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Biconformal symmetry and static Maxwell fields near higher-dimensional black holes

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We study an electric field created by a static electric charge near the higher-dimensional Reissner-Nordström black hole. The relation between the static Green functions on the D-dimensional Reissner-Nordström background and on the $(D+2)$-dimensional homogeneous Bertotti-Robinson spacetime is found. Using the biconformal symmetry we obtain a simple integral representation for the static Maxwell Green functions in arbitrary dimensions. We show that in a four-dimensional spacetime the static Green function obtained by the biconformal method correctly reproduces known results. We also find a closed form for the exact static Green functions and vector potentials in the five-dimensional Reissner-Nordström spacetime.

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I. INTRODUCTION

In this paper we continue studying fields created by static charges placed in the vicinity of a higher-dimensional static black hole. For this purpose we use the method of biconformal transformations, which was developed in our previous paper [1,2] in application to the case of scalar charges in the Schwarzschild-Tangherlini and Reissner-Nordström geometries.

A vector potential A_μ in a D-dimensional spacetime with metric $g_{\mu\nu}$ $(\mu,\nu=0,\dots,D-1)$ obeys the Maxwell equations

$$F^{\mu\nu}{}_{;\nu}=4\pi J^\mu,\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu. \tag{1.1}$$

Here coefficient 4π comes from the definition of the Maxwell action in higher dimensions. In four dimensions our choice of units corresponds to the conventional Gaussian units. There is an ambiguity in generalizations of the Maxwell equations to higher dimensions that depends on a system of units and the definition of an electric charge. In this paper charges are normalized in such a way that the interaction force between two charges in D dimensions reads

$$f=\frac{4\pi}{\Omega_{(D-2)}}\frac{e_1e_2}{r^{D-2}},\qquad \Omega_{(k)}=\frac{2\pi^{(k+1)/2}}{\Gamma((k+1)/2)}. \tag{1.2}$$

Different choices of units in higher dimensions are also used in the literature. For example, in the paper [3] authors work in a different system of units, such that the force between two charges $\tilde{e}_1$ and $\tilde{e}_2$ in the D-dimensional Minkowski spacetime is given by

$$f=\frac{\tilde{e}_1\tilde{e}_2}{r^{D-2}}. \tag{1.3}$$

Thus, our normalization of the charges and that of the paper [3] are related as

$$e^2=\frac{\Omega_{(D-2)}}{4\pi}\tilde{e}^2, \tag{1.4}$$

where the volume $\Omega_{(k)}$ of k-dimensional sphere S^k is given by (1.2).

In the Lorentz gauge $\nabla^\mu A_\mu=0$ the Maxwell equations become

$$\Box A_\mu-R^\nu_\mu A_\nu=-4\pi J_\mu. \tag{1.5}$$

Let us consider the potential created by a static electric source $J^\mu=j(x)\delta^\mu_0$, where $j(x)$ is the charge density in a static spacetime described by the metric

$$ds^2=-\alpha^2dt^2+g_{ab}dx^adx^b,\qquad a=1,\dots,D-1,$$
$$X^\mu=(t,x^a),\qquad \alpha=\alpha(x),\qquad g_{ab}=g_{ab}(x). \tag{1.6}$$

In the static case one can choose $A_a=0$ and $A_0=A_0(x)$, that is

$$A_\mu=A_0\delta^0_\mu. \tag{1.7}$$

Then the Maxwell equations (1.1) for the potential A_μ boil down to

$$\hat{O}A_0=4\pi j,\qquad \hat{O}=\frac{1}{\alpha\sqrt{g}}\partial_a\left(\frac{1}{\alpha}\sqrt{g}g^{ab}\partial_b\right). \tag{1.8}$$

Here $g=\det(g_{ab})$. The redshift factor α is connected with the norm of the static Killing vector $\boldsymbol{\xi}$ as follows: $\alpha=\sqrt{-\boldsymbol{\xi}^2}=\sqrt{-g_{tt}}$.

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We define the static Green function for the operator $\hat{O}$ as the solution of the following equation:

$$\hat{O}G_{00}(x,x') = \frac{1}{\alpha\sqrt{g}}\delta(x-x'). \tag{1.9}$$

Equation (1.8) is invariant under the following *biconformal* transformations:

$$A_0 = \bar{A}_0, \qquad g_{ab} = \Omega^2\bar{g}_{ab},$$
$$\alpha = \Omega^n\bar{\alpha}, \qquad j = \Omega^{-2n-2}\bar{j}, \tag{1.10}$$

where $n \equiv D-3$ and Ω is an arbitrary function of spatial coordinates x^a.

These biconformal transformations can be used to relate solutions of the Maxwell equations on a physical metric to solutions on a some other "reference" geometry. If the reference spacetime is more symmetrical than the original one, then there is a good chance to simplify the problem of finding the Green function exactly. This approach, for example, enabled us [4] to compute the Green functions of static scalar and Maxwell fields on the background of the Majumdar-Papapetrou spacetime, which describes a set of extremally charged black holes. It was possible because the higher-dimensional Majumdar-Papapetrou metric is biconformally related to the flat Minkowski metric.

In this paper we use biconformal transformations to compute the static Green functions of the Maxwell field on the background of a generic higher-dimensional Reissner-Nordström black hole.

