is important to study the complete classification of orbits for this solution.

There are already some papers written in the literature in this direction; Bretón discussed in a series of papers the test particle trajectories for the static-charged EBI black hole [20,21]. Properties of null geodesics of static charged black holes in EBI gravity were presented by Sharmanthie [22] very recently. The aim of our paper is to complete the discussion about the classification of orbital motion by analyzing the problem in a more systematic way and by addressing some issues that had not been discussed before.

This paper is organized as follows: In Sec. II, we review the EBI solution and discuss the conditions for the existence of an EBI extreme black hole. The equations of motion for a test particle with both electric and magnetic charge moving in EBI spacetime are derived in Sec. III using the Hamilton-Jacobi formalism and the complete classification of the trajectories is presented in Sec. IV. Analytic explicit solutions are given in Sec. V, for both the radial and angular differential equations of the orbital motion and in Sec. VI we discuss the issue of observables. We end up with some remarks in the Conclusions.

Throughout this paper we will use geometrical units $G = c = 1$.

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II. EBI SPACETIME

The history of finding solutions to the Einstein equations of motion coupled to the energy momentum tensor of the nonlinear electrodynamics of BI [2] goes back to the first attempt made by Pellicer and Torrence [23]. They found a static spherical symmetric solution for a point charge source, which approaches the RN solution at large distances from the source.

Some years later, Morales [24] found that the Bertotti-Robinson solution admitted an interpretation in terms of nonlinear electrodynamics. Soon after, García *et al.* [16] found all type-D solutions in the Petrov classification of the EBI system of equations [16]. Among the solutions they obtained was the generalized RN black hole metric again, usually called EBI black hole. In this section we give a short summary of the way in which the solution is obtained (for a detailed derivation see [16]).

A. EBI black hole

The action for the gravitational field coupled to a generic nonlinear electrodynamics is

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi} - \mathcal{L}(F)\right). \tag{1}$$

Here R denotes the curvature scalar obtained from the metric coefficients $g_{\mu\nu}$, $g \equiv \det|g_{\mu\nu}|$ and $\mathcal{L}(F)$ is the nonlinear electrodynamics Lagrangian density, which depends in a nonlinear way on the two invariants of the electromagnetic tensor F. For the BI nonlinear electrodynamics we have explicitly

$$\mathcal{L}_{BI} = b^2\left(1 - \sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{2b^2} - \frac{(F_{\mu\nu}\tilde{F}^{\mu\nu})^2}{4b^4}}\right), \tag{2}$$

where

$$\tilde{F}_{\mu\nu} = -\frac{1}{2\sqrt{-g}}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{3}$$

denotes the dual tensor of the electromagnetic tensor and $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor. The parameter b is the maximum electromagnetic field intensity and has dimensions of $[\text{length}]^{-2}$. Notice that this Lagrangian reduces to the Maxwell one in the strong field limit ($b \to \infty$)

$$\mathcal{L}_{BI}(F) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(F^4). \tag{4}$$

The full system of equations of motion derived from the action Eq. (1) is given by the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \tag{5}$$

and the electromagnetic field equations

$$\nabla_\mu(F^{\mu\nu}\mathcal{L}_{,F}) = 0. \tag{6}$$

In the Einstein field equations, Eq. (5), the energy momentum tensor is given by

$$T_{\mu\nu} = \mathcal{L}_{BI}g_{\mu\nu} - F_{\mu\sigma}F^{\sigma}_{\nu}, \tag{7}$$

and in the conservation laws, Eq. (6), $\mathcal{L}_{,F}$ represents the partial derivative of $\mathcal{L}_{BI}(F)$ with respect to F.

The static electrically charged black hole solution with spherical symmetry for the EBI system of equations is well known, it is given by the metric

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{8}$$

and the radial electric field

$$F_{\mu\nu} = \frac{Q}{\sqrt{r^4 + Q^2/b^2}}(\delta^r_\mu\delta^t_\nu - \delta^t_\mu\delta^r_\nu). \tag{9}$$

The function $\Delta = \Delta(r)$ in Eq. (8) is given by

$$\Delta := 1 - \frac{2M}{r} + \frac{2}{3}b^2r^2\left(1 - \sqrt{1 + Q^2/b^2r^4}\right) + \frac{4Q^2}{3r}\int_r^\infty \frac{ds}{\sqrt{s^4 + Q^2/b^2}}. \tag{10}$$

The last term is an elliptic integral of the first kind, which in the literature can be found written either in terms of the Legendre's elliptic integral: $F(\beta,\kappa) \equiv \int_\beta^\infty (1 - k^2\sin^2 s)^{-1/2}ds$, or in terms of the hypergeometric function ${}_2F_1(a,b;c;x)$ as follows

$$\int_r^\infty \frac{ds}{\sqrt{s^4 + Q^2/b^2}} = \frac{1}{2}\sqrt{\frac{b}{Q}}F\left[\arccos\left(\frac{br^2/Q - 1}{br^2/Q + 1}\right), \frac{1}{\sqrt{2}}\right] = \frac{1}{r}{}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; -\frac{Q^2}{b^2r^4}\right). \tag{11}$$

For a detailed deduction of Eq. (11) see [25].

In the following we write down the function Δ in terms of Legendre's elliptic function [20]

$$\Delta = 1 - \frac{2M}{r} + \frac{2}{3}b^2r^2\left(1 - \sqrt{1 + Q^2/b^2r^4}\right) + \frac{2Q^2}{3r}\sqrt{\frac{b}{Q}}F\left[\arccos\left(\frac{br^2/Q - 1}{br^2/Q + 1}\right), \frac{1}{\sqrt{2}}\right]. \tag{12}$$

The physical interpretation of the parameters in the function Δ is the following: M is the mass and Q is the electric charge of the black hole. The parameter b is the BI parameter which corresponds to the magnitude of the electric field at $r = 0$. The solution can have either zero (naked singularity), one or two horizons depending on the values of these parameters. This conclusion is obtained by simple inspection of the condition $\Delta = 0$.

To have a better understanding on the nature of the horizons, we have plotted in Fig. 1 the mass M as a function of the horizon radius. For the sake of clarity we have fixed the value of the BI parameter b and the value of the electric charge Q as well.

As can be seen from this plot, there is a critical value $M^\star$ for the mass of the black hole that leads to different physical scenarios: first, for values of $M < M^\star$ we have a naked singularity; we will not discuss this case any further in this paper but see [26] for an interesting physical consequence. For $M = M^\star$ we have a black hole solution with one horizon (dashed line) and for values of $M > M^\star$ we have a black hole with two horizons; we will denote by $r_{h_\pm}$ the inner (outer) radii respectively in this case.

The metric in Eq. (8) has the expected limits. In the strong field limit, $b \to \infty$, we recover the RN black hole solution in agreement with the Maxwell limit Eq. (4). As expected in this limit, the radial electric field (9) approaches the Maxwellian expression of the electric field $E = Q/r$, which diverges at the origin. For the function Δ we have

$$\lim_{b\to\infty} \Delta = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{13}$$

As usual, by setting further $Q = 0$, we obtain also the Schwarzschild black hole solution. Additionally, we can obtain the Schwarzschild black hole solution by taking the weak field limit $b \to 0$ and then $Q = 0$.

For large values of r with $b \neq 0$ and finite, the function Δ becomes the unity and we obtain a flat metric, meaning that the EBI black hole is an asymptotically flat solution. It is clear that for small values of the b parameter ($b \ll 1$) we have a black hole solution that looks like very similar to the Schwarzschild black hole and for large values of b ($b \gg 1$) we have a solution that is very similar to the RN black hole.

