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Analytical discussion on strong gravitational lensing for a massive source with a $f(R)$ global monopole

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Here we investigate the gravitational lensing in the strong field limit of a Schwarzschild black hole with a solid deficit angle owing to a global monopole within the context of the $f(R)$ gravity theory. We obtain the expressions of the deflection angle and time delay in the forms of elliptic integrals and discuss the asymptotic behavior of the elliptic integrals to find the explicit formulas of the angle and time difference in the strong field limit. We show that the deflection angle and the time delay between multiple images are related not only to the monopole but also to the $f(R)$ correction ψ_0 by taking the cosmological boundary into account. Some observables such as the minimum impact parameter, the angular separation, the relative magnification, and the compacted angular position are estimated as well. It is intriguing that a tiny modification on standard general relativity will make a remarkable deviation on the angle and the time lag, offering a significant way to explore some possible distinct signatures of the topological soliton and the correction of Einstein's general relativity.

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I. INTRODUCTION

Gravitational lensing is an important astrophysical application of general relativity and a powerful probe to the gravitational source, lens object, and spacetime structure [1–6]. We can make use of gravitational lensing to investigate the distant stars no matter if they are bright or dim. If the lens is massive enough, like a black hole, electromagnetic radiation can get very close to the object while the deflection angle exceeds $3\pi/2$ [7], and it encodes the information from the strong field caused by a compact body. In this circumstance, a sequence of images (called relativistic images) is formed on both sides of the optical axis due to large deflections of light more than 2π apart from the so-called primary and secondary images observed in the weak gravitational field and formed due to a small deflection of light rays [8].

In general, the deflection angle of photons passing close to a compact and massive source is expressed in integral forms, so it is difficult to discuss the detailed relation between the angle and the gravitational source or the spacetime geometry. Alternatively, we perform the calculation of the integral expression in the strong field limit where the minimum distance a photon is able to approach the black hole. The analytic method proposed by Bozza [9–14] is to expand the integral expression towards the photon sphere which showed that there exists a logarithmic divergence of the deflection angle in the proximity of the photon sphere. The strong gravitational lensing was treated in a Schwarzschild black hole and a Schwarzschild black hole and a Reissner-Nordström black hole [9], a GMGHS charged black hole [15], a spinning black hole [16], a braneworld black hole [17], the deformed Horava-Lifshitz black hole [18], and the black hole with a global monopole [19]. In addition to Bozza's scheme, the explicit calculation of elliptic integrals is also valid and powerful and was used in the description of the strong deflection of a massive particle around the supermassive black hole [20].

If photons propagate from the emitter to the observers along different rays, then the light travel time corresponding to every image is obviously different. The lag time between the multiple images is called the time delay. Generally, travel time is not observable. However, in the situation of an appearance of multiple images, the time delay can be observed if the intrinsic luminosity of the source is time dependent. Therefore, the luminosity variations can be used to describe the geometry of the lens which is related to the images as a relative temporal phase. As an advantage, time delay is a one-dimensional quantity and can be used to test the underlying cosmological expansion [5]. The measurement of time delay provides a probe of the Hubble constant [3,21]. The general approximative expressions of the time delay between relativistic images in strong field limits for asymptotically flat spacetime without a cosmological horizon has been presented in [22].

Several types of topological objects such as domain walls, cosmic strings, and a monopole may have been formed during the vacuum phase transition in the early Universe [23,24]. These topological defects appeared due to a breakdown of local or global gauge symmetries. A global monopole is a spherically symmetric topological defect formed in the phase transition of a system composed by a self-coupling triplet of a scalar field whose original global O(3) symmetry is spontaneously broken to U(1).

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The properties of the metric outside a monopole are investigated in [25], which also shows that the monopole exerts practically no gravitational force on nonrelativistic matter, but the space around it has a solid angle, and all light rays are deflected at the same angle. We have considered the gravitational lens equation for the massive global monopole in the strong field limit to exhibit the correlation between the deflection angle and the deficit solid angle subject to the monopole model parameters in standard general relativity [19].

However, subject to the fact of the accelerated expansion of the Universe, the metric with $f(R)$ modification describes the spacetime more completely. The theory of $f(R)$ gravity is a type of modified gravity theory first proposed by Buchdahl [26] and has been applied to explain the accelerated-inflation problem instead of adding dark energy or dark matter [27–29]. The gravitational field of a global monopole in the modified gravity theory has been discussed [30]. Further, in [31], they find that the presence of the parameter associated with the modification of gravity is indispensable in providing stable circular orbits for particles. The nonvanishing modified parameter ψ_0 also bring a cosmological horizon as a boundary of the Universe to the spacetime described by the $f(R)$ monopole metric, but the spacetime without gravity modification is asymptotically flat. It should be noticed that the asymptotically flat spacetime is an essential condition for the derivation of gravitational lensing in both weak field limits [32] and strong field limits [9]. Here we use the elliptic integrals to rewrite the expressions of the deflection angle and the time delay which contain polynomials for three or higher order. This method further presents the analytic results when the test particle is close to the photon sphere whether or not the size and scale of the observable universe exist.

In this paper, we plan to probe the strong gravitational lensing in terms of the deflection angle of the light and time delay of multiple images in the strong field limit on the massive source swallowing a global monopole governed by $f(R)$ theory. In the next section, we give a brief introduction about the metric considered here. In Sec. III, the integral form of the deflection angle of the light ray is derived. We discuss the asymptotic behavior of the elliptic integrals to present the expression of the deflection angle at the position close to the photon sphere, and we perform the numerical estimation of observables as well. In Sec. IV, we put forward the time delay in this background with double horizons as elliptic integrals. Further, we calculate the strong-limit time delay by means of the series representations of elliptic integrals. Finally, we discuss our result in Sec. V.

