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Stability analysis for new theories of massive spin-two particle and black hole entropy of new bigravity

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In [Phys. Rev. D 90, 043006 (2014)], we proposed a new ghost-free massive spin-two model in flat spacetime. Furthermore, as some extension, we coupled the new model with a nondynamical curved background in [Phys. Rev. D 90, 123013 (2014)] and constructed new interaction terms without the appearance of an extra mode. The characteristic property of the new model is the existence of nonlinear potential terms which give the nontrivial vacua. The presence of the nontrivial vacua, however, does not mean that the particle can be defined around all vacua. Therefore, in this paper, we discuss the conditions for the new model to have stable vacua in flat spacetime and curved spacetime. Then, we couple this spin-two theory with a dynamical background and obtain the solutions. Moreover, we investigate the effect of this new spin-two model on the Einstein gravity by calculating the black hole entropy, since the gravity coupled with massive spin-two theory admits a black hole solution in addition to the (anti-)de Sitter space solution.

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I. INTRODUCTION

The consistent free massive spin-two theory was first established by Fierz and Pauli [1]. The mass term for spin-two particles generally leads to a ghost mode, but they preserve the consistency of the theory by tuning coefficients of the mass term. Since the Fierz-Pauli theory does not have any gauge symmetry, it seems that arbitrary interactions can be added to the theory. Contrary to this naive expectation, Boulware and Deser [2] showed that nonlinear terms generally lead to another ghost called the Boulware-Deser (BD) ghost. There was another problem—that is, the appearance of the van Dam–Veltman–Zakharov (vDVZ) discontinuity [3] in the massless limit, $m \to 0$, although the discontinuity can be screened by the Vainstein mechanism [4] (see, for example, Ref. [5]).

After these indications, the study of massive spin-two fields had not progressed until 2003. In 2003, Arkani-Hamed, Georgi, and Schwartz [6], however, revealed a cutoff scale of the theory by introducing the Stuckelberg field. They considered a limit which focuses on the cutoff, and they have shown that the special choice of the coefficients in the potential terms makes the cutoff scale larger. As the potential-tuned theory consists of infinite terms, it was unclear whether the theory contains the BD ghost or not. After that, de Rham, Gabadadze, and Tolley [7,8] succeeded in the resummation of the potential terms, and Hassan and Rosen [9] proved that the theory with the resummed potential terms does not contain any ghost. This theory is called dRGT massive gravity. The most important point in this theory is that special forms of the fully nonlinear potential terms eliminate the extra mode. Although the massive gravity models have a nondynamical background metric, they have been extended to the models with a dynamical metric [10–12], which are called bigravity models.

Hinterbichler [13] (see also Ref. [14]) pointed out the possibility of new derivative interaction terms in dRGT massive gravity. It was shown that new derivative interactions can be added to the Fierz-Pauli theory by taking specific linear combinations of interactions and conjectured fully nonlinear counterparts of these interaction terms in dRGT massive gravity. In this context, it was also shown that the leading term of the dRGT potential term does not generate the ghost in the Fierz-Pauli theory. Thus, we constructed a new massive spin-two model in a flat spacetime by adding the leading terms to Fierz-Pauli free theory in Ref. [15]. Furthermore, we extend the theory to the rigid curved background and show that the theory is ghost-free on the Einstein manifold [16].

In this paper, we investigate the stability of the potential extrema of the new model in flat spacetime and curved spacetime. Furthermore, we consider the model where the field of massive spin-two particles couples with gravity by assuming, for simplicity, that the spin-two field is proportional to the background metric. The other kind of solutions has been found [17–19] in the context of the Hassan-Rosen bigravity model [10–12]. A reason why we consider this model is an application to the cosmology and black hole (BH) physics. We often consider the models of scalar fields to explain the expanding Universe not violating the isotropy, while the condensation of the vector field violates the isotropy in general, except for the case in which the model has a non-Abelian gauge symmetry.[1] The field of the massive spin-two particle is given by a rank-2 symmetric tensor. We should note that the condensation of the trace part of the rank-2 symmetric tensor [or (t, t) component, or the trace of the spacial part] does not violate the isotropy, and therefore we can use the rank-2 symmetric tensor in order to explain the expansion of the Universe. Such a cosmology has been studied in the massive gravity models [20] by considering the decoupling limit where the models reduce to scalar-tensor theories. After that, there follow several activities in the massive gravity models [21–24] and in the bimetric gravity models [19,25–31].

As for black hole physics, the effect of massive spin-two particles on the black hole entropy has already been calculated in the Hassan-Rosen bigravity model [32,33]. Since the gravity coupled with the massive spin-two model presented in this paper is essentially different from the Hassan-Rosen bigravity model [10–12], it is quite interesting to see how the results change depending on the model.

II. NEW MODEL OF MASSIVE SPIN-TWO PARTICLES

The Lagrangian of the Fierz-Pauli theory is given by [1]

$$\mathcal{L}_{\rm FP} = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2). \tag{1}$$

The relative sign of the mass term is tuned to eliminate a ghost. Hinterbichler pointed out that new interaction terms can be added to this model without any ghost by taking the specific linear combination [13,14]. In four dimensions, there are two kinds of nonderivative interactions:

$$\mathcal{L}_3 \sim \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3}, \tag{2}$$

$$\mathcal{L}_4 \sim \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4}. \tag{3}$$

Here $\eta^{\mu_1\nu_1\cdots\mu_n\nu_n}$ is given by the product of $n\eta_{\mu\nu}$ and antisymmetrizing the indexes ν_1, ν_2, …, and ν_n, for example,

$$\begin{aligned} \eta^{\mu_1\nu_1\mu_2\nu_2} &\equiv \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} - \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1}, \\ \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} &\equiv \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3} - \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_3\nu_2} \\ &\quad + \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_3}\eta^{\mu_3\nu_1} - \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1}\eta^{\mu_3\nu_3} \\ &\quad + \eta^{\mu_1\nu_3}\eta^{\mu_2\nu_1}\eta^{\mu_3\nu_2} - \eta^{\mu_1\nu_3}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_1}. \end{aligned} \tag{4}$$

In Ref. [15], we proposed the new model of massive spin-two particles by adding the two terms (2) and (3) to the Fierz-Pauli Lagrangian:

$$\begin{aligned} \mathcal{L}_{h0} &= -\frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(\partial_{\mu_1}\partial_{\nu_1}h_{\mu_2\nu_2})h_{\mu_3\nu_3} + \frac{m^2}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} - \frac{\mu}{3!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} \\ &\quad - \frac{\lambda}{4!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4} \\ &= -\frac{1}{2}(h\Box h - h^{\mu\nu}\Box h_{\mu\nu} - h\partial^\mu\partial^\nu h_{\mu\nu} - h_{\mu\nu}\partial^\mu\partial^\nu h + 2h_\nu{}^\rho\partial^\mu\partial^\nu h_{\mu\rho}) + \frac{m^2}{2}(h^2 - h_{\mu\nu}h^{\mu\nu}) \\ &\quad - \frac{\mu}{3!}(h^3 - 3hh_{\mu\nu}h^{\mu\nu} + 2h_\mu{}^\nu h_\nu{}^\rho h_\rho{}^\mu) - \frac{\lambda}{4!}(h^4 - 6h^2h_{\mu\nu}h^{\mu\nu} + 8hh_\mu{}^\nu h_\nu{}^\rho h_\rho{}^\mu - 6h_\mu{}^\nu h_\nu{}^\rho h_\rho{}^\sigma h_\sigma{}^\mu + 3(h_{\mu\nu}h^{\mu\nu})^2). \end{aligned} \tag{5}$$

Here m and μ are parameters with the dimension of mass, and λ is a dimensionless parameter. We assume that μ always takes a positive value but cannot decide the sign of λ, because it is nontrivial to learn which sign for λ stabilizes this system.

