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Hidden symmetry of the Galileon

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We show that there is a special choice of parameters for which the Galileon theory is invariant under an enhanced shift symmetry whose nonlinear part is quadratic in the coordinates. This symmetry fixes the theory to be equivalent to one with only even powers of the field, with no free coefficients, and accounts for the improved soft-limit behavior observed in the quartic Galileon *S*-matrix.

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I. INTRODUCTION

Effective theories with derivative interactions have been of great interest recently. Much of this work has focused on a particular family of scalar field theories, the Galileons [1]. These theories have primarily been of interest in cosmology, where they arise in various infrared modifications of gravity, but they have potential applications in many corners of high-energy physics, cosmology and even condensed matter. In this paper, we show that for a specific choice of Galileon parameters, the theory enjoys an enhanced symmetry, with associated improved physical properties. Aside from the consequences we explore, this unexpected additional structure of the Galileon may aid the understanding of the fundamental physical origin of these theories.

Galileons were first seen to arise in the decoupling limit of the Dvali-Gabadadze-Porrati model [2–4], and have since been seen to also appear in massive gravity [5,6]. As such, they are of great interest to cosmologists as effective descriptions of various modifications of gravitational physics. In these applications, the idea is that new physics residing within the gravitational sector may be responsible for cosmic acceleration. The discovery of the Galileons has galvanized new interest in derivatively coupled scalar theories, and a proliferation of models has ensued. (For reviews, see Refs. [7–9].) However, the Galileons retain a distinguished position amongst this collection, owing to their enhanced symmetries. In what follows, we will see that particular Galileon theories have an even greater enhancement of their symmetries.

Though Galileons first arose in a gravitational context, it is not obvious that this is the most natural place for them to appear. It may be that in the real world Galileons most aptly describe low-energy or condensed matter systems in a nonrelativistic context. One concrete argument for this position is that Galileons appear in the nonrelativistic limit of theories describing fluctuating hypersurfaces [10–13]—this is clearly of import in high-energy physics—which may be of interest for biophysics or soft condensed matter applications, where the study of thin films and membranes is of great importance. The Galileons and their higher-shift analogues may also be useful for describing Goldstone bosons near multicritical points [14–16], where their symmetries can stabilize exotic dispersion relations which are seen in some physical systems, e.g., Refs. [17–20]. Elucidating the full symmetry structure underlying the Galileons may aid the search for such a laboratory example of a Galileon system.

Galileons possess a number of interesting field-theoretic properties: they obey a nonrenormalization theorem [2,21], which indicates that they may be employed to address naturalness problems. They can exhibit classically nonlinear behavior without losing control of quantum corrections—this is the essence of the Vainshtein mechanism which can screen the presence of the Galileon from Solar System tests [22,23]. Additionally, the Galileons can be interpreted as Wess-Zumino terms for a particular spontaneously broken spacetime symmetry [24].

The Galileon has two essential defining properties: second-order equations of motion (which ensures that the theory does not propagate an Ostrogradsky-type ghost) and invariance under the symmetry

$$\delta\phi = c + b_\mu x^\mu, \tag{1}$$

where c is a constant, b_μ is a constant vector and x^μ is the spacetime coordinate. There are a finite number of terms with these properties; $D+1$ of them in D dimensions.

In this paper, we show that, up to field redefinitions, there is a single choice of coefficients for the Galileon terms for which the theory is additionally invariant under a higher-shift symmetry. Up to field redefinitions, this

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symmetry fixes all the coefficients of the Galileon, and the resulting theory is equivalent to one with only even powers of the field. For example, in four dimensions, the theory containing only the quartic Galileon term,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{1}{12\Lambda^6}(\partial\phi)^2[(\Box\phi)^2 - (\partial_\mu\partial_\nu\phi)^2], \quad (2)$$

where Λ is the strong coupling scale, is invariant under

$$\delta\phi = s_{\mu\nu}x^\mu x^\nu + \frac{1}{\Lambda^6}s^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \quad (3)$$

where $s_{\mu\nu}$ is a traceless symmetric constant tensor $s_{\mu\nu} = s_{\nu\mu}$, $s^\mu{}_\mu = 0$.