II. THE SCHWARZSCHILD BLACK HOLE WITH A $f(R)$ GLOBAL MONOPOLE

The simplest model that gives rise to the global monopole is describes by the Lagrangain [26]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) - \frac{1}{4}\lambda(\phi^a\phi^a - \eta^2)^2. \tag{1}$$

The triplet of the field configuration showing a monopole is

$$\phi^a = \eta h(r)\frac{x^a}{r}, \tag{2}$$

where $x^a a^a = r^2$. Here, λ and η are model parameters. This model has a global O(3) symmetry, which is spontaneously broken to U(1). In order to couple this matter field to the gravitational field equation in the $f(R)$ theory and obtain their spherically symmetric solution, we adopt the line element as follows:

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{3}$$

In the $f(R)$ gravity theory, the action is given by [30]

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S_m, \tag{4}$$

where $f(R)$ is an analytical function of the Ricci scalar R and $\kappa = 8\pi G$, G is the Newton constant, g is the determinant of the metric tensor, and S_m is the action associated with the matter fields. According to the metric formalism, the field equation leads to

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}F(R) = \kappa T_{m\mu\nu}, \tag{5}$$

where $F(R) = df(R)/dR$, and $T_{m\mu\nu}$ is the minimally coupled energy-momentum tensor. Under the weak field approximation, which assumes the components of a metric tensor like $A(r) = 1 + a(r)$ and $B(r) = 1 + b(r)$ with $|a(r)|$ and $|b(r)|$ being smaller than unity, the field equation is solved in [31]. Finally, the metric is found,

$$A(r) = B^{-1}(r) = 1 - 8\pi G\eta^2 - \frac{2GM}{r} - \psi_0 r. \tag{6}$$

Here, the modification theory of gravity corresponds to a small correction on the general relativity like $F(R(r)) = 1 + \psi(r)$ with $\psi(r) \ll 1$. It can also be taken as the simplest analytical function of the radial coordinate $\psi(r) = \psi_0 r$. In this case, the factor ψ_0 reflects the deviation of standard general relativity. Here, the correction $\psi_0 r$ in the metric is linear, which is different from those in cases such as de Sitter spacetime and the Reissner-Nordström metric, etc. It should be noted that for a typical grand unified theory, the monopole parameter η is of the order 10^{16} GeV, which means $8\pi G\eta^2 \approx 10^{-5}$. The mass parameter is $M \sim M_{\rm core}$, which is very small.

We choose that both the observer and the gravitational source lie in the equatorial plane with condition $\theta = \frac{\pi}{2}$. The whole trajectory of the photon is limited to the same plane. On the equatorial plane, the metric reads

$$ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)d\varphi^2, \tag{7}$$

where

$$C(r) = r^2. \tag{8}$$

We note that with the presence of a nonzero ψ_0, a cosmological horizon

$$r_c = \frac{1}{\psi_0}\left(1 - 8\pi G\eta^2 + \sqrt{(1 - 8\pi G\eta^2)^2 - 8GM\psi_0}\right) \tag{9}$$

appears beside an event horizon

$$r_h = \frac{1}{\psi_0}\left(1 - 8\pi G\eta^2 - \sqrt{(1 - 8\pi G\eta^2)^2 - 8GM\psi_0}\right). \tag{10}$$

The nonvanishing modified parameter ψ_0 also brings a cosmological horizon as a boundary of the Universe to the spacetime described by the $f(R)$ monopole metric, but the spacetime without gravity modification is asymptotically flat.

III. THE DEFLECTION ANGLE OF A MASSIVE SOURCE WITH A $f(R)$ GLOBAL MONOPOLE

The deflection angle for the electromagnetic ray coming from the source to the observer can be expressed as a function of the closest approach [33],

$$\alpha = I(r_0) - \pi, \tag{11}$$

where

$$I(r_0) = I_{\rm OL}(r_0) + I_{\rm LS}(r_0) = \int_{r_0}^{D_{\rm OL}}\left|\frac{d\varphi}{dr}\right|dr + \int_{r_0}^{D_{\rm LS}}\left|\frac{d\varphi}{dr}\right|dr, \tag{12}$$

and

$$\frac{d\varphi}{dr} = \frac{\sqrt{B(r)}}{\sqrt{C(r)}\sqrt{\frac{C(r)}{C(r_0)}\frac{A(r_0)}{A(r)} - 1}}. \tag{13}$$

Here, r_0 is the minimum distance from the photon path to the source, $D_{\rm OL}$ is the distance of the lens from the observer, and $D_{\rm LS}$ is the distance of the lens from the source. We should note that $r_0 < D_{\rm OL} < r_c$ and $r_0 < D_{\rm LS} < r_c$. It requires that the deflection angle be infinite, meaning that the denominator of expression (13) is equal to zero. To achieve this aim, we solve the equation $\frac{C'(r)}{C(r)} = \frac{A'(r)}{A(r)}$ to obtain the radius of the photon sphere [34,35]. Certainly, the closest approach distance r_0 must be larger than the radius of the photon sphere or the photon will move around the gravitational source forever instead if escaping from the source. The radius of the photon sphere in the $f(R)$ global monopole metric is given by

$$r_m = \frac{1 - 8\pi G\eta^2 - \sqrt{(1 - 8\pi G\eta^2)^2 - 6GM\psi_0}}{\psi_0} \approx \frac{3GM}{1 - 8\pi G\eta^2} + O(\psi_0). \tag{14}$$

When we neglect the influence from $f(R)$ theory $\psi_0 = 0$, the photon sphere radius (14) will recover to that of the metric by a massive object involving a global monopole within the frame of Einstein's general relativity [19]. It can be checked as $r_h < r_m < r_c$, which indicates that a photon sphere will survive for the spacetime with two horizons.