Although the model (5) is power-counting renormalizable, the model is not renormalizable, because the propagator behaves as $\mathcal{O}(p^2)$ for large momentum p instead of the naive expectation $\mathcal{O}(p^{-2})$. In fact, the propagator has the following form:

$$D^m_{\alpha\beta,\rho\sigma} = -\frac{1}{2(p^2+m^2)}\left\{P^m_{\alpha\rho}P^m_{\beta\sigma} + P^m_{\alpha\sigma}P^m_{\beta\rho} - \frac{2}{3}P^m_{\alpha\beta}P^m_{\rho\sigma}\right\}, \tag{6}$$

$$P^m_{\mu\nu} \equiv \eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}. \tag{7}$$

Then, when p^2 is large, the propagator behaves as $D^m_{\alpha\beta,\rho\sigma} \sim \mathcal{O}(p^2)$ due to the projection operator $P^m_{\mu\nu}$, which

[1]Non-Abelian gauge always contains $SU(2)$ or $SO(3)$ as a subgroup. The condensation of the vector field breaks both the isotropy or rotational invariance and the gauge symmetry. Because the rotational symmetry is $SO(3)$, even in the vector field condensate, there remains the diagonal symmetry in the product of the rotational symmetry $SO(3)$ and the gauge symmetry $SO(3)$, and we can regard the diagonal symmetry as a new rotational symmetry.

makes the behavior for large p^2 worse, and therefore the model should not be renormalizable.

Since this theory has no symmetry and is already nonrenormalizable, it seems that there is no reason why we only consider the potential terms up to the quartic order. However, introducing higher-order potential terms breaks the consistency as quantum field theory in four dimensions. The potential terms described above do not generate any ghost due to the antisymmetric property. Therefore, in four dimensions, we cannot construct similar ghost-free potential terms. Needless to say, we can add higher-order terms in five or higher dimensions.

III. CLASSICAL SOLUTION IN THE NEW THEORY OF THE MASSIVE SPIN-TWO FIELD

Because the potential of the new theory of the massive spin-two field has a structure like the potential of the Higgs field, it could be interesting to investigate the classical solutions, which may correspond to the extrema of the potential. The nonvanishing value of the potential for the classical solution may give an energy of the vacuum.

By the variations of $h_{\mu\nu}$, we obtain the equations of motion for $h_{\mu\nu}$,

$$\begin{aligned}\frac{\delta S}{\delta h^{\mu\nu}} &= \Box h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + g_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - g_{\mu\nu}\Box h \\ &\quad - m^2(h_{\mu\nu} - g_{\mu\nu}h) - \frac{\mu}{3!}(3g_{\mu\nu}h^2 - 3g_{\mu\nu}h_{\rho\sigma}h^{\rho\sigma} - 6hh_{\mu\nu} + 3h_\nu{}^\rho h_{\rho\mu} + 3h_\mu{}^\rho h_{\rho\nu}) \\ &\quad - \frac{1}{4!}(4g_{\mu\nu}h^3 - 12g_{\mu\nu}hh_{\rho\sigma}h^{\rho\sigma} - 12h^2h_{\mu\nu} + 8g_{\mu\nu}h^\rho{}_\sigma h^\sigma{}_\kappa h^\kappa{}_\rho + 12hh_\nu{}^\rho h_{\rho\mu} + 12hh_\mu{}^\rho h_{\rho\nu} \\ &\quad - 12h_\mu{}^\rho h_\rho{}^\sigma h_{\sigma\nu} - 12h_\nu{}^\rho h_\rho{}^\sigma h_{\sigma\mu} + 12h_{\mu\nu}(h^{\rho\sigma}h_{\rho\sigma})) = 0. \end{aligned} \tag{8}$$

We assume the solution of equation (8) is given by

$$h_{\mu\nu} = C\eta_{\mu\nu}. \tag{9}$$

Here C is a constant. Substituting (9) into equation (8) gives

$$(3m^2C - 3\mu C^2 - \lambda C^3)\eta_{\mu\nu} = 0. \tag{10}$$

The solutions for (10) are given by

$$C = 0, \qquad \frac{-3\mu \pm \sqrt{9\mu^2 + 12m^2\lambda}}{2\lambda}. \tag{11}$$

Because the solution should be a real number, the parameters are constrained to be

$$\begin{cases} \lambda \geq -\frac{3\mu^2}{4m^2} & \text{for } m^2 > 0 \\ \lambda \leq \frac{3\mu^2}{4|m^2|} & \text{for } m^2 < 0 \end{cases}.$$

Note that the parameter m^2 is not required to be positive definite due to the presence of the potential terms. By assuming (9), the Lagrangian (5) is reduced to

$$\mathcal{L}_{h0} = V(C) \equiv -6m^2C^2 + 4\mu C^3 + \lambda C^4. \tag{12}$$

We may regard $V(C)$ as a potential for C. Then Eq. (10) is nothing but the condition $V'(C) = 0$. We should note that when $\mu = \lambda = 0$, which corresponds to the Fierz-Pauli model, the potential $V(C)$ is not unbounded below, and $C = 0$ corresponds to the local maximum instead of the local minimum. As we know, however, the massive spin-two field is stable on the local maximum. On the other hand, on the local minimum of C, the fluctuation of the massive spin-two field becomes tachyonic and unstable.

Such a contradiction to intuition occurs because C does not correspond to the propagating mode and C should be a constant. In fact, if we assume (9) and that C could not be a constant, (8) tells us

$$0 = \eta^{\mu\nu}(2\Box C + 3m^2C - 3\mu C^2 - \lambda C^3) - 2\partial^\mu\partial^\nu C. \tag{13}$$

Then, when $\mu \neq \nu$ in (13), it gives

$$\partial_\mu\partial_\nu C = 0, \tag{14}$$

which tells us that C is given by a sum of the functions of each of coordinates $C = \sum_\mu C^{(\mu)}(x^\mu)$. Eq. (13) also gives

$$\eta^{\mu\mu}\partial_\mu^2 C = \eta^{\nu\nu}\partial_\nu^2 C. \tag{15}$$

In Eq. (15), the indices μ on the lhs and ν on the rhs are not summed up. Eq. (15) tells us that C takes the following form: $C = \sum_{\mu,\nu}\frac{c}{2}\eta_{\mu\nu}x^\mu x^\nu + \sum_\mu c_\mu x^\mu + C_0$. Here c, c_μ's, and C_0 are constants. By substituting this expression into (13), we find $c = 0$ and $c_\mu = 0$, which means that C should be surely a constant. This tells us that even if C is on the local maximum of the potential (12), C does not roll down.

If $V(C)$ does not vanish, the potential $V(C)$ could be the vacuum energy and might play the role of the cosmological constant when we couple the model with gravity. Then it could be interesting to investigate the signature of the potential and the (in)stability of the classical solution corresponding to the extrema of the potential.