II. POINT CHARGE NEAR A HIGHER-DIMENSIONAL REISSNER-NORDSTRÖM BLACK HOLE

Let us consider a static spherically symmetric D-dimensional metric of the form

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\omega_{n+1}^2, \tag{2.1}$$

where $n = D-3$ and $d\omega_{n+1}^2$ is the line element on a $(n+1)$-dimensional unit sphere

$$d\omega_{n+1}^2 = d\theta_n^2 + \sin^2\theta_n d\omega_n^2, \qquad d\omega_0^2 = d\phi^2, \qquad \theta_n = \theta. \tag{2.2}$$

We denote $\theta_0 \equiv \phi \in [0, 2\pi]$. The other angular coordinates $\theta_{i>0} \in [0,\pi]$.

$$f = 1 - \frac{2M}{r^n} + \frac{Q^2}{r^{2n}}. \tag{2.3}$$

For real positive M and real Q, which satisfy the condition $|Q| \le M$, the metric (2.1)–(2.3) describes the geometry of a higher-dimensional generalization of a spherically symmetric electrically charged black hole [5]. The parameters M and Q are proportional to the Arnowitt-Deser-Misner mass and charge of the black hole, respectively. The coefficients of proportionality (see, e.g., [6]) depend on the dimensionality of the spacetime and on the choice of units.

It is convenient to introduce a new radial variable ρ related to the radial coordinate r as follows:

$$\rho = \frac{r^n - M}{\mu}, \qquad \mu = \sqrt{M^2 - Q^2}, \qquad r^n = M + \mu\rho. \tag{2.4}$$

Then the Reissner-Nordström metric (2.1)–(2.3) takes the form

$$ds^2 = -\frac{\mu^2(\rho^2-1)}{(M+\mu\rho)^2}dt^2 + (M+\mu\rho)^{2/n}\left[\frac{1}{n^2(\rho^2-1)}d\rho^2 + d\omega_{n+1}^2\right]. \tag{2.5}$$

The horizon corresponds to $\rho = 1$, and its (gravitational) radius r_{g} is given by the expression $r_{\mathrm{g}}^n = M + \mu$. The surface gravity at the horizon is

$$\kappa = \frac{n\mu}{r_{\mathrm{g}}^{n+1}}. \tag{2.6}$$

Taking into account that

$$\alpha = \frac{\mu\sqrt{\rho^2-1}}{M+\mu\rho}, \qquad g^{\rho\rho} = n^2(\rho^2-1)(M+\mu\rho)^{-2/n},$$
$$g^{\theta\theta} = (M+\mu\rho)^{-2/n},$$
$$\sqrt{-g^{\mathrm{D}}} = \alpha\sqrt{g} = \frac{1}{n}\mu(M+\mu\rho)^{2/n}\sqrt{g_\omega},$$
$$\sqrt{g_\omega} = \prod_{k=1}^{n}(\sin\theta_k)^k, \tag{2.7}$$

the equation for the Green function (1.9) takes the form

$$\left[n^2\partial_\rho(M+\mu\rho)^2\partial_\rho + \frac{(M+\mu\rho)^2}{\rho^2-1}\Delta_\omega^{n+1}\right]G_{00}(x,x') = n\mu\delta(\rho-\rho')\delta(\omega,\omega'). \tag{2.8}$$

Here Δ_ω^{n+1} stands for the Laplace operator defined on the unit $(n+1)$-dimensional sphere S^{n+1}.

If we use the ansatz

$$G_{00}(x,x') = -\frac{\mu^2(\rho^2-1)(\rho'^2-1)}{(M+\mu\rho)(M+\mu\rho')}H(x,x'), \tag{2.9}$$

then the equation for $H(x,x')$ becomes

$$\{n^2[(\rho^2-1)\partial_\rho^2+4\rho\partial_\rho+2]+\triangle_\omega^{n+1}\}H(x,x') = -\frac{n}{\mu(\rho'^2-1)}\delta(\rho-\rho')\delta(\omega,\omega'). \tag{2.10}$$

In order to solve this equation we use the following trick. We first consider another equation for the static Green function of a massive scalar operator $\Box - m^2$ with the mass

$$m^2 = -2n^2/a^2 = -2/b^2 \tag{2.11}$$

on a $(D+2)$-dimensional homogeneous spacetime, which is a direct product of the $(n+1)$-dimensional sphere of a radius a and a four-dimensional anti–de Sitter spacetime or, in the Euclidean version, the hyperboloid H^4 of a radius $b = a/n$.

$$d\tilde{s}^2 = d\ell_{H^4}^2 + a^2 d\omega_{n+1}^2, \qquad b = \frac{a}{n},$$
$$X_{\rm E} = (\rho,\sigma,\bar\theta,\bar\phi,\theta,\theta_{n-1},\dots,\phi), \tag{2.12}$$

$$d\ell_{H^4}^2 = b^2\left[\frac{1}{\rho^2-1}d\rho^2 + (\rho^2-1)d\bar\omega_3^2\right], \tag{2.13}$$

$$d\bar\omega_{n+1}^2 = d\bar\theta_n^2 + \sin^2\bar\theta_n d\bar\omega_n^2,$$
$$d\bar\omega_0^2 = d\bar\phi^2, \qquad \bar\theta_n = \sigma. \tag{2.14}$$