As with all global symmetries, this symmetry has consequences for correlation functions and scattering amplitudes. We show that the soft-ϕ theorem associated to this extended shift symmetry implies that the soft limit of scattering amplitudes starts at $\mathcal{O}(q^3)$, higher than the $\mathcal{O}(q^2)$ behavior of a generic Galileon. This explains a phenomenon seen recently in Ref. [25]. In addition, the exact tree-level S-matrix for this special Galileon theory was recently constructed in Ref. [26] through a type of dimensional reduction of graviton scattering amplitudes; understanding that this theory has additional symmetry may shed light on why it is precisely this theory which appears in this context. Further, in four dimensions, this special Galileon theory enjoys self-duality under the Legendre transformation of Ref. [27].

Conventions: We use the mostly plus signature. D is the number of spacetime dimensions. $\simeq$ denotes equality up to a total derivative.

II. GALILEON LAGRANGIANS AND USEFUL QUANTITIES

The Galileon Lagrangians can be conveniently written in terms of certain total derivative combinations. We define the matrix of second derivatives: $\Phi^\mu{}_\nu \equiv \partial^\mu\partial_\nu\phi$. At each order in ϕ, there is a unique combination of Φ's that is a total derivative [1,28],

$$\mathcal{L}_n^{\rm TD} = \sum_p (-1)^p \eta^{\mu_1 p(\nu_1)}\eta^{\mu_2 p(\nu_2)}\ldots\eta^{\mu_n p(\nu_n)} \times (\Phi_{\mu_1\nu_1}\Phi_{\mu_2\nu_2}\ldots\Phi_{\mu_n\nu_n}). \quad (4)$$

The sum is over all permutations of the ν indices, where $(-1)^p$ is the sign of the permutation. The first few cases are

$$\begin{aligned}
\mathcal{L}_1^{\rm TD} &= [\Phi],\\
\mathcal{L}_2^{\rm TD} &= [\Phi]^2 - [\Phi^2],\\
\mathcal{L}_3^{\rm TD} &= [\Phi]^3 - 3[\Phi][\Phi^2] + 2[\Phi^3],\\
\mathcal{L}_4^{\rm TD} &= [\Phi]^4 - 6[\Phi^2][\Phi]^2 + 8[\Phi^3][\Phi] + 3[\Phi^2]^2 - 6[\Phi^4],\\
&\vdots
\end{aligned} \quad (5)$$

where the brackets are traces of the enclosed matrix product. We also define $\mathcal{L}_0^{\rm TD} = 1$. Since one cannot antisymmetrize more than D indices in the definition (4), the term $\mathcal{L}_n^{\rm TD}$ vanishes identically when $n > D$, so there are only a finite number of nontrivial such combinations.

The Galileon terms are given by

$$\mathcal{L}_n = -\frac{1}{2}(\partial\phi)^2\mathcal{L}_{n-2}^{\rm TD} \simeq \frac{1}{n}\phi\mathcal{L}_{n-1}^{\rm TD}, \quad (6)$$

with the last equality up to integration by parts [29]. $\mathcal{L}_1 = \phi$ is a tadpole and $\mathcal{L}_2 = -\frac{1}{2}(\partial\phi)^2$ is the kinetic term. The general Galileon theory is a linear combination of the terms (6) with coefficients $c_1, \ldots, c_{D+1}$

$$\mathcal{L} = \sum_{n=1}^{D+1} c_n\mathcal{L}_n. \quad (7)$$

There is an energy scale, Λ, which suppresses the terms relative to each other, and at which the theory becomes strongly coupled. We have chosen units such that $\Lambda = 1$.

An important ingredient will be the tensors $X_{\mu\nu}^{(n)}$ constructed out of $\Phi_{\mu\nu}$ as follows[1]:

$$X_{\mu\nu}^{(n)} = \frac{1}{n+1}\frac{\delta}{\delta\Phi_{\mu\nu}}\mathcal{L}_{n+1}^{\rm TD}. \quad (8)$$

The first few are

$$\begin{aligned}
X_{\mu\nu}^{(0)} &= \eta_{\mu\nu},\\
X_{\mu\nu}^{(1)} &= [\Phi]\eta_{\mu\nu} - \Phi_{\mu\nu},\\
X_{\mu\nu}^{(2)} &= ([\Phi]^2 - [\Phi^2])\eta_{\mu\nu} - 2[\Phi]\Phi_{\mu\nu} + 2\Phi_{\mu\nu}^2,\\
X_{\mu\nu}^{(3)} &= ([\Phi]^3 - 3[\Phi][\Phi^2] + 2[\Phi^3])\eta_{\mu\nu}\\
&\quad - 3([\Phi]^2 - [\Phi^2])\Phi_{\mu\nu} + 6[\Phi]\Phi_{\mu\nu}^2 - 6\Phi_{\mu\nu}^3,\\
&\vdots
\end{aligned} \quad (9)$$

