

Siklos waves in Poincaré gauge theoryM. Blagojević^{*} and B. Cvetković[†]*Institute of Physics, University of Belgrade, Pregrevica 118, 11080 Belgrade, Serbia*

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A class of Siklos waves, representing exact vacuum solutions of general relativity with a cosmological constant, is extended to a new class of Siklos waves with torsion, defined in the framework of the Poincaré gauge theory. Three particular exact vacuum solutions of this type, the generalized Kaigorodov, the homogeneous solution and the exponential solution, are explicitly constructed.

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I. INTRODUCTION

The first complete formulation of the idea of (internal) gauge invariance was given in Weyl's classic paper [1]. Significant progress in this direction was achieved somewhat later by Yang, Mills and Utiyama [2,3]. It opened a new perspective for understanding gravity as a gauge theory, the perspective that was realized by Kibble and Sciama [4] in their proposal of a new theory of gravity, known as the Poincaré gauge theory (PGT). The PGT is a gauge theory of the Poincaré group, with an underlying Riemann-Cartan (RC) geometry of spacetime [5,6]. In this approach, basic gravitational variables are the tetrad field b^i and the Lorentz connection ω^{ij} (1-forms), and the related field strengths are the torsion $T^i = db^i + \omega^i_m \wedge b^{mj}$ and the curvature $R^{ij} = d\omega^{ij} + \omega^i_m \wedge \omega^{mj}$ (2-forms). At a more physical level, the source of gravity in PGT is matter possessing both the energy-momentum and spin currents. The importance of the Poincaré symmetry in particle physics leads one to consider PGT as a favorable framework for describing the gravitational phenomena.

Based on the experience stemming from Einstein's general relativity, it is known that exact solutions play a crucial role in developing our understanding of the geometric and physical content of a gravitational theory; for a review, see Refs. [7–10]. An important set of these solutions refers to exact gravitational waves, the structure of which has been studied also in PGT [11]. In the present work, we focus on a particular class of the gravitational waves, the class of Siklos waves that are vacuum solutions of general relativity with a cosmological constant (GR_Λ), propagating on the anti-de Sitter (AdS) background [12]. By generalizing the ideas developed in three dimensions [13], we construct here a class of the four-dimensional Siklos waves with torsion as vacuum solutions of PGT.

The paper is organized as follows. In Sec. II, we give a short account of the Siklos waves in the tetrad formulation of GR_Λ . In Sec. III, we show that Siklos waves are torsion-free vacuum solutions of PGT. In Sec. IV, we introduce new

vacuum solutions of PGT, the Siklos waves with torsion, by modifying the Siklos geometry in a manner that preserves the radiation nature of the original configuration. That is achieved by an ansatz for the RC connection that produces only the tensorial irreducible mode of the torsion with $J^P = 2^+$. The PGT field equations are simplified and shown to depend only on three parameters, including the mass of the torsion mode. In Secs. V–VII, we describe three different vacuum solutions belonging to the class of Siklos waves with torsion: the generalized Kaigorodov, the homogeneous solution and the exponential solution. Section VII is devoted to concluding remarks, and two appendixes contain some technical details.

Our conventions are as follows. We use the Poincaré coordinates $x^\mu = (u, v, x, y)$ as the local coordinates; the latin indices (i, j, \dots) refer to the local Lorentz (co)frame and run over $(+, -, 2, 3)$, b^i is the tetrad (1-form), and h_i is the dual basis (frame), such that $h_i \rfloor b^k = \delta_i^k$; the volume 4-form is $\hat{\epsilon} = b^+ \wedge b^- \wedge b^2 \wedge b^3$, the Hodge dual of a form α is $*\alpha$, with $*1 = \hat{\epsilon}$, and the totally antisymmetric tensor is defined by $*(b_i \wedge b_j \wedge b_k \wedge b_m) = \epsilon_{ijklm}$ and normalized to $\epsilon_{+-23} = 1$; in the rest of the paper, the exterior product of forms is implicit.

II. SIKLOS WAVES IN GR_Λ

Siklos waves were introduced as a class of exact gravitational waves propagating on the AdS background [12]. In the Poincaré coordinates $x^\mu = (u, v, x, y)$, the Siklos metric is given by

$$ds^2 = \frac{\ell^2}{y^2} [2du(Hdu + dv) - dx^2 - dy^2], \quad (2.1)$$

with $H = H(u, x, y)$. It admits the null Killing vector field ∂_v that is not covariantly constant; the wave fronts are surfaces of constant u and v , and the case $H = 0$ corresponds to the AdS background. The metric (2.1) coincides with a special subclass of the Kundt class [9,10], and is obviously conformal to pp waves. The physical interpretation of the Siklos waves was investigated by Podolský [14,15].

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Now we give a short account of the Siklos waves in the tetrad formulation of GR_Λ , which allows for a simpler generalization to PGT. First, we choose the tetrad field in the form

$$\begin{aligned} b^+ &:= \frac{\ell}{y} du, & b^- &:= \frac{\ell}{y} (H du + dv), \\ b^2 &:= \frac{\ell}{y} dx, & b^3 &:= \frac{\ell}{y} dy, \end{aligned} \quad (2.2)$$

so that the line element becomes $ds^2 = 2b^+b^- - (b^2)^2 - (b^3)^2 \equiv \eta_{ij}b^ib^j$, where η is the half-null Minkowski metric,

$$\eta_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The dual frame h_i is given by

$$\begin{aligned} h_+ &= \frac{y}{\ell} (\partial_u - H \partial_v), & h_- &= \frac{y}{\ell} \partial_v, \\ h_2 &= \frac{y}{\ell} \partial_x, & h_3 &= \frac{y}{\ell} \partial_y. \end{aligned} \quad (2.3)$$

