Neutron interference in the gravitational field of a ring laser

Robert D. Fischetti and Ronald L. Mallett
Department of Physics, University of Connecticut, 2152 Hillside Road, Storrs, Connecticut 06269, USA
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The neutron split-beam interferometer has proven to be particularly useful in measuring Newtonian gravitational effects such as those studied by Colella, Overhauser, and Werner (COW). The development of the ring laser has led to numerous applications in many areas of physics including a recent general relativistic prediction of frame dragging in the gravitational field produced by the electromagnetic radiation in a ring laser. This paper introduces a new general technique based on a canonical transformation of the Dirac equation for the gravitational field of a general linearized spacetime. Using this technique it is shown that there is a phase shift in the interference of two neutron beams due to the frame-dragging nature of the gravitational field of a ring laser.

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I. INTRODUCTION

Neutron interferometry has opened up a whole new domain of research utilizing the wavelike nature of neutrons to explore quantum mechanical effects. The general technique uses, at its core, some of the more basic fundamental principles of quantum mechanics involving the de Broglie wavelength and the interference of neutrons. The effect of the Newtonian gravitational potential on the interference of two neutron beams was tested in the Colella-Overhauser-Werner (COW) experiment and found to agree with predictions [1]. There are currently other interference techniques, such as atom interferometry experiments, that are affected by gravity. An example of this is the work by Kasevich and Chu [2].

The gravitational field of a light beam was first considered in 1931 by Tolman *et al.* [3] where a test particle was theoretically shown to deflect towards a thin beam of passing light. Later, M. O. Scully (1979) [4] theoretically considered an experiment where a test beam of light would deflect towards an intense beam of light as a result of the metric of the intense beam.

The development of the ring laser has led to numerous applications in many areas of physics. One of us [5] solved the linear Einstein field equations to obtain the gravitational field produced by the electromagnetic radiation of a unidirectional ring laser. It was shown that a massive neutral spinning classical particle at the center of the ring laser exhibited gravitational inertial frame dragging. The post-Newtonian phenomenon of inertial frame dragging is usually associated with the gravitational field generated by rotating matter. An example of this is the prediction that a satellite in a polar orbit around the earth should be dragged around by the gravitational field generated by the rotation of the earth. The recent results of the Gravity Probe B [6] and LAGEOS experiments [7] have successfully indicated the existence of this effect for matter. A comprehensive survey of the many proposed experiments to test gravitational frame dragging by matter can be found in Ciufolini and Wheeler [8].

Tolman *et al.* [3] have emphasized that the gravitational field generated by light has a number of significant differences compared with the gravitational field of matter. To our knowledge no analysis has yet been carried out of the gravitational frame-dragging effect of circulating light on the phase shift of two neutron beams. In this paper, we present a new general technique giving the Foldy-Wouthuysen transformed Hamiltonian for a Dirac particle in the most general linearized spacetime metric. This technique is then used for a theoretical analysis of the interference shift of two neutron beams in the gravitational field of a ring laser. This will illustrate the inertial frame-dragging effect due to light in a quantum mechanical interferometry context.

II. NEW GENERAL APPROACH TO THE DIRAC EQUATION IN A LINEARIZED GRAVITATIONAL FIELD

A. The Dirac equation in curved space

To begin to solve the problem of how a spinning neutron will propagate in the curved space of the ring laser, we need a covariant set of quantum equations. By starting from the covariant Dirac equation and casting it into a Hamiltonian wave equation form in the rest frame of the interferometer apparatus, we can solve the wave equation, as Greenberger and Overhauser did, using a similar perturbed solution that is space dependent but not time dependent. By using the Dirac equation, rather than the Klein-Gordon wave equation, we are not assuming that spin effects are negligible.

The Dirac equation in flat space is a Lorentz covariant equation with a Dirac spinor type wave function as a solution which can be written as [9]

$$i\hbar\gamma^{(\alpha)}\frac{\partial}{\partial x^{(\alpha)}}\psi - mc\psi = 0 \tag{1}$$

or in a more convenient form,

$$\left(\gamma^{(\alpha)}\frac{\partial}{\partial x^{(\alpha)}}+k\right)\psi=0, \tag{2}$$

defining k. It has been shown by Weinberg [10] and Parker [11] that the generally covariant form of Eq. (2) is

$$\left\{\gamma^{(\alpha)}\xi^{\nu}_{(\alpha)}\left(\frac{\partial}{\partial x^{\nu}}-\frac{i}{4}\sigma^{(\beta)(\gamma)}\xi^{\mu}_{(\beta)}\xi_{(\gamma)\mu:\nu}\right)+k\right\}\psi=0, \tag{3}$$

where the object,

$$D_{(\alpha)}=\xi^{\nu}_{(\alpha)}\left(\frac{\partial}{\partial x^{\nu}}-\frac{i}{4}\sigma^{(\beta)(\gamma)}\xi^{\mu}_{(\beta)}\xi_{(\gamma)\mu:\nu}\right), \tag{4}$$

is a generally covariant spinor derivative, where

$$\xi^{\nu}_{(\beta)}\xi^{\mu}_{(\alpha)}g_{\nu\mu}=\eta_{(\alpha)(\beta)}\quad \xi^{\nu}_{(\beta)}v_{\nu}=v_{(\beta)}\quad \xi^{(\beta)}_{\nu}v^{\nu}=v^{(\beta)} \tag{5}$$

are the tetrad fields with the flat space indices α raised and lowered by η and the general coordinate indices by g, while the σ's are objects with the commutation property

$$\begin{aligned}[\sigma^{(\alpha)(\beta)},\sigma^{(\gamma)(\delta)}]&=\eta^{(\gamma)(\beta)}\sigma^{(\alpha)(\delta)}-\eta^{(\gamma)(\alpha)}\sigma^{(\beta)(\delta)}\\&\quad+\eta^{(\delta)(\beta)}\sigma^{(\gamma)(\alpha)}-\eta^{(\delta)(\alpha)}\sigma^{(\gamma)(\beta)}.\end{aligned} \tag{6}$$

It can be shown that a suitable form for the σ's are

$$\sigma^{(\alpha)(\beta)}=\frac{i}{2}[\gamma^{(\alpha)},\gamma^{(\beta)}]. \tag{7}$$

Equation (3) is a field equation which is form invariant under general coordinate transformations of the ν indices, with all other quantities being treated as scalars, and form invariant under a Lorenz transformation of the locally flat (α) coordinates.