As it has been shown in [16], because the BI theory has the freedom of electromagnetic duality rotations, the EBI black hole solution can include also a magnetic charge G; the corresponding solution is obtained simply from the electric case by the substitution $Q \to \sqrt{Q^2 + G^2}$, i.e., the general case can be obtained by making this replacement in all the relevant expressions for the function $\Delta(r)$ that appears in the EBI metric, namely Eqs. (10)–(12).

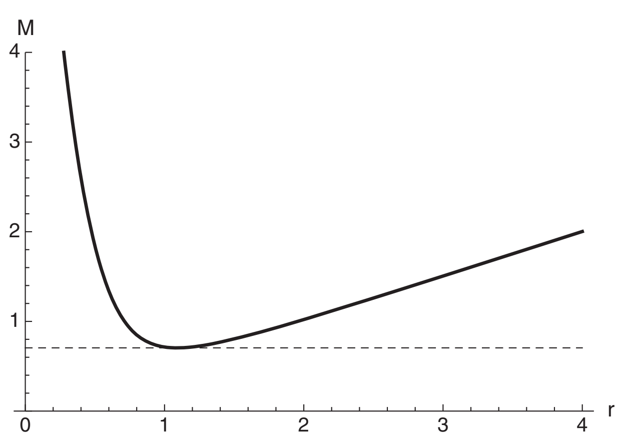

FIG. 1. The mass M of the EBI black hole as a function of r ($Q = 2, b = 5$); at $M = 0.7$ there is only one horizon.

B. Extreme black holes

The necessary and sufficient conditions to have an extreme EBI black hole solution ($r_{h_+} = r_{h_-} \equiv r_{ex}$) are $\Delta = 0$ and $d\Delta/dr = 0$. Combining both conditions, we obtain from Eq. (12) a constraint that determines the horizon radii r_{ex} for the extreme black hole in terms of the electric charge

$$1 + 2\left(b^2 r_{ex}^2 - \sqrt{b^4 r_{ex}^4 - Q^2 b^2}\right) = 0, \tag{14}$$

its solution being given by

$$r_{ex}^2 = Q^2 - \frac{1}{4b^2}. \tag{15}$$

Hence the horizon belonging to the extreme EBI black hole is determined by the positive root of Eq. (15), i.e.,

$$r_{ex} = \sqrt{Q^2 - \frac{1}{4b^2}}. \tag{16}$$

This solution is meaningful only if the radicand is positive, i.e., if $Q > 1/2b$. When the radicand is zero, we have the case of a spacetime singularity. In the case $Q < 1/2b$ we obtain a naked singularity.

It is possible to express the extremality condition as a function of the form $M = M(b, Q)$ by substituting back the expression of r_{ex} in the condition $\Delta = 0$. This gives

$$M(r_{ex}) = \frac{r_{ex}}{2} - \frac{b^2 r_{ex}^3}{3}\left(1 - \sqrt{1 + Q^2/b^2 r_{ex}^4}\right) + \frac{Q^2}{3}\sqrt{\frac{b}{Q}} F\left(\arccos\left\{\frac{b r_{ex}^2/Q - 1}{b r_{ex}^2/Q + 1}\right\}, \frac{1}{\sqrt{2}}\right). \tag{17}$$

In the strong field limit, $b \to \infty$, this condition reduces to the well-known condition for the extremal RN black hole solution $r_{ex} = M = Q$. Figure 2 shows $M(r_{ex})$ as a function of b for Q fixed. It is clear that for a given value of Q, the horizon size depends strongly on the choice of b.

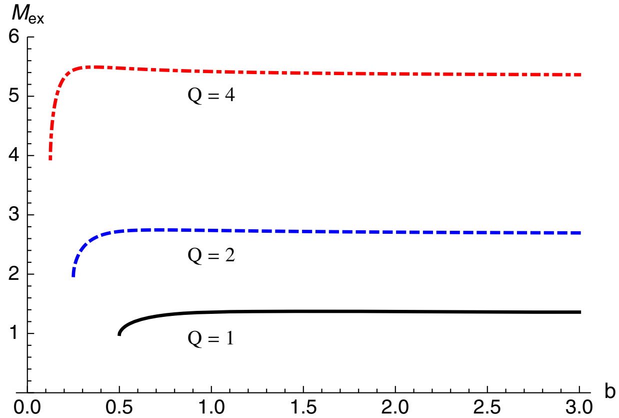

FIG. 2 (color online). The mass M_{ex} of the extreme EBI black hole as a function of b.

III. THE ORBITAL EQUATIONS OF MOTION

The EBI solution is described by the metric in Eq. (8). In the following we will be concerned with the dyon case where both the electric charge Q and the magnetic charge G of the source are nonvanishing. In this scenario, the field strength $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, and its dual $\tilde{F}_{\mu\nu} = \tilde{A}_{\nu,\mu} - \tilde{A}_{\mu,\nu}$ are derived respectively from the vector potentials A_μ and $\tilde{A}_\mu$; their explicit forms are known from the analysis in [16] of type-D solutions in EBI spacetime

$$A_t = Q\int_r^\infty \frac{ds}{\sqrt{s^4 + Q^2/b^2}}, \qquad A_\phi = -G\cos\theta,$$
$$\tilde{A}_t = iG\int_r^\infty \frac{ds}{\sqrt{s^4 + Q^2/b^2}}, \qquad \tilde{A}_\phi = iQ\cos\theta. \tag{18}$$

The equations of motion of test particles in the EBI spacetime can be analyzed using the Hamilton-Jacobi (HJ) equation, which can be constructed from a constant of motion that we always have at our disposal for geodesics: metric compatibility implies that along the geodesic path the quantity

$$\delta = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}, \tag{19}$$

is constant. Of course, for a massive particle we typically choose $\lambda = \tau$ (proper time), $\delta = 1$, and the above relation simply becomes $m^2 = -g^{\mu\nu}p_\mu p_\nu$. For a massless particle we always have $\delta = 0$. We will also be concerned with spacelike trajectories (even though they do not correspond to paths of particles), for which we will choose $\delta = -1$.

The Hamilton-Jacobi (HJ) equation is given by [27]

$$m^2\delta = -g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu}\right)\left(\frac{\partial S}{\partial x^\nu}\right). \tag{20}$$

As we have mentioned, we are interested in studying the orbital motion of charged particles, both electrically and magnetically, in EBI spacetime. For this purpose, we use the minimal coupling prescription defined by $p_\mu \to p_\mu - qA_\mu + ig\tilde{A}_\mu$ to account for all electromagnetic interactions; the HJ equation for a test particle with electric charge q and magnetic charge g is then

$$m^2\delta = -g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu} - qA_\mu + ig\tilde{A}_\mu\right)\left(\frac{\partial S}{\partial x^\nu} - qA_\nu + ig\tilde{A}_\nu\right). \tag{21}$$

In our case, the Hamiltonian does not depend explicitly on the coordinates τ and ϕ, i.e., these coordinates are cyclical and thus there are conserved quantities. This allows us to consider the following Ansatz

$$S = -Et + L\phi + S_1(r) + S_2(\theta), \tag{22}$$

for the action S. The parameter δ, as mentioned before, is equal to 0 for a massless particle and equal to 1 for a massive particle. On the other hand, the constants E and L are identified respectively with the energy and the angular momentum, along the z direction, of the test particle. As noted from Eq. (18), the terms qA_ϕ and $q\tilde{A}_\phi$ in Eq. (21) have a nontrivial dependence on the angular variable θ; in consequence, even though the EBI metric Eq. (8) is spherically symmetric, the motion followed by a massive test particle possesses axial symmetry.