Next, we introduce the Riemannian connection ω^{ij} by imposing the condition of vanishing torsion, $\nabla b^i := db^i + \omega^i_m b^m = 0$, which yields

$$\begin{aligned} \omega^{+-}, \omega^{+2} &= 0, & \omega^{+3} &= \frac{1}{\ell} b^+, \\ \omega^{23} &= \frac{1}{\ell} b^2, & \omega^{-2} &= -\frac{y}{\ell} \partial_x H b^+, \\ \omega^{-3} &= \frac{1}{\ell} b^- - \frac{y}{\ell} \partial_y H b^+. \end{aligned} \quad (2.4a)$$

The wave nature of the Siklos wave is clearly seen by rewriting ω^{ij} in the form

$$\omega^{ij} = \bar{\omega}^{ij} + k^i (h^j] H) b^+, \quad (2.4b)$$

where $\bar{\omega}^{ij} = \omega^{ij}(H=0)$ refers to the AdS background, and the second term is the radiation piece, characterized by the null vector $k^i = (k^+, k^-, k^2, k^3) = (0, 1, 0, 0)$.

Now one can calculate the Riemannian curvature:

$$\begin{aligned} R^{+j} &= \frac{1}{\ell^2} b^+ b^j, & R^{23} &= \frac{1}{\ell^2} b^2 b^3, \\ R^{-2} &= \frac{1}{\ell^2} b^- b^2 + \frac{1}{\ell^2} (y^2 \partial_{xx} H - y \partial_y H) b^+ b^2 \\ &\quad + \frac{1}{\ell^2} (y^2 \partial_{xy} H) b^+ b^3, \\ R^{-3} &= \frac{1}{\ell^2} b^- b^3 + \frac{1}{\ell^2} (y^2 \partial_{yy} H - y \partial_y H) b^+ b^3 \\ &\quad + \frac{1}{\ell^2} (y^2 \partial_{xy} H) b^+ b^2, \end{aligned} \quad (2.5)$$

where we use $\partial_{xx} := \partial^2 / \partial x^2$ etc. The Ricci curvature $Ric^i = h_m] R^{mi}$ and the scalar curvature $R = h_i] Ric^i$ are found to be

$$\begin{aligned} Ric^m &= \frac{3}{\ell^2} b^m, & m &= +, 2, 3, \\ Ric^- &= \frac{3}{\ell^2} b^- + \frac{1}{\ell^2} (y^2 \partial_{xx} H + y^2 \partial_{yy} H - 2y \partial_y H) b^+, \\ R &= \frac{12}{\ell^2}. \end{aligned} \quad (2.6)$$

Dynamical structure of GR_Λ is defined by the action $I_\Lambda = - \int d^4x \sqrt{-g} (a_0 R + 2\Lambda)$. The corresponding vacuum field equations can be suitably written in the traceless form as

$$Ric^i - \frac{1}{4} R b^i = 0. \quad (2.7)$$

As a consequence, the metric function H must obey

$$y^2 (\partial_{xx} H + \partial_{yy} H) - 2y \partial_y H = 0. \quad (2.8)$$

The profile (u -dependence) of the Siklos wave may be arbitrary.

We display here three special solutions of (2.8) discussed by Siklos [12]:

$$\begin{aligned} H_1 &= y^3, & \text{Kaigorodov's solution (1963);} \\ H_2 &= \arctan(x/y) + xy/(x^2 + y^2), & \tilde{H}_2 &= (x^2 + y^2) H_2; \\ H_3 &= C_1 e^x (\cos y + y \sin y) + C_2 e^x (\sin y - y \cos y). \end{aligned}$$

Note that Defrise's metric (1969), with $H = 1/y^2$, is *not* a vacuum solution of GR_Λ [15].

III. SIKLOS WAVES AS TORSION-FREE SOLUTIONS OF PGT

In this section, we show that the Siklos spacetime of the previous section is an exact Riemannian solution of PGT in vacuum.

Starting from the general PGT dynamics described in Appendix B, one can easily derive its reduced form in the Riemannian sector of PGT, characterized by $T^i = 0$. First, we note that the only nonvanishing irreducible components of the Riemannian curvature are ${}^{(1)}R^{ij}$, ${}^{(4)}R^{ij}$ and ${}^{(6)}R^{ij}$, defined in Appendix A. And second, the condition $T^i = 0$ implies that dynamical evolution of the Riemannian solutions in PGT is described by a reduced form of the general field equations (B3):

$$\begin{aligned} (1ST) \quad E_i &= 0, \\ (2ND) \quad \nabla H_{ij} &= 0. \end{aligned} \quad (3.1a)$$

Here, the Riemannian expressions for E_i and H_{ij} are obtained directly from the corresponding PGT formulas (see Appendix B) in the limit $T^i = 0$:

$$\begin{aligned} H_{ij} &= -2a_0^*(b^i b^j) + 2^*(b_1^{(1)}R_{ij} + b_4^{(4)}R_{ij} + b_6^{(6)}R_{ij}), \\ E_i &:= h_i]L_G - \frac{1}{2}(h_i]R^{mn})H_{mn}. \end{aligned} \quad (3.1b)$$

As shown in Ref. [5], the field equations (3.1) are satisfied for any configuration in which the traceless symmetric Ricci tensor vanishes:

$$Ric_{(ij)} - \frac{1}{4}\eta_{ij}R = 0. \quad (3.2)$$

Comparing this result with the GR_Λ field equation (2.7), one concludes that any vacuum solution of GR_Λ is automatically a torsion-free solution of PGT. In particular, this is true for the Siklos metric.