B. Foldy-Wouthuysen transformation

It has been discussed that the original Dirac theory in flat space contains results that are difficult to relate to classical physics. In fact, Foldy and Wouthuysen [12] found that by applying a unitary transformation on the system these problems can be alleviated and the theory is then capable of independent 2-component solutions having definite energy. For experiments, this means that you can properly interpret the de Broglie wavelength in terms of the classical particle velocity.

For a perturbed Hamiltonian $H=\beta mc^2+(c\vec{\alpha}\cdot\vec{p})+\epsilon$ where $\epsilon=\epsilon_E+\epsilon_O$ may have both "even" and "odd" parts, the Foldy-Wouthuysen transformation is

$$\begin{aligned}H&=\beta mc^2+(\epsilon_E)+(c\vec{\alpha}\cdot\vec{p}+\epsilon_O)\\&=\beta mc^2+E+O\\E&=\epsilon_E\end{aligned} \tag{8}$$

$$O=(c\vec{\alpha}\cdot\vec{p}+\epsilon_O) \tag{9}$$

$$\psi'=e^{iS}\psi\quad H'=e^{iS}He^{-iS}\quad S=-i\frac{\beta}{2mc^2}O. \tag{10}$$

If the odd term is smaller than mc^2, we can expand the exponential to low order. It can be shown that after the transformation, keeping terms of order $\frac{1}{m^3c^6}$, the new Hamiltonian has the form

$$\begin{aligned}H'&=\beta\left(mc^2+\frac{O^2}{2mc^2}-\frac{O^4}{8m^3c^6}\right)+E\\&\quad-\frac{1}{8m^2c^4}[O,[O,E]]+\frac{\beta}{2mc^2}[O,E]-\frac{O^3}{3m^2c^4}\\&=\beta mc^2+E'+O'.\end{aligned} \tag{11}$$

The effect of the transformation is such that odd terms appear in the new Hamiltonian of order $\frac{1}{mc^2}$ or higher. By applying successive transformations, one can achieve higher order relativistic corrections with the ability to drop the lowest order odd terms leaving behind a purely even Hamiltonian capable of 2-component spinor solutions with definite energies. After two more transformations the Hamiltonian achieves a purely even form of order $\frac{1}{m^2c^4}$,

$$H'''=\beta\left(mc^2+\frac{O^2}{2mc^2}\right)+E-\frac{1}{8m^2c^4}[O,[O,E]]. \tag{12}$$

It is this final form which we wish to use for spinning particles in an interferometer.

C. New general technique

We have a general procedure for expressing the Dirac equation in a generally covariant form, Eq. (3), which is influenced by the metric in which you are working $h_{\mu\nu}$. Further, in the event that all $h_{\mu\nu}$ go to zero, we then have the flat space free particle Hamiltonian. This leads us to interpret the additional terms arising from the metric perturbations as a perturbing potential. We can rearrange Eq. (3) to the form

$$i\hbar\frac{\partial}{\partial t}\psi=H\psi=[mc^2+c\vec{\alpha}\cdot\vec{p}+f(h_{\mu\nu})]\psi. \tag{13}$$

If we now apply three FW transformations to the Hamiltonian in Eq. (13), we will have a physical theory in which we can find it easy to interpret observables such as average position, momentum, velocity, and spin for solutions that have positive definite energy.

Previous authors have been interested in how GR can be incorporated into the dynamics of a Dirac particle in a particular metric. For comparison, we will briefly discuss the goals and techniques of two such authors which together illustrate the benefit of a more general approach.

Varju and Ryder [13] analyzed the Schwarzschild solution in Cartesian coordinates and obtained the result

$$\begin{aligned} ds^2 = (1-2\phi)c^2dt^2 - \Bigg[\left(1+2\frac{g_1x^1}{c^2}\right)(dx^1)^2 \\ + \left(1+2\frac{g_2x^2}{c^2}\right)(dx^2)^2 + \left(1+2\frac{g_3x^3}{c^2}\right)(dx^3)^2\Bigg] \\ - \frac{2}{c^2}[(g_1x^2+g_2x^1)dx^1dx^2 + (g_2x^3+g_3x^2)dx^2dx^3 \\ + (g_3x^1+g_1x^3)dx^3dx^1], \end{aligned} \tag{14}$$

where $g_i = -\frac{\partial\phi}{\partial x^i}c^2$ and $\phi = \frac{GM_s}{rc^2}$.

In this example, they were able to ignore all time-space components in the metric $h_{0i} = 0$ while applying the generalization procedure.

Hehl and Ni [14] studied a spinning particle in an accelerating and rotating frame and obtained a line element of the form

$$\begin{aligned} ds^2 = (dx^0)^2\Bigg[1 + \frac{2\vec{a}\cdot\vec{x}}{c^2} + \left(\frac{\vec{a}\cdot\vec{x}}{c^2}\right)^2 \\ + \left(\frac{\vec{\omega}\cdot\vec{x}}{c}\right)^2 - \left(\frac{\vec{\omega}\cdot\vec{\omega}}{c^2}\right)(\vec{x}\cdot\vec{x})\Bigg] \\ - \frac{2}{c}dx^0 d\vec{x}\cdot\vec{\omega}\times\vec{x} - d\vec{x}\cdot d\vec{x}, \end{aligned} \tag{15}$$

where $\vec{a}$ is the proper 3-acceleration and $\vec{\omega}$ the proper rotation experienced by the moving frame. In this example, they were able to avoid complications of all spacial perturbation components of the type h_{ij} in the generalization procedure for the Dirac equation.