There are two Killing vectors associated with the stationarity and axisymmetry of the EBI spacetime:

$$\xi^\mu_{(t)} \equiv (\partial_t)^\mu = (1,0,0,0),$$
$$\xi^\mu_{(\phi)} \equiv (\partial_\phi)^\mu = (0,0,0,1). \tag{23}$$

The EBI spacetime also has an irreducible Killing tensor given by

$$K_{\mu\nu} \equiv 2r^2 l_{(\mu}n_{\nu)} + r^2 g_{\mu\nu} = 2r^2 m_{(\mu}\bar{m}_{\nu)}, \tag{24}$$

with the null tetrad defined by

$$l^\mu \equiv (r^2, \Delta_r, 0, 0)/\Delta_r$$
$$n^\mu \equiv (r^2, -\Delta_r, 0, 0)/2r^2,$$
$$m^\mu \equiv (0, 0, 1, i/\sin\theta)/\sqrt{2}r, \tag{25}$$

where $l^\mu n_\mu = -1$ and $m^\mu \bar{m}_\mu = 1$ while all the other inner products vanish. The metric $g_{\mu\nu}$ can be written in terms of the null vectors as $g_{\mu\nu} = -2l_{(\mu}n_{\nu)} + 2m_{(\mu}\bar{m}_{\nu)}$. Here we have defined $\Delta_r \coloneqq r^2\Delta$.

Leaving aside the effect of self-interaction, a charged particle in EBI spacetime can be regarded as a test particle that moves along a path obtained from Eq. (21). The coordinates of the trajectory $t(\tau)$, $r(\tau)$, $\theta(\tau)$ and $\phi(\tau)$ are all parametrized by the proper time τ. Furthermore, there are three integrals of motion from the symmetries of the EBI spacetime: the energy E, angular momentum L and Carter constant K [28], respectively. These are expressed as

$$E \equiv -\xi^{\mu}_{(t)} p_{\mu} = m \frac{\Delta_r}{r^2}\frac{\partial t}{\partial \tau} + \Delta_q I(r),$$
$$L \equiv \xi^{\mu}_{(\phi)} p_{\mu} = m r^2 \sin^2\theta \frac{\partial \phi}{\partial \tau} - \Delta_g \cos\theta,$$
$$K \equiv K^{\mu\nu} p_{\mu} p_{\mu} = p_{\theta}^2 + L^2 \left(\frac{\cos\theta}{\sin\theta}\right)^2. \tag{26}$$

Here $I(r)$ is the integral in Eq. (11); the fact that we are analyzing a charged particle in the vector potential of EBI spacetime is reflected on the presence of the factors [18] $\Delta_q \equiv \tilde{G}q - \tilde{Q}g$ and $\Delta_g \equiv \tilde{Q}q + \tilde{G}g$. Because functions of conserved quantities are also conserved, any function of K and the two other constants of the motion can be used as a third constant in place of K. This results in some confusion as to the form of Carter's constant. For example, it is sometimes more convenient to use $k \coloneqq K + L^2$ as the conserved quantity of motion.

For later convenience, we define dimensionless quantities ($r_s \coloneqq 2M$)

$$\tilde{r} \coloneqq \frac{r}{r_s}, \qquad \tilde{t} \coloneqq \frac{t}{r_s}, \qquad \tilde{\tau} \coloneqq \frac{\tau}{r_s},$$
$$\tilde{Q} \coloneqq \frac{Q}{r_s}, \qquad \tilde{G} \coloneqq \frac{G}{r_s}, \qquad \tilde{L} \coloneqq \frac{L}{r_s}. \tag{27}$$

The use of the Ansatz Eq. (22) in the HJ equation leads to a differential equation for each coordinate. At this stage, it is more convenient to parametrize the particle orbit with the so-called *Mino time* γ, which is related to the parameter $\tilde{\tau}$ as $d\tilde{\tau} \equiv \tilde{r}^2 d\gamma$ [29]. In terms of the Mino time, the first set of equations of motion are

$$\left(\frac{d\tilde{r}}{d\gamma}\right)^2 = R(\tilde{r}), \qquad \left(\frac{d\theta}{d\gamma}\right)^2 = \Theta(\theta), \tag{28}$$

where

$$R(\tilde{r}) \coloneqq \frac{\tilde{r}^4}{m^2}[E + \Delta_q I(\tilde{r})]^2 - \frac{\tilde{\Delta}_r}{m^2}(m^2\delta\tilde{r}^2 + k), \tag{29}$$

$$\Theta(\theta) \coloneqq \frac{k}{m^2} - \frac{1}{m^2\sin^2\theta}(\tilde{L} + \Delta_g \cos\theta)^2, \tag{30}$$

meanwhile the second set is

$$\frac{d\phi}{d\gamma} = \frac{1}{m\sin^2\theta}(\tilde{L} + \Delta_g\cos\theta), \tag{31}$$

$$\frac{d\tilde{t}}{d\gamma} = \frac{\tilde{r}^4}{m\tilde{\Delta}_r}[E - \Delta_q I(\tilde{r})]. \tag{32}$$

In the above expressions the following notation was used [18]: $\tilde{\Delta}_r \equiv \Delta_r / r_s^2$ and $I(\tilde{r})$ is the integral in Eq. (11) after rescaling. Notice that the θ- and ϕ-equations of motion are the same as in the RN case.

IV. CLASSIFICATION OF TEST PARTICLE TRAJECTORIES

We now proceed to solve the HJ Eqs. (28)–(32). They are rather complicated due to the function R in Eq. (29) and the function Θ in Eq. (30). These functions depend strongly on the constants of motion, the metric coefficients and the charges of the test particle and this in turn will influence the possible types of orbits that a particle may follow.

A. The θ–motion

The polar angle θ should certainly take only real values. From Eq. (30), we see that real solutions are allowed if the condition $\Theta \geq 0$ holds. This means that $k \geq 0$. Using now the new variable $\xi \coloneqq \cos\theta$, the θ-equation of motion in Eq. (30) becomes

$$\left(\frac{d\xi}{d\gamma}\right)^2 = \Theta_{\xi} \quad \text{with} \quad \Theta_{\xi} = a\xi^2 + b\xi + c, \tag{33}$$

where

$$a = -(k + \Delta_g^2), \quad b = -2\tilde{L}\Delta_g, \quad c = k - \tilde{L}^2. \tag{34}$$

It should be noticed that we obtain a quadratic polynomial on the right-hand side of this equation. It follows that $a < 0$ since $k \geq 0$. The turning points where Θ_{ξ} vanishes define the angles of two cones and the motion of test particles is confined to this region; it has been pointed out before that a similar feature appears in Taub-NUT and Kerr spacetimes [30,31]. In the special case when Δ_g vanishes, the motion takes place on a plane, as exemplified by the orbits of electrically charged or neutral particles in RN spacetime.