The $X_{\mu\nu}^{(n)}$ are symmetric, identically conserved $\partial^\mu X_{\mu\nu}^{(n)} = 0$, and satisfy the recursion relation [6]

[1] These are the same tensors which appear in the decoupling limit of massive gravity [6] (see the appendix of Ref. [30] for more on their properties).

$$X^{(n)}_{\mu\nu} = -n\Phi_\mu{}^\lambda X^{(n-1)}_{\lambda\nu} + \mathcal{L}^{\rm TD}_n \eta_{\mu\nu}, \qquad (10)$$

as well as the contraction property $\Phi^{\mu\nu} X^{(n)}_{\mu\nu} = \mathcal{L}^{\rm TD}_{n+1}$.

The most important property for what follows is that these tensors satisfy dimension-dependent identities: $X^{(n)}_{\mu\nu}$ vanishes identically for $n \geq D$,

$$X^{(n)}_{\mu\nu} = 0, \qquad n \geq D. \qquad (11)$$

This is because $\mathcal{L}^{\rm TD}_n$ vanishes for $n > D$.

III. THE SYMMETRY

The general transformation we will consider includes a part with no fields, a part with one power of the field, and a part with two powers of the field,

$$\delta\phi = \delta_0\phi + 2\beta\delta_1\phi + (\alpha + \beta^2)\delta_2\phi, \qquad (12)$$

where α, β are constant parameters (the specific parametrization is chosen for later convenience), and

$$\delta_0\phi = s_{\mu\nu}x^\mu x^\nu, \qquad \delta_1\phi = s_{\mu\nu}\partial^\mu\phi x^\nu,$$
$$\delta_2\phi = s^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \qquad (13)$$

where $s_{\mu\nu}$ is a constant, symmetric, traceless tensor,

$$s_{\mu\nu} = s_{\nu\mu}, \qquad s^\mu{}_\mu = 0. \qquad (14)$$

The Euler-Lagrange derivatives of the variation of the general Galileon terms under the three pieces (13) take a simple form in terms of the tensors (8) (see the Appendix for the proof),

$$\frac{\delta}{\delta\phi}(\delta_0\mathcal{L}_n) = 2(n-1)s^{\mu\nu}X^{(n-2)}_{\mu\nu},$$
$$\frac{\delta}{\delta\phi}(\delta_1\mathcal{L}_n) = -2s^{\mu\nu}X^{(n-1)}_{\mu\nu},$$
$$\frac{\delta}{\delta\phi}(\delta_2\mathcal{L}_n) = \frac{2}{n}s^{\mu\nu}X^{(n)}_{\mu\nu}. \qquad (15)$$

Looking at the form of Eq. (15), we see that if we choose relative coefficients properly, the terms of various order in the Galileon Lagrangian (7) can be made to cancel against each other under the action of Eq. (12), up to a total derivative. To accomplish this, we demand

$$c_n(\alpha+\beta^2)\delta_2\mathcal{L}_n + 2c_{n+1}\beta\delta_1\mathcal{L}_{n+1} + c_{n+2}\delta_0\mathcal{L}_{n+2} \simeq 0, \qquad (16)$$

which yields the recursion relation

$$(n+1)c_{n+2} - 2\beta c_{n+1} + (\alpha+\beta^2)\frac{1}{n}c_n = 0. \qquad (17)$$

This relation determines all the coefficients of Eq. (7) in terms of c_1, c_2 and the parameters α, β of the transformation.

To establish invariance of the action, we must also show that the lowest-order terms are invariant under the lowest-order parts of the symmetry, and that the highest terms are invariant under the highest parts of the symmetry. For the lower part, it is straightforward to see that the kinetic term and tadpole terms are invariant up to a total derivative under the lowest-order parts of the symmetry,

$$\delta_0\mathcal{L}_2 \simeq 0, \qquad \delta_0\mathcal{L}_1 \simeq 0, \qquad \delta_1\mathcal{L}_1 \simeq 0. \qquad (18)$$

The highest terms are invariant under the higher-order parts of the symmetry because of the dimension-dependent identity (11)

$$\delta_2\mathcal{L}_D \simeq 0, \qquad \delta_2\mathcal{L}_{D+1} \simeq 0, \qquad \delta_1\mathcal{L}_{D+1} \simeq 0. \qquad (19)$$