It is clear that the techniques of both authors [13] and [14] share similarities in finding a physical Dirac Hamiltonian in that gravity is incorporated into the theory through the metric field, then the resulting Hamiltonian is subjected to a FW transformation. It would prove useful for future investigations of various metrics of different forms to carry out the procedure for the most general form, meaning to keep all 16 components (10 independent) of the metric perturbation field $h_{\mu\nu}$ and express the FW transformed Hamiltonian in curved space in terms of $h_{\mu\nu}$ and its derivatives.

Since we are not eliminating any parts of the metric $h_{\mu\nu}$, we proceed to separate the Dirac equation (3) according to whether an object contains the energy term h_{00}, the rotation time-space terms h_{0i}, or the space-space cross terms h_{ij} including the diagonal $i = j$.

To start, we will make use of the following properties of the tetrad fields Eq. (5) with a linearized metric,

$$\begin{aligned} \xi^\nu_{(\beta)}\xi^\mu_{(\alpha)}g_{\nu\mu} &= \xi^\nu_{(\beta)}\xi^\mu_{(\alpha)}(\eta_{\nu\mu} + h_{\mu\nu}) \\ &= \eta_{(\alpha)(\beta)} \\ \xi^\mu_{(\alpha)} &= \delta^\mu_\alpha - \frac{1}{2}h^\mu_\alpha \\ \xi^{(\alpha)}_\mu &= \delta^{(\alpha)}_\mu + \frac{1}{2}h^{(\alpha)}_\mu \\ \xi_{(\alpha)\mu} &= \eta_{(\alpha)\mu} + \frac{1}{2}h_{(\alpha)\mu} \\ \xi^{(\alpha)\mu} &= \eta^{(\alpha)\mu} - \frac{1}{2}h^{(\alpha)\mu}, \end{aligned} \tag{16}$$

and find the covariant derivative of the tetrad field that appears in Eq. (3),

$$\begin{aligned} \xi_{(\beta)\mu;\alpha} &= \frac{1}{2}\left(\frac{\partial h_{\alpha\mu}}{\partial x^\beta} - \frac{\partial h_{\beta\alpha}}{\partial x^\mu}\right) \\ -\frac{i}{4}\sigma^{(\sigma)(\beta)}\xi^\mu_{(\sigma)}\xi_{(\beta)\mu;\alpha} &= -\frac{i}{4}\sigma^{(\sigma)(\beta)}\frac{\partial h_{\sigma\alpha}}{\partial x^\beta}, \end{aligned} \tag{17}$$

with the last step owing to the antisymmetric nature of $\sigma^{(\sigma)(\beta)}$.

Inserting Eq. (17) into Eq. (3), we get

$$\begin{aligned} 0 = \Bigg\{\left(\gamma^{(0)}\left(1-\frac{1}{2}h^0_0\right) - \frac{1}{2}\gamma^{(j)}h^0_j\right)\frac{\partial}{\partial x^0} \\ + \left(\gamma^{(i)} - \frac{1}{2}\gamma^{(\gamma)}h^i_\gamma\right)\frac{\partial}{\partial x^i} - \frac{i}{4}\gamma^{(\alpha)}\sigma^{(\sigma)(\beta)}\frac{\partial h_{\sigma\alpha}}{\partial x^\beta} + k\Bigg\}\psi. \end{aligned} \tag{18}$$

Rearranging to obtain a field equation of the form $i\hbar\frac{\partial\psi}{\partial t} = H\psi$, we rewrite Eq. (18) as

$$\begin{aligned} H = \left(1+\frac{1}{2}h_{00}\right)c\vec{\alpha}\cdot\vec{p} - c\vec{h}\cdot\vec{p} \\ + \frac{c}{2}\vec{\alpha}\cdot\overleftrightarrow{h}\cdot\vec{p} - \frac{ic}{2}\vec{\sigma}\cdot\vec{h}\times\vec{p} + \frac{i\hbar c}{4}[\vec{\alpha}\cdot\vec{\nabla}h_{00} - \vec{\nabla}\cdot\vec{h} \\ - (\vec{\alpha}\cdot\vec{\nabla})(\vec{\alpha}\cdot\overleftrightarrow{h}\cdot\vec{\alpha}) + \vec{\nabla}\cdot\overleftrightarrow{h}\cdot\vec{\alpha}] \\ + \beta mc^2\left[\left(1+\frac{1}{2}h_{00}\right) + \frac{\vec{\alpha}\cdot\vec{h}}{2}\right], \end{aligned} \tag{19}$$

where $\overleftrightarrow{h} = h^{ij}$, $\vec{h} = h^{0i}$ and we have introduced a common convention that a dot product of two 3-vectors $\vec{A}\cdot\vec{B}$ is taken to be the negative of the sum of a contravariant and a covariant index,

$$\vec{A}\cdot\vec{B} = -A^iB_i = \Sigma A^iB^i, \tag{20}$$

and by raising and lowering all indices with $\eta_{\mu\nu}$, the momentum vector is defined as

$$p^i = i\hbar\frac{\partial}{\partial x_i} = -i\hbar\frac{\partial}{\partial x^i}. \tag{21}$$

Algebraically, Eq. (19) is exactly the same as a flat space free particle piece plus a small perturbing energy. It, therefore, inherits all of the issues of the mixing of energy states and the inconsistent operator definitions due to the persistent odd terms with odd powers of the $\vec{\alpha}$ matrices. In order to interpret any real particle mechanics from this Hamiltonian, it is necessary to perform a FW transformation to attain an acceptable working theory.