We now assume the parameter μ is positive, because the signature of μ can always be absorbed into the redefinition of $h_{\mu\nu}$. As the signs of λ and m^2 are undetermined while μ takes a positive value, we consider the following cases:

(a) $\lambda > 0$ and $m^2 > 0$:

Besides the trivial solution $C = 0$, there are nontrivial solutions for C, which are given by

$$C_1 = \frac{-3\mu + \sqrt{9\mu^2 + 12m^2\lambda}}{2\lambda} > 0,$$
$$C_2 = \frac{-3\mu - \sqrt{9\mu^2 + 12m^2\lambda}}{2\lambda} < 0. \tag{16}$$

We now consider which solution corresponds to the positive- (negative-) energy solution under the assumption $\mu > 0$, $\lambda > 0$, and $m^2 > 0$. For this purpose, we have to solve the inequalities

$$V(C) = -6m^2C^2 + 4\mu C^3 + \lambda C^4 > 0,$$
$$V(C) = -6m^2C^2 + 4\mu C^3 + \lambda C^4 < 0. \tag{17}$$

The solutions are given by

$$C < C_- \quad \text{or} \quad C_+ < C \quad \text{for positive energy}, \tag{18}$$

$$C_- < C < C_+ \quad \text{for negative energy}. \tag{19}$$

Here C_+ and C_- are defined by

$$C_+ = \frac{-2\mu + \sqrt{4\mu^2 + 6\lambda m^2}}{\lambda} > 0,$$
$$C_- = \frac{-2\mu - \sqrt{4\mu^2 + 6\lambda m^2}}{\lambda} < 0. \tag{20}$$

In order for $C_\pm$ to be real numbers, λ should be larger than $-2\mu^2/3m^2$, but we assume the positivity of λ and m^2 here. C_1 and C_+ are both positive, and C_2 and C_- are both negative. Thus, what we should do is to compare C_1 to C_+ and C_2 to C_-:

(1) C_+ and C_1.

We now consider the following quantity:

$$C_+ - C_1 = \frac{\mu}{2\lambda}\left(-1 + 4\sqrt{1 + \frac{3m^2\lambda}{2\mu^2}} - 3\sqrt{1 + \frac{4m^2\lambda}{3\mu^2}}\right)$$
$$> \frac{\mu}{2\lambda}\left(-1 + \sqrt{1 + \frac{4m^2\lambda}{3\mu^2}}\right) > 0. \tag{21}$$

Thus, we obtain the relation $C_+ > C_1$. Since C_+ is positive while C_- is negative, we find $C_- < C_1 < C_+$, which means the solution C_1 corresponds to the negative energy.

(2) C_2 and C_-.

Similarly, we investigate the difference between C_- and C_2:

$$C_- - C_2 = \frac{\mu}{2\lambda}\left(-1 - 4\sqrt{1 + \frac{3m^2\lambda}{2\mu^2}} + 3\sqrt{1 + \frac{4m^2\lambda}{3\mu^2}}\right)$$
$$< \frac{\mu}{2\lambda}\left(-1 - \sqrt{1 + \frac{3m^2\lambda}{2\mu^2}}\right) < 0. \tag{22}$$

This means $C_- < C_2$. Since C_2 has a negative value, $C_- < C_2 < C_+$ holds.

Therefore, we see that both of solutions satisfy the condition for the negative-energy solution.

(b) $-3\mu^2/4m^2 < \lambda < 0$ and $m^2 > 0$:

We continue the similar analysis. However, $C_{1,2}$ and $C_\pm$ are given as follows in this case:

$$C_1 = \frac{3\mu - \sqrt{9\mu^2 - 12m^2|\lambda|}}{2|\lambda|} > 0,$$
$$C_2 = \frac{3\mu + \sqrt{9\mu^2 - 12m^2|\lambda|}}{2|\lambda|} > 0,$$
$$C_- = \frac{2\mu - \sqrt{4\mu^2 - 6m^2|\lambda|}}{|\lambda|} > 0,$$
$$C_+ = \frac{2\mu + \sqrt{4\mu^2 - 6m^2|\lambda|}}{|\lambda|} > 0,$$
$$C_2 > C_1, \quad C_- < C_+. \tag{23}$$

Note that C_- and C_+ in (23) correspond to C_+ and C_- in (20), respectively. Since λ is the coefficient of C^4, the solutions for the inequality also change:

$$C_- < C < C_+ \quad \text{for positive energy}, \tag{24}$$

$$C < C_- \quad \text{or} \quad C_+ < C \quad \text{for negative energy}. \tag{25}$$

The condition for $C_\pm$ to be real is given by $-2\mu^2/3m^2 < \lambda < 0$. As we assume the reality of C in this analysis, $C_\pm$ does not exist for the case $-3\mu^2/4m^2 < \lambda < -2\mu^2/3m^2$. Therefore, we divide the parameter region $-3m^2/4\mu^2 < \lambda < 0$ into $-3\mu^2/4m^2 < \lambda < -2\mu^2/3m^2$ and $-2\mu^2/3m^2 < \lambda < 0$.

(**b1**) $-2\mu^2/3m^2 < \lambda < 0$ and $m^2 > 0$.

Let us compare $C_{1,2}$ with $C_\pm$:

(1) C_- and C_1.

As in the previous case, we consider the quantity

$$C_- - C_1 = \frac{\mu}{2|\lambda|}\left(1 - 4\sqrt{1 - \frac{3m^2|\lambda|}{2\mu^2}} + 3\sqrt{1 - \frac{4m^2|\lambda|}{3\mu^2}}\right)$$
$$> \frac{\mu}{2|\lambda|}\left(1 - \sqrt{1 - \frac{3m^2|\lambda|}{2\mu^2}}\right) > 0. \quad (26)$$

Thus, we find $C_1 < C_- < C_+$, and C_1 turns out to be the negative-energy solution.

(2) C_- and C_2.

$$C_- - C_2 = \frac{\mu}{2|\lambda|}\left(1 - 4\sqrt{1 - \frac{3m^2|\lambda|}{2\mu^2}} - 3\sqrt{1 - \frac{4m^2|\lambda|}{3\mu^2}}\right)$$
$$< \frac{\mu}{2|\lambda|}\left(1 - 4\sqrt{1 - \frac{3m^2|\lambda|}{2\mu^2}} - 3\cdot\frac{1}{3}\right) < 0. \quad (27)$$

In the second line, we use the fact that $1/3 < \sqrt{1 - 4m^2|\lambda|/3\mu^2} < 1$. From (27), we obtain the result $C_- < C_2$.

(3) C_+ and C_2.

$$C_+ - C_2 = \frac{\mu}{2|\lambda|}\left(1 + 4\sqrt{1 - \frac{3m^2|\lambda|}{2\mu^2}} - 3\sqrt{1 - \frac{4m^2|\lambda|}{3\mu^2}}\right)$$
$$> \frac{\mu}{2|\lambda|}\left\{1 + 3\left(\sqrt{1 - \frac{3m^2|\lambda|}{2\mu^2}} - \sqrt{1 - \frac{4m^2|\lambda|}{3\mu^2}}\right)\right\}. \quad (28)$$

The quantity $\sqrt{1 - 3m^2|\lambda|/2\mu^2} - \sqrt{1 - 4m^2|\lambda|/3\mu^2}$ is always negative if we assume case (**b1**). Thus, (28) is rewritten as follows:

$$C_+ - C_2 > \frac{\mu}{2|\lambda|}\left\{1 - 3\left|\sqrt{1 - \frac{3m^2|\lambda|}{2\mu^2}} - \sqrt{1 - \frac{4m^2|\lambda|}{3\mu^2}}\right|\right\}. \quad (29)$$

The maximum value of the second term is given by

$$\left|\sqrt{1 - \frac{3m^2|\lambda|}{2\mu^2}} - \sqrt{1 - \frac{4m^2|\lambda|}{3\mu^2}}\right| < \frac{1}{3} \quad (30)$$

in the assumed parameter region. Therefore, C_+ is larger than C_2.

According to these analyses, C_2 and C_1 correspond to the positive-energy solution and the negative-energy solution, respectively.

(**b2**) $-3\mu^2/4m^2 < \lambda < -2\mu^2/3m^2$.