It is useful to explore this general statement in detail. Using the geometry of the Siklos spacetime found in the previous section, the content of Eqs. (3.1a) is found to be

$$\begin{aligned} (1ST) \quad (b_4 + b_6 - a_0\ell^2)y[y(\partial_{xx}H + \partial_{yy}H) - 2\partial_yH] &= 0, \\ 3a_0 + \ell^2\Lambda &= 0, \\ (2ND) \quad (b_1 + b_4)y^2\partial_x[y(\partial_{xx}H + \partial_{yy}H) - 2\partial_yH] &= 0, \\ (b_1 + b_4)y^2\partial_y[y(\partial_{xx}H + \partial_{yy}H) - 2\partial_yH] &= 0. \end{aligned} \quad (3.3)$$

For the generic values of the Lagrangian parameters a_0, b_1, b_4, b_6 , dynamical content of these equations is obviously the same as in GR_Λ , since the metric function H must be such that

$$\hat{S}H := y(\partial_{xx}H + \partial_{yy}H) - 2\partial_yH = 0. \quad (3.4)$$

Thus, although PGT has a rather different dynamical structure as compared to GR_Λ , the class of Riemannian Siklos spacetimes is still an exact vacuum solution of PGT.

IV. SIKLOS WAVES WITH TORSION

We are now ready to generalize the previous results by constructing a new, non-Riemannian class of Siklos waves, the Siklos waves with torsion.

A. Geometry of the ansatz

We wish to introduce torsion in a manner that preserves the radiation nature of the Riemannian Siklos waves of GR_Λ , relying on the approach proposed in [13].

We start the construction by assuming that the tetrad field in PGT retains its Riemannian form (2.2). Then, by noting that the radiation piece of the Riemannian connection (2.4) has the form $(\omega^{ij})^R = k^i(h^{j\mu}\partial_\mu H)b^+$, we assume that the new RC connection is given by

$$\omega^{ij} = \bar{\omega}^{ij} + k^i h^{j\mu}(\partial_\mu H + K_\mu)b^+, \quad (4.1a)$$

where the form of K_μ is defined by

$$\begin{aligned} K_\mu &= (0, 0, K_x, K_y), \\ K_x &= K_x(u, x, y), \quad K_y = K_y(u, x, y). \end{aligned} \quad (4.1b)$$

This ansatz modifies only two components of the Riemannian connection (2.4):

$$\begin{aligned} \omega^{-2} &= -\frac{y}{\ell}(\partial_x H + K_x)b^+, \\ \omega^{-3} &= \frac{1}{\ell}b^- - \frac{y}{\ell}(\partial_y H + K_y)b^+. \end{aligned}$$

The new terms in the connection are related to the torsion of spacetime:

$$T^- = \frac{y}{\ell}(K_x b^+ b^2 + K_y b^+ b^3), \quad T^+, T^2, T^3 = 0. \quad (4.2)$$

The only nonvanishing irreducible torsion piece is the tensor piece ${}^{(1)}T^i$, with ${}^{(1)}T^- = T^-$.

Denoting the Riemannian curvature found in Sec. II by \tilde{R}^{ij} , the new RC curvature is found to have the form

$$\begin{aligned} R^{+j} &= \frac{1}{\ell^2}b^+ b^j, \quad R^{23} = \frac{1}{\ell^2}b^2 b^3, \\ R^{-2} &= \tilde{R}^{-2} + \frac{1}{\ell^2}(y^2\partial_x K_x - yK_y)b^+ b^2 + \frac{1}{\ell^2}(y^2\partial_y K_x)b^+ b^3, \\ R^{-3} &= \tilde{R}^{-3} + \frac{1}{\ell^2}(y^2\partial_y K_y)b^+ b^3 + \frac{1}{\ell^2}(y^2\partial_x K_y + yK_x)b^+ b^2. \end{aligned} \quad (4.3a)$$

Note that the radiation piece of R^{ij} is proportional to the null vector $k^i = (0, 1, 0, 0)$. The corresponding Ricci and scalar curvatures are

$$\begin{aligned}
Ric^m &= \frac{3}{\ell^2} b^a, \quad m = +, 2, 3, \\
Ric^- &= \widetilde{Ric}^- + \frac{1}{\ell^2} (y^2 \partial_x K_x + y^2 \partial_y K_y - y K_y) b^+, \\
R &= \frac{12}{\ell^2}.
\end{aligned} \tag{4.3b}$$

The nonvanishing irreducible components of the curvature are ${}^{(n)}R^{ij}$ for $n = 1, 4, 6$ (as in GR_Λ) and $n = 2$. Quadratic invariants of the field strengths are regular:

$$R^{ij*} R_{ij} = \frac{12}{\ell^4} \hat{\epsilon}, \quad T^i * T_i = 0.$$

B. Field equations

Dynamical content of our ansatz is effectively described by the RC Lagrangian (B1) with nonvanishing parameters $(a_0, \Lambda; a_1, b_1, b_2, b_4, b_6)$ and the associated PGT field equations (B3). Explicit calculation of the second field equation in (B3), denoted shortly by \mathcal{F}^{ij} , is shown to have two nontrivial components, \mathcal{F}^{-2} and \mathcal{F}^{-3} . After introducing the quantity $\hat{S}H$ as in Eq. (3.4), these components take the respective forms