Three consecutive FW transformations applied to the Hamiltonian of Eq. (19) give

$$\begin{aligned} H''' =& \beta mc^2\left(1+\frac{1}{2}h_{00}\right)+(1+h_{00})\frac{\beta}{2m}(p)^2-\frac{c}{2}\vec{h}\cdot\vec{p} \\ &-\frac{ic}{2}\vec{\sigma}\cdot\vec{h}\times\vec{p}-\frac{c}{4}\hbar\vec{\sigma}\cdot\vec{\nabla}\times\vec{h}+\frac{3\beta}{8m}i\hbar\vec{\alpha}\cdot\vec{\nabla}h_{00}\vec{\alpha}\cdot\vec{p} \\ &+\frac{\beta}{8m}i\hbar\vec{\nabla}h_{00}\cdot\vec{p}-\frac{\hbar^2\beta}{16m}(\nabla)^2h_{00}+\frac{\beta}{4m}i\hbar\vec{\alpha}\cdot\vec{\nabla}\vec{\alpha}\cdot\overset{\leftrightarrow}{h}\cdot\vec{p} \\ &+\frac{\beta}{2m}\overset{\leftrightarrow}{h}\cdot\vec{p}\cdot\vec{p}+\frac{\hbar^2\beta}{8m}(\nabla)^2(\vec{\alpha}\cdot\overset{\leftrightarrow}{h}\cdot\vec{\alpha}) \\ &-\frac{i\hbar\beta}{4m}\vec{\nabla}(\vec{\alpha}\cdot\overset{\leftrightarrow}{h}\cdot\vec{\alpha})\cdot\vec{p} \\ &-\frac{\hbar^2\beta}{8m}\vec{\alpha}\cdot\vec{\nabla}\vec{\nabla}\cdot\overset{\leftrightarrow}{h}\cdot\vec{\alpha}+\frac{i\hbar\beta}{4m}\vec{\nabla}\cdot\overset{\leftrightarrow}{h}\cdot\vec{p}. \end{aligned} \tag{22}$$

III. APPLICATION TO NEUTRON INTERFERENCE IN THE GRAVITATIONAL FIELD OF A RING LASER

A. Gravitational field of a ring laser

The linearized Einstein gravitational field equations in the Hilbert gauge $\partial_\mu(h^{\mu\nu}-\frac{1}{2}\eta^{\mu\nu}h)=0$ are

$$\partial_\lambda\partial^\lambda\left(h^{\mu\nu}-\frac{1}{2}\eta^{\mu\nu}h\right) = -\kappa\tau^{\mu\nu}, \tag{23}$$

where for electromagnetic radiation,

$$\tau^{\mu\nu} = -\frac{1}{4\pi}\left(f^{\mu\alpha}f_\alpha^\nu-\frac{1}{4}\eta^{\mu\nu}f^{\alpha\beta}f_{\alpha\beta}\right), \tag{24}$$

and $f^{\alpha\beta}$ is the Maxwell field tensor. Since the trace of Eq. (24) τ_μ^μ is zero, Eq. (23) can be rewritten as

$$\partial_\lambda\partial^\lambda h^{\mu\nu} = -\kappa\tau^{\mu\nu}. \tag{25}$$

For a thin laser beam in the configuration of Fig. 1 with a polarization in the z direction, the Maxwell tensor $f^{\mu\nu}$ components are

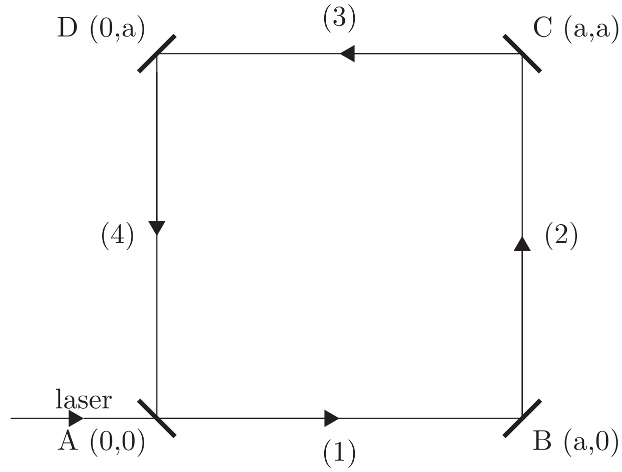

FIG. 1. Ring laser with side length a.

$$f^{30}_{(1)} = f^{30}_{(2)} = f^{30}_{(3)} = f^{30}_{(4)} = E_z \tag{26}$$

$$f^{13}_{(1)} = -f^{13}_{(3)} = -B_y \tag{27}$$

$$f^{32}_{(2)} = -f^{32}_{(4)} = B_x, \tag{28}$$

while all other components are zero.

Assuming a thin laser beam of linear energy density ρ, the nonzero metric components for the ring laser are shown [5] to be

$$h^{00} = -\frac{\kappa\rho}{4\pi}[\phi_{(1)}+\phi_{(2)}+\phi_{(3)}+\phi_{(4)}] \tag{29}$$

$$h^{01} = -\frac{\kappa\rho}{4\pi}[\phi_{(1)}-\phi_{(3)}] \tag{30}$$

$$h^{02} = -\frac{\kappa\rho}{4\pi}[\phi_{(2)}-\phi_{(4)}] \tag{31}$$

$$h^{11} = -\frac{\kappa\rho}{4\pi}[\phi_{(1)}+\phi_{(3)}] \tag{32}$$

$$h^{22} = -\frac{\kappa\rho}{4\pi}[\phi_{(2)}+\phi_{(4)}], \tag{33}$$

with the definitions

$$\phi_{(1)} = \ln\left\{\frac{-x+a+[(x-a)^2+y^2+z^2]^{\frac{1}{2}}}{-x+[x^2+y^2+z^2]^{\frac{1}{2}}}\right\} \tag{34}$$

$$\phi_{(2)} = \ln\left\{\frac{-y+a+[(x-a)^2+(y-a)^2+z^2]^{\frac{1}{2}}}{-y+[(x-a)^2+y^2+z^2]^{\frac{1}{2}}}\right\} \tag{35}$$

$$\phi_{(3)} = \ln\left\{\frac{-x+a+[(x-a)^2+(y-a)^2+z^2]^{\frac{1}{2}}}{-x+[x^2+(y-a)^2+z^2]^{\frac{1}{2}}}\right\} \tag{36}$$

$$\phi_{(4)} = \ln\left\{\frac{-y+a+[x^2+(y-a)^2+z^2]^{\frac{1}{2}}}{-y+[x^2+y^2+z^2]^{\frac{1}{2}}}\right\}, \tag{37}$$

with subscripts in parentheses indicating the particular beam path in Fig. 1.