Let us now focus on the requirement $\Theta_{\xi} \geq 0$. We have first that the zeroes of this polynomial are given by

$$\xi_{1,2} = -\frac{\tilde{L}\Delta_g \pm \sqrt{k\kappa}}{k + \Delta_g^2}, \tag{35}$$

where $\kappa \coloneqq k - \tilde{L}^2 + \Delta_g^2$. Since $k \geq 0$, for these zeroes to be real we must have $\kappa \geq 0$. It can be easily seen that $\xi \in [-1, 1]$ and $\Theta_{\xi} \geq 0$ are then guaranteed. On the other hand, the maximum of Θ_{ξ} is at $(-\frac{\tilde{L}\Delta_g}{k+\Delta_g^2}, \frac{k\kappa}{k+\Delta_g^2})$.

If $\tilde{L}$ or Δ_g were vanishing, then the zeroes would be symmetric with respect to the line $\xi = 0$. Physically this means that only for vanishing $\tilde{L}$ or Δ_g, the motion has a symmetry with respect to the equatorial plane. As mentioned before, motion on a cone is also permissible when electric and/or magnetic charges are considered. This follows from the condition $\Theta_\xi = 0$, which in the general case admits the two solutions given in Eq. (35). These values determine the angles $\theta_{\rm min}$ and $\theta_{\rm max}$ from which the conicity of the orbit, $\Delta_{\rm conicity} \coloneqq \pi - (\theta_{\rm min} + \theta_{\rm max})$, can be obtained. The essential features of the orbits of the θ–motion can then be analyzed in quite a similar way as that given in [18] for the RN spacetime and we shall not dwell on this.

B. The $\tilde{r}$-motion

We now explore the dynamics on the $\tilde{r}$-coordinate for massive particles and shall start as for the θ-motion, namely, we require real values for $\tilde{r}$. Clearly this implies $R \geq 0$. Now, the regions where this condition is satisfied are bounded by the zeroes of R and we can further analyze these regions by looking for roots of multiplicity 2 of the function R. More specifically, we consider the locus determined by the conditions

$$R = \frac{\tilde{r}^4}{m^2}[E + \Delta_q I(\tilde{r})]^2 - \frac{\tilde{\Delta}_r}{m^2}(m^2\delta\tilde{r}^2 + k) = 0,$$
$$\frac{dR}{d\tilde{r}} = 0. \tag{36}$$

In a similar way as for the RN case [18], parametric plots on the (E, k)-plane can be done. For comparison purposes, we focus only on the $Q = 0.3$, $G = 0.1$, $q = 0.1$, $g = 0$ situation for different values of the BI parameter b.

As we see in Fig. 3, for b large we recover the RN curves (dashed red line) and for b small the Schwarzschild limit (solid black line); the general features of these curves are similar to the RN case discussed in [18] and can be analyzed in a similar fashion.

Furthermore, along the lines of [18], we can also determine the turning points of the orbits followed by massive particles. From the $\tilde{r}$-equation of motion in Eq. (28) we have that the constraint

$$0 = \left(\frac{d\tilde{r}}{d\gamma}\right)^2 = \tilde{r}^4(E - V_{\rm eff}^+)(E - V_{\rm eff}^-), \tag{37}$$

defines an effective potential of the form

$$V_{\rm eff}^{\pm} = -\Delta_q I(\tilde{r}) \pm \frac{1}{\tilde{r}^2}\sqrt{\tilde{\Delta}_r(\delta\tilde{r}^2 + k)}. \tag{38}$$

In Figs. 4–7 we show this potential for some values of the parameters $\tilde{Q}, \tilde{G}, \tilde{q}, \tilde{g}, k$ and the BI parameter $\tilde{b} \coloneqq r_s^2 b$. The red area shows the Schwarzschild limit ($\tilde{b} \ll 1$), the green area corresponds to the RN limit ($\tilde{b} \gg 1$) and the yellow region is associated to a generic EBI case ($\tilde{b} \sim 0.1$–3). It can be remarked the absence of a barrier wall near the origin for EBI in Figs. 4 and 7.

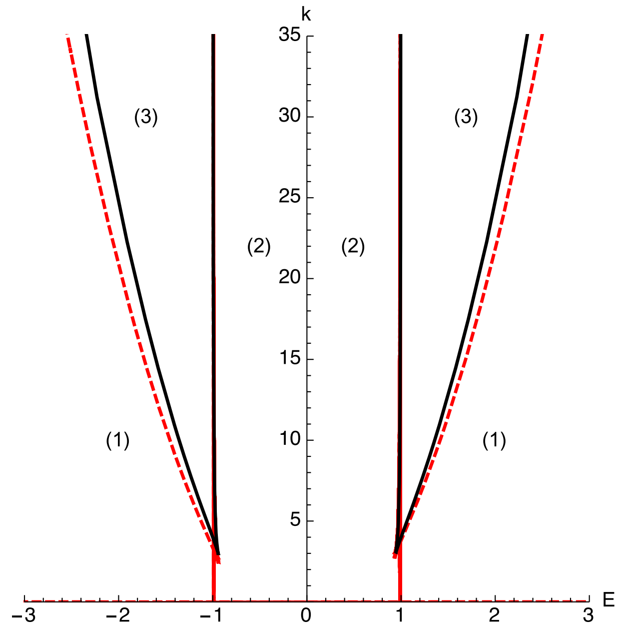

FIG. 3 (color online). Distribution of roots of the function R on the $k - E$ plane showing the transition from Schwarzschild spacetime ($\tilde{b} = 1 \times 10^{-6}$, solid line) to RN spacetime ($\tilde{b} = 1 \times 10^{6}$, dashed line); the regions with 4 zeroes are not indicated.

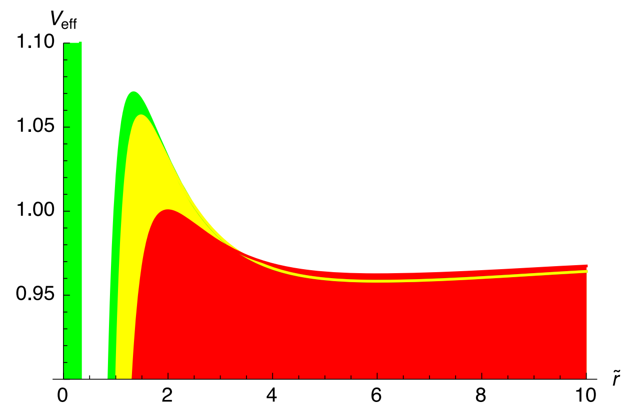

FIG. 4 (color online). $\tilde{Q} = 0.4$, $\tilde{G} = 0.25$, $\tilde{q} = 0.05$, $\tilde{g} = 0.1$, $k = 4$.

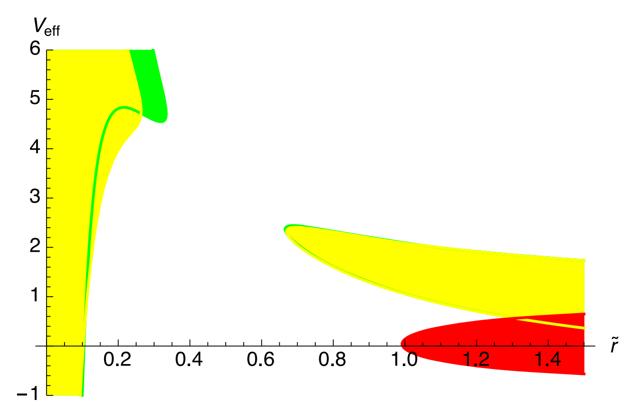

FIG. 5 (color online). $\tilde{Q} = 0.4$, $\tilde{G} = 0.25$, $\tilde{q} = -4$, $\tilde{g} = 0.1$, $k = 0.2$.