As mentioned above, $C_\pm$ is no longer real in this case. Thus, $V(C)$ takes negative values only, which means that C_1 and C_2 produce the negative-energy solution.

(c) $\lambda < 0$ and $m^2 < 0$:

In this parameter region, we have

$$C_1 = \frac{3\mu - \sqrt{9\mu^2 + 12|m^2||\lambda|}}{2|\lambda|} < 0, \qquad C_2 = \frac{3\mu + \sqrt{9\mu^2 + 12|m^2||\lambda|}}{2|\lambda|} > 0,$$
$$C_- = \frac{2\mu - \sqrt{4\mu^2 + 6|m^2||\lambda|}}{|\lambda|} < 0, \qquad C_+ = \frac{2\mu + \sqrt{4\mu^2 + 6|m^2||\lambda|}}{|\lambda|} > 0. \quad (31)$$

The conditions for the negative- and positive-energy solutions are given by

$$\begin{aligned} &C_- < C < C_+ \quad \text{for positive energy,} \\ &C < C_- \quad \text{or} \quad C_+ < C \quad \text{for negative energy.} \end{aligned} \tag{32}$$

We repeat the analysis similarly to that presented above. Thus, we will only give the results in the following sections:

(1) C_+ and C_2. By taking the difference between C_+ and C_2, we find

$$\begin{aligned} C_+ - C_2 &= \frac{\mu}{2|\lambda|}\left[1 + 4\sqrt{1 + \frac{3|m^2||\lambda|}{2\mu^2}} - 3\sqrt{1 + \frac{4|m^2||\lambda|}{3\mu^2}}\right] \\ &> \frac{\mu}{2|\lambda|}\left[1 + \sqrt{1 + \frac{4|m^2||\lambda|}{3\mu^2}}\right] > 0. \end{aligned} \tag{33}$$

This means that C_2 is a positive-energy solution, because C_2 takes a positive value.

(2) C_- and C_1.

By taking the difference between C_- and C_1, we find

$$\begin{aligned} C_- - C_1 &= \frac{\mu}{2|\lambda|}\left[1 - 4\sqrt{1 + \frac{3|m^2||\lambda|}{2\mu^2}} + 3\sqrt{1 + \frac{4|m^2||\lambda|}{3\mu^2}}\right] \\ &< \frac{\mu}{2|\lambda|}\left[1 - \sqrt{1 + \frac{3|m^2||\lambda|}{2\mu^2}}\right] < 0. \end{aligned} \tag{34}$$

Therefore, C_1 is also a positive-energy solution, as $C_1 < 0$ is obviously smaller than $C_+ > 0$.

(d) $0 < \lambda < 3\mu^2/4|m^2|$ and $m^2 < 0$:

C_1, C_2, and $C_\pm$ are given by

$$\begin{aligned} C_1 &= \frac{-3\mu + \sqrt{9\mu^2 - 12|m^2|\lambda}}{2\lambda} < 0, \qquad C_2 = \frac{-3\mu - \sqrt{9\mu^2 - 12|m^2|\lambda}}{2\lambda} < 0, \\ C_- &= \frac{-2\mu - \sqrt{4\mu^2 - 6|m^2|\lambda}}{\lambda} < 0, \qquad C_+ = \frac{-2\mu + \sqrt{4\mu^2 - 6|m^2|\lambda}}{\lambda} < 0. \end{aligned} \tag{35}$$

The energy conditions are

$$\begin{aligned} &C_- < C < C_+ \quad \text{for negative energy,} \\ &C < C_- \quad \text{or} \quad C_+ < C \quad \text{for positive energy.} \end{aligned} \tag{36}$$

Since $C_\pm$ are not real in the case of $2\mu^2/3|m^2| < \lambda < 3\mu^2/4|m^2|$, we divide the parameter region as in the previous case:

(**d1**) $0 < \lambda < 2\mu^2/3|m^2|$.

(1) C_+ and C_1.

$$\begin{aligned} C_+ - C_1 &= \frac{\mu}{2\lambda}\left[-1 + 4\sqrt{1 - \frac{3|m^2|\lambda}{2\mu^2}} - 3\sqrt{1 - \frac{4|m^2|\lambda}{3\mu^2}}\right] \\ &< \frac{\mu}{2|\lambda|}\left[-1 + \sqrt{1 - \frac{3|m^2||\lambda|}{2\mu^2}}\right] < 0. \end{aligned} \tag{37}$$

This means that C_1 is a positive-energy solution.

(2) C_+ and C_2.

$$C_+ - C_2 = \frac{\mu}{2\lambda}\left[-1 + 4\sqrt{1 - \frac{3|m^2|\lambda}{2\mu^2}} + 3\sqrt{1 - \frac{4|m^2|\lambda}{3\mu^2}}\right]$$
$$> \frac{\mu}{2\lambda}\left[-1 + 4\sqrt{1 - \frac{3|m^2|\lambda}{2\mu^2}} + 3\cdot\frac{1}{3}\right] > 0. \tag{38}$$

This is because $1/3 < \sqrt{1 - 4|m^2|\lambda/3\mu^2} < 1$.

(3) C_- and C_2.

$$C_- - C_2 = \frac{\mu}{2\lambda}\left[-1 - 4\sqrt{1 - \frac{3|m^2|\lambda}{2\mu^2}} + 3\sqrt{1 - \frac{4|m^2|\lambda}{3\mu^2}}\right]$$
$$< \frac{\mu}{2\lambda}\left[-1 - 3\left(\sqrt{1 - \frac{3|m^2|\lambda}{2\mu^2}} - \sqrt{1 - \frac{4|m^2|\lambda}{3\mu^2}}\right)\right]. \tag{39}$$

As in the case of (30), the maximum value of the second term is given by

$$\left|\sqrt{1 - \frac{3|m^2|\lambda}{2\mu^2}} - \sqrt{1 - \frac{4|m^2|\lambda}{3\mu^2}}\right| < \frac{1}{3}. \tag{40}$$

Hence, we find $C_- < C_2$. (2) and (3) mean C_2 is a negative-energy solution.

(**d2**) $2\mu^2/3|m^2| < \lambda < 3\mu^2/4|m^2|$.

$C_\pm$ are not real in this parameter region. Thus, both solutions $C_{1,2}$ correspond to the positive energy.

These results are summarized in Tables I and II. The former and the latter correspond to the cases of $m^2 > 0$ and $m^2 < 0$, respectively.

As we mentioned, the Fierz-Pauli theory is stable on the local maximum. Therefore, it is plausible to assume that the theory is stable on the local maximum even though the parameters μ and λ take nonvanishing values. Under this assumption, we check the stability of the solution $C_{1,2}$. For this purpose, we have to obtain the expression of the second derivative of the potential:

$$V''(C) = -12m^2 + 24\mu C + 12\lambda C^4. \tag{41}$$

We find the stability by substituting the solutions into (41) for each parameter region.

(a) $\lambda > 0$ and $m^2 > 0$:

In this case, both solutions correspond to the negative-energy solutions. Plugging in these solutions yields

(1) $C = C_1$.

$$V''(C = C_1) = \frac{3}{\lambda}\left(6\mu^2 - 2\mu\sqrt{9\mu^2 + 12m^2\lambda}\right) + 24m^2 > 0. \tag{42}$$

TABLE I. Relation between $C_{1,2}$, the vacuum energy, and the stability of the solutions when $m^2 > 0$.

	Parameters		
Energy	$\lambda > 0$	$-2\mu^2/3m^2 < \lambda < 0$	$-3\mu^2/4m^2 < \lambda < -2\mu^2/3m^2$
Positive energy	no solution	C_2 (stable)	no solution
Negative energy	C_1 (unstable) and C_2 (unstable)	C_1 (unstable)	C_1 (unstable) and C_2 (stable)

TABLE II. Relation between $C_{1,2}$, the vacuum energy, and the stability of the solutions when $m^2 < 0$.