B. Neutron interference in the gravitational field of a ring laser

Now that we have an "even" Hamiltonian for a Dirac particle perturbed by a general gravitational metric perturbation Eq. (22), we can calculate the phase shift of a free particle due to this perturbation. We do this in the same manner as Greenberger and Overhauser [15] with the purpose of finding the phase shift of two particle paths of a split beam interferometer for comparison at a recombination point. The phase shift is of the form

$$\Psi = \Psi_0 e^{i\delta\phi} = M_0 e^{i(\vec{k}_0\cdot\vec{x}+\delta\phi-\omega_0 t)}, \tag{38}$$

where M_0 is a 2-component spinor column matrix with norm $M_0^\dagger M_0 = 1$.

This gives a solution for $\delta\phi$ in the same manner as the Schrödinger case, in the form of a time integral over the path of a free particle trajectory

$$\delta\phi(\vec{x}) = -\frac{1}{\hbar}\int_0^{T(\vec{x})} dt' H'(t'), \tag{39}$$

where the H' is the perturbed energy function Eq. (22) and $T(\vec{x})$ is the elapsed time for the free particle to reach $\vec{x}$. We will use Eq. (39) to find the phase shift of each particle beam and then their difference at the recombination point.

Our procedure then is to use H''' from Eq. (22) for the specific metric of the ring laser, Eq. (29) through Eq. (37). This H''' will be used as the perturbing potential in Eq. (39). Considering this metric does not have off-diagonal space-space components h^{ij}, all $\overleftrightarrow{h}$ terms in Eq. (22) can be simplified. Using the first-order effect $h^{\mu\nu}p^i = h^{\mu\nu} i\hbar\nabla^i i(k_0^1x^1 + k_0^2x^2 + k_0^3x^3) = h^{\mu\nu}\hbar k_0^i$, we also drop first and second derivatives of $h^{\mu\nu}$ for large k_0, meaning, $\frac{\partial^2 h^{\mu\nu}}{\partial(|k_0|x^i)\partial(|k_0|x^j)} \ll \frac{\partial h^{\mu\nu}}{\partial(|k_0|x^k)} \ll h^{\mu\nu}$.

With these approximations, the perturbed energy function can be written in the simplified form

$$\begin{aligned} H' &= mc^2\frac{1}{2}h_{00} + h_{00}\frac{\beta}{2m}(p)^2 - \frac{c}{2}\vec{h}\cdot\vec{p} \\ &\quad - \frac{ic}{2}\vec{\sigma}\cdot\vec{h}\times\vec{p} + \frac{\beta}{2m}\overleftrightarrow{h}\cdot\vec{p}\cdot\vec{p} \\ &= \frac{mc^2}{2}h_{00} + \frac{\beta}{2m}h_{00}(p)^2 - \frac{c\hbar}{2}(h^{01}k_0^1 + h^{02}k_0^2) \\ &\quad - \frac{ic\hbar}{2}\vec{\sigma}\cdot\vec{h}\times\vec{k}_0 + \frac{\beta\hbar^2}{2m}\{h^{11}(k_0^1)^2 + h^{22}(k_0^2)^2\}. \end{aligned} \tag{40}$$

For a particle initially moving in the $+x$ direction or the $+y$ direction, this can be expressed as

$$\begin{aligned} H'_x &= \frac{mc^2}{2}h_{00} + \frac{\hbar^2}{2m}h_{00}(k_0)^2 - \frac{c\hbar}{2}h^{01}k_0 \\ &\quad + \frac{ic\hbar}{2}(\sigma^3 h^{02}k_0) + \frac{\hbar^2}{2m}h^{11}(k_0)^2 \end{aligned} \tag{41a}$$

$$\begin{aligned} H'_y &= \frac{mc^2}{2}h_{00} + \frac{\hbar^2}{2m}h_{00}(k_0)^2 - \frac{c\hbar}{2}h^{02}k_0 \\ &\quad - \frac{ic\hbar}{2}\sigma^3 h^{01}k_0 + \frac{\hbar^2}{2m}h^{22}(k_0)^2. \end{aligned} \tag{41b}$$

Since, to first order, the integral in Eq. (39) will be taken over the path of the free particle, it can be rewritten as

$$\begin{aligned} \delta\phi(\vec{x}) &= -\frac{1}{\hbar}\int_0^{T(\vec{x})} dt' H'(t')\frac{k_0}{k_0} \\ &= -\frac{1}{\hbar k_0}\int_0^{T(\vec{x})} dt' H'(t')\frac{m}{\hbar}\frac{dx'}{dt'} \\ &= -\frac{m}{\hbar^2 k_0}\int_0^{\vec{x}} dx' H'(x'). \end{aligned} \tag{42}$$

We now aim to find the phase difference of two paths of a split beam interferometer for the scenario depicted in Fig. 2. Here we have a neutron beam split at point A, at which point we will assume the free particle will enter the perturbing region and is the point where we consider the two beams to be coherent. The two beams will then travel their separate paths, accumulating unique phase shifts found using the line integral of Eq. (42), where at the recombination point, D, we are interested in the measurable difference

$$\begin{aligned} &\delta\phi_{ABD} - \delta\phi_{ACD} \\ &\quad = -\frac{m}{\hbar^2 k_0}\left\{\int_{ABD} dx' H'(x') - \int_{ACD} dx' H'(x')\right\}. \end{aligned} \tag{43}$$

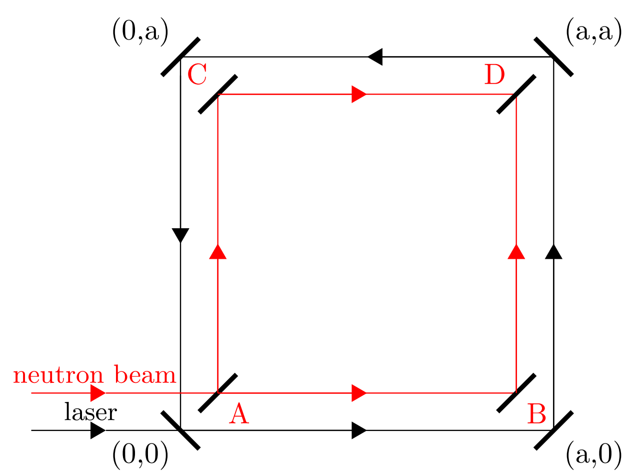

FIG. 2 (color online). Ring laser with side length a and interferometer.