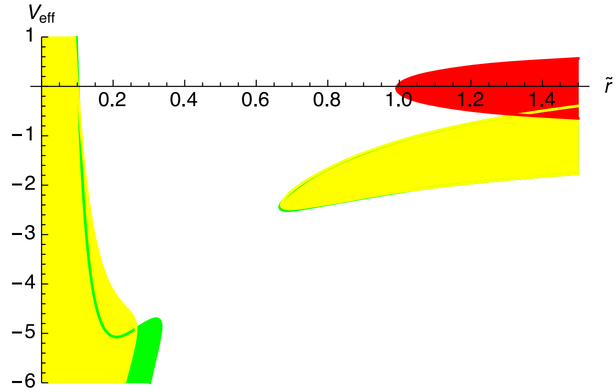

FIG. 6 (color online). $\tilde{Q} = 0.4$, $\tilde{G} = 0.25$, $\tilde{q} = 4$, $\tilde{g} = 0.1$, $k = 0.2$.

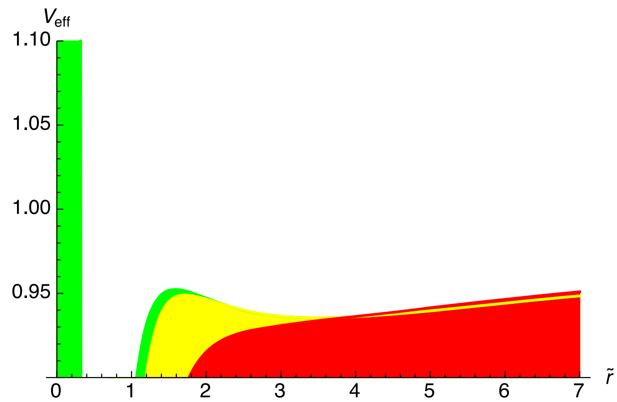

FIG. 7 (color online). $\tilde{Q} = 0.4$, $\tilde{G} = 0.25$, $\tilde{q} = 0.025$, $\tilde{g} = 0.1$, $k = 2.7$.

V. SOLUTION OF THE ORBITAL EQUATIONS OF MOTION

We now proceed to discuss some analytical solutions to the equations of motion Eqs. (28)–(32). For simplicity we consider only the electric case knowing that the general case can be discussed using the replacement $Q \to \sqrt{Q^2 + G^2}$ in the final expressions.

A. Solution of the $\tilde{r}$-equation of motion

Given the complexity of the functions $R(\tilde{r})$ and $\Theta(\theta)$, analytical solutions in terms of elementary functions are not known, in fact, since the expression for the metric coefficients g_{tt} and g_{rr} involve hypergeometric functions, the standard procedure of studying the roots of a polynomial to determine a solution to the radial equation of motion cannot be implemented here. However, there are two cases for which it is possible to simplify the equations in such a way that explicit analytical solutions can be found using Weierstrass' function; in the following we discuss these particular situations where explicit details can be worked out: the Schwarzschild and the RN limit.

1. Case $Q/b \gg r^2$

This case corresponds to the Schwarzschild limit $b \to 0$. The function Δ_r has the expression

$$\Delta_r = r^2 - 2Mr + 2b^2 r \int_r^\infty ds \left(\sqrt{s^4 + Q^2/b^2} - s^2 \right). \tag{39}$$

Using now the relation [25]

$$\begin{aligned} &\int_r^\infty \left(\sqrt{s^4 + r_0^4} - s^2 \right) ds \\ &\quad = \frac{1}{3} r^3 \left(1 - \sqrt{1 + r_0^4/r^4} \right) + \frac{2}{3} r_0^4 \int_r^\infty \frac{ds}{\sqrt{s^4 + r_0^4}} \\ &\quad \simeq \frac{r_0^3}{6\sqrt{\pi}} \Gamma\left(\frac{1}{4}\right)^2 - r_0^2 r - \frac{1}{10} \frac{r^5}{r_0^2} + \mathcal{O}(r^9), \end{aligned} \tag{40}$$

with $r_0^4 \coloneqq Q^2/b^2$, we obtain

$$\Delta_r = -2(M - M_m) r + (1 - 2bQ) r^2 + \mathcal{O}(r^6), \tag{41}$$

where we have defined

$$M_m \coloneqq \frac{1}{6} \sqrt{\frac{b}{\pi}} Q^{3/2} \Gamma\left(\frac{1}{4}\right)^2. \tag{42}$$

It follows that

$$\tilde{\Delta}_r = \tilde{r}(\tilde{M} - 1) + (1 - 2\tilde{b}\,\tilde{Q}) \tilde{r}^2, \tag{43}$$

with $\tilde{M} = M_m/2M$. Let us now define

$$b_1 \coloneqq \tilde{M} - 1, \qquad b_2 \coloneqq 1 - 2\tilde{b}\,\tilde{Q}, \tag{44}$$

then the $\tilde{r}$-equation of motion becomes

$$\left(\frac{d\tilde{r}}{d\gamma}\right)^2 = \tilde{r}^4 \left(E + \frac{\Delta_q}{\tilde{r}} \right)^2 - (b_1 \tilde{r} + b_2 \tilde{r}^2)(\delta \tilde{r}^2 + k), \tag{45}$$

or equivalently

$$\left(\frac{d\tilde{r}}{d\gamma}\right)^2 = a_1 \tilde{r} + a_2 \tilde{r}^2 + a_3 \tilde{r}^3 + a_4 \tilde{r}^4, \tag{46}$$

where

$$\begin{aligned} a_1 &= -b_1 k, \qquad a_2 = \Delta_q^2 - b_2 k, \\ a_3 &= 2\Delta_q E - b_1 \delta, \qquad a_4 = E^2 - b_2 \delta. \end{aligned} \tag{47}$$

Notice the dependence of the coefficients $a_i's$ on the BI parameter: for $\tilde{b}=0$ we have $b_1=-1$ and $b_2=1$. By making now the change of variable $\tilde{r}=1/x$, Eq. (46) can be cast as the differential equation

$$\left(\frac{dx}{d\gamma}\right)^2 = a_1x^3 + a_2x^2 + a_3x + a_4. \tag{48}$$

We have obtained then a polynomial on the right-hand side of this equation of third order on x. Finally, through the defining relation

$$x =: 4y/a_1 - a_2/3a_1, \tag{49}$$

we obtain the standard form of the Weierstrass differential equation

$$\left(\frac{dy}{d\gamma}\right)^2 = 4y^3 - g_2y - g_3, \tag{50}$$

with corresponding parameters

$$g_2 = \frac{a_2^2}{12} - \frac{a_1a_3}{4}, \qquad g_3 = \frac{a_1a_2a_3}{48} - \frac{a_2^3}{216} - \frac{a_0a_3^2}{16}. \tag{51}$$

Equation (50) is of elliptic type and its solution is well known, it is given by the Weierstrass function

$$y(\gamma) = \wp(\gamma - \gamma'_{\rm in}; g_2, g_3), \tag{52}$$

and hence, the $\tilde{r}$-equation of motion has the solution

$$\tilde{r} = \frac{a_3}{4\wp(\gamma - \gamma'_{\rm in}; g_2, g_3) - \frac{a_2}{3}}. \tag{53}$$

2. Case $\tilde{b} \to \infty$

The situation here corresponds to the RN limit. In this case we proceed as above but before doing it we must analyze the behavior of the function $\tilde{\Delta}_r$. Figure 8 shows this function for small and large values of the parameter $\tilde{b}=r_s^2b$ and $\tilde{Q}=\sqrt{0.2}$.