The Green function of a Euclidean massive operator

$$\hat{O} = \Box_{\rm E} - m^2 \tag{2.15}$$

defined on a $(D+2)$-dimensional Bertotti-Robinson spacetime can be obtained by the heat kernel method. In order to calculate the function $H(x,x')$ one has to find at first the $(D+2)$-dimensional Green function of the operator (2.15), which satisfies the equation

$$\hat{O}\mathbb{G}_{\hat{O}} = -\delta^{D+2}(X_{\rm E},X'_{\rm E}). \tag{2.16}$$

The hyperboloid H^4 is spherically symmetric. We show that integration of $\mathbb{G}_{\hat{O}}$ over all angle coordinates $\bar\theta_k$ on H^4 gives

$$H(x,x') = \frac{a^{n+3}}{n^3\mu}\int d^3\bar\omega\,\sqrt{g_{\bar\omega}}\,\mathbb{G}_{\hat{O}}(X_{\rm E},X'_{\rm E}). \tag{2.17}$$

This approach is similar to the case of a scalar field [1,2]. The difference is that the required static Green function in the spacetime of a black hole is generated by the Green function of the massive scalar operator, which is defined on the $H^4\times S^{n+1}$ geometry.

III. GREEN FUNCTIONS AND HEAT KERNELS

A. General formulas

The Euclidean Green function for the self-adjoint operator $\hat{O}_{\rm E}$ in any dimensions can be written in terms of the heat kernel of this operator,

$$\mathbb{G}_{\hat{O}}(X_{\rm E},X'_{\rm E}) = \int_0^\infty ds K_{\hat{O}}(s|X_{\rm E},X'_{\rm E}). \tag{3.1}$$

Here the heat kernel $K_{\hat{O}}(s|X_{\rm E},X'_{\rm E})$ is the solution of the problem

$$(\partial_s - \hat{O})K_{\hat{O}}(s|X_{\rm E},X'_{\rm E}) = 0,$$
$$K_{\hat{O}}(0|X_{\rm E},X'_{\rm E}) = \delta(X_{\rm E},X'_{\rm E}), \tag{3.2}$$

which satisfies the same boundary conditions with respect to its arguments $X_{\rm E}$ and $X'_{\rm E}$ as the Green function in question.

Because the geometry of the $(D+2)$-dimensional Bertotti-Robinson spacetime has the form of a direct sum of two homogeneous spaces, the heat K has a form of a product of the heat kernels K_{H^4} and $K_{S^{n+1}}$ for the reduced box operators defined on the hyperboloid H^4 and on the sphere S^{n+1}, respectively,

$$K_{\hat{O}}(s|\rho,\bar\theta_k,\theta_n;\rho',\bar\theta'_k,\theta'_n) = e^{-m^2 s}K_{H^4}(s|\rho,\bar\theta_k;\rho',\bar\theta'_k)K_{S^{n+1}}(s|\theta_n;\theta'_n). \tag{3.3}$$

Both spaces H^4 and S^{n+1} are homogeneous and isotropic and the corresponding heat kernels are known explicitly [7].

In the case of the operator $\Box_{\rm E} - m^2$ defined on the $n+5$-dimensional Bertotti-Robinson metric (2.12) the Euclidean Green function satisfies the equation

$$(\Box_{\rm E} - m^2)\mathbb{G}_{\hat{O}}(X_{\rm E},X'_{\rm E}) = -\delta(X_{\rm E},X'_{\rm E}). \tag{3.4}$$

Because of the symmetry of the metric, this Green function is, in fact, the function of only two geodesic distances: χ between points on the hyperboloid and γ on the sphere

$$\mathbb{G}_{\hat{O}}(X_{\rm E},X'_{\rm E}) = \mathbb{G}_{\hat{O}}(\chi,\gamma). \tag{3.5}$$

Explicitly, the equation for the $(D+2)$-dimensional Green function $\mathbb{G}_{\hat{O}}$ reads

$$\left\{n^2\left[(\rho^2-1)\partial_\rho^2 + 4\rho\partial_\rho + \frac{1}{\rho^2-1}\triangle_{\bar\omega}^3\right] - a^2m^2 + \triangle_\omega^{n+1}\right\} \times\mathbb{G}_{\hat{O}}(\chi,\gamma) = -\frac{n^4\delta(\rho-\rho')\delta(\bar\omega,\bar\omega')\delta(\omega,\omega')}{a^{n+3}(\rho'^2-1)}. \tag{3.6}$$

B. Heat kernel on H^4

The heat kernel of the four-dimensional Laplace operator defined on the hyperboloid (2.13) of the radius b reads

$$K_{H^4}(s|\chi) = -e^{-2s/b^2}\left(\frac{1}{2\pi b^2}\frac{\partial}{\partial\cosh\chi}\right)K_{H^2}(s|\chi). \quad (3.7)$$

Here $K_{H^2}(s|\chi)$ is the heat kernel of the Laplace operator defined on the two-dimensional hyperboloid H^2 of the same radius

$$d\ell^2_{H^2} = b^2[(\rho^2-1)^{-1}d\rho^2 + (\rho^2-1)d\sigma^2]. \quad (3.8)$$

It reads [7]

$$K_{H^2}(s|\chi) = \frac{\sqrt{2}b}{(4\pi s)^{3/2}}e^{-s/(4b^2)} \times \int_\chi^\infty dy\frac{ye^{-b^2y^2/(4s)}}{(\cosh y - \cosh(\chi))^{1/2}}. \quad (3.9)$$