It should be noted that an additional effect, shown by Mannheim [16], due to the acceleration of the reflecting surfaces at points B and C has been considered but found to not to contribute to lowest order in this setup.

To understand the results of these integrals, it will be useful to look at the position dependence of the functions in the integrands. According to Eq. (41a) and Eq. (41b), whether the particle is traveling in the x direction or the y direction, it will always experience a phase shift due to the $\{\frac{mc^2}{2} + \frac{\hbar^2}{2m}(k_0)^2\}h_{00}$ term in Eq. (42). That this term will go to zero can be shown visually by expressing the line integral in Eq. (42) as an average of the integrand. This can be written

$$\int_0^{x_0} dx' H'(x) = \bar{H}' X_{\mathrm{tot}}, \tag{44}$$

where X_{tot} is the total length of the line of the integral, and a barred quantity will be taken as the length average of that quantity over the path of the integral. This means that we can express the measurable phase difference between the paths due to the h_{00} term as

$$\left\{\int_{ABD} dx' h_{00}(x') - \int_{ACD} dx' h_{00}(x')\right\} = X_{\mathrm{tot}}\{\bar{h}_{00-(ABD)} - \bar{h}_{00-(ACD)}\}. \tag{45}$$

Looking now at the function h_{00} in Fig. 3, we can see from the symmetry of the function that the average value is the same along the two paths, and the difference is zero in a symmetrically designed interferometer. The h_{00} part of the metric, in this experiment, will cause no measurable phase difference.

It is a simple extension now to analyze the contribution in Eq. (43) from the h^{11} and h^{22} parts of the metric. We can write their contribution as

$$\left[\int_{ABD} dx'(h^{11} + h^{22}) - \int_{ACD} dx'(h^{11} + h^{22})\right] = \{X_{AB}(\bar{h}^{11}_{AB} - \bar{h}^{11}_{CD}) - Y_{BD}(\bar{h}^{22}_{BD} - \bar{h}^{22}_{AC})\} = 0, \tag{46}$$

from the symmetry of the functions shown in Figs. 4 and 5.

The functions h^{01} and h^{02} will contribute to the phase difference differently whether the particle is travelling along the x direction or the y direction, as seen from Eq. (41a) and Eq. (41b). We can follow the integrals more carefully by explicitly showing the paths to be followed and show that the only terms left are

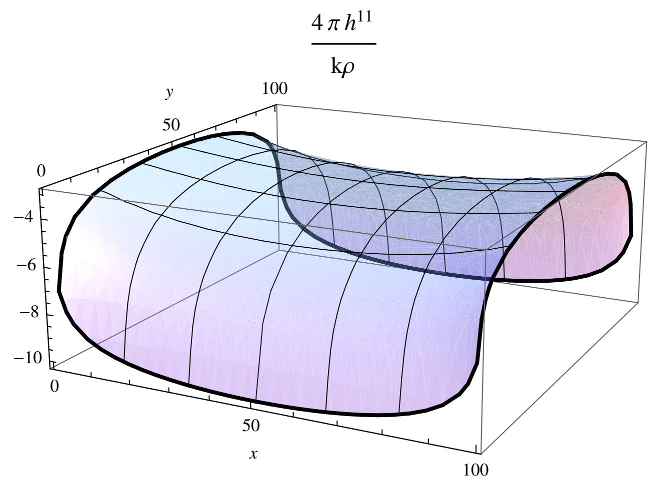

FIG. 4 (color online). The function $\frac{4\pi}{\kappa\rho}h^{11} = -[\phi_{(1)} + \phi_{(3)}]$ for a ring laser of side length $a = 100$ units. The bold contour outlining the plotted area is at a distance of 1 unit from each side of the ring laser.

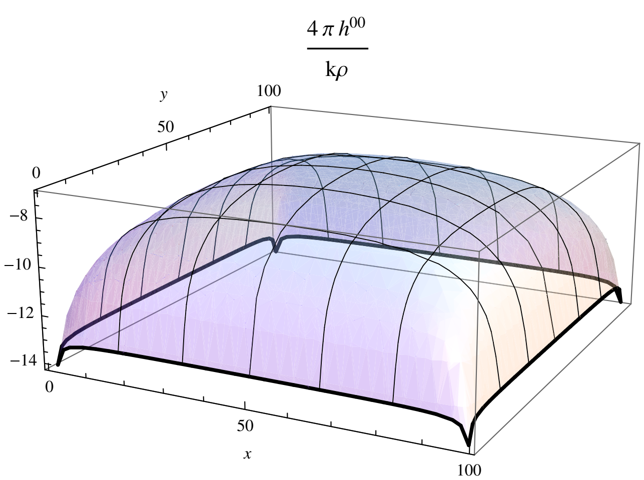

FIG. 3 (color online). Function $\frac{4\pi}{\kappa\rho}h^{00} = -[\phi_{(1)} + \phi_{(2)} + \phi_{(3)} + \phi_{(4)}]$ for a ring laser of side length $a = 100$ units. The bold contour outlining the plotted area is at a distance of 1 unit from each side of the ring laser.