For the conservation of angular momentum, at $r = r_0$, we define the impact parameter related to the minimum approach by [33]

$$y = \sqrt{\frac{C(r_0)}{A(r_0)}}. \tag{15}$$

The angular separation can be approximately expressed by $\theta = \frac{y}{D_{\rm OL}}$. The minimum impact parameter corresponding to the radius of the photon sphere will, thus, be

$$y_m = \sqrt{\frac{r_m^3}{-\psi_0 r_m^2 + (1 - 8\pi G\eta^2)r_m - 2GM}}. \tag{16}$$

Now we rewrite the integral expression for the deflection angle (16) with the help of elliptic functions. First, we introduce the notation like $u = \frac{1}{r}$ leading to

$$u_0 = \frac{1}{r_0}, \qquad u_m = \frac{1}{r_m}, \qquad u_{\rm OL} = \frac{1}{r_{\rm OL}}, \qquad u_{\rm LS} = \frac{1}{r_{\rm LS}}. \tag{17}$$

The deflection angle (11) becomes

$$\alpha = \int_{u_{\rm OL}}^{u_0}\frac{du}{\sqrt{2GM(u - u_0)(u - u_1)(u - u_2)}} + \int_{u_{\rm LS}}^{u_0}\frac{du}{\sqrt{2GM(u - u_0)(u - u_1)(u - u_2)}} - \pi, \tag{18}$$

where

$$u_{1,2} = \frac{1}{4GM}\left(1 - 8\pi G\eta^2 - 2GMu_0 \pm \sqrt{(1-8\pi G\eta^2)^2 + 4(1-8\pi G\eta^2)GMu_0 - 8GM\psi_0 - 12(GMu_0)^2}\right), \quad (19)$$

where the upper "+" sign belongs to u_1 and the lower "−" sign belongs to u_2. According to Ref. [36], the two integral parts of Eq. (18) can be written as

$$\int_{u_{\mathrm{LS}}(u_{\mathrm{OL}})}^{u_0} \frac{du}{\sqrt{2GM(u-u_0)(u-u_1)(u-u_2)}} = \frac{1}{\sqrt{2GM}}\frac{2}{\sqrt{u_1-u_2}}F(\delta_{\mathrm{LS(OL)}}, q). \quad (20)$$

Here, $F(\delta_{\mathrm{LS(OL)}}, q)$ is an elliptic integral of the first kind [36],

$$F(\delta_{\mathrm{LS(OL)}}, q) = \int_0^{\delta_{\mathrm{LS(OL)}}} \frac{d\alpha}{\sqrt{1-q^2\sin^2\alpha}} = \int_0^{\sin\delta_{\mathrm{LS(OL)}}} \frac{dx}{\sqrt{(1-x^2)(1-q^2x^2)}}, \quad (21)$$

where

$$\sin\delta_{\mathrm{LS(OL)}} = \sqrt{\frac{(u_1-u_2)(u_0-u_{\mathrm{LS(OL)}})}{(u_0-u_2)(u_1-u_{\mathrm{LS(OL)}})}}, \quad (22)$$

$$q = \sqrt{\frac{u_0-u_2}{u_1-u_2}}. \quad (23)$$

When the photon travel paths are near the source, the deflection angle will be bigger. If the angle is greater than 2π, the photons will circle the massive source several times before they reach the observers. When the minimum distance from the photon travel paths to the source r_0 approaches the radius of the photon sphere, the parameters (22) and (23) will be

$$\sin\delta_{\mathrm{LS(OL)}}|_{u_0=u_m} = 1 \quad (24)$$

and

$$q|_{u_0=u_m} = 1. \quad (25)$$

The asymptotic behavior of an elliptic integral of the first kind is [36]

$$\lim_{q\to 1} F(\delta_{\mathrm{LS(OL)}}, q) = \ln\frac{4}{\sqrt{1-q^2}} - \ln\cot\frac{\delta_{\mathrm{LS(OL)}}}{2} + O(1-q^2). \quad (26)$$

The result is independent of the position of the source or the observer under the strong field condition, thus, $I_{\mathrm{OL}}(r_0 \to r_m) = I_{\mathrm{LS}}(r_0 \to r_m)$.

Within the region just containing the photon sphere, we expand the deflection angle expression (19) in virtue of the properties of elliptic functions,

$$\alpha(\theta) = -a ln\left(\frac{\theta D_{\mathrm{OL}}}{y_m} - 1\right) + b + O(y - y_m), \quad (27)$$

where the coefficients of the deflection angle are

$$a = \frac{1}{[(1-8\pi G\eta^2)^2 - 6GM\psi_0]^{\frac{1}{4}}}, \quad (28)$$

$$b = 2a\left[\frac{1}{2}ln\sigma + 3ln2 + ln3 - ln\frac{1-8\pi G\eta^2 + \sqrt{(1-8\pi G\eta^2)^2 - 6GM\psi_0}}{\sqrt{(1-8\pi G\eta^2)^2 - 6GM\psi_0}}\right] - \pi. \quad (29)$$

Here we have used

$$\frac{r_0}{r_m} - 1 = \left[\frac{1}{\sigma}\left(\frac{y}{y_m} - 1\right)\right]^{\frac{1}{2}}, \quad (30)$$

where

$$\sigma = \frac{12G^2M^2 - 20GM\psi_0 r_m^2 + 4(1-8\pi G\eta^2)r_m^3\psi_0 - \psi_0^2 r_m^4}{8[-(1-8\pi G\eta^2)r_m + 2GM + \psi_0 r_m^2]^2}. \quad (31)$$

Figure 1 shows the deflection angle in the strong field limit, $y = y_m + 0.003GM$, for various values of ψ_0 and η. We see that α increases as $GM\psi_0$ increases.