Here χ is the geodesic distance between two points on the H^2 of the unit radius $b=1$. It is given by the relation

$$\cosh(\chi) = \rho\rho' - \sqrt{\rho^2-1}\sqrt{\rho'^2-1}\cos(\sigma-\sigma'). \quad (3.10)$$

C. Heat kernel on S^{n+1}

The heat kernel on a two-dimensional sphere S^2 of the radius a reads [7]

$$K_{S^2}(s|\gamma) = \frac{\sqrt{2}a}{(4\pi s)^{3/2}}e^{s/(4a^2)} \times \sum_{k=-\infty}^{\infty}(-1)^k\int_\gamma^\pi d\phi\frac{(\phi+2\pi k)e^{-a^2(\phi+2\pi k)^2/(4s)}}{(\cos\gamma-\cos\phi)^{1/2}}. \quad (3.11)$$

Another equivalent representation of this kernel is

$$K_{S^2}(s|\gamma) = \frac{1}{4\pi a^2}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)e^{-\frac{sl(l+1)}{a^2}}. \quad (3.12)$$

Here γ is the geodesic distance between two points on the unit S^2 $(a=1)$,

$$\cos\gamma = \cos(\theta_1)\cos(\theta_1') + \sin(\theta_1)\sin(\theta_1')\cos(\phi-\phi'). \quad (3.13)$$

The heat kernel on the three-dimensional sphere S^3 of the radius a reads [7]

$$K_{S^3}(s|\gamma) = \frac{1}{(4\pi s)^{3/2}}e^{s/a^2} \times \sum\sum_{k=-\infty}^{\infty}\frac{(\gamma+2\pi k)e^{-a^2(\gamma+2\pi k)^2/(4s)}}{\sin\gamma}, \quad (3.14)$$

where γ is the geodesic distance between two points on the unit S^3 $(a=1)$,

$$\cos\gamma = \cos(\theta_2)\cos(\theta_2') + \sin(\theta_2)\sin(\theta_2') \times [\cos(\theta_1)\cos(\theta_1') + \sin(\theta_1)\sin(\theta_1')\cos(\phi-\phi')]. \quad (3.15)$$

The heat kernels on all higher-dimensional spheres S^{n+1} can be derived from K_{S^2} and K_{S^3} using the relations [see [7], Eqs. (8.12)–(8.13)]

$$K_{S^{n+1}}(s|\gamma) = e^{\frac{(n^2-1)s}{4a^2}}\left(\frac{1}{2\pi a^2}\frac{\partial}{\partial\cos\gamma}\right)^{\frac{(n-1)}{2}}K_{S^2}(s|\gamma),$$

n odd, (3.16)

$$K_{S^{n+1}}(s|\gamma) = e^{\frac{(n^2-4)s}{4a^2}}\left(\frac{1}{2\pi a^2}\frac{\partial}{\partial\cos\gamma}\right)^{\frac{(n-2)}{2}}K_{S^3}(s|\gamma),$$

n even. (3.17)

D. Heat kernel and Green functions on $H^4 \times S^{n+1}$

For computational reasons, we also use another Green function, which is the Green function of the Laplace operator defined on the D-dimensional Euclidean Bertotti-Robinson space $H^2 \times S^{n+1}$.

$$d\bar{s}^2 = d\ell^2_{H^2} + a^2 d\omega^2_{n+1}, \qquad b = \frac{a}{n}, \quad (3.18)$$

$$d\ell^2_{H^2} = b^2\left[\frac{1}{\rho^2-1}d\rho^2 + (\rho^2-1)d\sigma^2\right]. \quad (3.19)$$

In the latter case the corresponding Euclidean Green function, which we denote $\bar{\mathbb{G}}$, satisfies the equation

$$\bar{\Box}_{\mathrm{E}}(X,X')\bar{\mathbb{G}} = -\delta(X,X'), \qquad X^\alpha = (\rho,\sigma,\theta_n,\ldots,\phi). \quad (3.20)$$

Because of the symmetries of the Bertotti-Robinson spacetime, the Green function $\bar{\mathbb{G}}$ and the heat kernel $\bar{K}$ of the operator $\bar{\Box}_{\mathrm{E}}$ are functions of only the geodesic distances γ and χ between the points on the sphere and on the hyperboloid, respectively. One can write explicitly

$$\left\{n^2\left[(\rho^2-1)\partial_\rho^2 + 2\rho\partial_\rho + \frac{1}{\rho^2-1}\partial_\sigma^2\right] + \triangle_\omega^{n+1}\right\}\bar{\mathbb{G}}(\chi,\gamma) = -\frac{n^2}{a^{n+1}}\delta(\rho-\rho')\delta(\sigma-\sigma')\delta(\omega,\omega'). \quad (3.21)$$

Using (2.11), (3.7), (3.16), and (3.3) one can express the heat kernel for the $(D+2)$-dimensional massive scalar operator $\hat{O}$ in terms of that of the D-dimensional massless operator $\bar{\Box}_{\mathrm{E}}$,

$$K_{\hat{O}}(s|\chi,\gamma) = -\left(\frac{n^2}{2\pi a^2}\frac{\partial}{\partial \cosh\chi}\right)\bar{K}(s|\chi,\gamma), \tag{3.22}$$

where we put $b = a/n$. The heat kernel $\bar{K}(s|\chi,\gamma)$ is that of the massless scalar Euclidean D'Alembert operator. It has been calculated in [1,2].