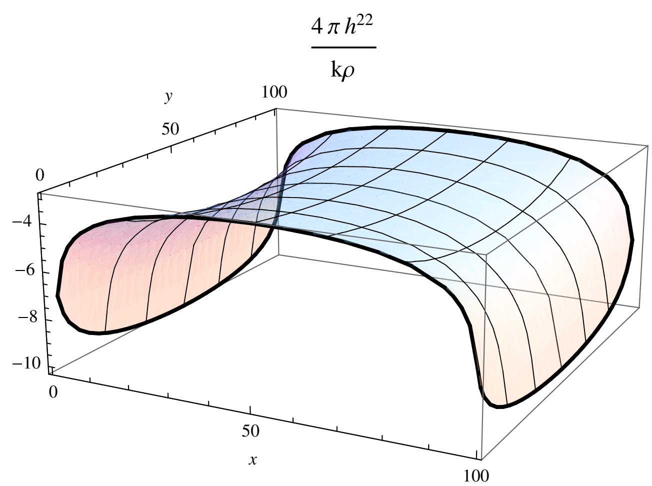

FIG. 5 (color online). The function $\frac{4\pi}{\kappa\rho}h^{22} = -[\phi_{(2)} + \phi_{(4)}]$ for a ring laser of side length $a = 100$ units. The bold contour outlining the plotted area is at a distance of 1 unit from each side of the ring laser.

$$
\begin{aligned}
\delta\phi_{ABD} - \delta\phi_{ACD} &= -\frac{m}{\hbar^2 k_0}\left\{\int_{ABD} dx' H'(x') - \int_{ACD} dx' H'(x')\right\} \\
&= -\frac{m}{\hbar^2 k_0}\left\{\int_{AB} dx'\left[-\frac{c\hbar}{2}h^{01}k_0 + \frac{ic\hbar}{2}(\sigma^3 h^{02}k_0)\right] + \int_{BD} dy'\left[-\frac{c\hbar}{2}h^{02}k_0 - \frac{ic\hbar}{2}\sigma^3 h^{01}k_0\right]\right. \\
&\quad \left. + \int_{AC} dy'\left[-\frac{c\hbar}{2}h^{02}k_0 - \frac{ic\hbar}{2}(\sigma^3 h^{01}k_0)\right] + \int_{CD} dx'\left[-\frac{c\hbar}{2}h^{01}k_0 + \frac{ic\hbar}{2}\sigma^3 h^{02}k_0\right]\right\} \\
&= \frac{m}{\hbar^2 k_0}\frac{c\hbar}{2}k_0\{X_{AB}\bar{h}^{01}_{AB} + Y_{BD}\bar{h}^{02}_{BD} - X_{CD}\bar{h}^{01}_{CD} - Y_{AC}\bar{h}^{02}_{AC}\} \\
&\quad + \frac{m}{\hbar^2 k_0}\frac{ic\hbar}{2}(\sigma^3 k_0)\{-X_{AB}\bar{h}^{02}_{AB} + Y_{BD}\bar{h}^{01}_{BD} + X_{CD}\bar{h}^{02}_{CD} - Y_{AC}\bar{h}^{01}_{AC}\} \\
&= \frac{mc}{2\hbar}\{X_{AB}(\bar{h}^{01}_{AB} - \bar{h}^{01}_{CD}) + Y_{BD}(\bar{h}^{02}_{BD} - \bar{h}^{02}_{AC})\} + \frac{imc\sigma^3}{2\hbar}\{-X_{AB}(\bar{h}^{02}_{AB} - \bar{h}^{02}_{CD}) + Y_{BD}(\bar{h}^{01}_{BD} - \bar{h}^{01}_{AC})\}. \qquad (47)
\end{aligned}
$$

On comparing our final result Eq. (47) with Figs. 6 and 7, we can see that the momentum terms are reinforced due to the antisymmetric nature of those metric components. The spin terms go to zero, not from the additive nature of the expression in Eq. (47), but the figure shows that the values $\bar{h}^{02}_{AB}, \bar{h}^{02}_{CD}, \bar{h}^{01}_{BD}, \bar{h}^{01}_{AC}$ go to zero individually. From Eq. (41a) and Eq. (41b), this evidently comes from the result of the spin being precessed by the frame-dragging momentum energy of the gravity generating source traveling perpendicularly with respect to the momentum of the spinning particle. Therefore, during a trip along any one leg of the neutron beams path in Fig. 2, the particle feels the momentum of the ring laser section behind it for the first half of the leg, and in front of it for the last half, resulting in no net accumulation.

Turning now back to Eq. (47), we can express the only nonzero contribution to the total phase difference along the two paths of the interferometer as, from the symmetry of the functions, 4 times the magnitude of the effect along any one leg to, thus, give

$$
\begin{aligned}
\delta\phi_{ABD} - \delta\phi_{ACD} &= \frac{mc}{2\hbar}\{X_{AB}(\bar{h}^{01}_{AB} - \bar{h}^{01}_{CD}) + Y_{BD}(\bar{h}^{02}_{BD} - \bar{h}^{02}_{AC})\} \\
&= \frac{2mcL_n}{\hbar}\bar{h}^{01}_{AB} \\
&= -\frac{2mcL_n}{\hbar}\frac{\kappa\rho}{4\pi}\overline{[\phi_{(1)} - \phi_{(3)}]}_{AB} \\
&= \frac{4mG\rho}{\hbar c^3}\{-L_n\overline{[\phi_{(1)} - \phi_{(3)}]}_{AB}\}, \qquad (48)
\end{aligned}
$$

where L_n is the length of one leg of the neutron beam.