It can be remarked that for $\tilde{b}$ large and strictly positive values of $\tilde{r}$, the function $\tilde{\Delta}_r$ has a similar behavior as that of a quadratic polynomial on $\tilde{r}$ with two real roots. We write then the expression

$$\tilde{\Delta}_r = (\tilde{r} - r_{h_-})(\tilde{r} - r_{h_+}), \tag{54}$$

which corresponds to a quadratic polynomial, its roots being located at the inner and outer event horizons r_{h_-} and r_{h_+}, respectively. The dependence on the BI parameter b is coded in the locations of these two horizons; they become the standard RN horizons in the limit $b\to\infty$. Using this expression we have

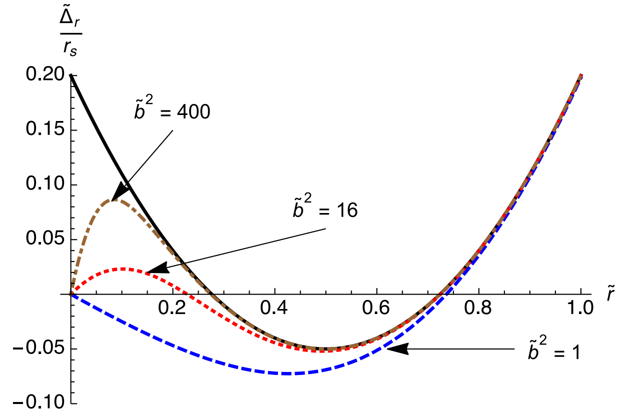

FIG. 8 (color online). The function $\tilde{\Delta}_r$ for different values of the dimensionless parameter $\tilde{b}=r_s^2b$; the solid line corresponds to the RN spacetime with $\tilde{Q}=\sqrt{0.2}$.

$$\left(\frac{d\tilde{r}}{d\gamma}\right)^2 = \tilde{r}^4E^2 + 2\tilde{r}^3\Delta_q + \tilde{r}^2\Delta_q^2 - (\tilde{r} - r_{h_-})(\tilde{r} - r_{h_+})(k + \delta\tilde{r}^2), \tag{55}$$

or equivalently

$$\left(\frac{d\tilde{r}}{d\gamma}\right)^2 = a_0 + a_1\tilde{r} + a_2\tilde{r}^2 + a_3\tilde{r}^3 + a_4\tilde{r}^4, \tag{56}$$

where

$$\begin{aligned} a_0 &= -kr_{h_-}r_{h_+}, \qquad a_1 = -k(r_{h_-} + r_{h_+}),\\ a_2 &= (\Delta_q^2 - k - \delta r_{h_-}r_{h_+}),\\ a_3 &= [2E\Delta_q - \delta(r_{h_-} + r_{h_+})],\\ a_4 &= (E^2 - \delta). \end{aligned} \tag{57}$$

We now proceed as before. First, we make the change of variable $\tilde{r} = \pm\frac{1}{x} + \tilde{r}_R$, where $\tilde{r}_R$ is one of the roots of the quartic polynomial

$$a_0 + a_1\tilde{r} + a_2\tilde{r}^2 + a_3\tilde{r}^3 + a_4\tilde{r}^4 = 0, \tag{58}$$

and in consequence we arrive to

$$\left(\frac{dx}{d\gamma}\right)^2 = b_0 + b_1x + b_2x^2 + b_3x^3, \tag{59}$$

where

$$\begin{aligned} b_0 &= a_4, \qquad b_1 = a_3 + 4a_4\tilde{r}_R,\\ b_2 &= a_2 + 3a_3\tilde{r}_R + 6a_4\tilde{r}_R^2,\\ b_3 &= a_1 + 2a_2\tilde{r}_R + 3a_3\tilde{r}_R^2 + 4a_4\tilde{r}_R. \end{aligned} \tag{60}$$

The further change of variable

$$x =: (4y - b_2/3)/b_3, \tag{61}$$

allows us to obtain

$$\left(\frac{dy}{d\gamma}\right)^2 = 4y^3 - g_2 y + g_3, \tag{62}$$

where

$$g_2 = \frac{b_2^2}{12} - \frac{b_1 b_3}{4}, \qquad g_3 = \frac{b_1 b_2 b_3}{48} - \frac{b_2^3}{216} - \frac{b_0 b_3^2}{16}. \tag{63}$$

Equation (62) is again Weierstrass' differential equation and therefore we can write immediately

$$\tilde{r} = \pm \frac{b_3}{4\wp(\gamma - \gamma'_{\rm in}; g_2, g_3) - \frac{b_2}{3}} + \tilde{r}_R, \tag{64}$$

as the solution to the $\tilde{r}$-equation of motion in this case.

Summarizing, in this subsection and the previous one we have considered the Schwarzschild and RN limit of the generic EBI black hole solution. To obtain these results, we could have proceeded in a different way. For instance, we could have performed a perturbative analysis in these two cases, taking the RN solution as the unperturbed solution. We did not follow this path because we know the exact solution in EBI spacetime and moreover, the equation of motion involving the correction can be cast into Weierstrass' differential equation. In both cases the dependence on the BI parameter is encoded in the coefficients of the cubic polynomial appearing in the Weierstrass equation and we certainly recover the known results in the literature [18] in the limits $b \to 0$ and $b \to \infty$.

It is worth mentioning that knowledge of the exact EBI black hole solution is not a trivial issue. For instance, as far as we know, it is still a challenge to obtain a rotating black hole solution for the EBI system, in such a way that in the proper limit, we can recover the Kerr-Neumann black hole. In this case only a perturbed solution for small rotations is known [32] but not the generic solution.

B. Solution of the $\theta(\gamma)$–equation

The solution of Eq. (33) with $a < 0$ and $D > 0$ can be obtained in a straightforward way and it is given by the elementary function

$$\theta(\gamma) = \arccos\left(\frac{1}{2a}(\sqrt{D}\sin(\sqrt{-a}\gamma - \gamma_{in}^{\vartheta}) - b)\right), \tag{65}$$

where $\gamma_{\rm in}^{\vartheta} = \sqrt{-a}\gamma_{\rm in} - \arcsin(\frac{\gamma_{\rm in}+b}{\sqrt{D}})$, $\gamma_{\rm in}$ is the initial value of γ and $D := 4k\kappa$.

C. Solution of the $\phi(\gamma)$–equation

Equation (31) can be simplified by using the θ-equation of motion and the change of variable $\xi = \cos\theta$. We have thus

$$d\phi = -\frac{d\xi}{\sqrt{\Theta_\xi}}\frac{\tilde{L}}{1-\xi^2} - \frac{\xi d\xi}{\sqrt{\Theta_\xi}}\frac{\Delta_g}{1-\xi^2}, \tag{66}$$

where Θ_ξ is given in Eq. (33). The resulting equation can be easily integrated and the solution for $a < 0$ and $D > 0$ is given by

$$\phi(\gamma) = \frac{1}{2}(I_+ + I_-)\Big|_{\xi_{\rm in}}^{\xi(\gamma)} + \phi_{\rm in}, \tag{67}$$

where

$$I_\pm := -sgn(\tilde{L} \pm \Delta_g)\arcsin\frac{f}{\sqrt{D}},$$
$$f := \frac{k + \kappa - (\tilde{L} \pm \Delta_g)^2 \mp (k + \kappa + (\tilde{L} \pm \Delta_g)^2)\xi}{\xi \mp 1}. \tag{68}$$

Here $sgn(z)$ means the sign function.