The Green function of the scalar operator is the integral over the proper time s of the corresponding heat kernel

$$\mathbb{G}_{\hat{O}}(\chi,\gamma) = \int_0^\infty ds K_{\hat{O}}(s|\chi,\gamma). \tag{3.23}$$

Therefore, one can write

$$\mathbb{G}_{\hat{O}}(\chi,\gamma) = -\left(\frac{n^2}{2\pi a^2}\frac{\partial}{\partial \cosh\chi}\right)\bar{\mathbb{G}}(\chi,\gamma). \tag{3.24}$$

Note that in this relation both $\mathbb{G}_{\hat{O}}$ and $\bar{\mathbb{G}}$ are considered as functions of χ and γ. On the other hand one has to keep in mind that the geodesic distances χ on H^4 and H^2 are quite different functions of coordinates on these hyperboloids. One should also remember that the corresponding static Green functions are defined as integrals of $\mathbb{G}_{\hat{O}}$ and $\bar{\mathbb{G}}$ over the Euclidean time σ with different measures. The scalar Green function $\bar{\mathbb{G}}(\chi,\gamma)$ has been calculated in our papers [1,2].

Taking into account that

$$\int d^3\bar{\omega}\sqrt{g_{\bar{\omega}}}\,\delta(\bar{\omega},\bar{\omega}') = 1, \qquad \int d^3\bar{\omega}\sqrt{g_{\bar{\omega}}}\,\triangle^3_{\bar{\omega}}(\ldots) = 0, \tag{3.25}$$

we obtain

$$H(x,x') = \frac{a^{n+3}}{n^3\mu}\int d^3\bar{\omega}\sqrt{g_{\bar{\omega}}}\,\mathbb{G}_{\hat{O}}(\chi,\gamma). \tag{3.26}$$

The spherical symmetry of the $\bar{\omega}$ allows one to choose the coordinates on this sphere such that χ depends only on the angle σ on the sphere. In these coordinates for any function $f(\chi)$,

$$\int d^3\bar{\omega}\sqrt{g_{\bar{\omega}}}\,f(\chi) == 2\pi\int_0^{2\pi} d\sigma \sin^2\sigma f(\chi). \tag{3.27}$$

Thus

$$\begin{aligned} H(x,x') &= 2\pi\frac{a^{n+3}}{n^3\mu}\int_0^{2\pi} d\sigma\sin^2\sigma\,\mathbb{G}_{\hat{O}}(\chi,\gamma) \\ &= -\frac{a^{n+1}}{n\mu}\int_0^{2\pi} d\sigma\sin^2\sigma\frac{\partial}{\partial\cosh\chi}\bar{\mathbb{G}}(\chi,\gamma). \end{aligned} \tag{3.28}$$

Thus one can formally express the static Green function for the Maxwell field in terms of the scalar Green function $\mathbb{G}(\chi,\gamma)$ of the scalar field, which has been calculated in [1,2].

$$G_{00} = \frac{\mu^2(\rho^2-1)(\rho'^2-1)}{(M+\mu\rho)(M+\mu\rho')}\frac{a^{n+1}}{n\mu} \times \int_0^{2\pi} d\sigma\sin^2\sigma\frac{\partial}{\partial\cosh\chi}\bar{\mathbb{G}}(\chi,\gamma). \tag{3.29}$$

E. Even dimensions

In even dimensions the exact static Green function can be represented in the form

$$\bar{\mathbb{G}}(\chi,\gamma) = \frac{1}{a^{n+1}}\frac{1}{2(2\pi)^{\frac{n+3}{2}}}\left(\frac{\partial}{\partial\cos\gamma}\right)^{(n+1)/2} A_n. \tag{3.30}$$

When $n \geq 2$, the functions $A_n(\sigma,\rho,\rho';\gamma)$ are given by the integral

$$A_n = \int_\chi^\infty dy\frac{1}{\sqrt{\cosh(y)-\cosh(\chi)}}\frac{\sinh(\frac{y}{n})}{\sqrt{\cosh(\frac{y}{n})-\cos(\gamma)}}. \tag{3.31}$$

At large values of y the integrand in (3.31) behaves like $\exp[-y(n-1)/(2n)]$. Therefore, (3.31) is convergent for any $n \geq 2$. In the case of the four-dimensional spacetime $(n = 1)$ the integrand has to be modified to guarantee convergence of the integral. For example, one can subtract the asymptotic of the integrand, which does not depend on γ. Since (3.30) contains the derivative of A_n over γ, the resulting Green function does not depend on the particular form of the subtracted γ-independent asymptotic. Thus, for $n = 1$ one can choose

$$A_1 = \int_\chi^\infty dy\frac{1}{\sqrt{\cosh(y)-\cosh(\chi)}} \times\left[\frac{\sinh(y)}{\sqrt{\cosh(y)-\cos(\gamma)}} - \frac{\sinh(y)}{\sqrt{\cosh(y)+1}}\right]. \tag{3.32}$$