	Parameters		
Energy	$\lambda < 0$	$0 < \lambda < 2\mu^2/3\vert m^2\vert$	$2\mu^2/3\vert m^2\vert < \lambda < 3\mu^2/4\vert m^2\vert$
Positive energy	C_1 (stable) and C_2 (stable)	C_1 (stable)	C_1 (stable) and C_2 (unstable)
Negative energy	no solution	C_2 (unstable)	no solution

(2) $C = C_2$.

$$V''(C = C_2) = \frac{3}{\lambda}\left(6\mu^2 + 2\mu\sqrt{9\mu^2 + 12m^2\lambda}\right) + 24m^2 > 0. \tag{43}$$

Equation (41) is positive in both cases. Therefore, these solutions are unstable.

(**b1**) $-2\mu^2/3m^2 < \lambda < 0$ and $m^2 > 0$:

C_1 and C_2 are linked with the positive-energy and negative-energy solutions, respectively. As in the above case, we find (41) for each solution.

(1) $C = C_1$.

$$V''(C = C_1) = \frac{3}{|\lambda|}\left(-6\mu^2 + 2\mu\sqrt{9\mu^2 - 12m^2|\lambda|}\right) + 24m^2 > 0. \tag{44}$$

(2) $C = C_2$.

$$V''(C = C_2) = \frac{3}{|\lambda|}\left(-6\mu^2 - 2\mu\sqrt{9\mu^2 - 12m^2|\lambda|}\right) + 24m^2 < 0. \tag{45}$$

This result means that C_1, corresponding to the negative-energy solution, is unstable; while C_2, corresponding to the positive-energy solution, is stable.

(**b2**) $-3\mu^2/4m^2 < \lambda < -2\mu^2/3m^2$ and $m^2 > 0$:

The negative-energy solution is realized for both solutions C_1 and C_2.

(1) $C = C_1$.

$$V''(C = C_1) = \frac{3}{|\lambda|}\left(-6\mu^2 + 2\mu\sqrt{9\mu^2 - 12m^2|\lambda|}\right) + 24m^2 > 0. \tag{46}$$

(2) $C = C_2$.

$$V''(C = C_2) = \frac{3}{|\lambda|}\left(-6\mu^2 - 2\mu\sqrt{9\mu^2 - 12m^2|\lambda|}\right) + 24m^2 < 0. \tag{47}$$

Although both solutions lead to the negative-energy solution, C_1 is unstable and the other solution is stable.

(c) $\lambda < 0$ and $m^2 < 0$:

Both solutions correspond to the positive energy.

(1) $C = C_1$.

$$V''(C = C_1) = \frac{3}{|\lambda|}\left(-6\mu^2 + 2\mu\sqrt{9\mu^2 + 12|m^2||\lambda|}\right) - 24|m^2| < 0. \tag{48}$$

(2) $C = C_2$.

$$V''(C = C_2) = \frac{3}{|\lambda|}\left(-6\mu^2 - 2\mu\sqrt{9\mu^2 + 12|m^2||\lambda|}\right) - 24|m^2| < 0. \tag{49}$$

Both the positive-energy solutions are stable.

(**d1**) $0 < \lambda < 2\mu^2/3|m^2|$ and $m^2 < 0$:

C_1 is a positive-energy solution, and C_2 is a negative-energy solution.

(1) $C = C_1$.

$$V''(C = C_1) = \frac{3}{|\lambda|}\left(6\mu^2 - 2\mu\sqrt{9\mu^2 - 12|m^2|\lambda}\right) - 24|m^2| < 0. \tag{50}$$

(2) $C = C_2$.

$$V''(C = C_2) = \frac{3}{|\lambda|}\left(6\mu^2 + 2\mu\sqrt{9\mu^2 - 12|m^2|\lambda}\right) - 24|m^2| > 0. \tag{51}$$

The positive-energy solution is stable, while the negative-energy solution is unstable.

(**d2**) $2\mu^2/3|m^2| < \lambda < 3\mu^2/4|m^2|$ and $m^2 < 0$:

In this parameter region, both solutions correspond to the positive energy. The stability analysis is the same as the previous case, because the expressions of C_1 and C_2 do not change from case (**d1**). Thus, we find that C_1 is stable and C_2 is unstable.

The above discussion tells us that in the solutions $C_{1,2}$ for both cases (a) and (c), the values of the potential have the same signature, but the stability is different. Both of the solutions are unstable in the case (a), while there exist stable solutions in the case (c). The cases (**b1**) and (**d1**) have one stable positive-energy solution and one unstable negative-energy solution. These results are summarized in Table I.

We also comment on the global structure of the potential and the global stability for the massive spin-two field. The special feature in the model of massive spin-two particles is that the vacuum where the potential is convex upward is stable, but the vacuum where the potential is convex downward is unstable. In the case where both C_1 and C_2 correspond to the stable vacua, however, the system also has the "trivial" vacuum $C = 0$, which realizes the lowest energy in the system, although the massive spin-two particle becomes tachyonic around the vacuum. We may think that the system could be ultimately rendered unstable by the quantum tunneling from the stable "false" vacua to the unstable "true" vacuum. In the case of scalar field theory, this speculation could be true. In the case of the massive spin-two field, however, it is not clear if the system is unstable or not, because the potential does not correspond to the propagating modes, which is not the scalar mode but the massive spin-two mode. If we consider the tunneling for the massive spin-two mode by, say, the WKB approximation, we need to consider inhomogeneous and anisotropic intermediate states, which makes the situation very complex. Therefore, at least at present, we do not know how we should discuss the global stability, and we only concentrate on the arguments about the local stability.

IV. NEW MODEL OF THE MASSIVE SPIN-TWO PARTICLE IN A CURVED SPACETIME

The naive extension to the theory in a curved spacetime is given by the minimal coupling model:

$$S = \int d^4x\sqrt{-g}\Big\{-\frac{1}{2}\nabla_\mu h_{\nu\rho}\nabla^\mu h^{\nu\rho} + \nabla_\mu h_{\nu\rho}\nabla^\rho h^{\nu\mu} - \nabla^\mu h_{\mu\nu}\nabla^\nu h + \frac{1}{2}\nabla_\mu h\nabla^\mu h + \frac{m^2}{2}g^{\alpha\beta\gamma\delta}h_{\alpha\beta}h_{\gamma\delta} - \frac{\mu}{3!}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} - \frac{\lambda}{4!}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}\Big\}. \tag{52}$$

Unfortunately, this minimal coupling model is not ghost free even in case of the free theory according to Refs. [34,35]. Therefore, we constructed a new ghost-free massive spin-two model coupled with gravity by adding nonminimal coupling terms [16]. Instead of (52), the action of the ghost-free model in arbitrary dimensions D is given by

$$S = \int d^Dx\sqrt{-g}\Big\{-\frac{1}{2}\nabla_\mu h_{\nu\rho}\nabla^\mu h^{\nu\rho} + \nabla_\mu h_{\nu\rho}\nabla^\rho h^{\nu\mu} - \nabla^\mu h_{\mu\nu}\nabla^\nu h + \frac{1}{2}\nabla_\mu h\nabla^\mu h + \frac{m^2}{2}g^{\alpha\beta\gamma\delta}h_{\alpha\beta}h_{\gamma\delta} + \frac{\xi}{D}Rh_{\mu\nu}h^{\mu\nu} + \frac{1-2\xi}{2D}Rh^2 - \frac{\mu}{3!}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} - \frac{\lambda}{4!}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}\Big\}. \tag{53}$$