$$\begin{aligned}
&b_1 (y \partial_x \hat{S}H + y^2 \partial_{xx} K_x + y^2 \partial_{yy} K_x - 2y \partial_x K_y) \\
&+ b_2 (y^2 \partial_{yy} K_x - y^2 \partial_{xy} K_y - y \partial_x K_y) \\
&+ b_4 (y \partial_x \hat{S}H + y^2 \partial_{xx} K_x + y^2 \partial_{xy} K_y - y \partial_x K_y) \\
&+ 2(b_6 - b_1 + a_1 \ell^2 - a_0 \ell^2) K_x = 0,
\end{aligned} \tag{4.4a}$$

and

$$\begin{aligned}
&b_1 (y \partial_y \hat{S}H + y^2 \partial_{xx} K_y + y^2 \partial_{yy} K_y + 2y \partial_x K_x) \\
&+ b_2 (-y^2 \partial_{xy} K_x + y^2 \partial_{xx} K_y + y \partial_x K_x) \\
&+ b_4 (y \partial_y \hat{S}H + y^2 \partial_{xy} K_x + y^2 \partial_{yy} K_y + y \partial_x K_x) \\
&+ 2(b_6 - b_1 + a_1 \ell^2 - a_0 \ell^2) K_y = 0.
\end{aligned} \tag{4.4b}$$

The content of the first field equation is much simpler. To have the smooth limit for vanishing torsion, we require $3a_0 + \ell^2 \Lambda = 0$, whereupon the first equation reads

$$\begin{aligned}
&(b_4 + b_6 - a_0 \ell^2) \hat{S}H \\
&+ (b_4 + b_6 - a_0 \ell^2 + a_1 \ell^2) (y \partial_x K_x + y \partial_y K_y - K_y) = 0.
\end{aligned} \tag{4.4c}$$

The form of the differential equations (4.4) appears to be rather complicated [16]. However, there exists a suitable reformulation that makes their content much more transparent. To see that, we first rewrite Eq. (4.4c) in the form

$$\begin{aligned}
\hat{S}H &= \sigma (y \partial_x K_x + y \partial_y K_y - K_y), \\
\sigma &:= - \left(1 + \frac{a_1 \ell^2}{b_4 + b_6 - a_0 \ell^2} \right).
\end{aligned} \tag{4.5a}$$

Then, by substituting the expressions for $y \partial_x \hat{S}H$ and $y \partial_y \hat{S}H$ into (4.4a)–(4.4b), and dividing the resulting equations by $(b_1 + b_4)(\sigma + 1)$, one obtains

$$\begin{aligned}
&(y^2 \partial_{xx} + \rho y^2 \partial_{yy} + 2\ell^2 \mu^2) K_x \\
&+ [(1 - \rho) y^2 \partial_{xy} - (1 + \rho) y \partial_x] K_y = 0,
\end{aligned} \tag{4.5b}$$

$$\begin{aligned}
&(y^2 \partial_{yy} + \rho y^2 \partial_{xx} + 2\ell^2 \mu^2) K_y \\
&+ [(1 - \rho) y^2 \partial_{xy} + (1 + \rho) y \partial_x] K_x = 0,
\end{aligned} \tag{4.5c}$$

where

$$\rho := \frac{b_1 + b_2}{(b_1 + b_4)(\sigma + 1)}, \quad \mu^2 := \frac{a_1 - a_0 + (b_6 - b_1)/\ell^2}{(b_1 + b_4)(\sigma + 1)}.$$

The final equations (4.5) contain only three independent parameters, σ, ρ and μ^2 , which makes it much easier to find some specific solutions for the Siklos waves with torsion.

The parameter μ^2 has a simple physical interpretation. As the linearized PGT analysis shows, possible torsion excitations around the Minkowski background are modes with spin parity $J^P = 0^\pm, 1^\pm, 2^\pm$ [17]. In particular, the spin-2⁺ state is associated to the tensorial piece of the torsion, and its mass is

$$\bar{\mu}^2 = \frac{a_0(a_1 - a_0)}{a_1(b_1 + b_4)}.$$

For $1/\ell^2 \rightarrow 0$, the coefficient μ^2 tends exactly to $\bar{\mu}^2$, whereas for finite (and positive) ℓ^2 , μ^2 is associated to the spin-2⁺ torsion excitation with respect to the AdS background.

In what follows, we present three exact solutions of the PGT field equations (4.5), enlightening thereby basic dynamical aspects of the Siklos waves with torsion. All the integration ‘‘constants’’ appearing in these solutions are functions of u .

V. KAIGORODOV-LIKE SOLUTION

Motivated by the form of the Kaigorodov solution of GR_Λ (Sec. II), we consider now a class of PGT configurations for which the functions H, K_x and K_y are x independent. Then the field equations (4.5) take a much simpler form:

$$(\rho y^2 \partial_{yy} + 2\mu^2 \ell^2) K_x = 0, \tag{5.1a}$$

$$(y^2 \partial_{yy} + 2\mu^2 \ell^2) K_y = 0, \tag{5.1b}$$

$$y\partial_{yy}H - 2\partial_y H = \sigma(y\partial_y - 1)K_y. \quad (5.1c)$$

The Euler-Fuchs differential equation (5.1a) is solved by the ansatz $K_x = y^\alpha$, where α is restricted by the requirement $\alpha^2 - \alpha + 2\mu^2\ell^2/\rho = 0$, which implies

$$\alpha_\pm = \frac{1}{2} \pm p, \quad p := \frac{1}{2} \sqrt{1 - 8\mu^2\ell^2/\rho}. \quad (5.2)$$