Taking into account (3.29) we obtain

$$G_{00} = \frac{\mu^2(\rho^2-1)(\rho'^2-1)}{(M+\mu\rho)(M+\mu\rho')}\frac{1}{2(2\pi)^{\frac{n+3}{2}}}\frac{1}{n\mu} \times\left(\frac{\partial}{\partial\cos\gamma}\right)^{(n+1)/2}\int_0^{2\pi} d\sigma\sin^2\sigma\tilde{A}_n, \tag{3.33}$$

where

$$\tilde{A}_n = \frac{\partial}{\partial\cosh\chi}A_n. \tag{3.34}$$

Thus we obtain

$$\tilde{A}_n = \int_\chi^\infty dy \frac{1}{\sqrt{\cosh(y) - \cosh(\chi)}} \times \frac{\partial}{\partial y}\left[\frac{1}{\sinh(y)} \frac{\sinh(\frac{y}{n})}{\sqrt{\cosh(\frac{y}{n}) - \cos(\gamma)}}\right]. \tag{3.35}$$

Note that (3.35) remains to be valid for all n including $n = 1$ case. In the latter case one has

$$\tilde{A}_1 = -\frac{\partial}{\partial \cos\gamma} A_1. \tag{3.36}$$

F. Odd dimensions

In odd-dimensional spacetimes we have

$$\bar{\mathbb{G}}(\chi,\gamma) = \frac{1}{a^{n+1}} \frac{1}{\sqrt{2}(2\pi)^{\frac{n+4}{2}}} \left(\frac{\partial}{\partial \cos\gamma}\right)^{n/2} \int_0^{2\pi} d\sigma B_n, \tag{3.37}$$

where

$$B_n = \int_\chi^\infty dy \frac{1}{\sqrt{\cosh y - \cosh\chi}} \frac{\sinh(\frac{y}{n})}{\cosh(\frac{y}{n}) - \cos\gamma}. \tag{3.38}$$

Taking into account (3.29) one can write

$$G_{00} = \frac{\mu^2(\rho^2-1)(\rho'^2-1)}{(M+\mu\rho)(M+\mu\rho')} \frac{1}{\sqrt{2}(2\pi)^{\frac{n+4}{2}}} \frac{1}{n\mu} \times \left(\frac{\partial}{\partial \cos\gamma}\right)^{n/2} \int_0^{2\pi} d\sigma \sin^2\sigma \tilde{B}_n, \tag{3.39}$$

where

$$\tilde{B}_n = \frac{\partial}{\partial \cosh\chi} B_n, \tag{3.40}$$

$$\tilde{B}_n = \int_\chi^\infty dy \frac{1}{\sqrt{\cosh y - \cosh\chi}} \times \frac{\partial}{\partial y}\left[\frac{1}{\sinh(y)} \frac{\sinh(\frac{y}{n})}{\cosh(\frac{y}{n}) - \cos\gamma}\right]. \tag{3.41}$$

IV. CLOSED FORM OF THE GREEN FUNCTION: EXAMPLES

A. Four dimensions

In four dimensions ($n = 1$) the integral (3.32) reads [1,2]

$$A_1 = \ln\left(\frac{\cosh(\chi) + 1}{\cosh(\chi) - \cos(\gamma)}\right). \tag{4.1}$$

Hence, according to (3.36)

$$\tilde{A}_1 = -\frac{1}{\cosh\chi - \cos\gamma}. \tag{4.2}$$

The integral over σ in (3.33) can be taken explicitly and we obtain the closed form for the static Green function

$$G_{00}(x,x') = -\frac{1}{4\pi(M+\mu\rho)(M+\mu\rho')} \times \left[\mu \frac{\rho\rho' - \cos\gamma}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\gamma - \sin^2\gamma}} - \mu\right]. \tag{4.3}$$

The last term $-\mu/[4\pi(M+\mu\rho)(M+\mu\rho')]$ in this formula describes a zero mode contribution, which satisfies a homogeneous equation. One should add an extra zero mode contribution

$$\frac{C}{4\pi(M+\mu\rho)(M+\mu\rho')} \tag{4.4}$$

with a coefficient C, such that the flux of the electric field across any surface surrounding the charge and the black hole does not depend on the position of the charge. This leads to the final result for the static Green function in four-dimensional Reissner-Nordström geometry,

$$G_{00}(x,x') = -\frac{1}{4\pi(M+\mu\rho)(M+\mu\rho')} \times \left[\mu \frac{\rho\rho' - \cos\gamma}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\gamma - \sin^2\gamma}} + M\right], \tag{4.5}$$

which satisfies the correct fall-off conditions at infinity. This four-dimensional Green function was obtained earlier in [8]. The vector potential created by a point charge e placed at the point x'

$$J^\mu = e\delta(x - x')\delta_0^\mu \tag{4.6}$$

reads

$$A_0(x) = 4\pi e G_{00}(x,x'). \tag{4.7}$$

B. Five dimensions

In five dimensions ($n = 2$) the Green function (3.39) takes the form

$$B_2 = \int_\chi^\infty dy \frac{1}{\sqrt{\cosh y - \cosh\chi}} \frac{\sinh(\frac{y}{2})}{\cosh(\frac{y}{2}) - \cos\gamma}. \tag{4.8}$$

$$B_2 = \frac{\sqrt{2}}{2}\frac{1}{(\cosh^2(\chi/2) - \cos^2\gamma)^{1/2}} \times \left[\arctan\left(\frac{\cos\gamma}{\sqrt{\cosh^2(\chi/2) - \cos^2\gamma}}\right) + \frac{\pi}{2}\right]. \quad (4.9)$$