We relate the position and the magnification to the strong field limit coefficients for the sake of comparing our results with the observable evidence. The position of the source and the images are related through the lens equation derived in [10] given by

$$\beta = \theta - \frac{D_{\mathrm{LS}}}{D_{\mathrm{OL}}}\triangle\alpha_n, \quad (32)$$

where β denotes the angular separation between the source and the lens, and θ is the angular separation between the lens and the image. The offset of the deflection angle is expressed as $\triangle\alpha_n = \alpha(\theta) - 2n\pi$ by subtracting all the times the photons run around the source. Because of $y_m \ll D_{\mathrm{OL}}$, the position of the nth relativistic image can be approximated as

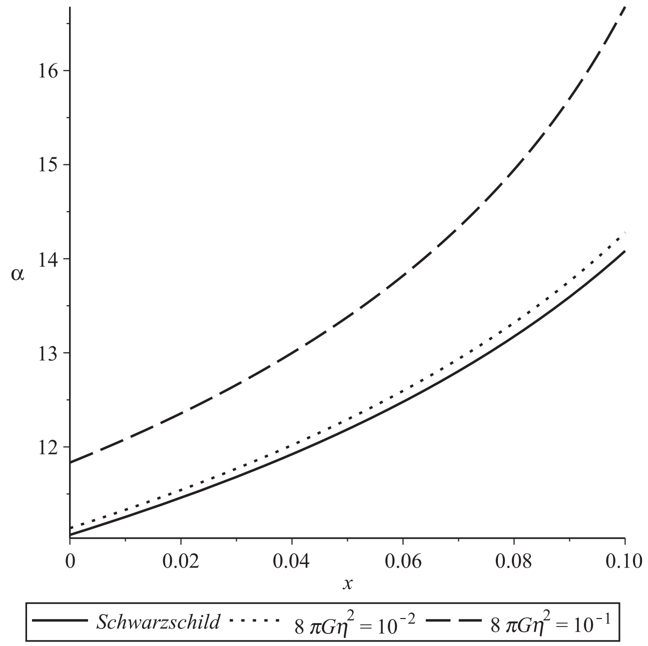

FIG. 1. The dependence of the deflection angle on the $f(R)$ parameter with variation $8\pi G\eta^2 = 10^{-5}, 10^{-2}, 10^{-1}$ in the strong field limit with $y = y_m + 0.003GM$.

$$\theta_n = \theta_n^0 + \frac{y_m e_n(\beta - \theta_n^0) D_{\rm OS}}{a D_{\rm LS} D_{\rm OL}}, \tag{33}$$

where

$$e_n = \exp\left(\frac{b - 2n\pi}{a}\right), \tag{34}$$

and $D_{\rm OS} = D_{\rm OL} + D_{\rm LS}$, while the second term on the right-hand side of Eq. (33) is much smaller than θ_n^0 and we introduce θ_n^0 as $\alpha(\theta_n^0) = 2n\pi$. The magnification of the nth relativistic image is the inverse of the Jacobian evaluated at the position of the image and is obtained as

$$y_n = \frac{y_m e_n(1 + e_n) D_{\rm OS}}{a\beta D_{\rm LS} D_{\rm OL}^2}. \tag{35}$$

In the limit $n \to \infty$, the asymptotic position of approached by a set of images θ_∞ relates to the minimum impact parameter as

$$y_m = D_{\rm OL}\theta_\infty. \tag{36}$$

As an observable, the angular separation between the first image and the others is defined as

$$s = \theta_1 - \theta_\infty = \theta_\infty e^{\frac{b-2\pi}{a}}, \tag{37}$$

where θ_1 represents the outermost image in the situation that the outermost one is thought of as a single image and all the remaining ones are packed together at θ_∞. The ratio of the flux from the first image and the flux of all the other images is

$$\mathcal{R} = \frac{\mu_1}{\sum_{n=2}^{\infty}\mu_n} = e^{\frac{2\pi}{a}}. \tag{38}$$

According to $e^{\frac{2\pi}{a}} \ll 1$ and $e^{\frac{b}{a}} \sim 1$, these observables can be written in terms of the deflection angle parameters, which are presented in Eqs. (42) and (43). As another observable, the magnification of the first image with the other images can be, thus, defined as $\mathcal{R}_m = 2.5 \log \mathcal{R}$. We take the black hole mass $M = 2.8 \times 10^6 M_\odot$ in the center of our Galaxy and $D_{\rm OL} = 8.5$ kpc as the distance between the Sun and the Galactic center.

It is important that the strong field limit coefficients such as a, b, and y_m are directly connected to the observables like $\mathcal{R}_m$ and s. It is then possible for us to probe whether the original general relativity needs to be generalized in virtue of the strong field gravitational lensing for a Schwarzschild black hole with a global monopole. From Figs. 2 and 3, by increasing the monopole parameter, all curves of the angular separation s and of the asymptotic position of the set of outer images θ_∞ rise while the $\mathcal{R}_m$ curves decrease. In Fig. 4, the magnification of the relativistic images decreases when $GM\psi_0$ increases and when $8\pi G\eta^2$ increases, which means that the difference of the flux from the first image and the flux of all the other images reduces