For the special case $k = \tilde{L}^2$ and $\tilde{L} = \pm\Delta_g$, the solution reduces to the simple form

$$\phi(\gamma) = \frac{1}{2}\left(sgn(\tilde{L})\arcsin\frac{1 \pm 3\xi}{\xi \mp 1}\right)\Big|_{\xi_{in}}^{\xi(\gamma)} + \phi_{\rm in}, \tag{69}$$

where $\phi_{\rm in} := \phi(\gamma_{\rm in})$. The θ and ϕ-motions are actually the same as those obtained by Grunau and Kagramanova [18].

VI. OBSERVABLES

In this section we calculate the periastron shift associated to the paths of test particles moving in EBI spacetime. Let us consider for simplicity the case of neutral test particles ($q = g = 0$) and, as a further reduction, we take $\theta = \pi/2$, i.e., we restrict the motion to the equatorial plane. Then from Eqs. (28)–(32), we deduce that the above value of θ is admissible provided that $k = \tilde{L}^2$; in the following we assume this relation.

We now follow the classical argument found in textbooks of general relativity leading to the perihelion shift. From the HJ equation we have

$$\left(\frac{\tilde{L}}{\tilde{r}^2}\frac{d\tilde{r}}{d\phi}\right)^2 = E^2 - \frac{\tilde{\Delta}_r}{\tilde{r}^2}\left(m^2 + \frac{\tilde{L}^2}{\tilde{r}^2}\right), \tag{70}$$

or, after the change of variable $\tilde{u} := 1/\tilde{r}$,

$$\left(L\frac{d\tilde{u}}{d\phi}\right)^2 = E^2 - F(\tilde{u})(m^2 + \tilde{L}^2\tilde{u}^2), \tag{71}$$

where

$$F(\tilde{u}) := 1 - \tilde{u} + \frac{2}{3}\frac{\tilde{b}^2}{\tilde{u}^2}\left(1 - \sqrt{1 + \frac{\tilde{Q}^2\tilde{u}^4}{\tilde{b}^2}}\right) + \frac{4}{3}\tilde{Q}^2\tilde{u}^2{}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; -\frac{\tilde{Q}^2\tilde{u}^4}{\tilde{b}^2}\right). \tag{72}$$

The derivative of Eq. (71) with respect to ϕ gives then

$$\frac{d^2\tilde{u}}{d\phi^2} = -\tilde{u}F(\tilde{u}) - \frac{1}{2}\frac{dF(\tilde{u})}{d\tilde{u}}\left(\frac{m^2}{\tilde{L}^2} + \tilde{u}^2\right). \tag{73}$$

The RN limit is obtained by letting $\tilde{b} \to \infty$; thus

$$F(\tilde{u}) \sim 1 - \tilde{u} + \tilde{Q}^2\tilde{u}^2 - \frac{\tilde{Q}^4\tilde{u}^6}{20\tilde{b}^2}, \tag{74}$$

to lowest order on negative powers of $\tilde{b}$. Using this expression we arrive to

$$\frac{d^2\tilde{u}}{d\phi^2} = \frac{m^2}{2\tilde{L}^2} - \tilde{u} + \frac{3}{2}\tilde{u}^2 - \frac{m^2\tilde{Q}^2}{\tilde{L}^2}\tilde{u} - 2\tilde{Q}^2\tilde{u}^3 + \frac{3m^2\tilde{Q}^4}{20\tilde{b}^2\tilde{L}^2}\tilde{u}^5 + \frac{\tilde{Q}^4}{5\tilde{b}^2}\tilde{u}^7, \tag{75}$$

or equivalently

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{h^2} + 3Mu^2 - \frac{Q^2}{h^2}u - 2Q^2u^3 + \frac{3}{80}\frac{Q^4}{h^2b^2M^2}u^5 + \frac{1}{20}\frac{Q^4}{b^2M^2}u^7, \tag{76}$$

where $h := L/m$ is the angular momentum per unit mass of the test particle and $u := 1/r$. The Newtonian orbit is obtained by solving this differential equation with only the constant term on the right-hand side; the standard relativistic correction is given by the second term. The RN correction is determined by the third and fourth terms involving only the charge $\tilde{Q}$ and the BI correction is given by the fifth and sixth terms.

To get a feeling of the modification to the perihelion of the test particle, let us consider the case of a test particle with h such that $\epsilon := 3M^2/h^2$ is a small number; as it is well known, this is the situation for Mercury's perihelion. Assuming the following Ansatz for the solution

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + O(\epsilon^3), \tag{77}$$

we obtain

$$\frac{d^2u_0}{d\phi^2} + u_0 = \frac{M}{h^2},$$
$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{h^2}{M}u_0^2 - \frac{1}{3}\xi^2u_0 - \frac{2}{3}\xi^2h^2u_0^3 + \frac{1}{80}\frac{\xi^4}{b^2}u_0^5 + \frac{1}{60}\frac{h^2\xi^4}{b^2}u_0^7$$
$$\frac{d^2u_2}{d\phi^2} + u_2 = \frac{1}{16}\frac{\xi^4u_0^4}{b^2}u_1 + \frac{7}{60}\frac{\xi^4h^2u_0^6}{b^2}u_1 - \frac{1}{3}\xi^2u_1 + \frac{2h^2}{M}u_0u_1 - 2h^2\xi^2u_0^2u_1, \tag{78}$$

where we have defined $\xi := Q/M$. In consequence the BI correction is of second order on ϵ, which is of higher order than the Newtonian and RN contribution.

Alternatively, we could have started from Eqs. (28)–(32), with $\theta = \pi/2$, to obtain

$$\left(\frac{d\tilde{r}}{d\phi}\right)^2 = \frac{\tilde{r}^4}{\tilde{L}^2}\left[E^2 - \frac{\tilde{\Delta}_r}{\tilde{r}^2}\left(m^2 + \frac{\tilde{L}^2}{\tilde{r}^2}\right)\right] =: R_\phi(\tilde{r}). \tag{79}$$

The periastron shift is then calculated from

$$\Omega_p = 2\int_{\tilde{r}_p}^{\tilde{r}_a}\frac{d\tilde{r}}{\sqrt{R_\phi(\tilde{r})}} - 2\pi. \tag{80}$$

This follows straightforwardly from Eq. (70). In the limit $b \to \infty$, we have for the function R_ϕ

$$R_\phi(\tilde{r}) = \frac{\tilde{r}^4}{\tilde{L}^2}\left[E^2 - \left(1 - \frac{1}{\tilde{r}} + \frac{\tilde{Q}^2}{\tilde{r}^2} - \frac{\tilde{Q}^4}{20\tilde{b}^2\tilde{r}^6}\right)\left(1 + \frac{\tilde{L}^2}{\tilde{r}^2}\right)\right]. \tag{81}$$

Let us find an explicit expression for the perihelion in the case of small charge Q with $b \gg 1$; we look then for a solution to Eq. (76) of the form

$$u = \frac{M}{h^2}(1 - e\cos\Omega\phi). \tag{82}$$

Then, by demanding that after substitution the coefficients of $\cos\Omega\phi$ on the left and right-hand side of the differential equation coincide, we obtain

$$1 - \Omega = 3\frac{M^2}{h^2} - \frac{1}{2}\frac{Q^2}{h^2} - 3\frac{M^2Q^2}{h^4} + \frac{3}{32}\frac{M^2Q^4}{b^2h^{10}} + \frac{7}{40}\frac{M^4Q^4}{b^2h^{12}}$$
$$= \epsilon - \frac{1}{6}\frac{Q^2}{M^2}\epsilon - \frac{1}{3}\frac{Q^2}{M^2}\epsilon^2 + \frac{1}{32\cdot 3^4}\frac{Q^4}{M^4}\frac{1}{M^4b^2}\epsilon^5 + \frac{7}{40\cdot 3^6}\frac{Q^4}{M^4}\frac{1}{M^4b^2}\epsilon^6. \tag{83}$$