In addition to two nonminimal coupling terms, we also found the following nonderivative interaction terms in four dimensions [16].

$$\begin{aligned}
&C^{\mu_1\mu_2\nu_1\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2},\\
&\delta^{\mu_1}{}_{\rho_1}{}^{\mu_2}{}_{\rho_2}{}^{\mu_3}{}_{\rho_3}\delta^{\nu_1}{}_{\sigma_1}{}^{\nu_2}{}_{\sigma_2}{}^{\nu_3}{}_{\sigma_3}C^{\rho_1\rho_2\sigma_1\sigma_2}g^{\rho_3\sigma_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3},\\
&\delta^{\mu_1}{}_{\rho_1}{}^{\mu_2}{}_{\rho_2}{}^{\mu_3}{}_{\rho_3}{}^{\mu_4}{}_{\rho_4}\delta^{\nu_1}{}_{\sigma_1}{}^{\nu_2}{}_{\sigma_2}{}^{\nu_3}{}_{\sigma_3}{}^{\nu_4}{}_{\sigma_4}C^{\rho_1\rho_2\sigma_1\sigma_2}g^{\rho_3\sigma_3\rho_4\sigma_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}.
\end{aligned} \tag{54}$$

Here $C_{\mu\nu\rho\sigma}$ is the Weyl tensor defined by

$$C_{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma}) + \frac{1}{6}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{55}$$

Note that the interaction terms containing the scalar curvature like $R^n g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}$ can be added because R is constant on the Einstein manifold, but we ignore such a redundant term here.

V. CLASSICAL SOLUTIONS AND STABILITY

In the previous section, we revealed the parameter region which allows the system to have stable solutions. Although the result is also important, the analysis is not enough because of the appearance of the nonminimal coupling term.

In this analysis, we assume the four-dimensional (anti-) de Sitter spacetime as a background metric where the nonminimal coupling terms containing the Weyl tensor (54) vanish. Therefore, we consider the action (53) in four dimensions:

$$S = \int d^4x\sqrt{-g}\left\{-\frac{1}{2}\nabla_\mu h_{\nu\rho}\nabla^\mu h^{\nu\rho} + \nabla_\mu h_{\nu\rho}\nabla^\rho h^{\nu\mu} - \nabla^\mu h_{\mu\nu}\nabla^\nu h + \frac{1}{2}\nabla_\mu h\nabla^\mu h + \frac{m^2}{2}g^{\alpha\beta\gamma\delta}h_{\alpha\beta}h_{\gamma\delta}\right.$$
$$\left.+\frac{\xi}{4}Rh_{\mu\nu}h^{\mu\nu} + \frac{1-2\xi}{8}Rh^2 - \frac{\mu}{3!}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} - \frac{\lambda}{4!}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}\right\}. \tag{56}$$

As in the case of the flat spacetime, vacuum solutions have to be invariant under the isometry of the spacetime:

$$h_{\mu\nu} = Cg_{\mu\nu}, \qquad \mathcal{L}_\xi g_{\mu\nu} = 0. \tag{57}$$

Here C is a constant and ξ is the Killing vector for the (anti-)de Sitter spacetime. Substituting the ansatz (57) into the equations of motion gives

$$-2\{6m^2 + (2-3\xi)R\}C + 12\mu C^2 + 4\lambda C^3 = 0. \tag{58}$$

This is the equation determining the extrema of the potential for the system. The solutions are given as follows:

$$C = 0, \qquad C = \frac{-6\mu \pm \sqrt{36\mu^2 + 48m^2\lambda + 8\lambda(2-3\xi)R}}{4\lambda}. \tag{59}$$

The condition for the existence of the nontrivial solutions is

$$9\mu^2 + 12m^2\lambda + 2\lambda(2-3\xi)R > 0. \tag{60}$$

In order to investigate the stability around the vacuum solution, we consider the fluctuation

$$h_{\mu\nu} = Cg_{\mu\nu} + f_{\mu\nu} \tag{61}$$

and rewrite the action in terms of $f_{\mu\nu}$:

$$S = \int d^4x\sqrt{-g}\left\{-\frac{1}{2}\nabla_\mu f_{\nu\rho}\nabla^\mu f^{\nu\rho} + \nabla_\mu f_{\nu\rho}\nabla^\rho f^{\nu\mu} - \nabla^\mu f_{\mu\nu}\nabla^\nu f + \frac{1}{2}\nabla_\mu f\nabla^\mu f - V(C)\right\}. \tag{62}$$

Here $V(C)$ takes the following form:

$$\begin{aligned}
V(C) &= \sum_{n=0}^{4} V_n(C),\\
V_0(C) &= -\{6m^2 + (2-3\xi)R\}C^2 + 4\mu C^3 + \lambda C^4,\\
V_1(C) &= -3m^2Cf + 3\mu C^2 f + \lambda C^3 f - \left(1 - \frac{3}{2}\xi\right)RCf = \frac{V'_0(C)}{4}f,\\
V_2(C) &= \left(-\frac{m^2}{2} + \mu C + \frac{\lambda}{2}C^2\right)g^{\alpha\beta\gamma\delta}f_{\alpha\beta}f_{\gamma\delta} - \frac{\xi}{4}Rf_{\alpha\beta}f^{\alpha\beta} - \frac{1-2\xi}{8}Rf^2.\\
&\ \ \vdots
\end{aligned} \tag{63}$$

Here $V_n(C)$ expresses the term including n—the power of $f_{\mu\nu}$. We should note that $V_2(C)$ is proportional to the Fierz-Pauli mass term in (1) due to the following identity:

$$g^{\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}\mu_n}{}_{\mu_n} = (D-n+1)g^{\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}. \tag{64}$$

Here D denotes the dimensions of the spacetime. We should also note that $V_3(C)$ and $V_4(C)$ are also given by the pseudolinear terms in (2) and (3).

We now define an effective mass M of $f_{\mu\nu}$ by $M^2 \equiv m^2 - 2\mu C - \lambda C^2$. As the vacuum solutions satisfy the equation $V'_0(C) = 0$, the linear term in $f_{\mu\nu}$ vanishes:

$$S = \int d^4x\sqrt{-g}\left\{-\frac{1}{2}\nabla_\mu f_{\nu\rho}\nabla^\mu f^{\nu\rho} + \nabla_\mu f_{\nu\rho}\nabla^\rho f^{\nu\mu} - \nabla^\mu f_{\mu\nu}\nabla^\nu f + \frac{1}{2}\nabla_\mu f\nabla^\mu f\right.$$
$$\left.+\frac{M^2}{2}g^{\alpha\beta\gamma\delta}f_{\alpha\beta}f_{\gamma\delta} + \frac{\xi}{4}Rf_{\alpha\beta}f^{\alpha\beta} + \frac{1-2\xi}{8}Rf^2 - V_0(C) + \mathcal{O}(f^3, f^4)\right\}. \tag{65}$$

Because the purpose is to investigate the stability around the vacua, we need to keep the terms including the second power of $f_{\mu\nu}$. In a curved spacetime, the stability of the free massive spin-two field is determined by the Higuchi bound [36]. In the case where $\lambda = \mu = 0$ and $\xi = 1$, it is well known that by assuming $M^2 = m^2 > 0$, if $R > 6m^2$, the vacuum is unstable and if $R \leq 6m^2$, it is stable [36,37]. On the boundary $R = 6m^2$, the theory is invariant under the gauge transformation

$$\delta f_{\mu\nu} = \nabla_\mu \nabla_\nu \Gamma + \frac{1}{2} M^2 \Gamma,$$

where Γ is a gauge parameter. For this reason, the theory satisfying the condition $R = 6m^2$ with $\lambda = \mu = 0$ and $\xi = 1$ is called partially massless. The stability has not been investigated when $\xi \neq 1$, but as we see below, the deviation is not very important when the curvature of the spacetime is constant. Let us see the quadratic term in the potential of (56):

$$-\frac{m^2}{2}(h^{\mu\nu}h_{\mu\nu} - h^2) + \frac{\xi}{4} R h_{\alpha\beta} h^{\alpha\beta} + \frac{1 - 2\xi}{8} R h^2. \tag{66}$$