(a1) For $8\mu^2\ell^2/\rho < 1$ (real p),

$$K_x = \sqrt{y}(A_1 y^p + A_2 y^{-p}). \quad (5.3a)$$

(a2) For $8\mu^2\ell^2/\rho > 1$ (imaginary p , $q := |p|$),

$$K_x = \sqrt{y}[A_3 \cos(q \ln y) + A_4 \sin(q \ln y)]. \quad (5.3b)$$

(a3) For $8\mu^2\ell^2/\rho = 1$ ($p = 0$),

$$K_x = \sqrt{y}(A_5 + A_6 \ln y). \quad (5.3c)$$

Equation (5.1b) follows from (5.1a) in the limit $\rho \rightarrow 1$. Hence, using the notation

$$\bar{\alpha}_\pm = \frac{1}{2} \pm \bar{p}, \quad \bar{p} := \frac{1}{2} \sqrt{1 - 8\mu^2\ell^2}, \quad \bar{q} = |\bar{p}|, \quad (5.4)$$

the solutions for K_y can be obtained from Eqs. (5.3) by the replacements $p \rightarrow \bar{p}$, $q \rightarrow \bar{q}$.

(b1) For $8\mu^2\ell^2 < 1$,

$$K_y = \sqrt{y}(B_1 y^{\bar{p}} + B_2 y^{-\bar{p}}). \quad (5.5a)$$

(b2) For $8\mu^2\ell^2 > 1$,

$$K_y = \sqrt{y}[B_3 \cos(\bar{q} \ln y) + B_4 \sin(\bar{q} \ln y)]. \quad (5.5b)$$

(b3) For $8\mu^2\ell^2 = 1$,

$$K_y = \sqrt{y}(B_5 + B_6 \ln y). \quad (5.5c)$$

Knowing the form of K_y , one can integrate Eq. (5.1c) to obtain the metric function H . Let us first find a particular solution $H_{(i)}$ of the inhomogeneous equation (5.1c).

(c1) For $8\mu^2\ell^2 < 1$,

$$H_{(i)} = \sigma y^{3/2} \left(\frac{(\bar{\alpha}_+ - 1)}{(\bar{\alpha}_+ + 1)(\bar{\alpha}_+ - 2)} B_1 y^{\bar{p}} + \frac{(\bar{\alpha}_- - 1)}{(\bar{\alpha}_- + 1)(\bar{\alpha}_- - 2)} B_2 y^{-\bar{p}} \right). \quad (5.6a)$$

(c2) For $8\mu^2\ell^2 > 1$,

$$H_{(i)} = \frac{2\sigma}{9 + 4\bar{q}^2} y^{3/2} [(B_3 - 2B_4\bar{q}) \cos(\bar{q} \ln y) + (B_4 + 2B_3\bar{q}) \sin(\bar{q} \ln y)]. \quad (5.6b)$$

(c3) For $8\mu^2\ell^2 = 1$,

$$H_{(i)} = \frac{2\sigma}{9} y^{3/2} (B_5 - 2B_6 + B_6 \ln y). \quad (5.6c)$$

Adding to $H_{(i)}$ the solution of the homogeneous equation (5.1c), that is the Kaigorodov solution H_1 from Sec. II, one obtains the complete solution:

$$H = H_1 + H_{(i)}, \quad H_1 = Dy^3. \quad (5.7)$$

Thus, the existence of torsion has a direct influence on the form of metric.

The above solutions for K_x, K_y and H define a Kaigorodov wave with torsion as a vacuum solution of PGT.

A. Asymptotic AdS limit

It is interesting to note that the Kaigorodov solution in GR_Λ is asymptotically AdS, as follows from the asymptotic relation $H = \mathcal{O}(y^3)$ for $y \rightarrow 0$, and the form of the Riemannian curvature (2.5). In PGT, the presence of torsion makes the situation not so simple. Namely, the condition that the RC curvature R^{ij} in (4.3) has the AdS asymptotics produces two types of requirements: the first one is obtained from the non-Riemannian piece of R^{ij} ,

$$yK_x \rightarrow 0, \quad yK_y \rightarrow 0, \quad (5.8a)$$

$$y^2\partial_y K_x \rightarrow 0, \quad y^2\partial_y K_y \rightarrow 0, \quad (5.8b)$$

and the second from the Riemannian piece:

$$y\partial_y H_{(i)} \rightarrow 0, \quad y^2\partial_{yy} H_{(i)} \rightarrow 0. \quad (5.8c)$$

Further analysis goes as follows.

(i) In the sector with $8\mu^2\ell^2/\rho \geq 1$ and $8\mu^2\ell^2 \geq 1$, one can directly verify that the solutions for K_x, K_y and $H_{(i)}$ satisfy the requirements (5.8).

(ii) In the complementary sector with $8\mu^2\ell^2/\rho < 1$ and $8\mu^2\ell^2 < 1$, one finds that the requirements (5.8) are valid for $p < 1$ and $\bar{p} < 1$, or equivalently, for

$$8\mu^2\ell^2/\rho > -1 \quad \text{and} \quad 8\mu^2\ell^2 > -1. \quad (5.9)$$

Continuing with exploring the asymptotic properties of the torsion, we see that (5.8a) implies $T^i \rightarrow 0$ for $y \rightarrow 0$. Thus, the choice of parameters described in (5.9) ensures that the Kaigorodov-like solution has an AdS asymptotic behavior, with vanishing torsion. Clearly, in the physical

sector with $\mu^2 \geq 0$, the second condition in (5.9) is automatically satisfied.