The Green function is

$$G_{00} = \frac{\mu(\rho^2-1)(\rho'^2-1)}{(M+\mu\rho)(M+\mu\rho')}\frac{1}{2\sqrt{2}(2\pi)^3} \times \left(\frac{\partial}{\partial\cos\gamma}\right)\int_0^{2\pi} d\sigma \sin^2\sigma \tilde{B}_2, \quad (4.10)$$

where

$$\tilde{B}_2 = \int_\chi^\infty dy \frac{1}{\sqrt{\cosh y - \cosh\chi}} \times \frac{\partial}{\partial y}\left[\frac{1}{\sinh(y)}\frac{\sinh(\frac{y}{2})}{\cosh(\frac{y}{2}) - \cos\gamma}\right]. \quad (4.11)$$

$$\tilde{B}_2 = -\frac{\sqrt{2}}{4}\frac{1}{(\cosh^2(\chi/2) - \cos^2\gamma)^{3/2}} \times \left[\arctan\left(\frac{\cos\gamma}{\sqrt{\cosh^2(\chi/2) - \cos^2\gamma}}\right) + \frac{\pi}{2}\right] - \frac{\sqrt{2}}{4}\frac{\cos\gamma}{\cosh^2(\chi/2)(\cosh^2(\chi/2) - \cos^2\gamma)}. \quad (4.12)$$

The result of integration over σ in (4.10),

$$\int_0^{2\pi} d\sigma \sin^2\sigma \tilde{B}_2 = \frac{4\sqrt{2}\pi}{(\rho^2-1)(\rho'^2-1)}Q, \quad (4.13)$$

can be expressed in terms of the elliptic integrals. The result reads

$$G_{00} = \frac{\mu}{4\pi^2(M+\mu\rho)(M+\mu\rho')}\frac{\partial}{\partial\cos\gamma}Q, \quad (4.14)$$

where

$$Q = -q[\mathbf{E}(\eta,\varkappa) - 2\vartheta(\cos\gamma)\mathbf{E}(\varkappa)] + \frac{q^2+p^2}{2q}[\mathbf{F}(\eta,\varkappa) - 2\vartheta(\cos\gamma)\mathbf{K}(\varkappa)] + \frac{q_0 - p_0}{q_0}\cos\gamma. \quad (4.15)$$

Here $\vartheta(x)$ is the Heaviside step function and

$$p = \sqrt{\rho\rho' - \sqrt{\rho^2-1}\sqrt{\rho'^2-1} + 1 - 2\cos^2\gamma}/\sqrt{2},$$

$$q = \sqrt{\rho\rho' + \sqrt{\rho^2-1}\sqrt{\rho'^2-1} + 1 - 2\cos^2\gamma}/\sqrt{2},$$

$$p_0 = \sqrt{\rho\rho' - \sqrt{\rho^2-1}\sqrt{\rho'^2-1} + 1}/\sqrt{2},$$

$$q_0 = \sqrt{\rho\rho' + \sqrt{\rho^2-1}\sqrt{\rho'^2-1} + 1}/\sqrt{2},$$

$$\varkappa = \frac{\sqrt{q^2-p^2}}{q}, \qquad \sin\eta = \frac{q}{q_0}\mathrm{sign}(\cos\gamma). \quad (4.16)$$

Note that in spite of the appearance of the Heaviside step function $\vartheta(\cos\gamma)$ in the expressions (4.15) the function Q is continuous and smooth at $\gamma = \pi/2$, so that the Green function (4.21) is also continuous and smooth everywhere.

On has to add to G_{00} a zero mode contribution of the form

$$\frac{C(\rho')}{4\pi^2(M+\mu\rho)}, \quad (4.17)$$

in order to satisfy the boundary condition at infinity, meaning that the total flux of the electric field through the surface surrounding the electric charge should not depend on the position of the charge. This condition uniquely fixes the function $C(\rho')$.

When $\rho \to \infty$ we get

$$\frac{\partial}{\partial\cos\gamma}Q|_{\rho\to\infty} = 1 - \rho'. \quad (4.18)$$

Therefore, one has to add the zero mode (4.17) with

$$C = -\frac{M+\mu}{M+\mu\rho'} \quad (4.19)$$

to get a proper asymptotic of the Green function at infinity,

$$G_{00}|_{\rho\to\infty} = -\frac{1}{4\pi^2 r^2} = -\frac{1}{4\pi^2(M+\mu\rho)}. \quad (4.20)$$

Finally we obtain the closed form for the static Green function, satisfying the correct fall-off conditions at infinity,

$$G_{00} = -\frac{M + \mu\frac{p_0}{q_0} - \mu\frac{\partial}{\partial\cos\gamma}W}{4\pi^2(M+\mu\rho)(M+\mu\rho')}, \quad (4.21)$$

where

$$W = -q[\mathbf{E}(\eta,\varkappa) - 2\vartheta(\cos\gamma)\mathbf{E}(\varkappa)] + \frac{q^2+p^2}{2q}[\mathbf{F}(\eta,\varkappa) - 2\vartheta(\cos\gamma)\mathbf{K}(\varkappa)], \tag{4.22}$$

and the other parameters are defined by (4.16). This closed form for the static Green function of the Maxwell field is new. The vector potential created by a point charge e placed at the point x',

$$J^\mu = e\delta(x-x')\delta_0^\mu, \tag{4.23}$$

is equal to

$$A_0(x) = 4\pi e G_{00}(x,x'). \tag{4.24}$$

Here coefficient 4π comes from the normalization of the charge in the higher-dimensional Maxwell equations (1.1).