Without loss of generality, we may take $c_1 = 0$ by expanding around a background solution $\phi \propto x^2$ [1] and, assuming the background is stable, we may canonically normalize the kinetic term to set $c_2 = 1$ [the form of the ansatz (12) is unchanged under these field redefinitions]. Taking these values as initial conditions, the recursion relation (17) can be solved to give

$$c_n = \frac{(\beta+\sqrt{-\alpha})^{n-1} - (\beta-\sqrt{-\alpha})^{n-1}}{2\sqrt{-\alpha}(n-1)!}. \qquad (20)$$

IV. BEHAVIOR UNDER GALILEON DUALITY

Galileon duality [31,32] gives a one-parameter redundancy of the Galileon Lagrangians. By performing a field redefinition

$$\phi' = e^{\theta\delta}\phi, \qquad \delta\phi = -\frac{1}{2}(\partial\phi)^2, \qquad (21)$$

we transform a Galileon theory with one set of parameters into a Galileon theory with a different set of parameters which are related by [33],

$$\sum_{n=1}^{D+1} c_n\mathcal{L}_n(\phi') = \sum_{n=1}^{D+1} d_n(\theta)\mathcal{L}_n(\phi),$$
$$d_n(\theta) = \sum_{m=1}^{n}\frac{\theta^{n-m}}{(n-m)!}c_m.$$

Under this duality, the symmetry (3) with $\phi \to \phi'$ becomes

$$\delta\phi = s_{\mu\nu}x^\mu x^\nu + 2(\beta+\theta)s_{\mu\nu}x^\mu\partial^\nu\phi$$
$$+ (\alpha + (\beta+\theta)^2)s^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \qquad (22)$$

Given our canonically normalized theory with no tadpole, the coefficient c_3 simply shifts by θ under duality, so a convenient way to fix the duality ambiguity is to choose θ so that $c_3 = 0$. From Eq. (20), we have $c_3 = \beta$, so we choose $\theta = -\beta$, after which the symmetry (22) takes the form

$$\delta\phi = s_{\mu\nu}x^\mu x^\nu + \alpha s^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \tag{23}$$

and the Lagrangian coefficients (20) become

$$c_n = \frac{(-\alpha)^{\frac{n}{2}-1}}{(n-1)!}, \qquad n = 2, 4, 6, \ldots \tag{24}$$

with the odd c_n's vanishing.

We see that once the duality ambiguity is removed, the theory with the symmetry (23) contains only even powers of the field, with the coefficients completely fixed in terms of one parameter α, which, furthermore, can be reabsorbed into the energy scale Λ by changing units. Thus, the theory is completely fixed, with no free parameters other than the energy scale of strong coupling.

V. SYMMETRY ALGEBRA

The generators of the Galileon symmetry (1) are

$$C\phi = 1, \qquad B^\mu\phi = x^\mu. \tag{25}$$

Along with the standard linear Poincaré generators $P_\mu\phi = -\partial_\mu\phi$, $J_{\mu\nu}\phi = (x_\mu\partial_\nu - x_\nu\partial_\mu)\phi$, they close to form the Galileon algebra [24], whose nonzero commutators are

$$[P_\mu, B_\nu] = \eta_{\mu\nu}C, \qquad [J_{\mu\nu}, B_\lambda] = \eta_{\mu\lambda}B_\nu - \eta_{\nu\lambda}B_\mu,$$

along with the standard commutators $[J_{\mu\nu}, P_\lambda] = \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu$, $[J_{\mu\nu}, J_{\lambda\sigma}] = \eta_{\mu\lambda}J_{\nu\sigma} - \eta_{\nu\lambda}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\lambda} - \eta_{\mu\sigma}J_{\nu\lambda}$ of the Poincaré algebra.