To address the deviation from the $\xi = 1$ case, we express the ξ parameter in terms of δ:

$$\xi = 1 + \delta. \tag{67}$$

Then, we rewrite (66) as follows:

$$-\frac{m^2}{2}(h^{\mu\nu}h_{\mu\nu} - h^2) + \frac{1}{4} R h^{\mu\nu} h_{\mu\nu} - \frac{1}{8} R h^2 + \frac{\delta}{4} R (h^{\mu\nu}h_{\mu\nu} - h^2). \tag{68}$$

This means that the deviation from $\xi = 1$ is equivalent to the shift in the mass parameter since R is constant, and we can set $\xi = 1$ without loss of generality. (We should note that this is just a mathematical equivalence. Because the mass parameter is strongly related with the stability of the system, the deviation from $\xi = 1$ is physically important.) Hence, we impose the following conditions for the stability:

$$M^2 \geq 0, \qquad R \leq 6M^2. \tag{69}$$

Therefore, the stable, nontrivial solutions have to satisfy both of the conditions (60) and (69). For example, let us consider the simple case where $\mu = 0$. The nontrivial solution is simplified as follows:

$$C = \pm \sqrt{\frac{6m^2 - R}{2\lambda}}. \tag{70}$$

The condition for the existence of (70) is $\lambda < 0$ and $R > 6m^2$, or $\lambda > 0$ and $R < 6m^2$. On the other hand, the effective mass M around the vacuum is given by

$$M = m^2 - \lambda C^2 = -2m^2 + \frac{R}{2}. \tag{71}$$

From (69) and (71), we have the stability condition around the vacuum expectation value (VEV) as follows:

$$R > 4m^2, \qquad R > 6m^2. \tag{72}$$

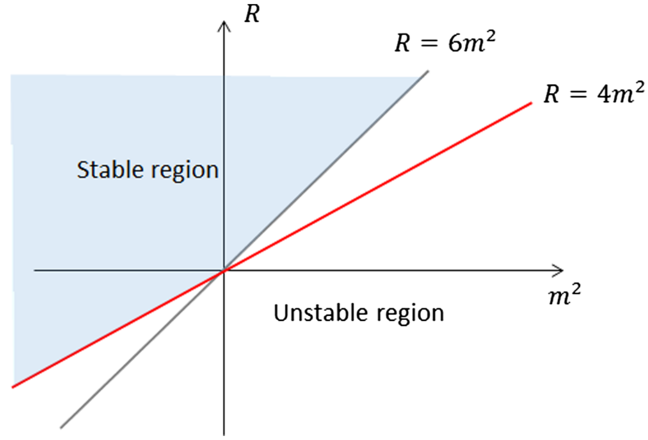

FIG. 1 (color online). The parameter region for the constant curvature spacetime.

The stable region is shown in Fig. 1. Hence, the solution satisfying the stability condition exists when $\lambda < 0$ and $R > 6m^2$.

VI. NEW BIGRAVITY

The bigravity model can be regarded as a model where a massive spin-two field couples with gravity. Then we may consider the model where $h_{\mu\nu}$ couples with gravity, which can be regarded as a new bigravity model because there appear two symmetric tensor fields $g_{\mu\nu}$ and $h_{\mu\nu}$, as follows:

$$S = \int d^4x \sqrt{-g} \left\{ -\frac{1}{2} \nabla_\mu h_{\nu\rho} \nabla^\mu h^{\nu\rho} + \nabla_\mu h_{\nu\rho} \nabla^\rho h^{\nu\mu} - \nabla^\mu h_{\mu\nu} \nabla^\nu h + \frac{1}{2} \nabla_\mu h \nabla^\mu h + \frac{m^2}{2} g^{\alpha\beta\gamma\delta} h_{\alpha\beta} h_{\gamma\delta} + \frac{\xi}{4} R h_{\mu\nu} h^{\mu\nu} + \frac{1 - 2\xi}{8} R h^2 - \frac{\mu}{3!} g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} - \frac{\lambda}{4!} g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4} \right\} + S_{\mathrm{EH}}. \tag{73}$$

Here $h_{\mu\nu}$ is not the perturbation in $g_{\mu\nu}$, but $h_{\mu\nu}$ is a field independent of $g_{\mu\nu}$, and $S_{\rm EH}$ is given by the Einstein-Hilbert action without the cosmological constant. We have to note that the action is constructed up to the dimension-4 operators, and it is not obvious whether or not this system has a ghost, since the matter part of the Lagrangian is constructed for the field $h_{\mu\nu}$ to be ghost free only on the Einstein manifold. We show, however, that there are classical solutions which realize the spacetime which has constant curvature.

We also stress that the ξ parameter is not redundant here, unlike the rigid curved background. In Sec. V, we see that the deviation from $\xi = 1$ means the appearance of the Fierz-Pauli tuned term (68) proportional to the constant curvature R. Thus, we concluded such a term is redundant and can be ignored without loss of generality. On the other hand, in (73), as R is not constant but a dynamical variable, we cannot regard the ξ as a redundant parameter.

Finally, we note that the mass parameter m, the cubic coupling μ, and the quartic coupling λ take an arbitrary real value.

VII. COSMOLOGICAL AND BLACK HOLE SOLUTIONS

In this section, we obtain the cosmological solution and the black hole solutions. We now assume, for simplicity, that the solution has the following form as in Sec. V:

$$h_{\mu\nu} = C g_{\mu\nu}. \tag{74}$$

Here C is a constant. The equations of motion for $h_{\mu\nu}$ are given by

$$-2\{6m^2 + (2-3\xi)R\}C + 12\mu C^2 + 4\lambda C^3 = 0. \tag{75}$$

Furthermore, we obtain the following action by substituting (74) into (73):

$$\begin{aligned} S &= -\int d^4x\sqrt{-g}V_0(C) + \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R \\ &= \left\{(2-3\xi)C^2 + \frac{1}{2\kappa^2}\right\}\int d^4x\sqrt{-g}[R - 2\Lambda_{\rm eff}]. \end{aligned} \tag{76}$$

Here we assume $(2-3\xi)C^2 + \frac{1}{2\kappa^2} \neq 0$; otherwise the gravity completely decouples. The effective cosmological constant $\Lambda_{\rm eff}$ is defined by

$$\Lambda_{\rm eff} \equiv \frac{\kappa^2(-6m^2C^2 + 4\mu C^3 + \lambda C^4)}{2\kappa^2C^2(2-3\xi) + 1}. \tag{77}$$

Thus, the Einstein equation given by the variation of the action with respect to $g_{\mu\nu}$ has the following form:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_{\rm eff}g_{\mu\nu} = 0. \tag{78}$$

Since (78) admits the Einstein manifolds as solutions, we can consider the fluctuation around the solution of (75) without any ghost as long as the fluctuation is small enough. By multiplying (78) with $g^{\mu\nu}$, we obtain

$$R = 4\Lambda_{\rm eff}. \tag{79}$$

By substituting (79) into the expression of (75), we find

$$4C\{-2\mu\zeta C^3 + (\lambda + 6\zeta m^2)C^2 + 3\mu C - 3m^2\} = 0. \tag{80}$$