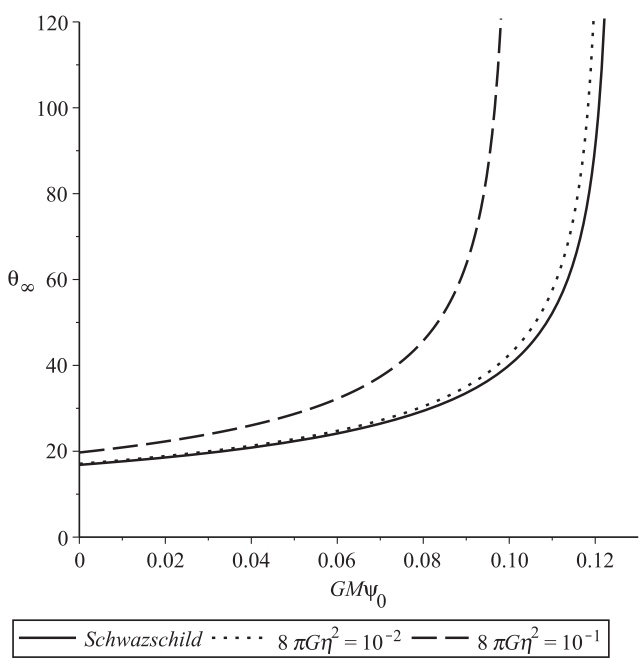

FIG. 2. The behavior of the compacted images' position θ_∞ (μ arc sec) on the dependence of the $f(R)$ parameter as $8\pi G\eta^2 = 10^{-5}, 10^{-2}, 10^{-1}$ when varying the dimensionless modification parameter $GM\psi_0$. We assume the black hole is located in our Galactic center, so $M = 2.8 \times 10^6 M_\odot$ and $D_{\rm OL} = 8.5$ kpc.

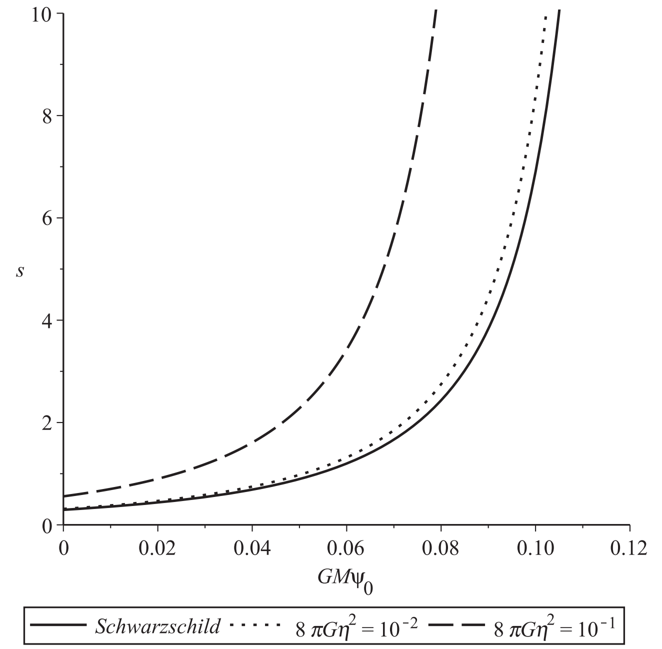

FIG. 3. The angular separation between the first image and the other compacted images s (μ arc sec) as increasing functions of $GM\psi_0$. Here, $M = 2.8 \times 10^6 M_\odot$ and $D_{\rm OL} = 8.5$ kpc.

due to the modification of the gravity theory. We present the estimations in Table I to show how the appearance of the deviation of standard general relativity enhances the observables for strong gravitational lensing. For example,

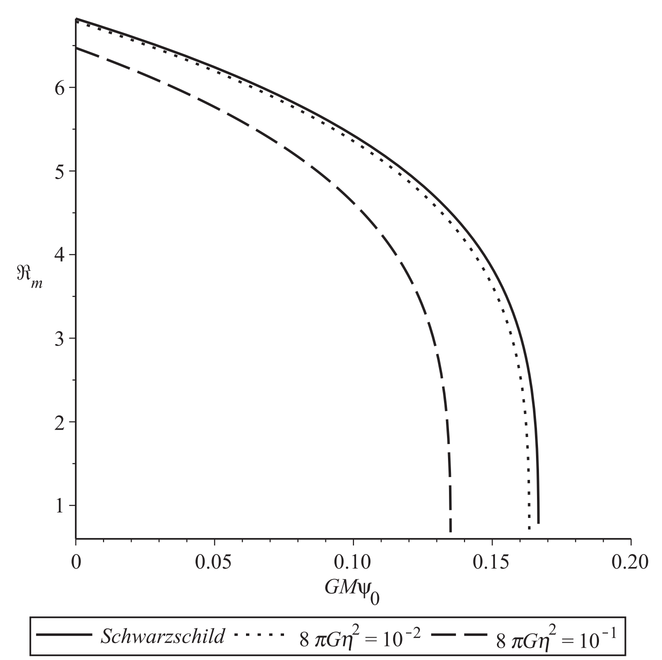

FIG. 4. The magnification of the first image and all the other images $\mathcal{R}_m$ as decreasing functions of $GM\psi_0$. The relationship between the ratio of the flux from the first image and the flux of all the other images and the magnification of the first image with the other images is $\mathcal{R}_m = 2.5 \log \mathcal{R}$.

the angular separation between the first image and all the other images for the $f(R)$ Schwarzschild lens is 310 times larger than the value for Schwarzschild lens, which means it is unambiguous to distinguish the relativistic images. Once observational devices catch such multiple images, it is indispensable evidence to study the $f(R)$ gravity. Hence, the strong gravitational lensing for a massive source with a global monopole is an efficient probe to enlarge the effect of the deviation owing to the correction to Einstein's general relativity, although the correction itself is small.