There is a new symmetric traceless generator associated with the new symmetry (23)

$$S_{\mu\nu}\phi = x_\mu x_\nu - \frac{1}{D}x^2\eta_{\mu\nu} + \alpha\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{D}(\partial\phi)^2\eta_{\mu\nu}\right].$$

This generator closes with the Galileon algebra to form an enlarged symmetry algebra whose new nonzero commutators are

$$\begin{aligned}
&[P_\mu, S_{\nu\lambda}] = \eta_{\mu\nu}B_\lambda + \eta_{\mu\lambda}B_\nu - \frac{2}{D}B_\mu\eta_{\nu\lambda},\\
&[B_\mu, S_{\nu\lambda}] = -\alpha\left(\eta_{\mu\nu}P_\lambda + \eta_{\mu\lambda}P_\nu - \frac{2}{D}P_\mu\eta_{\nu\lambda}\right),\\
&[S_{\mu\nu}, S_{\lambda\sigma}] = \alpha(\eta_{\mu\lambda}J_{\nu\sigma} + \eta_{\nu\lambda}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\lambda} + \eta_{\mu\sigma}J_{\nu\lambda}),\\
&[J_{\mu\nu}, S_{\lambda\sigma}] = \eta_{\mu\lambda}S_{\nu\sigma} - \eta_{\nu\lambda}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\lambda\nu} - \eta_{\nu\sigma}S_{\lambda\mu}.
\end{aligned} \tag{26}$$

When $\alpha \to 0$, this reduces to the algebra of traceless $N = 2$ extended shift symmetries studied in Ref. [15].

VI. SOFT LIMIT

Recently, the authors of Ref. [25] studied the behavior of soft limits of scattering amplitudes in various scalar field theories in $D = 4$. In particular, they found that Dirac-Born-Infeld has better behavior [with amplitudes scaling as $\mathcal{O}(q^2)$ in the soft limit] than a generic $P(X)$ theory, and that the general Galileon has the best soft behavior among theories whose terms have N fields and $2(N-1)$ derivatives. Additionally, they found a scalar field theory which has even better soft behavior, with its scattering amplitudes scaling as $\mathcal{O}(q^3)$ in the soft limit, which they conjectured to be the quartic Galileon. We are now in a position to see that this is a consequence of invariance under the extended shift symmetry (23).

This symmetry leads to a "soft-ϕ" theorem of the following form (which is analogous to the soft pion theorems of chiral perturbation theory [34,35] or the soft-ζ theorems in cosmology [36–38]):

$$\lim_{q\to 0}\partial_{q^{(\mu}}\partial_{q^{\nu)T}}\left(\frac{\langle\phi_q\phi_{k_1}\ldots\phi_{k_N}\rangle'}{\langle\phi_q\phi_{-q}\rangle'}\right) = \hat{\mathcal{D}}\langle\phi_{k_1}\ldots\phi_{k_N}\rangle', \tag{27}$$

which says that the traceless part of the $\mathcal{O}(q^2)$ soft limit of the $(N+1)$-point correlation function is given by some differential operator $\hat{\mathcal{D}}$ acting on the N-point correlator (the prime denotes a correlation function without the momentum-conserving delta function). The precise form of $\hat{\mathcal{D}}$ is not important for our present purposes. Now, our theory contains only Galileons with even numbers of fields, and this $\mathbb{Z}_2$ symmetry causes all odd-point amplitudes to vanish. Therefore, if $(N+1)$ is even, the right-hand side of the identity (27) is zero, and Eq. (27) tells us that the traceless part of the $\mathcal{O}(q^2)$ part of the soft limit vanishes. Since the Galileon is massless, its 4-momentum is null, so the trace part vanishes as well. Therefore, in the theory of the quartic Galileon, the soft limit of amplitudes starts at $\mathcal{O}(q^3)$, in agreement with the findings of Ref. [25] in explicit computations. Note that in higher dimensions, the theory (24) we have identified with this extended symmetry will also enjoy this improved soft-limit behavior, as will all the theories related to it by Galileon duality. This is also the same special Galileon theory for which an exact S-matrix was conjectured in Ref. [26]. In $D = 4$ it is the Legendre self-dual model described in Ref. [27].

VII. CONCLUSIONS

We have identified a family of Galileon theories which are invariant under an extended symmetry consisting of a shift quadratic in spacetime coordinates and a shift quadratic in the field. The presence of this symmetry explains the soft behavior of scattering amplitudes in these theories. It is possible that this structure generalizes to higher shifts in both spacetime coordinates and fields, which would lead to theories with even better soft behavior in higher dimensions. We note that a zeroth-order requirement that makes this plausible is that the kinetic term is invariant under an arbitrarily high-order traceless shift symmetry [15].