B. Defrise-like solution as a special case

It is interesting to observe that the form of $H_{(i)}$ in (5.6a) allows us to obtain a generalized Defrise solution, defined in Sec. II, as a special case of the Kaigorodov wave with torsion. Namely, by choosing $D = 0$ one eliminates H_1 from H , whereupon the term $H_{(i)}$, specified by $B_1 = 0$ and $\bar{p} = 7/2$, becomes identical to the Defrise metric function:

$$H = H_{(i)} \sim 1/y^2. \quad (5.10)$$

The restriction $\bar{p} = 7/2$ refers to the tachyonic sector of the 2^+ torsion mode, with $\mu^2 \ell^2 = -6$. The above result for H , combined with the corresponding expressions for K_x and K_y , defines the Defrise solution with torsion as a *vacuum* solution of PGT. In contrast to that, the corresponding solution in GR_Λ exists only in the presence of *matter*. One should stress that the metric function H originates purely from the torsional term $H_{(i)}$.

VI. HOMOGENEOUS SOLUTION

Let us now look for a solution in which K_x, K_y, H are homogeneous functions of y and x :

$$\begin{aligned} K_x &= f_x(t), & K_y &= f_y(t), \\ H &= h(t), & t &:= y/x. \end{aligned}$$

As a consequence, the field equations (4.5) become

$$(t^4 + \rho t^2)f_x'' + 2t^3 f_x' + 2\mu^2 f_x - (1 - \rho)t^3 f_y'' + 2\rho t^2 f_y' = 0, \quad (6.1a)$$

$$(t^2 + \rho t^4)f_y'' + 2\rho t^3 f_y' + 2\mu^2 f_y - (1 - \rho)t^3 f_x'' - 2t^2 f_x' = 0, \quad (6.1b)$$

$$\hat{S}H = \sigma(-t^2 f_x' + t f_y' - f_y), \quad (6.1c)$$

where $\hat{S}H = y[2t(t^2 - 1)h' + (t^4 + t^2)h'']$.

The set of equations (6.1) represents a system of ordinary, second-order, linear differential equations. The system is significantly simplified by assuming that the metric function H retains the same form as in GR_Λ , so that $\hat{S}H = 0$. Consequently, the right-hand side of Eq. (6.1c) vanishes, $-t^2 f_x' + t f_y' - f_y = 0$, which implies

$$f_x = \frac{1}{t} f_y + B, \quad (6.2)$$

where $B = B(u)$. Substituting this expression into (6.1a)–(6.1b), one obtains

$$\rho t^2(t^2 + 1)f_y'' + 2\rho t(t^2 - 1)f_y' + 2(\rho + \mu^2 \ell^2)f_y + 2\mu^2 t B = 0, \quad (6.3a)$$

$$\rho t^2(t^2 + 1)f_y'' + 2\rho t(t^2 - 1)f_y' + 2(\rho + \mu^2 \ell^2)f_y = 0. \quad (6.3b)$$

Taking the difference of these two equations yields

$$\mu^2 B = 0.$$

Hence, either μ^2 or B has to vanish.

A. Case $\mu^2 = 0$

Assuming $\rho \neq 0$, the set of equations (6.3) reduces to

$$t^2(t^2 + 1)f_y'' + 2t(t^2 - 1)f_y' + 2f_y = 0.$$

Hence, the general solution for f_y is given by

$$f_y = C_1 \frac{t}{t^2 + 1} + C_2 \frac{t^2}{t^2 + 1}, \quad (6.4)$$

f_x is determined by (6.2), and the metric function has the same form as in GR_Λ :

$$h = C_3 \left(-\arctan t + \frac{t}{1 + t^2} \right) + C_4. \quad (6.5)$$

As before, all the integration constants are functions of u .

B. Case $B = 0$

In this case, the set of equations (6.3) reduces to

$$t^2(t^2 + 1)f_y'' + 2t(t^2 - 1)f_y' + 2 \left(1 + \frac{\mu^2 \ell^2}{\rho} \right) f_y = 0.$$

(d1) For $8\mu^2 \ell^2 / \rho \neq 1$,

$$\begin{aligned} f_y &= C_5 t^{\frac{3}{2} - \xi} {}_2F_1 \left(\frac{3}{4} - \frac{\xi}{2}, \frac{5}{4} - \frac{\xi}{2}; 1 - \xi; -t^2 \right) \\ &+ C_6 t^{\frac{3}{2} + \xi} {}_2F_1 \left(\frac{3}{4} + \frac{\xi}{2}, \frac{5}{4} + \frac{\xi}{2}; 1 + \xi; -t^2 \right) \end{aligned} \quad (6.6a)$$

where $\xi = \frac{1}{2} \sqrt{1 - 8\mu^2 \ell^2 / \rho}$ and ${}_2F_1(a, b; c; z)$ is the hypergeometric function [18].

(d2) For $8\mu^2 \ell^2 / \rho = 1$,

$$f_y = C_7 t^{3/2} {}_2F_1 \left(\frac{3}{4}, \frac{5}{4}; 1; -t^2 \right) + C_8 G_{20}^{20} \left(-t^2 \left| \begin{matrix} 1/2, 1 \\ 3/4, 3/4 \end{matrix} \right. \right), \quad (6.6b)$$

where G_{pq}^{mn} is the Meijer G function [18]. In both cases, the associated solution for f_x is given by $f_x = f_y/t$, see (6.2), and the metric function h remains the same as in (6.5).