The average function $-L_n\overline{[\phi_{(1)} - \phi_{(3)}]}$, which is the average value along the outside contour near the x axis in Fig. 6, is a function of the relative dimensions of the ring laser and the interferometer. This is a tunable parameter in our result, Eq. (48). Also tunable is the power of the ring

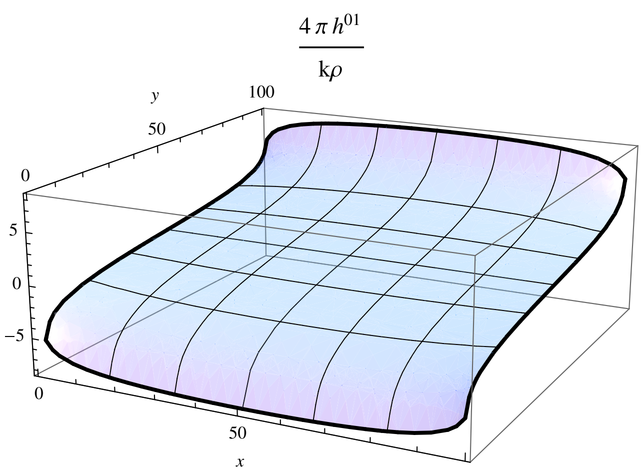

FIG. 6 (color online). The function $\frac{4\pi}{\kappa\rho}h^{01} = -[\phi_{(1)} - \phi_{(3)}]$ for a ring laser of side length $a = 100$ units. The bold contour outlining the plotted area is at a distance of 1 unit from each side of the ring laser.

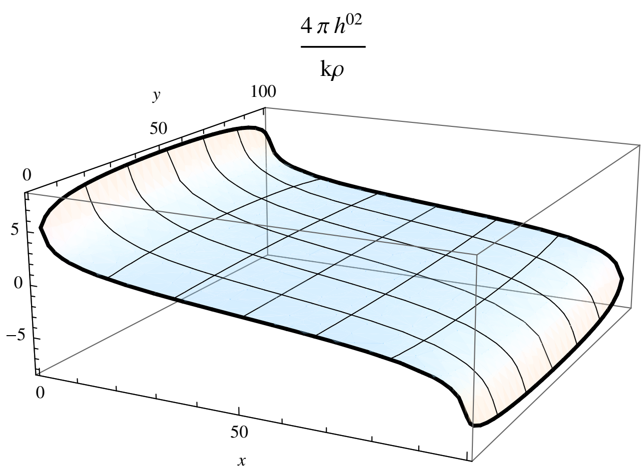

FIG. 7 (color online). The function $\frac{4\pi}{\kappa\rho}h^{02} = -[\phi_{(2)} - \phi_{(4)}]$ for a ring laser of side length $a = 100$ units. The bold contour outlining the plotted area is at a distance of 1 unit from each side of the ring laser.

laser which is generating the gravitational field. This is a satisfying result for our theoretical work which shows that there is an interference effect on two neutron beams in the gravitational field of a ring laser.

A quick numerical estimate can be made. With typical diffraction crystal sizes on the order of $L_n = 10^{-1}$ m and assuming that the neutron beam can travel within 10^{-3} m of the laser (same relative size as in Figs. 3 through 7) we can estimate the term $L_n\overline{[\phi_{(1)} - \phi_{(3)}]}$. Using a largest number approach for the other tunable parameters, we will use the power output of cutting edge CW lasers now exceeding $P = 10^5$ W [17]. Pulse lasers, while having high peak-energy densities, do not provide a beam length long enough to allow the neutron to complete its traversal through the interferometer. Therefore, with these assumptions, an estimated phase shift would be on the order of

$$\rho = \frac{P}{c} = \frac{10^5\ \mathrm{W}}{3.0\times 10^8\ \frac{\mathrm{m}}{\mathrm{s}}} \approx 10^{-3}\ \frac{\mathrm{J}}{\mathrm{m}}$$

$$L_n\overline{[\phi_{(1)} - \phi_{(3)}]} \approx 10^0\ \mathrm{m}$$

$$\frac{4mG\rho}{\hbar c^3}\{-L_n\overline{[\phi_{(1)} - \phi_{(3)}]}_{AB}\} = \frac{4(10^{-27}\ \mathrm{kg})(10^{-10}\ \frac{\mathrm{Nm}^2}{\mathrm{kg}^2})(10^{-3}\ \frac{\mathrm{J}}{\mathrm{m}})}{(10^{-33}\ \frac{\mathrm{kgm}^2}{\mathrm{s}})(10^{25}\ \frac{\mathrm{m}^3}{\mathrm{s}^3})} \approx 10^{-32}\ \mathrm{rad}. \tag{49}$$

Current neutron interference techniques have a phase shift resolution on the order of 10^{-12} rad [18]. The order of magnitude approximation in Eq. (49) using current experimental design parameters shows that this effect is not currently within the range of detection and that significant increases in the size of the interferometer and the light energy density, possibly by sending the laser many times around the ring pattern, are needed. As Scully [4] put it when reflecting on his beam deflection magnitude, “This *Gedanken* experiment is clearly not an experimental call to arms, but rather an argument that such experiments are ‘thinkable.’”

C. Conclusions

Motivated by the predictions and results of the COW experiment for the phase shift of a Schrödinger particle due to a Newtonian potential, we set out to find the analogous procedure for a Dirac particle in the gravitational field of a unidirectional ring laser. In pursuit of this goal, it was necessary to evaluate the methods previously published in calculating gravitational perturbations to the Dirac wave function.

As pointed out in this paper, the procedure for expressing the Dirac Hamiltonian in curved space used in our work has been utilized by a number of authors for special cases [13,14]. In our opinion, our presentation of the procedure is much more general. It has often been the topic in these papers to compare the Hamiltonian resulting from their chosen metric to the work of others using different metrics. The general form of Eq. (22) represents the final result of any of those papers, but without the specific metric components included or excluded. The method developed in this paper provides an alternative general approach for future investigations.

In Einstein’s metric theory, all energy, whether electromagnetic or material, will generate a gravitational field. Although, as pointed out earlier, there are intrinsic differences between the gravitational field generated by light and that of matter [3]. In this paper, new techniques were developed and applied to the calculation of the interference of two neutron beams in the gravitational field of a ring laser. The prediction of the interference shift of neutrons due to the gravitational field of a ring laser in Eq. (48) is new and provides, theoretically, an additional test of general relativity and gravitational frame dragging by light.

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