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APPENDIX: PROOF OF EQ. (15)

We write the Galileon Lagrangians in the form $\mathcal{L}_n = \frac{(n-1)!}{n}\phi\Phi_{\mu_1}{}^{[\mu_1}\cdots\Phi_{\mu_{n-1}}{}^{\mu_{n-1}]}$, with antisymmetrization of weight one, consistent with Eqs. (6) and (4). Using the antisymmetrization of the derivatives, any variation can be written as

$$\delta\mathcal{L}_n \simeq (n-1)!\delta\phi\Phi_{\mu_1}{}^{[\mu_1}\cdots\Phi_{\mu_{n-1}}{}^{\mu_{n-1}]}. \tag{A1}$$

The variation under δ_0 is

$$\delta\mathcal{L}_n \simeq (n-1)!s_{\mu\nu}x^\mu x^\nu\Phi_{\mu_1}{}^{[\mu_1}\cdots\Phi_{\mu_{n-1}}{}^{\mu_{n-1}]}, \tag{A2}$$

which upon integration by parts and using antisymmetry becomes

$$\delta\mathcal{L}_n \simeq 2(n-1)!s^\mu{}_\nu\phi\delta_\mu{}^{[\nu}\Phi_{\mu_2}{}^{\mu_2}\cdots\Phi_{\mu_{n-1}}{}^{\mu_{n-1}]}. \tag{A3}$$

Taking the variation with respect to ϕ, again using antisymmetry,

$$\begin{aligned}\frac{\delta}{\delta\phi}(\delta_0\mathcal{L}_n) &= 2(n-1)(n-1)!s^\mu{}_\nu\delta_\mu{}^{[\nu}\Phi_{\mu_2}{}^{\mu_2}\cdots\Phi_{\mu_{n-1}}{}^{\mu_{n-1}]}\\ &= 2(n-1)s^{\mu\nu}X^{(n-2)}_{\mu\nu},\end{aligned} \tag{A4}$$

giving the first line of Eq. (15).

The variation under δ_2 is

$$\delta\mathcal{L}_n \simeq (n-1)!s^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Phi_{\mu_1}{}^{[\mu_1}\cdots\Phi_{\mu_{n-1}}{}^{\mu_{n-1}]}. \tag{A5}$$

Taking the variation with respect to ϕ, all the contributions with three or four derivatives on any ϕ cancel out, and what remains are the two-derivative contributions,

$$\begin{aligned}\frac{\delta}{\delta\phi}(\delta_2\mathcal{L}_n) &= -2(n-1)!s^{\mu\nu}\partial_\mu\partial_\nu\phi\Phi_{\mu_1}{}^{[\mu_1}\cdots\Phi_{\mu_{n-1}}{}^{\mu_{n-1}]}\\ &= 2(n-1)(n-1)!s^{\mu\nu}\partial_\mu\partial^\lambda\phi\partial_\nu\partial_\sigma\phi\delta_\lambda{}^{[\sigma}\Phi_{\mu_2}{}^{\mu_2}\cdots\Phi_{\mu_{n-1}}{}^{\mu_{n-1}]}\\ &= -2s^{\mu\nu}[\Phi_{\mu\nu}\mathcal{L}^{\mathrm{TD}}_{n-1} - (n-1)\Phi_\mu{}^\lambda\Phi_\nu{}^\sigma X^{(n-2)}_{\lambda\sigma}].\end{aligned} \tag{A6}$$

Now we use the recursion relation (10) twice in order to reduce the second term in the brackets:

$$\begin{aligned}\Phi_\mu{}^\lambda\Phi_\nu{}^\sigma X^{(n-2)}_{\lambda\sigma} &= \frac{1}{n-1}\Phi_\mu{}^\lambda[-X^{(n-1)}_{\nu\lambda} + \mathcal{L}^{\mathrm{TD}}_{n-1}\eta_{\nu\lambda}]\\ &= \frac{1}{n-1}\left[\frac{1}{n}(X^{(n-1)}_{\mu\nu} - \mathcal{L}^{\mathrm{TD}}_{n-1}\eta_{\mu\nu}) + \Phi_{\mu\nu}\mathcal{L}^{\mathrm{TD}}_{n-1}\right].\end{aligned}$$

There is a cancellation between the final term here and the first term in the brackets of Eq. (A6), and we may ignore the term proportional to $\eta_{\mu\nu}$ because of the tracelessness of $s_{\mu\nu}$, leaving the result in the last line of Eq. (15), $\frac{\delta}{\delta\phi}(\delta_2\mathcal{L}_n) = \frac{2}{n}s^{\mu\nu}X^{(n)}_{\mu\nu}$. The proof of the second line of Eq. (16) follows similarly.

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