IV. TIME DELAY IN STRONG GRAVITATIONAL LENSING FOR THE MASSIVE SOURCE WITH A $f(R)$ GLOBAL MONOPOLE

For equatorial geodesics in spherically symmetric spacetime, the equation for the time and radial position for the motion of photons around the gravitational source is given by [37]

$$\frac{dt}{dr} = \pm \frac{\sqrt{B(r)}}{\sqrt{A(r)}\sqrt{1 - \frac{A(r)C(r_0)}{A(r_0)C(r)}}}. \tag{39}$$

The duration that a photon emitted by a star or cluster passes the lens and reaches the receiver is

$$\begin{aligned} T &= \int_{D_{\rm LS}}^{D_{\rm OL}} \frac{dt}{dr} dr \\ &= \int_{r_0}^{D_{\rm LS}} \left|\frac{dt}{dr}\right| dr + \int_{r_0}^{D_{\rm OL}} \left|\frac{dt}{dr}\right| dr. \end{aligned} \tag{40}$$

We substitute the metric (8) into Eq. (40) to find that

$$\begin{aligned} &\int_{r_0}^{D_{\rm LS}(D_{\rm OL})} \left|\frac{dt}{dr}\right| dr \\ &= \frac{\sqrt{(1-8\pi G\eta^2)u_0^2 - 2GMu_0^3 - \psi_0 u_0}}{(2GM)^{\frac{3}{2}}} \\ &\quad \times \int_{u_0}^{u_{\rm LS}(u_{\rm OL})} \frac{du}{u(u-u_3)(u-u_4)\sqrt{(u-u_0)(u-u_1)(u-u_2)}}, \end{aligned} \tag{41}$$

with

$$u_{\rm LS(OL)} = \frac{1}{D_{\rm LS(OL)}}, \tag{42}$$

while

$$u_{3,4} = \frac{1}{4GM}\left[(1 - 8\pi G\eta^2) \pm \sqrt{(1 - 8\pi G\eta^2)^2 - 8GM\psi_0}\right], \tag{43}$$

TABLE I. Estimations for the variable observables such as the compacted angular position, the angular separation and magnification of the first image and the other images, and the minimum impact parameter for a Schwarzschild lens in the center of the Milky Way or for the lens with $f(R)$ modification. The global monopole parameter is zero.

	Schwarzschild $f(R)$ $(\eta = 0)$							
$GM\psi_0$	0	0.001	0.01	0.05	0.08	0.1	0.11	0.12
θ_∞ (μ arc sec)	16.8	16.9	17.6	22.3	29.4	40.1	52.1	91.1
s (μ arc sec)	0.29	0.30	0.36	0.90	2.43	6.90	16.06	90.15
$\frac{y_m}{GM}$	5.196	5.220	5.445	6.896	9.080	12.369	16.103	28.147
$\mathcal{R}_m$	6.82	6.46	6.35	5.76	5.17	4.62	4.25	3.74

where the $+$ and $-$ signs are subject to u_3 and u_4, respectively. According to Ref. [36], the integral parts of Eq. (41) can be rewritten as

$$\begin{aligned}\int_{u_0}^{u_{\rm LS}(u_{\rm OL})}&\frac{du}{u(u-u_3)(u-u_4)\sqrt{(u-u_0)(u-u_1)(u-u_2)}}\\ =&\frac{2}{u_3(u_3-u_4)(u_1-u_3)(u_0-u_3)\sqrt{u_1-u_2}}\times\left[(u_0-u_1)\Pi\left(\delta_{\rm LS(OL)},q^2\frac{u_3-u_1}{u_3-u_0},q\right)+(u_3-u_0)F(\delta_{\rm LS(OL)},q)\right]\\ &+\frac{2}{u_3u_4u_1u_0\sqrt{u_1-u_2}}\times\left[(u_0-u_1)\Pi\left(\delta_{\rm LS(OL)},q^2\frac{u_1}{u_0},q\right)+(-u_0)F(\delta_{\rm LS(OL)},q)\right]\\ &-\frac{2}{u_4(u_3-u_4)(u_1-u_4)(u_0-u_4)\sqrt{u_1-u_2}}\times\left[(u_0-u_1)\Pi\left(\delta_{\rm LS(OL)},q^2\frac{u_4-u_1}{u_4-u_0},q\right)+(u_4-u_0)F(\delta_{\rm LS(OL)},q)\right].\end{aligned}\tag{44}$$

Here, $\Pi(\delta, n, q)$ is an elliptic integral of the third kind [36],

$$\begin{aligned}\Pi(\delta,n,q)&=\int_0^{\delta}\frac{d\alpha}{(1-n\sin^2\alpha)\sqrt{1-q^2\sin^2\alpha}}\\ &=\int_0^{\sin\delta}\frac{dx}{(1-nx^2)\sqrt{(1-x^2)(1-q^2x^2)}}.\end{aligned}\tag{45}$$

In the case of a large deflection angle,

$$\sin\delta_{\rm LS(OL)}|_{u_0=u_m}=1,\tag{46}$$

while

$$\lim_{r_0\to r_m}\int_{r_0}^{D_{\rm LS}}\left|\frac{dt}{dr}\right|dr=\lim_{r_0\to r_m}\int_{r_0}^{D_{\rm OL}}\left|\frac{dt}{dr}\right|dr.\tag{47}$$

Hence,

$$T=-\bar{a}ln\left(\frac{y}{y_m}-1\right)+\bar{b}+O(y-y_m),\tag{48}$$

where

$$\begin{aligned}\bar{a}=&\frac{\sqrt{u_m[(1-8\pi G\eta^2)u_m-2GMu_m^2-\psi_0]}}{(2GM)^{\frac{3}{2}}\sqrt{u_m-u_{2m}}}\\ &\times\left(\frac{-1}{u_mu_3u_4}+\frac{1}{u_3(u_3-u_4)(u_3-u_m)}\right.\\ &\left.-\frac{1}{u_4(u_3-u_4)(u_4-u_m)}\right),\end{aligned}\tag{49}$$

and $\bar{b}$ is a constant irrelevant to the impact parameter belonging to the variance relativistic images. In the strong gravitational field limit for a Schwarzschild black hole with a $f(R)$ global monopole, the time delay of two images on the same side of the lens is given,

$$\Delta T^s_{n,m}=2\pi y_m(n-m),\tag{50}$$

and for the two images lying on the opposite side of the lens,

$$\triangle T^o_{n,m}=2y_m[\pi(n-m)-\gamma],\tag{51}$$

where

$$\frac{\bar{a}}{a}=y_m,\tag{52}$$

TABLE II. Time delay between the first and second relativistic images for the black hole at the center of different galaxies in the case of Schwarzschild spacetime or $f(R)$ spacetime. The global monopole parameter is vanished here. All the masses and distances are taken from [7,22,39].