The first two terms on the right-hand side of this expression are recognized as the standard perturbations in RN spacetime; the remaining three terms are corrections of higher order on ϵ and those depending on the BI parameter b vanish in the limit $b \to \infty$. This agrees with our remarks after Eq. (78). For a Sun-like star we have the estimate [33]

$$\frac{Q}{M} = \frac{2\pi\epsilon_0 G(m_p - m_e)}{q_e}, \tag{84}$$

where ϵ_0 is the vacuum permittivity, G is Newton's constant, m_p the mass of the proton, m_e the mass of the electron and q_e its electric charge. For $\epsilon = 3M^2/h^2$ we take the value 7.98765×10^{-8} which gives Mercury's relativistic perihelion shift of 42.9818 seconds of arc per century. Using this we see from Eq. (83) that the coefficient of ϵ in the second term is of order 10^{-38}, the coefficient of ϵ^2 is also of order 10^{-38} and the coefficient of ϵ^5 is of order $10^{-77} \times (M^4 b^2)^{-1}$, being the same for the coefficient of ϵ^6. Due to the smallness of ϵ, the corrections due to the BI parameter will be negligible, even if $(M^4 b^2)^{-1} \sim 10^{69}$; the same considerations apply to the modifications on the radius orbit due to the BI parameter.

Motion on the equatorial plane is also allowed in the presence of electric and magnetic charges. For this to happen the value $\theta = \pi/2$, should be an admissible value of the polar angle consistent with the equations of motion. From Eq. (30), the θ-equation of motion, we see that this will be true provided $k = \tilde{L}^2$, the same condition as for neutral particles. Actually, the above results can be generalized following [18,34]; we have first the general expression

$$\Delta_P = (\Upsilon_\phi - \Upsilon_{\tilde{r}})/\Gamma, \tag{85}$$

for the perihelion shift where

$$\begin{aligned} \Lambda_{\tilde{r}} &\coloneqq 2\int_{\tilde{r}_p}^{\tilde{r}_a} \frac{d\tilde{r}}{\sqrt{R(\tilde{r})}}, \\ \Lambda_\theta &\coloneqq 2\int_{\theta_{\min}}^{\theta_{\max}} \frac{d\theta}{\sqrt{\Theta(\theta)}}, \\ \Upsilon_{\tilde{r}} &\coloneqq \frac{2\pi}{\Lambda_{\tilde{r}}}, \\ \Upsilon_\phi &\coloneqq \frac{2}{\Lambda_\theta}\int_{\theta_{\min}}^{\theta_{\max}} \frac{\Phi_\theta(\theta)}{\sqrt{\Theta(\theta)}} d\theta, \\ \Gamma &\coloneqq \frac{2}{\Lambda_{\tilde{r}}}\int_{\tilde{r}_p}^{\tilde{r}_a} \frac{T_{\tilde{r}}(\tilde{r})}{\sqrt{R(\tilde{r})}} d\tilde{r}. \end{aligned} \tag{86}$$

The integrals on the radial variable r are evaluated from the periapsis r_p to the apoapsis r_a and correspondingly, those involving the polar angle θ from $\theta_{\min}$ to $\theta_{\max}$. The functions $R(r)$ and $\Theta(\theta)$ in the above integrals were defined previously; the functions $\Phi_\theta(\theta)$ and $T_r(r)$ are given by the right-hand sides of Eqs. (31)–(32), respectively, i.e.

$$\begin{aligned} \Phi_\theta(\theta) &\coloneqq \frac{1}{m\sin^2\theta}(\tilde{L} + \Delta_g \cos\theta), \\ T_{\tilde{r}}(\tilde{r}) &\coloneqq \frac{\tilde{r}^4}{m\tilde{\Delta}_r}[E - \Delta_q I(\tilde{r})]. \end{aligned} \tag{87}$$

The values

$$\Lambda_\theta = \frac{2\pi}{\sqrt{k + \Delta_g^2}}, \qquad \Upsilon_\phi = \sqrt{k + \Delta_g^2}, \tag{88}$$

can readily be found using the change of variable $\xi = \cos\theta$ and thus

$$\Delta_P = \left(\sqrt{k + \Delta_g^2} - \frac{2\pi}{\Lambda_{\tilde{r}}}\right)\frac{1}{\Gamma}. \tag{89}$$

No Lense-Thirring effect is present in BI spacetime since physically the solution is describing a nonrotating black hole; this conclusion can also be reached by looking at the θ- and ϕ-equations of motion and noting that they are the same as in the RN case, where this effect is absent. More precisely, we have that

$$\Delta_{LT} \coloneqq (\Upsilon_\phi - \Upsilon_\theta)/\Gamma, \tag{90}$$

vanishes because

$$\Upsilon_\theta \coloneqq \frac{2\pi}{\Lambda_\theta} = \sqrt{k + \Delta_g^2}, \tag{91}$$

has the same value as Υ_ϕ.

VII. CONCLUSIONS

In this work we have analyzed the orbital motion of charged test particles for the EBI spacetime. We have seen that even though BI electrodynamics has a more complicated Lagrangian, the orbits followed by charged massive particles admit a decomposition similar to that of the RN spacetime.

The main difference between these two spacetimes is encoded into the function $\tilde{\Delta}_r$, which in the EBI case involves a hypergeometric function. This has a nontrivial influence on the motion of particles.

As seen from Figs. 4–7, the RN effective potential is closer in appearance to the typical EBI effective potential, however some differences are noticeable depending on the value of the parameter k. In particular from Figs. 4 and 7, we can deduce that a charged particle can fall to the origin in the EBI case since there is no barrier wall near the origin, meanwhile this will not happen in the RN scenario where there is always a barrier wall for small values of $\tilde{r}$. Obviously in the Schwarzschild case a massive particle can fall to the origin but in that case there is not electric or magnetic charge involved.

Another noteworthy feature from the orbital motion in EBI spacetime can be inferred from Figs. 5 and its mirror image Fig. 6. In the RN spacetime, there are some values of V_{eff} in the interval 4.6–4.8 for which a closed orbit is possible. These orbits disappear in the EBI case, being replaced by unbounded trajectories. Furthermore, for some orbits a turning point in the RN spacetime can be closer to the origin than the corresponding one in EBI spacetime and vice versa.

We have also analyzed two extremes cases where analytical results can be obtained. In both situations the radial equation of motion is amenable, after a series of transformations, to the differential equation satisfied by the Weierstrass function. In this way a full explicit solution is obtained. Furthermore, the periastron shift in BI spacetime was analyzed in the limit $b \to \infty$ following standard calculations and the corrections due to the BI parameter are very small for Sun-like stars. The Lense-Thirring effect was also shown to vanish as expected in a nonrotating metric.

The above results may be generalized to the case of a slowly rotating BI black hole [32], where the Lense-Thirring effect should be present; in this case since we do not have a generic black hole solution for any angular velocity, a perturbative treatment of the equations of motion will be necessary.

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