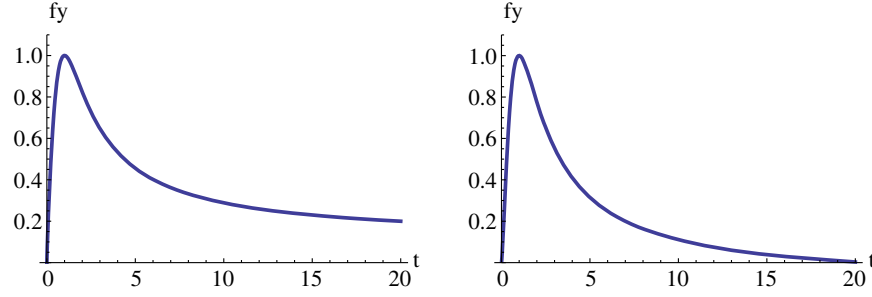


FIG. 1 (color online). The plots of the torsion function f_y in (6.6a), $8\mu^2\ell^2/\rho = -1$, $f_y[1] = 1$, $f'_y[1] = 0$ (left), and in (6.6b), $f_y[1] = 1$, $f'_y[1] = 0$ (right).

In the above two cases (d1)–(d2), the forms of the corresponding torsion functions f_y are illustrated in Fig. 1.

VII. EXPONENTIAL SOLUTION

In this section, we start with

$$K_x = e^x f_x(y), \quad K_y = e^x f_y(y), \quad H = e^x h(y), \quad (7.1)$$

whereupon the field equations (4.5) become

$$(y^2 + \rho y^2 \partial_{yy} + 2\mu^2 \ell^2) f_x + [(1 - \rho)y^2 \partial_y - (1 + \rho)y] f_y = 0, \quad (7.2a)$$

$$(y^2 \partial_{yy} + \rho y^2 + 2\mu^2 \ell^2) f_y + [(1 - \rho)y^2 \partial_y + (1 + \rho)y] f_x = 0, \quad (7.2b)$$

$$\hat{S}H = \sigma(y f_x + y \partial_y f_y - f_y), \quad (7.2c)$$

and $\hat{S}H = e^x[y(h + h'') - 2h']$.

As in the previous section, we assume that H coincides with the vacuum solution of GR_Λ , defined by $\hat{S}H = 0$. This imposes an extra condition on f_x and f_y :

$$y f_x + y \partial_y f_y - f_y = 0 \Rightarrow \frac{f_x}{y} + \left(\frac{f_y}{y}\right)' = 0. \quad (7.3)$$

By introducing a change of variables, given by

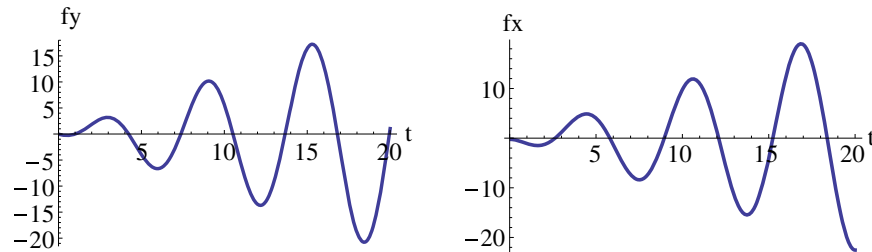


FIG. 2 (color online). The plots of the torsion functions (7.7) for $D_1 = D_2 = 1$, $8\mu^2\ell^2/\rho = -1$.

$$f_x = y g_x, \quad f_y = y g_y, \quad (7.4a)$$

the condition (7.3) takes a simple form:

$$g_x + g'_y = 0. \quad (7.4b)$$

As a consequence, Eqs. (7.2a)–(7.2b) are transformed into

$$\rho y^2 g_y^{(3)} + 2\rho y g_y'' + (\rho y^2 + 2\mu^2 \ell^2) g_y' + 2\rho y g_y = 0, \quad (7.5a)$$

$$\rho y^2 g_y'' + (\rho y^2 + 2\mu^2 \ell^2) g_y = 0. \quad (7.5b)$$

One can note that (7.5a) is equal to the derivative (with respect to y) of (7.5b). The solution of (7.5b) reads

$$g_y = \sqrt{y}[D_1 J_\nu(y) + D_2 Y_\nu(y)], \quad (7.6)$$

where $\nu = \frac{1}{2}\sqrt{1 - 8\mu^2\ell^2/\rho}$, and J_ν , Y_ν are the Bessel functions of the first and second kind, respectively [18]. Hence,

$$f_y = y^{\frac{3}{2}}(D_1 J_\nu(y) + D_2 Y_\nu(y)), \quad (7.7a)$$

and $f_x = -y g_y'$ yields

$$f_x = \sqrt{y}[D_1(y J_{\nu+1}(y) - (\nu + 1/2)J_\nu(y)) + D_2(y Y_{\nu+1}(y) - (\nu + 1/2)Y_\nu(y))]. \quad (7.7b)$$

The forms of the torsion functions (7.7) are illustrated in Fig. 2. They are of the same type as the GR_Λ metric function H_3 , defined in Sec. II. Together, they define our third specific Siklos wave with torsion.

VIII. CONCLUDING REMARKS

In this paper, we introduced a new class of exact vacuum solutions of PGT, the Siklos waves with torsion. The solution is constructed in a way that respects the radiation nature of the original Siklos configuration in GR_Λ . This is achieved by an ansatz for the RC connection that produces only the tensorial irreducible mode of the torsion, propagating on the AdS background. A compact form of the PGT field equations is used to find three particular vacuum solutions belonging to the class of Siklos waves with torsion; they generalize the Kaigorodov, the homogeneous solution and the exponential solution of GR_Λ .