Black hole in the galaxy	Mass ($M_\odot$)	Distance (Mpc)	Schwarzschild $\triangle T_{2,1}^{\mathrm{Sch}}$ (min)	$GM\psi_0 = 0.12$ $\triangle T_{2,1}^{0.12}$ (min)	$GM\psi_0 = 0.1$ $\triangle T_{2,1}^{0.1}$ (min)
Milky Way	2.8×10^6	0.0085	7.5	40.6	17.9
NGC4486 (M87)	3.3×10^9	15.3	8839.3	47880.5	21040.4
NGC3115	2.0×10^9	8.4	5357.1	29018.5	12751.8
NGC4374 (M84)	1.4×10^9	15.3	3745.0	20312.9	8926.2
NGC4594	1.0×10^9	9.2	2678.6	14509.2	6375.9
NGC4486B (M104)	5.7×10^8	15.3	1526.8	8470.3	3634.2
NGC4261	4.5×10^8	27.4	1205.4	6529.2	2869.1
NGC7052	3.3×10^8	58.7	883.9	4788.1	2104.0
NGC4342 (IC3256)	3.0×10^8	15.3	803.6	4352.8	1912.8
NGC3377	1.8×10^8	9.9	482.1	2611.7	1147.7
NGC0221 (M32)	3.4×10^6	0.7	9.1	49.3	21.7
NGC0224 (M31)	3.0×10^7	0.7	80.4	435.3	191.3

and n and m are the different times of the photons winding around the black hole, and γ is the angular separation between the source and the optical axis. The expression of deflection angle (27) has been considered to obtain Eq. (50). More commonly, if the source is highly aligned with the lens, the gravitational lensing effects become more prominent so that $\gamma \sim D_{\mathrm{OL}}^{-1} \ll 2\pi$ [22,38]. Then Eqs. (50) and (51) are reduced to the same result provided in Ref. [22] if we recover the physical units and consider the time delay between the first image,

$$\triangle T_{2,1} = \frac{2\pi}{c} D_{\mathrm{OL}} \theta_\infty, \tag{53}$$

where c is the speed of light. However, we deduce Eqs. (50) and (51) under the strong field approximation using the elliptic integral without rejection of the exponential terms [22]. It should be noticed that the analytical results from Eq. (53) were shown to have large percentage errors in [37]. We present our results in Table II. Since

$$\frac{\triangle T_{2,1}^{f(R)}}{\triangle T_{2,1}^{\mathrm{Sch}}} = \frac{y_m^{f(R)}}{y_m^{\mathrm{Sch}}}, \tag{54}$$

where the superscripts $f(R)$ and Sch represent the case of spacetime with $f(R)$ modification involved and Schwarzschild spacetime. We find $\triangle T_{2,1}^{f(R)} = 5.4 \times \triangle T_{2,1}^{\mathrm{Sch}}$, if $GM\psi_0 = 0.12$. When $GM\psi_0 = 0.11$, $\triangle T_{2,1}^{f(R)} = 3.1 \times \triangle T_{2,1}^{\mathrm{Sch}}$. When $GM\psi_0 = 0.01$, $\triangle T_{2,1}^{f(R)} = 1.05 \times \triangle T_{2,1}^{\mathrm{Sch}}$. From Fig. 5, either the deviation from standard general relativity or the topological defect can enhance the time delay, although the deviations are fairly tiny.

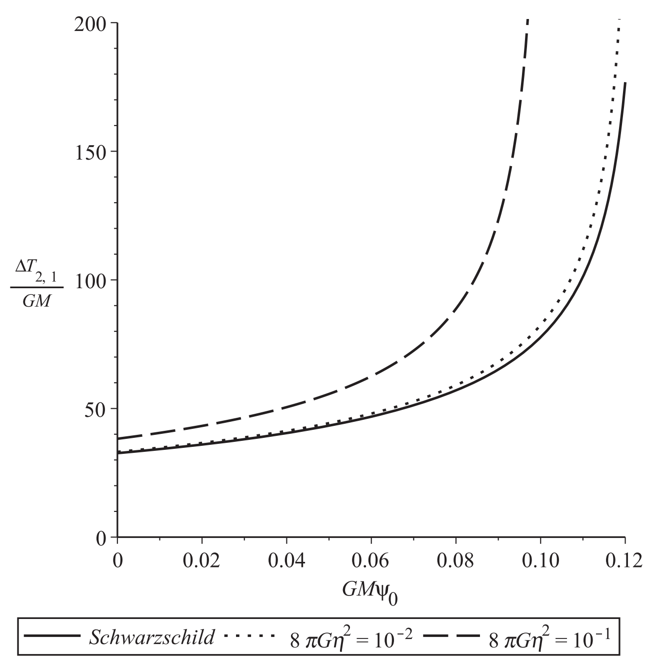

FIG. 5. The figure shows the time delay between the first and second relativistic images as increasing functions of $GM\psi_0$.

V. DISCUSSION

In this paper, we analyzed the gravitational lensing in the strong field limit for the Schwarzschild black hole spacetime with a solid deficit angle owing to a global monopole in the context of $f(R)$ gravity theory which produces one cosmological boundary because the expansion of the Universe is currently undergoing a period of acceleration. We employed several kinds of elliptic integrals to show the deflection angle and time delay and further discussed the integrals in the limiting case to reveal the dependence of the large angle and time difference on the spacetime structure and the generalization of standard general relativity. We found the $f(R)$ correction has significant effects on the gravitational lensing.