V. NEAR-HORIZON LIMIT

In the vicinity of the horizon the gravitational field becomes approximately homogeneous. In the limit of an infinite gravitational radius the static Green function should reproduce the result [9] for the Green function in the Rindler spacetime up to the zero mode contribution, because the topology of the Rindler horizon differs from that of the black hole horizon. One should also take into account that for the black hole the Killing vector is normalized to unity at infinity, while in the Rindler spacetime it is usually normalized to unity at the position of an accelerated observer, located close to the horizon.

A near-horizon limit can be derived by using the expressions

$$\rho = 1 + \frac{n^2}{2r_g^2}z^2 + O(r_g^{-4}), \qquad t = \frac{a}{\kappa}\tilde{t},$$
$$\gamma = \frac{1}{r_g}|\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp| + O(r_g^{-3}), \qquad \kappa = \frac{n\mu}{r_g^{n+1}}, \tag{5.1}$$

and then taking the limit $r_g \to \infty$, while the parameter a is kept finite. The parameter a has a meaning of a proper acceleration of an observer. In the limit, when the size of the black hole goes to infinity, the region in the vicinity of the observer is described by a homogeneous gravitational field, i.e., by the Rindler spacetime. The Rindler time coordinate $\tilde{t}$ is chosen in such a way that the timelike Killing vector $\xi^\alpha = \delta^\alpha_{\tilde{t}}$ has a unit norm at the position of an observer $z = a^{-1}$.

Then the metric (2.5) near the horizon takes the form

$$ds^2 = -a^2z^2d\tilde{t}^2 + dz^2 + d\boldsymbol{x}_\perp^2 + O(r_g^{-2}). \tag{5.2}$$

The static Green function, corresponding to the rescaled time $\tilde{t}$, is $G_{\tilde{t}\tilde{t}} = \lim_{r_g\to\infty}(a/\kappa)G_{00}$. Let us introduce the notations

$$R = \sqrt{(z-z')^2 + |\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp|^2},$$
$$\bar{R} = \sqrt{(z+z')^2 + |\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp|^2},$$
$$\zeta = \frac{\bar{R}}{2\sqrt{zz'}}. \tag{5.3}$$

Then in four dimensions we get

$$G_{\tilde{t}\tilde{t}} = -\frac{a}{8\pi}\frac{R^2+\bar{R}^2}{R\bar{R}} - \frac{Ma}{4\pi\mu}, \tag{5.4}$$

where $G_{\tilde{t}\tilde{t}}$ is the static Green function corresponding to the rescaled time coordinate $\tilde{t}$. The last term in (5.4) is constant. It comes from the zero mode contribution. In any case this constant is a pure gauge. The boundary conditions at infinity of the black hole and in the Rindler spacetime are different; therefore, it is not surprising that zero mode contributions may also differ in the near-horizon limit (see discussion in [9]).

Similarly in five dimensions in the near-horizon asymptotic we obtain

$$G_{\tilde{t}\tilde{t}} = -\frac{a\sqrt{zz'}}{4\pi^2}\left[\frac{R^2+\bar{R}^2}{R^2\bar{R}^2}\mathbf{E}\left(\arcsin\zeta, \frac{1}{\zeta}\right) - \frac{1}{\bar{R}^2}\mathbf{F}\left(\arcsin\zeta, \frac{1}{\zeta}\right)\right] - \frac{a}{4\pi^2\kappa r_g^2}$$
$$= -\frac{3az^2z'^2}{8\pi}\frac{1}{R^5}F\left(\frac{5}{2},\frac{3}{2};3;-\frac{4zz'}{R^2}\right) - \frac{a}{4\pi^2\kappa r_g^2}. \tag{5.5}$$

When expressed in terms of the hypergeometric function, this formula exactly reproduces Eq. (3.29) of the paper [9], where the static Green function in a homogeneous gravitational field was derived. The last term here is constant and can be omitted, because it is a pure gauge.

VI. DISCUSSION

In this paper we found the relation between static solutions of the Maxwell field equation on the background of the D-dimensional Reissner-Nordström black hole and on the background of the $(D+2)$-dimensional homogeneous Bertotti-Robinson spacetime. Using the heat kernel technique we obtained a useful integral representation for the electric potential created by a point static charge in the Bertotti-Robinson spacetime and, hence, in the Reissner-Nordström spacetime too. The method is very similar to the method of biconformal transformations [1,2], where the Green function and the potential created by static scalar charges near Reissner-Nordström black holes have been calculated.

In four- and five-dimensional cases we obtained the exact static Green functions in the closed form (4.5) and (4.21)–(4.22). In four dimensions it correctly reproduces the well-known result [8]. To the best of our knowledge the closed form for the five-dimensional static Green function is new. As a test of the obtained results, we demonstrated that the derived static Green functions in a generic Reissner-Nordström spacetime obey a correct near-horizon limit [9]. The obtained integral representation and analytical expressions for exact Green functions can be used to study the problem of the self-energy and self-force of point electric charges in the background of higher-dimensional static black holes. An interesting observation is that the self-force and the self-energy of charged particles qualitatively differ in odd and even spacetime dimensions [3,9–12]. In odd dimensions the self-force and the self-energy of point charges contain terms logarithmic in the distance to the horizon. These terms are related to the biconformal anomalies (see discussion in [10–12]).

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