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APPENDIX A: IRREDUCIBLE DECOMPOSITION OF THE FIELD STRENGTHS

We present here formulas for the irreducible decomposition of torsion and curvature in four-dimensional Riemann-Cartan spacetime [5]; for general D, see [19].

It is convenient to start the exposition with the Bianchi identities:

$$\nabla T^i = R^i_m b^m, \quad \nabla R^{ij} = 0. \quad (\text{A1})$$

The torsion 2-form has three irreducible pieces:

$$\begin{aligned} (2)T^i &= \frac{1}{3}b^i \wedge (h_m] T^m), \\ (3)T^i &= -\frac{1}{3}\star[b^i \wedge \star(T^m \wedge b_m)] = \frac{1}{3}h^i](T^m \wedge b_m), \\ (1)T^i &= T^i - (2)T^i - (3)T^i. \end{aligned} \quad (\text{A2})$$

The RC curvature 2-form can be decomposed into six irreducible pieces:

$$\begin{aligned} (2)R^{ij} &= -\star(b^{[i} \wedge \Psi^{j]}), & (4)R^{ij} &= b^{[i} \wedge \Phi^{j]}, \\ (3)R^{ij} &= -\frac{1}{12}X^\star(b^i \wedge b^j), & (6)R^{ij} &= \frac{1}{12}Wb^i \wedge b^j, \\ (5)R^{ij} &= \frac{1}{2}b^{[i} \wedge h^{j]}](b^m \wedge W_m), \\ (1)R^{ij} &= R^{ij} - \sum_{a=2}^6 (a)R^{ij}, \end{aligned}$$

where

$$\begin{aligned} W^i &:= h_m]R^{mi} = Ric^i, & W &:= h_i]W^i = R, \\ X^i &:= \star(R^{ki} \wedge b_k), & X &:= h_i]X^i \end{aligned} \quad (\text{A3})$$

and

$$\begin{aligned} \Phi_i &:= W_i - \frac{1}{4}b_i W - \frac{1}{2}h_i](b^m \wedge W_m), \\ \Psi_i &:= X_i - \frac{1}{4}b_i X - \frac{1}{2}h_i](b^m \wedge X_m). \end{aligned} \quad (\text{A4})$$

The trace and symmetry properties of $(^n)R_{ij}$ can be found in Ref. [19], page 127. All these properties are satisfied by our ansatz.

For torsion-free solutions, the first Bianchi identity in (A1) implies $X^i = 0$; hence $(2)R^{ij}$ and $(3)R^{ij}$ vanish. Moreover, $Ric_{[ij]} = 0$ implies $(5)R^{ij} = 0$. The remaining three curvature parts, first, fourth and sixth, are the PGT analogues of the irreducible pieces of the Riemannian curvature. In Riemannian geometry, $(1)R^{ij}$ coincides with the Weyl (conformal) tensor,

$$C^{ij} := R^{ij} - \frac{1}{2}(b^i Ric^j - b^j Ric^i) + \frac{1}{6}Rb^i b^j,$$

but in the RC geometry, $(1)R^{ij}$ differs from C^{ij} by the presence of torsion terms. Thus, $(1)R^{ij}$ is a true extension of C^{ij} to the RC geometry. The fourth component is defined in terms of the symmetric traceless Ricci tensor,

$$\Phi_i = \left(Ric_{(ij)} - \frac{1}{4}\eta_{ij}R \right) b^j. \quad (\text{A5})$$

The above formulas are taken from Refs. [5,19] with one modification: the definition of W^i is taken with an additional minus sign (Landau-Lifshitz convention), and for consistency, the overall signs of the fourth through sixth curvature parts are also changed.

APPENDIX B: PGT FIELD EQUATIONS

The gravitational dynamics of PGT is determined by a Lagrangian $L_G = L_G(b^i, T^i, R^{ij})$ (4-form), which is assumed to be at most quadratic in the field strengths (quadratic PGT) and parity invariant. The form of L_G can be conveniently represented as

$$L_G = -\star(a_0 R + 2\Lambda) + \frac{1}{2}T^i H_i + \frac{1}{4}R^{ij} H'_{ij}, \quad (\text{B1})$$

where $H_i := \partial L_G / \partial T^i$ (the covariant momentum) and H'_{ij} define the quadratic terms in L_G :

$$H_i = 2 \sum_{n=1}^3 \star(a_n (^n)T_i), \quad H'_{ij} := 2 \sum_{n=1}^6 \star(b_n (^n)R_{ij}). \quad (\text{B2a})$$

Varying L_G with respect to b^i and ω^{ij} yields the PGT field equations in vacuum. After introducing the complete covariant momentum $H_{ij} := \partial L_G / \partial R^{ij}$ by

$$H_{ij} = -2a_0^*(b^i b^j) + H'_{ij}, \quad (\text{B2b})$$

these equations can be written in a compact form as

$$\begin{aligned} (1ST) \quad \nabla H_i + E_i &= 0, \\ (2ND) \quad \nabla H_{ij} + E_{ij} &= 0, \end{aligned} \quad (\text{B3})$$

where E_i and E_{ij} are the gravitational energy-momentum and spin currents:

$$\begin{aligned} E_i &:= h_i]L_G - (h_i]T^m)H_m - \frac{1}{2}(h_i]R^{mn})H_{mn}, \\ E_{ij} &:= -(b_i H_j - b_j H_i). \end{aligned} \quad (\text{B4})$$

The general field equations (B3) are used in Sec. IV to describe specific dynamical aspects of the Siklos waves with torsion. In the Riemannian sector with $T^i = 0$, we have $H_i = 0$ and $E_{ij} = 0$, and the field equations (B3) reduce to the form given in Sec. III.

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