Here $\zeta \equiv \kappa^2(2-3\xi)$. Then the solutions, except for the trivial solution $C = 0$, should satisfy the following condition:

$$-2\mu\zeta C^3 + (\lambda + 6\zeta m^2)C^2 + 3\mu C - 3m^2 = 0. \tag{81}$$

Dividing (81) by $-2\mu\zeta$ and changing the variable C by $C = x + \frac{\lambda+6\zeta m^2}{6\mu\zeta}$, we can rewrite (81) as follows:

$$\begin{aligned} &x^3 + px + q = 0, \qquad p = -\frac{1}{3}\left\{\left(\frac{\lambda + 6\zeta m^2}{2\mu\zeta}\right)^2 + \frac{9}{2\zeta}\right\}, \\ &q = \frac{2}{27}\left(\frac{\lambda + 6\zeta m^2}{2\mu\zeta}\right)^3 - \frac{\lambda}{4\mu\zeta^2}. \end{aligned} \tag{82}$$

Then, by putting $\omega \equiv e^{i2\pi/3}$, the solutions of (82) are expressed as

$$\begin{aligned} x = {}&\omega^k\sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \\ &+ \omega^{3-k}\sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}, \qquad k = 1, 2, 3. \end{aligned} \tag{83}$$

Now the determinant is given by

$$D = -27q^2 - 4p^3 = -2^2\cdot 3^3\left\{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3\right\}. \tag{84}$$

Except for the case that $q = p = 0$, there are the following three cases:

(1) $D > 0$. There are three different real solutions.
(2) $D < 0$. There is only one real solution.
(3) $D = 0$. There are three real solutions, but two of them are degenerate with each other.

Let us consider a little bit the simple case $\mu = 0$ in the following. The equation of motion (81) is reduced to be

$$(\lambda + 6\zeta m^2)C^2 - 3m^2 = 0. \tag{85}$$

Then the solutions are given by

$$C_1 = \sqrt{\frac{3m^2}{\lambda + 6\zeta m^2}}, \qquad C_2 = -\sqrt{\frac{3m^2}{\lambda + 6\zeta m^2}}. \tag{86}$$

The solutions become real and nontrivial ($C \neq 0$) when

$$\begin{cases} m^2 > 0, & \lambda + 6\zeta m^2 > 0 \\ m^2 < 0, & \lambda + 6\zeta m^2 < 0 \end{cases}. \tag{87}$$

Furthermore, by substituting $C_{1,2}$ into the effective cosmological constant (77), we obtain

$$\Lambda_{\rm eff}(C_{1,2}) = -\kappa^2 (C_{1,2})^2 m^2. \tag{88}$$

If the conditions in Eq. (87) are satisfied, $C_{1,2}$ are real numbers. Therefore, we find

$$\begin{cases} m^2 > 0, & \lambda + 6\zeta m^2 > 0 \quad \text{anti–de Sitter} \\ m^2 < 0, & \lambda + 6\zeta m^2 < 0 \quad \text{de Sitter} \end{cases}. \tag{89}$$

As we have obtained the solutions of this new bigravity theory, we have to investigate the stability of the obtained solutions as in the case of the rigid background. For this purpose, we need to consider both of the fluctuations in $h_{\mu\nu}$ and $g_{\mu\nu}$ simultaneously, but this leads to very complicated equations. Thus, we will analyze the stability in future work.

VIII. BLACK HOLE ENTROPY

Since the action (73) admits the Schwarzschild–anti-de Sitter black hole solution under the assumption $h_{\mu\nu} = C g_{\mu\nu}$, we can calculate the black hole entropy. Let us use the Wald formula to calculate the entropy for the system, which is also applicable for the spacetime having the asymptotically anti–de Sitter spacetime.

(We could also have the Schwarzschild–de Sitter solution, but there are several subtleties due to the presence of the two event horizons.) Note that the Wald formula is applicable for the asymptotically anti–de Sitter spacetime because the mass can be defined based on the asymptotic Killing vector.

The Wald formula is given by

$$S = -2\pi \oint_{\mathcal{H}} dA \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma}. \tag{90}$$

Here $\epsilon^{\mu\nu}$ is a binormal tensor, and $\mathcal{H}$ denotes the horizon of the black hole. The term contributing to the functional derivative is

$$\mathcal{L}_{\rm rel} = \frac{1}{2\kappa^2} R + \frac{\xi}{4} R h_{\mu\nu} h^{\mu\nu} + \frac{1 - 2\xi}{8} R h^2. \tag{91}$$

Therefore, we obtain

$$\frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} = \left(\frac{\xi}{4} h_{\mu\nu} h^{\mu\nu} + \frac{1 - 2\xi}{8} h^2 + \frac{1}{2\kappa^2} \right) \cdot \frac{1}{2} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\mu\rho}). \tag{92}$$

The substitution of the classical solution $C_{1,2}$ and the Schwarzschild–anti-de Sitter or Kerr–anti-de Sitter metric yields

$$S = \frac{A}{4G} + \frac{12\pi A (2 - 3\xi) m^2}{\lambda + 48\pi G (2 - 3\xi) m^2}. \tag{93}$$

The last term corresponds to the contribution from the condensation of the massive spin-two particle. The area of the event horizon for the Schwarzschild-type metric is given by

$$A = \frac{16\pi}{|\Lambda_{\rm eff}|} \sinh^2 \left[\frac{1}{3} \sinh^{-1} (3M \sqrt{|\Lambda_{\rm eff}|}) \right]. \tag{94}$$

Here M denotes the black hole mass.

These results can be compared with those [32,33] in the Hassan-Rosen bigravity model [10–12], where the entropy is given by the sum of the contributions from two metric sectors.

IX. SUMMARY

In this paper, we have investigated the classical solutions for the theories of massive spin-two particles in flat spacetime and curved spacetime, which were proposed in Refs. [15,16] by coupling the model with gravity.

In conflict with intuition, the massive spin-two particle becomes tachyonic on the local minimum of the potential, and the particle is stable on the local maximum—that is, the local minimum induces the instability, although the local maximum corresponds to the stability. Based on this analysis, we classified the stable or unstable parameter region for the massive spin-two particle with potential terms in a flat spacetime. Although the model is very similar to a scalar field theory with quartic and quadratic potential terms, it is remarkable that the relation between the stability and the vacuum energy is opposite to the model of the scalar field having similar potential.

We extend the stability analysis to the case of the rigid background. In this case, the vacuum solutions are invariant under the transformation induced by the Killing vector for (anti-)de Sitter spacetime. Since the stability condition called the Higuchi bound for the free massive spin-two particle is given by Higuchi [36], we apply the analysis to our model.

Finally, we consider the case where the background metric is dynamical due to the presence of the

Einstein-Hilbert term. Then we obtained solutions describing the (anti-)de Sitter spacetime. The obtained de Sitter spacetime might correspond to the inflation in the early Universe or the accelerating expansion in the present Universe. These solutions correspond to the extrema of the potential for the trace of the symmetric tensor field. As mentioned in the text, we do not carry out the stability analysis for this gravity coupled system. This could be a future work.

In addition to the solutions describing the (anti-)de Sitter spacetime, we find the solutions describing the black hole, which could be the (anti-)de Sitter–Schwarzschild or the (anti-)de Sitter–Kerr spacetime. By calculating the black hole entropy, furthermore, we find that the entropy contains the explicit contribution from the condensation of the massive spin-two particle. In the case of the Hassan-Rosen bigravity model [10–12], the entropy is given by the sum of the contributions from two metric sectors. On the other hand, the black hole entropy for the model in this paper is not unique because of arbitrary parameters appearing in the entropy.

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