For violating the asymptotic flatness, the spherically symmetric spacetime will not allow any particles to propagate from or to infinity. We command, thus, that the distance between the source and the lens should be restricted among the radius of the photon sphere and the radius of the cosmological horizon, $r_0 < D_{\mathrm{SL}} < r_c$, and so does the distance between the lens and the observer. If the minimum approach is close enough to the radius of the photon sphere, the photon will wind around the black hole several times before escaping. This phenomenon is well known as the gravitational lensing in strong field limits.

We presented the analytic expressions of the deflection angle α and the time delay between relativistic images $\triangle T_{n,m}$ in the strong field approximation and relationships between the geometry and the observables such as the angular separation s, the magnification of relativistic images $\mathcal{R}_m$, the compacted images position θ_∞, and the minimum impact parameter y_m. We found the deviation from standard general relativity enhances the effect of gravitational lensing. As the dimensionless variable $GM\psi_0$ increases, all of the deflection angles, the angular separation between the first image and the compacted images, the minimum impact parameter, and the time delay increase. We found the time delay between first two relativistic images for Schwarzschild spacetime dominated by $f(R)$ gravity can be several times larger than the time lag for the Schwarzschild lens. Considering the $f(R)$ lens located in the center of IC3256 as an example, the time delay of the first and second images is more than three days if the derivation of standard general relativity is large enough to $GM\psi_0 = 0.12$. The effect from the correction to the original gravity theory is evident, which provides us a way to confirm whether Einstein's general relativity needs to be generalized.

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[1] A. Einstein, Science **84**, 506 (1936).
[2] S. Liebes, Jr., Phys. Rev. **133**, B835 (1964).
[3] S. Refsdal, Mon. Not. R. Astron. Soc. **128**, 295 (1964).
[4] R. R. Bourassa and R. Kantowski, Astrophys. J. **195**, 13 (1975).
[5] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitional Lenses* (Springer-Verlag, Berlin, 1992).
[6] R. D. Blandford and R. Narayan, Annu. Rev. Astron. Astrophys. **30**, 311 (1992); S. Refsdal and J. Surdej, Rep. Prog. Phys. **57**, 117 (1994).
[7] K. S. Virbhadra, Phys. Rev. D **79**, 083004 (2009).
[8] K. S. Virbhadra and G. F. R. Eills, Phys. Rev. D **62**, 084003 (2000).
[9] V. Bozza, Phys. Rev. D **66**, 103001 (2002).
[10] V. Bozza, S. Capozziello, G. Iovane, and G. Scarpetta, Gen. Relativ. Gravit. **33**, 1535 (2001).
[11] P. T. Mutka and P. Mahonen, Astrophys. J. **581**, 1328 (2002).
[12] P. T. Mutka and P. Mahonen, Astrophys. J. **576**, 107 (2002).
[13] A. M. Beloborodov, Astrophys. J. **566**, L85 (2002).
[14] C. R. Keeton and A. O. Petters, Phys. Rev. D **72**, 104006 (2005).
[15] A. Bhadra, Phys. Rev. D **67**, 103009 (2003).
[16] V. Bozza, Phys. Rev. D **67**, 103006 (2003).
[17] R. Whisker, Phys. Rev. D **71**, 064004 (2005);E. F. Eiroa, Phys. Rev. D **71**, 083010 (2005).
[18] S. Chen and J. Jing, Phys. Rev. D **80**, 024036 (2009).
[19] H. Cheng and J. Man, Classical Quantum Gravity **28**, 015001 (2011).
[20] O. Yu. Tsupko, Phys. Rev. D **89**, 084075 (2014).
[21] M. Oguri, Astrophys. J. **660**, 1 (2007).
[22] V. Bozza and L. Mancini, Gen. Relativ. Gravit. **36**, 435 (2004).
[23] T. W. B. Kibble, J. Phys. A **9**, 1387 (1976).
[24] A. Vilenkin, Phys. Rep. **121**, 263 (1985).
[25] M. Barriola and A. Vilenkin, Phys. Rev. Lett. **63**, 341 (1989).
[26] H. A. Buchdahl, Mon. Not. R. Astron. Soc. **150**, 1 (1970).
[27] S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003).
[28] S. M. Carrol, V. Duvvuri, M. Trodden, and M. S. Turner, Phys. Rev. D **70**, 043528 (2004).
[29] S. Fay, R. Tavakol, and S. Tsujikawa, Phys. Rev. D **75**, 063509 (2007).
[30] T. R. P. Carames, E. R. B. de Mello, and M. E. X. Guimaraes, Int. J. Mod. Phys. Conf. Ser. **03**, 446 (2011).
[31] T. R. P. Carames, E. R. B. de Mello, and M. E. X. Guimaraes, Mod. Phys. Lett. A **27**, 1250177 (2012).
[32] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[33] K. S. Virbhadra, D. Narasimha, and S. M. Chitre, Astron. Astrophys. **337**, 1 (1998).
[34] K. S. Virbhadra and G. F. R. Ellis, Phys. Rev. D **65**, 103004 (2002).
[35] C.-M. Claudel, K. S. Virbhadra, and G. F. R. Ellis, J. Math. Phys. (N.Y.) **42**, 818 (2001).
[36] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Elsevier, New York, 2007).
[37] K. S. Virbhadra and C. R. Keeton, Phys. Rev. D **77**, 124014 (2008).
[38] A. S. Majumdar and N. Mukherjee, Mod. Phys. Lett. A **20**, 2487 (2005).
[39] D. Richstone, E. A. Ajhar, R. Bender, G. Bower, A. Dressler, S. M. Faber, A. V. Filippenko, K. Gebhardt, R. Green, L. C. Ho, J. Kormendy, T. R. Laver, J. Magorrian, and S. Tremaine, Nature (London) **395**, A14 (1998).